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A REVIEW OF INFINITE ELEMENT METHODS

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This work is devoted to review infinite element discretizations for the Helmholtz equation in exterior domains which have become popular in recent years and many research papers on this topic have appeared in the literature. The early contributions have been mostly motivated by engineering considerations and the variational formulations are in general not stated in a mathematically precise way. Only recently theoretical aspects of the infinite element methodology have been analyzed and helped to put the different formulations into a mathematical framework. We build upon this and present and compare the infinite element formulations within this context.

1. Introduction

Exterior domain problems appear naturally in many engineering applications and are important in the study of scattering problems. Of particular interest is the scattering of waves on bodies that are submerged in a fluid which extends infinitely. The main difficulty in solving exterior scattering problems arises then from the unboundedness of the domain that has to be discretized. It is evident that an exterior domain cannot be completely discretized with standard finite elements based on polynomial shape functions. A number of suggestions for the treatment of exterior domains have been presented and analyzed in many research papers in the last two decades and in this work we review the contributions that deal with Infinite Element Methods (IEM).

A popular method for the solution of exterior problems is the application of a Boundary Element Method[25] (BEM) in which the differential operator and the boundary conditions in the exterior domain are replaced by an equivalent boundary integral equation on the surface of the scatterer. Such a BEM can deliver robust results with convergence properties being practically insensitive to the wave number[23,24]. The main drawback of the BEM is that due to the non local boundary operator the computational cost and the memory demand in a given implementation can be prohibitive and may indeed prevent the computation of a reliable numerical solution.

An alternative approach is to truncate the computational domain at some distance away from the scatterer and to impose a boundary condition at this artificial boundary. In general, a so called absorbing boundary condition is used in such a strategy and the computational efficiency depends then on the locality of the boundary condition and the distance at which it is applied. For details on such techniques we refer to the books by Givoli[37] and Ihlenburg[41] and the references therein. Many other methodologies have appeared and can often be related to non local absorbing boundary conditions, as for example the scaled boundary finite-element method[57].

About two decades ago was the IEM introduced in the original work of Bettess[8,58] and has since then been refined[5,15,32,54] and applied to a variety of problems. Early applications included work of Pissanetzky[51,52] on magnetostatics[50] and Kim[44] addressed magnetic field computations. The original infinite element of Bettess was derived for a Laplace problem and is very similar to a finite element except for the element extending towards infinity in one direction and the corresponding shape functions being non polynomial but integrable over the infinite element. Evidently, infinite elements are local, similar to finite elements, and can also be used to discretize the whole exterior domain, i.e. an absorbing boundary condition is not needed. Alternatively, one can also view an infinite element as a local absorbing boundary condition[43].

The accuracy of an infinite element relies on the choice of the shape functions towards infinity and on the order of approximation[6,26,32,54]. In the acoustical exterior scattering problem that we are mainly interested in, we also need to take care in deriving a theoretically sound variational formulation[32,54]. We present a derivation of the variational formulations that have been most commonly used, but derivations of other variational principles are also possible[39].

The mapped infinite element developed by Bettess and Zienkiewicz[59,60] allowed to use polynomial shape functions on a reference element which are then mapped by a singular mapping onto the physical infinite element. This technique generates shape functions which are consistent with the form of the solution of the exterior scattering problem. The so generated shape functions can also be applied and integrated directly over the infinite element. This has been done in a number of references[15,31,33,34,54].

The original wave envelope element by Astley[1,2,3,4] and coworkers makes use of an artifical boundary at a larger distance from the scatterer and the resulting boundary integral is evaluated numerically. This procedure assures that the complete exterior domain is taken into account.

The issue of formulating a mathematically precise variational statement has only recently been addressed[32,54] and most of the existing research work can be related to a precise variational formulation and this will be explored further subsequently.

In the remaining sections we give an overview of the research that has been performed on the IEM and address in particular the choice of shape functions and the variational formulation. We also make an attempt to use a unified notation throughout this work instead of keeping the different notations of the past contributions to the field. We note that most of the earlier work on infinite elements has already been summarized and we refer to these references[12,13,14,53] for additional information. Additionally, various aspects relating numerical techniques and benchmarks for exterior domain problems are presented in the recently edited book by Geers[30], and Astley[6] provides a recent overview of accuracy aspects of IE techniques.

The content of this paper is outlined as follows. In section 2 we present the rigid scattering problem and a derivation of the various IE formulations. Section 3 is devoted to an overview of earlier research work on the IE methodology and more recent contributions are discussed in section 4. The generalization of IE to scatterers of general shape are presented in section 5 and we finish the presentation with conclusions in section 6.

2. Infinite Element Formulations

2.1. *The exterior Helmholtz problem*

The standard model problem in acoustics that is addressed by many authors[8,26,32,42] is the scattering of waves on a rigid obstacle Ω. The domain exterior to the obstacle is the exterior domain $\Omega^e = \mathbb{R}^3 \setminus \Omega$ in which the Helmholtz equation in conjunction with boundary conditions governs the form of the scattered waves. The classical problem statement is then to find a function $u = u(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega^e$, which satisfies:

- the Helmholtz equation in the exterior domain

$$-\Delta u - k^2 u = 0 \qquad \text{in} \quad \Omega^e, \tag{2.1}$$

  where k is the wave number,

- a Neumann boundary condition on the scatterer

$$\nabla_n u = g \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega, \tag{2.2}$$

- and the Sommerfeld radiation condition at infinity

$$\left| \frac{\partial u}{\partial n} - iku \right| = O\left(\frac{1}{r^2}\right). \tag{2.3}$$

The infinite element methodology can in particular be applied if the obstacle Ω has a very general shape[15,16] but for the sake of clarity of the presentation we assume Ω to be a sphere with radius $r = 1$.

2.2. *Variational formulations*

A variational statement for the exterior Helmholtz equation has to be carefully derived by introducing a truncated exterior domain $\Omega^e_\gamma = \Omega^e \cap \left\{ \boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < \gamma \right\}$. A limiting process with $\gamma \to \infty$ is used later on to extend Ω^e_γ to Ω^e. Next, we multiply the Helmholtz equation by a test function, integrate over Ω^e_γ, and integrate by parts. This leads then under observation of (2.2) and (2.3) to

$$\int_{\Omega^e_\gamma} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega^e_\gamma - k^2 \int_{\Omega^e_\gamma} u\, v \, d\Omega^e_\gamma - ik \int_{S_\gamma} uv \, dS_\gamma = \int_{\partial\Omega} g\, v \, dS + \int_{S_\gamma} \varphi v \, dS_\gamma, \tag{2.4}$$

where S_γ is the surface of the truncating sphere with radius γ and $\varphi = O(r^{-2})$ is an unknown function.

Remark 1 *An equation similar to (2.4) can also be derived by using the complex conjugate of a test function v.*

It is well known that the leading term in the solution u of (2.1) is of the form $r^{-1}\exp(ikr)$ and that therefore both u and its gradient ∇u are not L^2-integrable over the exterior domain. We see readily from this observations that the limit $\gamma \to \infty$ is not well defined for all terms in (2.4). Similar observation have been made and studied[5,15,21,22,34,36,54] and two remedies exist to make the limiting process well defined. The first remedy is to apply a r^{-2} weight to the test functions and the second one is to apply the Cauchy Principal Value instead of a weighting. In the case of weighted test functions we note immediately that the term in (2.4) which stems from the Sommerfeld radiation condition will disappear if $\gamma \to \infty$. This prevents the retention of the Sommerfeld radiation condition in the weak form and therefore it must be included directly in the trial space and this was originally proposed by Leis[45], with

$$H_{1,w}(\Omega^e) = \{u : \|u\|_{1,w} < \infty\} \tag{2.5}$$

and the norm $\|u\|_{1,w}$ corresponding to the inner product

$$(u,v)_{1,w} = \int_{\Omega^e} w\, u\, \overline{v} + w\, \nabla u \cdot \nabla \overline{v}\, d\Omega^e + \int_{\Omega^e} \left(\frac{\partial u}{\partial r} - iku\right) \overline{\left(\frac{\partial v}{\partial r} - ikv\right)}\, d\Omega^e. \tag{2.6}$$

In either case, the weight $w = r^{-2}$ is applied to the trial function space and the different weighting of the test functions is achieved by setting the "dual" weight $w^* = r^2$ and requiring a test function being a member of $H_{1,w*}$. This weighting approach had also been derived by Astley[5] from engineering considerations. Therefore, the resulting variational formulation is also called after Astley and Leis, i.e. the unconjugated Astley-Leis formulation reads as

$$\left\{ \begin{array}{l} \text{Find } u \in H_{1,w}(\Omega^e) \quad \text{such that} \\ \displaystyle\int_{\Omega^e} \nabla u \cdot \nabla v\, d\Omega^e - k^2 \int_{\Omega^e} u\, v\, d\Omega^e = \int_{\partial\Omega} g\, v\, dS \quad \forall\, v \in H_{1,w^*}(\Omega^e), \end{array} \right. \tag{2.7}$$

The conjugated Astley-Leis formulation is derived similarly, except for the complex conjugate over the test function v.

The alternative procedure involving the Cauchy Principal Value has been proposed by Burnett[15] and is based on postulating that trial and test functions are of the form

$$u(r,\theta,\phi) = \frac{\exp(ikr)}{r} u_0(\theta,\phi) + U(r,\theta,\phi), \tag{2.8}$$

where function $u_0(\theta,\phi)$ is frequently known as the radiation pattern, and U and its gradient are square integrable. Upon substituting (2.8) into (2.4) we obtain a different variational formulation. The unconjugated Burnett formulation is then written as

$$\left\{ \begin{array}{l} \text{Find } u \in H_{1,w}(\Omega^e) \text{ such that } \forall\, v \in H_{1,w}(\Omega^e) \\ \displaystyle\lim_{\gamma\to\infty} \left(\int_{\Omega^e_\gamma} \nabla u \cdot \nabla v \; - k^2\, u\, v\, d\Omega^e_\gamma - ik \int_{S_\gamma} uv\; dS_\gamma \right) = \int_{\partial\Omega} g\, v\, dS, \end{array} \right. \tag{2.9}$$

The conjugated Burnett formulation is obtained similarly if the complex conjugate is applied to test function v. We emphasize once more that the integral on the left hand side of (2.9) is understood in the Cauchy Principal Value sense, i.e. the integrands are first added up and integrated before the limit is applied. It is surprising that in this case all not well defined integrals disappear in (2.9), for details we refer to Gerdes[34]. However, we note that in the Astley-Leis formulation there are no formal differences between the conjugated and unconjugated versions but in the Burnett formulations there are different terms left over. In particular, the surface integral on the left hand side in (2.9) will disappear in the unconjugated Burnett formulation but contributes to the conjugated Burnett formulation.

In summary, we have derived four mathematically precise variational statements for the exterior Helmholtz problem, which differ in the test function space and in the complex conjugate being present or not.

2.3. *The infinite element*

The separation of variables procedure is the starting point for the definition of infinite elements. In the case of the Helmholtz equation it is possible to separate variables in spherical or spheroidal coordinates[48]. Therefore, an infinite element can be constructed by a tensor product of a one dimensional infinite element in radial direction and a finite element in angular direction.

In the case of spherical coordinates can the solution u to the exterior Helmholtz equation be represented in the form

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} h_n(kr)P_n^m(\cos\theta)\left(A_{nm}\cos(m\phi) + B_{nm}\sin(m\phi)\right), \tag{2.10}$$

where h_n are spherical Hankel functions of the first kind, P_n^m are the Legendre functions, and A_{nm}, B_{nm} are coefficients that can be determined if a Neumann boundary condition is known on the circumscribing sphere[32]. We emphasize that a similar but slightly simplified representation can be derived for the Laplace equation[32], i.e. for the case $k = 0$.

The radial expansion (2.10) converges outside the smallest sphere that circumscribes the scatterer to the solution[56] and similar results in general spheroidal coordinates were recently established by Holford[40]. We note that u in (2.10) depends radialy on the spherical Hankel functions[48] which are defined as

$$h_n(kr) = \sum_{m=0}^{n} \frac{\exp(ikr)}{r^{m+1}} \frac{\exp\left(-i\frac{\pi}{2}(n+1)\right)}{k(2k)^m} i^m \left(n+\frac{1}{2}, m\right) \tag{2.11}$$

with

$$\left(n+\frac{1}{2}, m\right) = \begin{cases} 1 & m = 0 \\ \prod_{k=1}^{m}(n+k)\cdot\prod_{k=1}^{m}\dfrac{(n-m+k)}{k} & m \geq 1. \end{cases}$$

From (2.11) it is evident that the solution behaves radialy like $r^{-j}\exp(ikr)$, $j \geq 1$, and this motivates the definition of the radial shape functions in the infinite element.

The infinite reference element can now be defined as a tensor product of a standard finite reference element[27] $\hat{K}$ and of a one dimensional element extending towards infinity. An infinite element shape function N_m is therefore given as

$$N_m(r, \boldsymbol{\xi}) = \psi_j(r) \cdot \varphi_i(\boldsymbol{\xi}), \tag{2.12}$$

where φ_i are standard polynomial finite element shape functions defined on $\hat{K}$ and ψ_j are radial shape functions defined by

$$\psi_j(r) = \frac{\exp(ikr)}{r^{j+l}}, \qquad j \geq 1. \tag{2.13}$$

We emphasize that $l = 2$ in (2.13) for test functions in the Astley-Leis formulation and that otherwise always $l = 0$ applies. The use of the complex conjugate over test functions simply modifies the radial shape functions to

$$\psi_j(r) = \frac{\exp(-ikr)}{r^{j+l}}, \qquad j \geq 1. \tag{2.14}$$

Remark 2 *The radial integrals in the conjugated formulations simplify because the exponential functions cancel out. This is not the case in the unconjugated formulations but the radial integrals can then still easily be computed and involve the evaluation of the so called exponential integral*[38]*.*

It is clear that the so defined IE can easily be used to extend standard finite element implementations, because it just involves an update of the element library and corresponding integration routines.

3. Early Infinite Element Developments

We review in the following the development of infinite elements in the late 70ies and in the 80ies and relate the different contributions in the field to the framework of infinite element formulations that we presented in Section 2. We emphasize that the large number of research papers that have appeared make it impossible to address every contribution here. Instead, we review a number of papers that contributed with new directions and refer to the review papers by Bettess[10,11], Resende[53], and the book on infinite elements by Bettess[12] for additional information and references on the development of infinite elements.

The work of Bettess[8] introduced infinite elements to the literature. This original contribution addressed the solution of the Laplace equation, i.e. (2.1) with $k = 0$. The infinite element is derived for a one dimensional model problem and radial shape functions are defined by

$$\psi_j(r) = \exp\left(\frac{r_j - r}{L}\right) \cdot l_j(r), \qquad 1 \leq j \leq N. \tag{3.15}$$

Here l_j is the standard Lagrange interpolation polynomial at the nodes r_j which are located at a finite radial distance, and L is a positive parameter that can be arbitrarily chosen. Evidently, it is possible to use (3.15) in a tensor product Ansatz similar to (2.12). The exponential decay term in (3.15) assures that products of shape functions and derivatives are integrable in radial direction and that numerical results can be obtained. The accuracy of these results depends on the choice of L and for general application it is necessary to callibrate L on a model problem.

The initial infinite element for the Laplace equation was immediately extended to the Helmholtz equation by Bettess and Zienkiewicz[9]. This modification involved simply the inclusion of a complex valued factor in (3.15) to define the radial shape functions as

$$\psi_j(r) = \exp(ikr) \cdot \exp\left(\frac{r_j - r}{L}\right) \cdot l_j(r) \ , \qquad 1 \leq j \leq N. \tag{3.16}$$

Otherwise, this infinite element is identical to the one in the previous contribution[8]. The variational formulation that is used does not involve complex conjugated shape functions and also a weighting of the test functions is not considered. Indeed, such a weighting is not necessary to make radial integrals well defined due to the exponential decay term in (3.16). We note that the radial shape functions in (3.15) and (3.16) are not consistent with the radial behavior of the solution (2.10) which is derived from separation of variables. Therefore, the radial shape functions in (3.15) and (3.16) lead to an infinite element which can only give some idea of the solution and the solution cannot be accurately represented in the exterior domain. Precisely this is confirmed by numerical results[32].

The work of Chow and Smith[18] uses radial shape functions similar to (3.16). The difference results from replacing the Lagrange interpolation polynomials l_j in radial direction with the so called Serendipity family of polynomials. Otherwise, this infinite element formulation is similar to Bettess' and the same conclusions apply.

Lynn and Hadid[46] defined an infinite element for the Laplace equation, i.e. (2.1) with $k = 0$, which incorporates the correct decay rates r^{-j}, $j \geq 1$. Their radial shape functions are basically defined similar to (2.12) with

$$\psi_j(r) = r^{-j} \ , \qquad 1 \leq j \leq N. \tag{3.17}$$

They actually used a linear combination of the decay functions in their definition and applied in their numerical results only the cases with $N \leq 2$. Obviously, (3.17) is defined consistently with the decay rates that are present in (2.10) and therefore such radial shape functions allow to correctly represent the solution in the exterior domain. We note, however, that the here studied case $k = 0$ is from a mathematical point of view a much easier problem regarding the Sobolev space setting, which is not addressed in the paper. Therefore, the extension to $k > 0$ is not directly available from this work.

Medina[47] noted that the asymptotic behavior of the solution has the leading order term r^{-1} and includes this in his infinite element definition, which is similar to the previous infinite element formulations. In particular, it is noted that the infinite element does not represent the true solution behavior in the far field, which is evident from the presence of multiple decay rates in (2.10).

The work of Beer and Meek[7] applies a singular mapping to define the infinite element on a standard finite reference element. The advantage of introducing a mapping is that the element computations can be done on the standard reference elements.

Zienkiewicz, Emson and Bettess[59] improved and further analyzed in their work the mapped infinite element. They introduced a mapping of the form

$$x(\xi) = \frac{-\xi}{1-\xi} x_0 + \left(1 + \frac{\xi}{1-\xi}\right) x_2 \tag{3.18}$$

which maps the interval $[-1,1]$ to $[x_1, x_3]$ where $x_3 = \infty$. The point x_2 corresponds to the midpoint in the reference interval and x_0 is given such that x_1 lies midway between x_0 and x_2. It is further shown that polynomials defined on the reference element are transformed by this mapping to functions with reciprocal powers of r, i.e. a polynomial $p(\xi)$ can be written as

$$p(\xi) = \sum_{i=0}^{N} \alpha_i \xi^i = \sum_{i=0}^{N} \beta_i r^{-i} \tag{3.19}$$

with $r = x - x_0$. Obviously, standard polynomial shape functions of order N represent the N leading order decay rates in (2.10). Therefore, this mapped infinite element leads to a formulation which can consistently represent the solution in the exterior domain. The mapping (3.18) is used to solve the exterior Laplace problem, but the same mapping also applies to the exterior Helmholtz problem, although the mathematical derivation of the variational formulation involves additional technicalities.

Astley and Eversman[2,3,4] solve the exterior Helmholtz problem and analyze what they call wave envelope element. This is basically an infinite element and the derivation is similar to (2.4) and (2.9). The radial shape function used is of the form $r^{-1}\exp(ikr)$, which is the leading order term in (2.13) for $l = 0$. The novelty in this work is the use of a complex conjugated trial function as test function, i.e. the radial test function is simply defined as $r^{-1}\exp(-ikr)$. This is justified by the fact that the radial integrals simplify significantly because the exponential functions cancel out, compare Remark 2. Although the variational formulation is not stated and derived in a strict mathematical setting, it can be related to the derivation of (2.9). We also note, that only the correct leading order decay rate of the solution is captured by this infinite element formulation, but that higher order radial shape functions can easily be included in the formulation.

4. Recent Infinite Element Developments

The early ideas on infinite elements have been refined and extended in this decade and we give an overview of the contributions that did lead to the understanding of infinite elements. These recent developments fall into the framework that we derived in Section 2 and can therefore also be compared to each other. An accuracy comparison of the different infinite element methods is currently being developed by Astley[6] and also provides additional information on recent contributions to the infinite element methodology.

Astley, Macaulay and Coyette[5] extend the original wave envelope formulation[2,3,4] and use a mapping technique[59] to define the element. The refined formulation is derived for the one, two, and three dimensional exterior Helmholtz problem. Higher order radial shape functions are included in the formulation and the radial trial functions are defined in the three dimensional case similar to (2.13) by

$$\psi_j(r) = \frac{\exp(ikr)}{r^j}, \qquad j \geq 1. \tag{4.20}$$

The radial test functions are defined similar to (2.14) with $l = 0$ as complex conjugates of the trial functions. Evidently, this definition of radial shape functions is consistent with the form of the exact solution (2.10) and can therefore lead to accurate solutions. However, it is noted that difficulties arise with the integration in radial direction. This is due to the fact that the product of the leading order shape functions is not integrable in radial direction. From engineering considerations this problem is resolved by applying an additional weight factor r^{-2}, which is similar to using $l = 2$ in (2.14). The so obtained variational formulation is identical to the Astley-Leis formulation in (2.7) although it is not derived from the concept of weighted Sobolev spaces. The numerical results show the superior performance of the infinite element methodology with respect to CPU time and accuracy. For practical purposes, it is noted that the presence of additional higher order terms besides the asymptotically leading order term determines the capacity of the infinite element to model discrepancies between the actual behavior of the computed solution and its asymptotic far field form. This means that if more higher order radial shape functions are included in the method, then one can obtain more accurate results in the near field, which eliminates the need for a larger near field finite element mesh.

In the work of Cremers, Fyfe and Coyette[21], and Cremers and Fyfe[22] is the formulation applied which has been introduced by Astley[5] and which is based on the Astley-Leis formulation (2.7). Therefore, the results are of similar quality. However, we emphasize that they[22] note that the solution of (2.1) can be written by separation of variables outside a sphere as

$$u(r,\theta,\phi) = \frac{\exp(ikr)}{r} \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^n} \tag{4.21}$$

where the coefficients f_n can be determined from the radiation pattern, compare (2.8). This derivation and the form of the solution are identical to the derivation of (2.10).

Burnett[15] derived a new variational formulation which differs from the Astley-Leis formulation (2.7). In the original work the derivation is performed in prolate spheroidal coordinates, which are more appropriate if a scatterer with more than one length scale is considered. Burnett noted that the form of the solution to the exterior Helmholtz problem (2.1) can be written as

$$u(r,\theta,\phi) = \frac{\exp(ikr)}{r} \sum_{n=0}^{\infty} \frac{g_n(\theta,\phi)}{r^n} \tag{4.22}$$

for spherical and for prolate spheroidal coordinates with appropriate definition of the coordinates and the coefficients g_n. This observation motivated the definition of radial shape functions

$$\psi_j(r) = \frac{\exp(ikr)}{r^j}, \qquad j \geq 1 \tag{4.23}$$

with r being the spherical or prolate spheroidal radius, respectively, and this is identical to (2.13) with $l = 0$. The radial test functions are similarly chosen, i.e. $l = 0$. The variational formulation in the original work is then derived similar to (2.9) but without the use of weighted Sobolev spaces. We emphasize that up to now there is no mathematical theory which assures the existence and uniqueness of the solution for this formulation although the numerical results are more than promising. The original Burnett formulation was derived with unconjugated test functions but a conjugated Burnett formulation can also be derived easily. The spheroidal coordinates generate some complication in computing the Cauchy Principle Value (2.9) in the Burnett formulation but a similar computation[34] in spherical coordinates reveals clearly which terms cancel out. The numerical results for the infinite element in prolate spheroidal coordinates are very efficient because the prolate spheroid circumscribing the scatterer allows to minimize the volume between scatterer and truncating surface. Therefore, reliable results with relative error below one percent can be achieved with infinite elements at a fraction of the cost of boundary elements.

The introduction of the Burnett formulation influenced other research activities. The disseration of Shirron[54] analyzed the Burnett IE formulation (2.9) and provided a comparison of the Burnett IE in conjugated and unconjugated form with absorbing boundary conditions. The results clearly indicate the advantages of infinite elements and also report on a surprinsingly supperior performance of the unconjugated Burnett formulation in the near field with only few radial shape functions. In acoustical applications the far field solution is also of relevance, although it can be computed from a near field solution trace via Green's representation. Shirron's far field results for the Burnett IE are opposite to the near field results because the conjugated version provides a stable approximation throughout the whole exterior domain, whereas the unconjugated version fails in representing the far field solution. This far field failure in the unconjugated Burnett formulation is attributed to the inherent ill conditioning in the unconjugated versions and brings up the question why the unconjugated Burnett formulation performs supperior in the near field. This is analyzed in an abstract setting for a canonical problem which results from a Fourier decomposition of the original problem.

$$\text{Find } u \in H \text{ such that } \quad b(u,v) = f(v) \quad \forall\, v \in H. \tag{4.24}$$

We refer to Shirron[54] for the precise definition of b and only note that in this case $f(v) = a\, v(a)$, with a a point on the boundary. By the particular definition is b complex valued but we have symmetry in the real sense, i.e. $b(u,v) = b(v,u)$ as opposed to symmetry in the complex sense $b(u,v) = \overline{b(v,u)}$. This real symmetry appears only in the unconjugated Burnett formulation and is used in the following estimates. Let now $H_N \subset H$ be a finite subspace and $u_N \in H_N$ then we have the usual Galerkin orthogonality and for $w_N \in H_N$ we get

$$\begin{aligned} a(u(a) - u_N(a)) &= b(u,u) - b(u_N,u_N) \\ &= b(u,u) - b(u_N,u) + b(u,u_N) - b(u_N,u_N) \\ &= b(u-u_N,u) + b(u-u_N,u_N) \\ &= b(u-u_N,u-w_N) \\ &\leq C\|u-u_N\|\|u-w_N\| \end{aligned} \tag{4.25}$$

In (4.25) we only used the Galerkin orthogonality and the symmetry and continuity of b. We note that for this canoncial problem of Shirron the left hand side in (4.25) simply defines a norm on the interior boundary by taking the absolut value, i.e.

$$|u(a) - u_n(a)| \leq c\|u - u_n\|\|u - w_n\| \tag{4.26}$$

In particular, if we take $\{w_n\}$ to be a sequence that converges to $u \in H$ and if $\|u - u_n\|$ stays bounded then we can conclude that u_n converges to u on the boundary $\partial\Omega$. We emphasize that the above assumptions need to be validated by a theoretical analysis. Nevertheless, the numerical results by Shirron suggest that the unconjugated Burnett formulation can converge exponentially in the nearfield and therefore overcomes the with N exponentially growing stability constant.

The convergence analysis of Shirron[54] draws on the particular form of $f(v)$ and we note that in a general situation it is not possible to construct a norm on the left hand side of (4.25). Therefore, we would like to present a different convergence analysis which applies in more general situations and is based on using the dual problem, which is derived by partial integration of the bilinear form in the unconjugated Burnett formulation. We obtain

$$\lim_{\gamma\to\infty}\left(\int_{\Omega^e_\gamma} -u\Delta v - k^2uv\; d\Omega^e_\gamma - ik\int_{S_\gamma} uv\; dS_\gamma + \int_{\partial\Omega^e_\gamma} u\frac{\partial v}{\partial n}\; d\partial\Omega^e_\gamma\right) \tag{4.27}$$

From (4.27) we arrive at the strong form of the dual problem which is

$$\begin{array}{rcll} -\Delta v - k^2 v &=& 0 & \text{in } \Omega^e \\ \dfrac{\partial v}{\partial n} &=& G & \text{on } \partial\Omega \qquad (G \text{ arbitrary}) \\ \left|\dfrac{\partial v}{\partial n} - ikv\right| &=& O\left(\dfrac{1}{r^2}\right) & \text{at } \infty. \end{array} \tag{4.28}$$

Evidently, the bilinear form $b(\cdot,\cdot)$ is identical in the unconjugated Burnett formulation and its dual. The weak form of the dual problem is therefore

$$\text{Find } \Phi \in H_{1,w} \text{ such that} \qquad b(z,\Phi) = (z,G) \qquad \forall\, z \in H_{1,w} \tag{4.29}$$

with $(z,G) = f(z)$. Let now $H_N \subset H_{1,w}$ and $u_N, v_N \in H_N$, then we have by the Galerkin orthogonality that $b(u - u_N, v_N) = 0$, and that $u - u_N = e \in H_{1,w}$. We can now choose $z = e$ in the dual problem and assume $\Phi_N \in H_N$, which leads to

$$\begin{array}{rcl} (e,G) &=& b(e,\Phi) \\ &=& b(e,\Phi - \Phi_N) \\ &\leq& c\|u - u_N\|_{1,w}\|\Phi - \Phi_N\|_{1,w} \end{array} \tag{4.30}$$

by the continuity of b. We now make an assumption on the regularity of the dual problem, i.e. we assume

$$\|\Phi - \Phi_N\|_{1,w} \leq \varepsilon_N \|G\|_{L^2(\partial\Omega)} \tag{4.31}$$

with $\varepsilon_N \to 0$ for $N \to \infty$. From (4.31) we conclude that

$$(e, G) \leq c\,\varepsilon_N \|u - u_N\|_{1,w} \|G\|_{L^2(\partial\Omega)} \tag{4.32}$$

Function G was in principle arbitrary and we specify now $G = e$ on $\partial\Omega$ which allows to estimate

$$\|e\|_{L^2(\partial\Omega)} \leq c\,\varepsilon_N \|u - u_N\|_{1,w}. \tag{4.33}$$

If we now let $N \to \infty$ and if we assume that then $\|u - u_N\|_{1,w}$ remains bounded then we can conclude that u_N converges to u on the boundary $\partial\Omega$. Evidently, the symmetry of b is not used in (4.30) and we may immediately ask if the same argumentation applies to the conjugated Burnett formulation. In this case we get similar to (4.27)

$$\lim_{\gamma\to\infty} \left(\int_{\Omega^e_\gamma} -u\Delta\overline{v} - k^2 u\overline{v}\, d\Omega^e_\gamma - ik \int_{S_\gamma} u\overline{v}\, dS_\gamma + \int_{\partial\Omega^e_\gamma} u\frac{\partial\overline{v}}{\partial n}\, d\partial\Omega^e_\gamma \right) \tag{4.34}$$

with

$$\int_{\partial\Omega^e_\gamma} u\frac{\partial\overline{v}}{\partial n}\, d\partial\Omega^e_\gamma = \int_{\partial\Omega} u\frac{\partial\overline{v}}{\partial n}\, d\partial\Omega + \int_{S_\gamma} u\frac{\partial\overline{v}}{\partial n}\, dS_\gamma. \tag{4.35}$$

We assume now $\partial\overline{v}/\partial n = G$ on $\partial\Omega$ and the Sommerfeld radiation condition as in (4.28) and can obtain the same dual problem with complex conjugated v. The sesquilinear form b is again identical for the conjugated Burnett formulation and its dual and the argumentation following (4.30) applies and therefore estimate (4.33) is preserved. We further note that the argumentation involving the dual problem does not build upon test and trial function spaces being identical. In the unconjugated Astley-Leis formulation we obtain an equation which is identical to (4.27) except for $u \in H_{1,w}$ and $v \in H_{1,w^*}$. The dual problem is identical to (4.28) and b is the same in both problems. Due to the different space setting we have to choose $\Phi_N \in H_{N^*} \subset H_{1,w^*}$ in (4.30) and we have to assume

$$\|\Phi - \Phi_N\|_{1,w^*} \leq \varepsilon_N \|G\|_{L^2(\partial\Omega)} \tag{4.36}$$

with $\varepsilon_N \to 0$ for $N \to \infty$. We obtain again (4.33) by choosing $G = e$. We immediately realize that (4.33) is also valid for the conjugated Astley-Leis formulation. Therefore, we formally have (4.33) for the conjugated and unconjugated formulations (2.7) and (2.9). However, we must note that we infer convergence of u_N on the boundary $\partial\Omega$ from a regularity estimate for the problem. We remark that this regularity assumption is motivated by (4.41) but that it remains to rigorously confirm (4.31) and (4.36). The different performance of the formulations on $\partial\Omega$, which is observed numerically[34,54], indicates that the regularity assumption may have different validity in the formulations.

The Astley-Leis variational formulation (2.7) was studied in the work of Gerdes and Demkowicz[32] and the weighted Sobolev spaces defined by Leis[45] were used to state a mathematically complete variational formulation. The separation of variables procedure was carried out in detail and motivated the definition of test and trial functions as in (2.13) and (2.14). In particular, this work was motivated by mathematical considerations and therefore was the conjugated Astley-Leis formulation being analyzed, i.e. the complex conjugate was applied over test functions in (2.7) which is equivalent to defining a sesquilinear form b instead of a bilinear form. In this context the theory by Leis[45] assures the existence and uniqueness of a solution. Nevertheless, this theory does not provide a convergence proof for infinite element formulations. The numerical results by Gerdes and Demkowicz are based on HP90[27] which is an hp finite element implementation that allows for variable approximation orders and irregular mesh refinements in angular direction. This code is actually a two dimensional code that is extended radially by the infinite element shape functions and allows to study convergence rates with respect to various parameters. Of particular interest are the number of degrees of freedom in angular direction, the number of radial shape functions N, and the wave number. We emphasize that the error is measured in a weighted H^1 norm which is consistent with the definition of the weighted Sobolev spaces. The obtained results for wave number $k = 10$ indicate that the conjugated Astley-Leis formulation indeed converges provided that the number of radial shape functions is sufficiently large, i.e. for $N = 4$ the error remains at about 15% independent of the number of angular degrees of freedom and only for $N = 6$ the error decreases to about 1%. Additional evidence for the convergence of the conjugated Astley-Leis formulation is gathered from a continuous stability analysis, which starts with restating (2.7) in an abstract setting similar to (4.24) by

$$\text{Find } u \in H_{1,w} \text{ such that } \quad b(u,v) = f(v) \quad \forall\, v \in H_{1,w^*} \tag{4.37}$$

where b is understood as a sesquilinear form which also defines a linear operator B on $H_{1,w}$ with values in the dual space of H_{w^*}. The stability constant γ is then defined by

$$\gamma = \inf_{u \neq 0} \frac{\|Bu\|_{H'_{1,w^*}}}{\|u\|_{H_{1,w}}}. \tag{4.38}$$

The introduction of the Riesz operator and the Lagrange multiplier technique allow to recast (4.38) as an eigenvalue problem which can be solved provided that test and trial functions are assumed to be axisymmetric. The analysis[32] shows that $\gamma = O(1/k)$ and indicates the well posedness of the conjugated Astley-Leis formulation.

The convergence properties of the infinite element formulations are further analyzed in the work of Demkowicz and Gerdes[26] which gives the first insight into the convergence mechanism behind infinite elements for separable geometries. In general, infinite element formulations based on (2.7) and (2.9) follow the idea of spectral approximation to retain the first N terms in the approximate solution, i.e. we look for an approximate solution of the form

$$u^N(r,\theta,\phi) = \sum_{n=0}^{N} h_n(kr) u_n^N(\theta,\phi) \tag{4.39}$$

which closely represents the exact solution (2.10). The functions u_n^N are approximated by the finite element discretization in angular direction and the error in radial direction is governed by the value of N in the particular formulation. An intermediate solution u^N is introduced for the theoretical analysis, where N represents the order of approximation in radial direction and it is assumed that u^N is exact in angular direction. Therefore, we can easily decompose the errors with the triangle inequality, i.e.

$$\|u - u_n^N\| \leq \|u - u^N\| + \|u^N - u_n^N\|. \tag{4.40}$$

From (4.40) we see that the infinite element error is governed by the model error resulting from the approximation order N in radial direction and by the finite element error in angular direction. The assumption of a separable geometry, namely a sphere, is essential in the analysis of the model error because the separation of variables procedure allows to rewrite the original three dimensional problem in a one dimensional radial approximation problem. This one dimensional problem basically describes how well lower order Hankel functions can approximate higher order Hankel functions, i.e. for a given wave number and approximation order the solution of this radial problem allows to compute the model error in (4.40). This analysis is performed for the conjugated Astley-Leis and Burnett formulation and the numerical results indicate that both methods converge. Further, the model error can be estimated as

$$\|u - u^N\|_{1,w}^2 \leq c(N+1)^{-2r-1}\|g\|_{H^r(\partial\Omega)}^2 \tag{4.41}$$

provided that $g \in H^r(\partial\Omega)$ for some r and that the approximation error for the approximation of higher order Hankel functions by lower order Hankel functions can in general be bounded independent of the order.

The work of Gerdes[34] extends the convergence analysis[26] to the unconjugated Astley-Leis and Burnett formulations. The analysis assumes a spherical scatterer and shows that the unconjugated formulations do also converge but that the conjugated methods perform better in the exterior domain. We note that the convergence analysis can in principle be extended to other then spherical coordinate systems but that complications in the algebraic manipulations may arise. Numerical results indicate that the unconjugated Burnett formulation performs best in the near field and that the conjugated Astley-Leis formulation delivers best results in the far field. A stability analysis for axisymmetric solutions, compare (4.38), shows that the stability constants of the unconjugated formulations behave like $O(k^{-2})$ as opposed to $O(k^{-1})$ in the conjugated formulations. These results are consistent with observations of Shirron[54] and explain the different behaviors of conjugated and unconjugated formulations. Further, the formal differences between the different formulations (2.7) and (2.9) are analyzed in context of conjugated and unconjugated test functions of the form (4.39). The explicit computation of the radial integrals in (2.9) reveals in detail which terms contribute to the variational formulation.

Demkowicz and Ihlenburg[28] recently extended the convergence analysis of Demkowicz and Gerdes[26,34] to more general scatterers. In their work, they study the effect of the coupled finite and infinite element methodology on the convergence of the infinite element. The analysis is based on applying the Dirichlet to Neumann (DtN) operator on the truncating separable surface which separates the finite and infinite element domain. The proofs are based on analyzing the spectrum of the continuous and discrete DtN operators, but the spectral characterization of the DtN operator is given as a conjecture from intensive numerical experiments.

Burnett and Holford[16] extended Burnett's[15] original work on spheroidal coordinates to more general ellipsoidal coordinates. The novelty of the extension is that the angular ellipsoidal coordinates are defined in a non standard way which appears to be new and which assures that the coordinate transformation is one to one with respect to cartesian coordinates. The infinite element derivation is then based on a radial expansion in ellipsoidal coordinates which has the same form as in (4.22) except for r, θ, and ϕ now being ellipsoidal coordinates. We emphasize that an expansion theorem is formulated and proved in these ellipsoidal coordinates which is similar to the original Atkinson and Wilcox[56] multipole expansion in spherical coordinates. The proof is not completely analytic but gaps are covered by extensive numerical evidence and it can be concluded that the radial expansion converges absolutely and uniformly outside the smallest ellipsoid which surrounds the scatterer. The unconjugated Burnett formulation (2.9) and shape functions similar to (2.13) with $l = 0$ are then used to construct the corresponding infinite element. We note that an ellipsoid can circumscribe an object more closely than a sphere or a prolate spheroid and that this can significantly reduce the computational domain for the finite elements in a coupled finite and infinite element methodology. Therefore, this most general form of infinite element can lead to superior performance although the convergence analysis[26,28,34] has not been carried out for this case.

Cipolla and Buttler[19] study the conjugated and unconjugated Burnett formulation (2.9) in the time domain in context of prolate spheroidal coordinates. Their transient work is motivated by the need to predict acoustic scattering on naval structures. The element operators for their transient infinite element are derived in detail and applied to industrial model problems. The numerical experiments are compared to experimental data and show the efficiency and accuracy of the infinite element methodology in the time domain.

Astley[6] recently carried out a convergence analysis which is similar to the analysis of Demkowicz and Gerdes[26,34]. The various infinite element formulations are reviewed and their performance is assessed and compared. This comparison is based on a spherical scatterer and on decoupling the radial components of the solution by using the orthogonality properties of the spherical harmonics in angular direction. The starting point are the variational formulations (2.7) and (2.9) although they are not stated in the context of weighted Sobolev spaces. The resulting systems allow then to compute the radial error, compare (4.40). It is observed that all formulations converge provided that sufficient radial shape functions are applied, which is in agreement with earlier results. Additionally, conditioning problems are seen as critical factors in all formulations and this error analysis which eliminates the angular direction by assuming exact angular shape functions can be used to study preconditioning of infinite elements. The proper choice of the radial shape functions can result in a better conditioned linear system and previously[54] it had already been suggested to use linear combinations of the radial shape functions defined in (2.13) and (2.14).

5. Generalizations

We have assumed a spherical scatterer in our derivation of the IE methodology for clarity of the derivation. The presented technique can easily be extended to more general shapes of scatterers by simply introducing an artificial surface surrounding the scatterer. Examples of such surfaces are a spherical surface[35] or a prolate spheroidal surface[15] which in general allows for a more efficient discretization. In either case, the domain between the scatterer and the surface has to be discretized with finite elements and infinite elements are on the surface attached to the finite elements. We remark that it is also possible to use other surfaces if the resulting infinite elements are mapped onto a computational domain which is finite. This is done in the implementations that involve mapped infinite elements[5,20,22,21] and provide more flexibility in the mesh generation process but the drawback is that the mapping has to be computed and that the integration in radial direction cannot be done analytically. In either case, the variational formulations (2.7) and (2.9) remain unchanged except for the definition of the weights. Let Γ_s be the artifical surface, Ω_s be the domain between the scatterer and Γ_s, and Ω_s^e be the domain exterior to Γ_s. Then we define the weights as follows

$$w(\boldsymbol{x}) = \begin{cases} 1 & \text{for } \boldsymbol{x} \in \Omega_s \\ r^{-2} & \text{for } \boldsymbol{x} \in \Omega_s^e \end{cases} \qquad w^*(\boldsymbol{x}) = \begin{cases} 1 & \text{for } \boldsymbol{x} \in \Omega_s \\ r^{2} & \text{for } \boldsymbol{x} \in \Omega_s^e \end{cases} \tag{5.42}$$

The extended variational formulations can then easily be discretized by a coupled finite and infinite element discretization provided that the radial shape functions in the infinite element are properly scaled[35] to enable the complex valued infinite element shape functions and the real valued finite element shape functions to coincide on the artificial surface.

6. Summary and Conclusions

In this work we described the basic principles behind the IE methodology. In particular, we emphasized the importance of deriving a correct variational formulation and choosing proper radial shape functions. We also reviewed contributions about the infinite elements which have influenced the development of the field. From the here presented review it is evident how the original idea in the initial work by Bettess[8] more than two decades ago was extended gradually and systematically refined. The guessed exponential decay type shape functions were soon replaced by the correct leading order radial shape function[3,4]. The next step was then to extend to higher order radial shape functions and was first used by Astley[5] and Burnett[15] and then adopted by all researchers working in the field. Similarly and parallel to the derivation of the correct radial shape functions was the development of theoretically sound variational formulations. The early contributions did not address this issue and then Astley[1] introduced the wave envelope concept and this can be seen as the starting point for deriving a precise variational formulation. A mathematically correct formulation was then derived by Astley[5] from engineering considerations and also Burnett[15] derived his variational formulation. These formulations were then stated by Gerdes and Demkowicz[32] in a mathematically complete setting using weighted Sobolev spaces. Subsequently, the convergence mechanism of infinite elements was studied[26,54] and accuracy assessments of the different IE formulations[6,34] are available.

The theoretical results that are available on IE formulations and the robust and efficient numerical performance explain the popularity that infinite elements have gained in recent years. The recent patent[49] on an oblate/prolate infinite element indicates the increased importance of infinite element formulations. The application of infinite elements is not only limited to acoustical problems and the extension of the methodology to electromagnetics[17,29,55] demonstrates the potential of the method.

In conclusion, we expect that infinite elements become an important tool in the solution of exterior domain problems in the engineering practice.

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