On the deterministic control of swirling jets

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Abstract

The theme of this paper is the control of certain large-scale structures in swirling jets, such as the opening angle of the near-conical jet often observed in swirling jets, which have undergone vortex breakdown. To this end, a theory for the control of conically self-similar free-vortex solutions is developed for domains bounded by a conical streamsurface. In order to satisfy an obvious prerequisite for deterministic control, it is proved that if in addition to the opening angle of the bounding conical streamsurface and the circulation thereon, either of the radial velocity, the radial tangential stress or the pressure on the bounding streamsurface are given then a conically self-similar free-vortex solution is uniquely determined in the entire conical domain. In addition, shown that for flows inside a cone the same conclusion holds for Yih *et al.*'s [24] parameter T, but for exterior flows it is shown by explicit construction that in this case nonuniqueness may occur. For given values of the opening angle of the bounding conical streamsurface and the circulation thereon the asymptotic analysis of Shtern & Hussain [17] is applied to obtain asymptotic formulae relating the opening angle of the cone along which the jet fans out and the radial tangential stress on the bounding surface to each other. However, these formulae are shown to be rather inaccurate for moderate values of the circulation at the bounding surface. To amend this shortcoming, an alternative, more accurate asymptotic analysis is developed to derive second order correction terms for both of these formulae, which considerably improve the accuracy. A striking property of these formulae is that the opening angle of the cone along which the jet is fanning out is independent of the value of the viscosity as long as it is small enough for the first order asymptotic formulae to apply.

1 Introduction

Swirling jets have many crucial engineering applications for instance in vortex burners, but they also occur naturally in tornados and in the air flow above a vortex sink in water. However, the underlying physical principles of swirling jets are poorly understood and theoretical models can only be used on rare occasions as design tools. In addition, accurate computation of swirling jets using CFD is time-consuming, due to the high turbulence level typically present. Consequently, control of swirling jets based on CFD is likely to consider only a limited region of the parameter space. Therefore, both large reductions in cost and improved designs can be obtained provided that we find simple models for the control of certain key features of swirling jets. Fortunately, there are some essential large-scale features of jets of almost laminar character, as is hinted by the linear expansion of the far field of a non-swirling jet with downstream distance, see e.g. [14] or [9]. The self-preserving character of these flows provides the possibility to determine the exact degree of the expansion, from knowledge of only the total axial momentum and the almost constant turbulent eddy viscosity.

For swirling flows it is well known that at a certain degree of swirl the jet near the axis splits up, sometimes resulting in an almost conical annular jet which surrounds a recirculation zone, see e.g. [2]. The opening angle of this spreading jet is of great importance in engineering applications and is another example of an almost laminar feature, since it is essentially determined by the balance of the centrifugal force and a pressure gradient. Whereas there is little hope to find a simple model for the prediction of the effects of changing the inlet configuration on turbulent mixing, there may be some chance to predict the effects on more laminar features such as the opening angle. Indeed, in this article our aim is to obtain such a model, but to achieve this we will restrict our attention to the conically self-similar solutions to the Navier–Stokes equations. Perhaps there is some self-preserving swirling jet, which is closely approximated by a conically self-similar solution. Indeed, the flow inside the cone shown in [2] is a natural candidate for such a flow. However, even if we will never find these solutions in nature, understanding how to control them remains relevant as an intermediate step before taking on the infinately more difficult quest of controlling fully turbulent flow.

The swirling conically self-similar solutions to the Navier–Stokes equations, which were originally discovered by Long [10, 11] and independently by Goldshtik [6], provide a framework where laminar-like features of swirling flows can be studied without the imposed complexity of a highly turbulent flow. However, one inherent difficulty with all conically self-similar solutions is that they may either satisfy the no-slip condition at some conical boundary $\theta = \theta_c$ (where θ is the polar coordinate in a spherical coordinate system (R, θ, ψ)), but then they will have a singularity at the symmetry axis $\theta = 0$ [15], or they may be perfectly regular at the symmetry axis, but then they cannot satisfy the no-slip condition at the conical boundary [20, 24, 16, 17]. The physical reason for this is simple, as pointed out in [17], since all conically self-similar solutions require that the total axial momentum is the same for each spherical section of the flow domain, whereas momentum decays at walls due to friction. In this paper we will be concerned with the second case, *i.e.* we will consider the conical boundary as a bounding streamsurface rather than a wall and our velocity field will be bounded throughout the entire flow domain except for the apex of the cone, which we will place in the origin. This case is often referred to as the free-vortex case.

Despite the failure of conically self-similar free-vortex solutions to satisfy all of the relevant boundary conditions, these solutions exhibit features typical of swirling jets. For example, it has been shown [24, 16, 17] that there are three kinds of conically self-similar free-vortex solutions:

1. A near-axis jet. The fluid flows downstream in the vicinity of the symmetry axis, $\theta = 0$, and upstream close to the cone $\theta = \theta_c$.

2. A surface jet. The fluid flows downstream along the cone $\theta = \theta_c$ and upstream along the symmetry axis.

3. A two-cell flow. The fluid flows downstream along some conical streamsurface $\theta = \theta_s$ and upstream both along the cone $\theta = \theta_c$ and along the symmetry axis.

The two-cell flow case is depicted in figure 1. The near-axis jet corresponds to the limit $\theta_s = 0$ and for the surface jet we have that $\theta_s = \theta_c$.

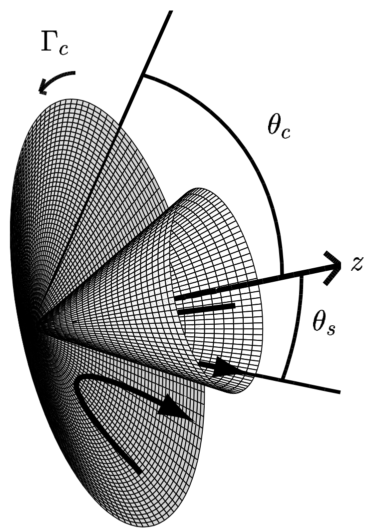

Figure 1: The geometry of the two-cell flow case. The symmetry axis is in the z direction, θ_c is the angle of the bounding streamsurface and Γ_c is the circulation thereon. θ_s is the angle of the streamsurface, which separates the two cones. The thick arrows indicate the direction of the radial velocity component.

The near-axis jet solution is very closely related to Long's vortex, which has attracted much attention as a fundamental flow problem see e.g. [3, 5, 4].

The surface jet solutions have attracted comparatively little interest, yet these may be the easiest to observe in nature. Indeed, the flow inside the conical jet observed by [2] seems to be a natural candidate for an approximately conically self-similar flow. In this paper, however, we shall mainly focus on the two-cell flows, and our primary aim is to find ways of determining θ_s from knowledge of flow at $\theta = \theta_c$ only.

Throughout this paper, we will assume that the opening angle of the bounding conical streamsurface θ_c and the circulation K on this streamsurface are given. Before deriving any formulae for the opening angle in terms of the flow parameters a more fundamental question must be addressed, namely what additional quantities are needed to enable unique control of conically self-similar solutions. From the theory of non-swirling jets we may expect that the total axial momentum J_z would do, but the numerical and asymptotic studies of Shtern & Hussain in [16, 17] (see also [18]) tell us that this situation is not so simple. There are combinations of values of θ_c, K and J_z for which several solutions exist. This important issue explains the observed bistability of tornados, but for control purposes it is highly undesirable to have several possible solutions for the same set of values of the control parameters, since that may allow the swirling jet to toggle between a desirable and an undesirable flow state at random. Consequently, alternative quantities which describe the intensity of the axial flow are needed. The axial velocity and the pressure at the bounding streamsurface are two candidates. A third option is to use the surface radial tangential stress as suggested by Goldshtik & Shtern [7]. Finally we will consider Yih *et al.*'s [24] parameter T for which no unambiguous physical interpretation was available until recently when it was shown that it can be related to the axial velocity and the pressure at the bounding streamsurface [21]. Below we will prove that for given values of θ_c and K, either of the three first quantities mentioned above will specify a conically self-similar solution uniquely. The sole exception is in the case of a bounding plane (*i.e.* when $\theta_c = 0$), in which case the pressure on the surface is always zero, and hence it contains no information of the axial flow. In addition, we will prove that for θ_c, K and Yih *et al.*'s parameter T we have uniqueness when $\theta_c \leq 90°$ (*i.e.* for flow inside a cone), but not necessarily when $\theta_c > 90°$(*i.e.* for flow outside a cone). We will also show by explicit construction that we may in fact have nonuniqueness for given values of $\theta_c < 0$, K and T, and we will explain this intricate behaviour physically.

The question of what uniquely determines a conically self-similar free-vortex solution has received virtually no attention in the literature. The major difficulty is the lack of appropriate mathematical techniques to approach questions of uniqueness for non-linear boundary value problems. Whereas a numerical approach may indicate the existence of a solution by explicit construction of the solution (approximately) and nonuniqueness of a problem by explicit construction of two different solutions for the same problem, the uniqueness issues are subtle in that numerics may easily be deceptive. Sometimes, this is due to the numerical algorithm, which may be biased to give only one of several possible solutions. Another difficulty is that nonuniqueness may occur only in small portions of parameter space in which case it requires both luck and perseverance to detect it. For example, even though Yih *et al.* formulated their problem 17 years ago, no previous numerical study has revealed the nonuniquenesss mentioned in the previous paragraph, and the common belief among workers in the field that the solution was uniquely specified by θ_c, K and T is manifested by the comment in [17] that conically self-similar free-vortex solutions do "not show any fold and non-uniqueness when different control parameters are used [24, 7]." On the same note, there are some implicit assumptions that uniqueness holds in [24]. For example, it is said that "The nondimensionalized momentum flux $M/\rho\nu^2$ is a function of the parameters Re[equivalent to our K] and T", which is not necessarily true in case of nonuniqueness since the same value of K and T can give two different values of M and hence we cannot speak of a function in the normal single-valued sense.

To the author's knowledge only one uniqueness result has been proven for conically self-similar free-vortex solutions, and it is a result in [22] which tells us that not only is such a solution real analytic outside the origin, but it is also uniquely determined by the radial velocity, the polar derivative of the circulation and the radial friction at the symmetry axis. This result is needed to justify mathematically the numerical method used in [16, 17] but it is of little use for control purposes, since we would clearly prefer to control the flow on the bounding surface.

To highlight the nontrivial nature of uniqueness considerations, a few words should be said about Serrin's problem. In this case we satisfy the no-slip condition at the bounding walls, but have a singularity along the symmetry axis. Serrin [15] described his solution (for which the bounding cone was a plane, *i.e.* $\theta_c = 90°$) by the circulation at the symmetry axis and a parameter related to the force driving the fluid at the symmetry axis. For these parameters Serrin failed to prove uniqueness, and mentioned this as the major mathematical problem left open for his problem. However, more than 15 years later Goldshtik & Shtern [7] were able to establish numerically that at least in some part of the parameter space the solution to Serrin's problem was indeed nonunique.

The uniqueness theorem provides us with a set of parameters, which we can use to control conically self-similar solutions uniquely. A major drawback of using the radial velocity is that it vanishes when the no-slip condition applies. This is clearly not the case for the surface pressure or the radial tangential stress, which are still finite even though the values may change. Primarily for this reason we will use the asymptotic analysis developed by Shtern & Hussain [16, 17] to derive a relation which for given values of θ_c and K determines the surface pressure or the the radial tangential stress at the surface in terms of θ_s. In addition we will derive a converse relation, which gives θ_s provided that the values of θ_c, K and either of the surface pressure or the radial tangential stress at the surface are known. In order to obtain reasonable accuracy for these formulae at moderate values of K/ν an alternative and more accurate asymptotic analysis will be developed and used to obtain second order correction terms to these formulae. Fortunately, both these formulae can still be given explicitly. Numerical calculations were used to confirm that for all but very large values of θ_c the presented formulae are accurate even for moderate values of K/ν.

In the next section we will formulate our problem mathematically, and state our uniqueness theorem. In Section 3 we will first show that in the case when $\theta_c < 0$, K and T are given, there may exist more than one solution, and we will explain this behaviour from a physical perspective. Secondly, we will use asymptotic analysis to derive formulae relating θ_s to θ_c, to K and to either of the surface pressure or the radial tangential stress at the surface. The final section will be devoted to the proof of the uniqueness theorem.

2 Formulation of the problem

The conically self-similar solutions to the Navier–Stokes equations is a class of nonlinear exact solutions to the Navier–Stokes equations, which retains both convective and diffusive terms. These solutions are defined in a conical domain $0 < \theta < \theta_c$, $r > 0$. In the origin the solutions are always singular, but if $\theta_c \neq 180°$ we may choose the boundary conditions so that there are no other singularities. The name conically self-similar solutions comes from the fact that the solutions are such that any quotient of two velocity components depends only on the polar angle of a properly aligned spherical coordinate system. The main physical idea behind these solutions is to seek solutions to the Navier–Stokes equations which are characterised by a streamfunction as well as by a circulation function, *i.e.* to seek solutions on the form:

$$\left.\begin{aligned} u_R = -\frac{\nu\psi'(x)}{R}, \quad u_\theta = -\frac{\nu\psi(x)}{R\sin\theta}, \quad u_\phi = \frac{\nu\Gamma(x)}{R\sin\theta}, \\ p - p_\infty = \frac{\rho\nu^2 q(x)}{R^2}, \quad \Psi = \nu R\psi(x), \quad x = \cos\theta, \end{aligned}\right\} \tag{1}$$

where (R, θ, ϕ) are spherical co-ordinates, (u_R, u_θ, u_ϕ) the corresponding velocity components, p the pressure, p_∞ the atmospheric pressure and Ψ a streamfunction. We have also let a prime denote differentiation with respect to x.

When (1) is substituted into the Navier–Stokes equations we obtain after some manipulations the following system of ODEs [15, 17]:

$$\left(1 - x^2\right)\psi' + 2x\psi - \frac{1}{2}\psi^2 = F, \tag{2}$$

$$\left(1-x^2\right)F''' = 2\Gamma\Gamma' \ , \tag{3}$$

$$\left(1-x^2\right)\Gamma'' = \psi\Gamma' \ . \tag{4}$$

2.1 Boundary conditions

In this article we are only interested in classical solutions to (2)-(4), *i.e.* solutions such that $\psi \in C^1\left((x_c,1)\right) \cup C\left([x_c,1]\right), F \in C^3\left((x_c,1)\right) \cup C^1\left([x_c,1]\right)$ and $\Gamma \in C^2\left((x_c,1)\right) \cup C\left([x_c,1]\right)$ which satisfy the boundary conditions at $x_c\,(=\cos\theta_c)$ and 1 which we are now about to specify.

The boundary conditions we use in this paper are obtained if we assume that we do not have any flow sources on the axis, except at the origin, which implies that

$$\Gamma(1) = \psi(1) = 0 \ . \tag{5}$$

For the radial velocity to be bounded outside a neighbourhood of the origin we require that

$$\lim_{x\to 1^-}\left|\psi'(x)\right| < \infty \ . \tag{6}$$

For F we specify the boundary conditions

$$F(1) = F'(1) = 0 \ . \tag{7}$$

The first of these conditions follows from (5), (6) and (2), but the second one requires in addition that

$$\lim_{x\to 1^-}\left(1-x^2\right)\psi''(x) = 0 \ , \tag{8}$$

which physically means that there is no line force acting along $x = 1$.

We furthermore assume that the swirling flow is driven by a constant circulation along some fixed conical streamsurface $x = x_c$ which implies that

$$\psi(x_c) = 0, \quad \Gamma(x_c) = \Gamma_c \ . \tag{9}$$

Since the system of equations (2)-(4) is symmetric with respect to the sign of Γ, the sign is immaterial, and we will henceforth assume that $\Gamma_c \geq 0$. The physical circulation K is related to Γ_c by $K = 2\pi\nu\Gamma_c$, and henceforth we will use Γ_c when discussing the circulation on the surface.

In addition we must somehow specify the intensity of the axial flow, and for the purposes of this paper it must be done in such a way that the solution to the problem is unique. In a series of articles [16, 17] Shtern & Hussain have showed that this condition is not fulfilled for the total axial flow force J_z. Instead we will consider three other alternative conditions. The first condition is to specify the radial velocity distribution at the streamsurface, which obviously must be of the form $1/r$ to be compatible with conical self-similarity. In terms of the parameters of the problem this condition is equivalent to prescribing $\psi^{'}(x_c)$. The second condition is to specify the surface radial tangential stress per unit length of the symmetry axis, which also must be of the form $1/r$ to be compatible with conical self-similarity. This is equivalent to specifying $\psi^{''}(x_c)$ as is seen from the formula

$$\int_0^{2\pi} -\tau_{r\theta} r \sin\theta \cos\theta d\psi = 2\pi x_c \left(1 - x_c^2\right) \psi^{''}(x_c) \rho \nu^2 r^{-1} \tag{10}$$

which was derived in [7]. The third alternative was to specify the pressure at the bounding streamsurface. This must be of the form $1/r^2$, and is given by (see e.g. [17])

$$q = \frac{2x\psi - xF^{'} - \psi^2}{1 - x^2} = -x\psi^{''} + \frac{x\psi\psi^{'} - \psi^2}{1 - x^2} \, . \tag{11}$$

Hence, using the boundary conditions we have that $q(x_c) = -x_c\psi^{''}(x_c)$. Consequently, if we specify $\psi^{''}(x_c)$ we specify both the pressure at the surface and the surface radial tangential stress. To obtain the parameter used by Yih *et al.* [24] we follow them and Serrin [15] and integrate (3) three times. After the enforcement of the conditions (7) this yields

$$F(x) = -(1-x)^2 \int_{x_c}^{x} \frac{t\Gamma^2 \mathrm{d}t}{\left(1-t^2\right)^2} - x \int_x^1 \frac{\Gamma^2 \mathrm{d}t}{(1+t)^2} - \frac{T\Gamma_c^2}{2}(1-x)^2 \, , \tag{12}$$

where T is an arbitrary parameter, which can be used to specify the strength of the axial flow.

3 Results and Discussion

We begin by presenting the fundamental result on which the control theory in this paper is based

Theorem 1 *Suppose that* $x_c \in (-1, 1)$ *then a conically self-similar free-vortex solution to the Navier–Stokes equations is uniquely determined by* x_c*,* Γ_c *and either of (a)* $\psi^{'}(x_c)$*, (b)* $\psi^{''}(x_c)$ *or in case* $x_c \geq 0$ *(c)* T*.*

Remark : The theorem does not necessarily hold in the case when $x_c = -1$.

The proof of this theorem is fairly long and technical and will accordingly be deferred until the next section.

The importance of Theorem 1 is that once we have found values of x_c, Γ_c and one of $\psi^{'}(x_c)$, $\psi^{''}(x_c)$ and T (if $x_c \geq 0$), which give us a desirable flow, then we can be sure that within the class of conically self-similar free-vortex solutions there is no other, less desirable, solution with the same values of these parameters. For deterministic control, this is a necessary condition we must impose on any control parameters we would like to use, and Theorem 1 thus shows that this criterion is satisfied when $\psi^{'}(x_c)$, $\psi^{''}(x_c)$ and, if

$x_c \geq 0$, T are used as control parameters alongside of x_c and Γ_c, which is not the case for, for example, J_z. It is noteworthy that for the non-swirling solutions we have uniqueness for J_z as well, whereas as the swirl increases uniqueness fails for J_z, but not for the parameters considered Theorem 1.

From a mathematical perspective, uniqueness is a necessary, but not a sufficient condition for a practical control theory. Specifically, the question of continuous dependence of parameters is important, yet still left open. The real analyticity of the solutions can perhaps be used to provide a partial answer, but the question is more complex since it is possible that a small change in the parameters may cause solution non-existence. However, in the parameter ranges studied in this paper, the numerical calculations performed do not hint at any discontinuities.

In the next subsection we will see that when $x_c < 0$ then there may exist more than one conically self-similar free-vortex solution with the same values of x_c , Γ_c and T. In the second subsection we will derive formulae for the two-cell flow solutions, which for given values of x_c and Γ_c allow us to calculate the angle of the separating cone for the two-cell solutions, $x_s\,(= \cos\theta_s)$ in terms of $\psi''\,(x_c)$ and vice-versa.

3.1 Nonuniqueness for the T-problem when $x_c < 0$.

When $x_c < 0$ the proof of Theorem 1(c) fails in only one step. Clearly, the failure could be due to a weakness of the method of proof, and uniqueness could still hold. To get some indication of the validity of this hypothesis we would like to make some numerical experiments. To perform such experiments, we must have some feeling for what kind of nonuniqueness we could possibly expect, and it seems that the most likely nonuniqueness is probably that for the same combination of values of $x_c < 0$, Γ_c and T there are two two-cell flow solutions with different values of x_s. To begin our search, we chose $x_c = -1/\sqrt{2}$ and $\Gamma_c = 30$, and various values of x_s, and used a numerical algorithm, which is an obvious modification of the one described in [17, p. 33] to solve our system numerically. For each value of x_s we calculated the corresponding value of T, and the results are shown in figure 2(a). From this figure it looks like T is monotone function of x_s and hence for each value of T there corresponds exactly one value of x_s. This clearly does not prove uniqueness, since we have confined our comparisons to a limited set of functions, but the evidence certainly does not suggest nonuniqueness.

However, to dismiss our failure to prove uniqueness in this case as a mathematical whim is highly premature, as becomes evident when we repeat the above procedure for the case $x_c = -0.15$ and $\Gamma_c = 30$. As seen from figure 2(b), T is no longer a monotone function of x_s, and there are several values of T for which at least two different solutions exist. For example, in the case when $T = 0.014$ we have one solution with $x_s \approx 0.915$ and one with $x_s \approx 0.355$. These solutions are presented in figure 3, and were calculated

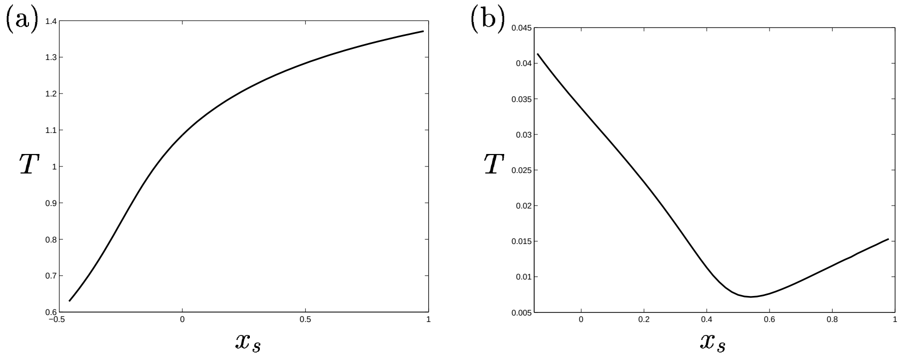

Figure 2: The variation of T as a function of x_s when $\Gamma_c = 30$ and (a) $x_c = -1/\sqrt{2}$ and (b) $x_c = -0.15$.

using the algorithm presented in [16, 17] with the tentative values given in table 1. A natural question to ask is precisely for what values of the parameters nonuniqueness occurs, but since it has no real bearing on the control issues which make up the main theme of this article we refrain from discussing it, lest this digression should be too long.

	$x_s \approx 0.355$	$x_s \approx 0.915$
$\psi'(1)$	10.1979067790	55.4283048307
$F''(1)$	-97.1263271411	-4393.10503451
$\Gamma'(1)$	-3.97582349423	-80.8505960463

Table 1: The tentative values to be used in the numerical method in [16, 17] to calculate two different solutions for the case when $x_c = -0.15$, $\Gamma_c = 30$ and $T = 0.014$. (The system is very sensitive to these values, and therefore many decimal places are needed to obtain reasonable precision for the solutions. However, it is not claimed that these values give the smallest error obtainable with this number of significant figures.)

These somewhat surprising results for the T problem call for a physical explanation, but first we must understand the physical meaning of T. In order to achieve this we may use the formula for the pressure

$$q = \frac{2x\psi - xF' - \psi^2}{1-x^2} = \frac{F''}{2} - \psi' - \frac{\psi^2 + \Gamma^2}{2\,(1-x^2)}\ , \tag{13}$$

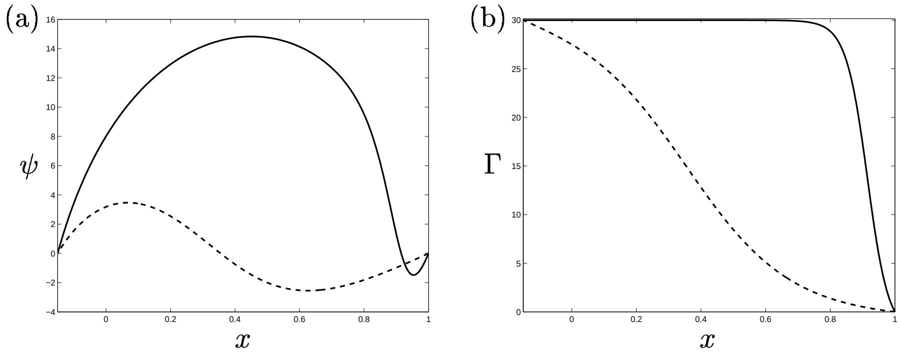

Figure 3: Two different numerical solutions which exist for $x_c = -0.15$, $\Gamma_c = 30$ and $T = 0.014$. The solid lines show the solution with $x_s \approx 0.915$ and the dashed lines show the solution with $x_s \approx 0.355$.

to derive the following formula [21]

$$T = \frac{1}{1-x_c^2} - \frac{F''(x_c)}{\Gamma_c^2} = -\frac{2}{\Gamma_c^2}\left(q(x_c) + \psi'(x_c)\right) . \tag{14}$$

Hence T is directly related to the pressure and the radial velocity at the conical streamsurface. For the special case when $x_c = 0$ it is clear that when (9) is inserted into the first expression in (13) we find that $q(0) = 0$ and thus T is completely determined by the radial velocity distribution on the planar streamsurface.

As the opening angle of the annular jet in a two-cell flow decreases, *i.e.* as x_s increases, we know from [17] that, at least asymptotically for large Γ_c, the magnitude of the radial velocity at the bounding streamsurface $x = x_c$ increases, and hence since $v_r = -\nu\psi'/r$, $\psi'(x_c)$ increases with increasing x_s.

On the other hand, the asymptotic value of the pressure at the bounding streamsurface can be obtained from the asymptotic solutions found by Shtern & Hussain [17]. (We present the formulae here since there are a couple of errors in the corresponding formulae in [17], as is seen by the fact that for the displayed formulae there neither of the equalities $q_1(x_s) = q_2(x_s) = -q_{s0}$ hold, as was claimed. [These errors as well as the one mentioned later have been acknowledged by Prof. Shtern in private communication. - This bit in the square brackets is not intended for publication.])

$$q = \begin{cases} -\Gamma_c^2 \dfrac{(x_s - x_c^2)\,x - [2x_s - (1+x_s)\,x_c]\,x_c}{(1+x_s)(1-x_c)^2(1-x^2)} + o\left(\Gamma_c^2\right), & \text{for } x_c \le x \le x_s \\ -\Gamma_c^2 \dfrac{(x_s - x_c)^2}{(1+x)(1-x_c)^2(1-x_s^2)} + o\left(\Gamma_c^2\right), & \text{for } x_s \le x \le 1 \end{cases} \tag{15}$$

Hence we have that

$$\frac{d}{dx_s}\left(q\left(x_c\right)\right) = \frac{x_c\Gamma_c}{\left(1+x_s\right)^2\left(1-x_c\right)^2}\ , \tag{16}$$

Consequently, $q\left(x_c\right)$ increases with increasing x_s when $x_c > 0$, and it decreases with increasing x_s when $x_c < 0$. To summarize, we have shown, at least asymptotically when Γ_c goes to infinity, that when $x_c > 0$, $q\left(x_c\right)$ and $\psi^{'}\left(x_c\right)$ both increase as x_s increases, whereas when $x_c < 0$ they move in opposite directions as x_s changes. This allows us to conclude that T is a monotone (decreasing) function of x_s when $x_c \geq 0$, but we cannot draw any such conclusion when $x_c < 0$.

To conclude this section we will say a few words about the numerical algorithm presented in [24]. In this algorithm one starts with $\Gamma_0 \equiv \Gamma_c$ and essentially one then iteratively calculates F_1, ψ_1, Γ_2, F_2 etc. from the formulae (3),(2) and (4). In practice the method seems to converge for the values of x_c, Γ_c and T for which a solution exists. (Yih *et al.* claimed to have proved convergence when $x_c \geq 0$, but in [21] it was argued that the proof is not complete.) Here it is of interest to study the performance of this algorithm in a case when the solution is not unique. To this end, this method was implemented for the case mentioned above when $x_c = -0.15$, $\Gamma_c = 30$ and $T = 0.014$, and it was found that this method seemed to converge towards the solution with $x_s \approx 0.355$. Hence, to detect nonuniqueness with this numerical method one would have to run the algorithm with several different starting functions Γ_0. This once again shows the danger of drawing conclusions about uniqueness exclusively from numerical simulations.

3.2 Asymptotic relations between $\psi^{''}\left(x_c\right)$ and x_s for given values of Γ_c and x_c.

In this subsection we assume that Γ_c and x_c are arbitrary but fixed. For any value of $\psi^{''}\left(x_c\right)$ (recall that this value can be obtained either from the tangential radial stress or if $x_c \neq 0$ from the surface pressure), Theorem 1 tells us that there is at most one conically self-similar free-vortex solution. If a solution exists for a given value of $\psi^{''}\left(x_c\right)$, Yih *et al.* [24] established that it must be of one of the three types mentioned in the introduction. Consequently, to each of the values of $\psi^{''}\left(x_c\right)$ which gives a two-cell solution there corresponds a unique value of x_s. In control terms this can be expressed as follows: Suppose that we for given values of x_c and Γ_c want to obtain a particular value of x_s, and that we have found a value of $\psi^{''}\left(x_c\right)$ which gives us this optimal value, then we can be sure that it does not correspond to any other solutions. However, at this point it should be stressed that we cannot say that this is the only optimal value, since we have not proved that a conically self-similar solution is uniquely determined by x_c, Γ_c and x_s. (Most of the numerical computations in this paper are calculated by a method of Shtern & Hussain [17] for which these values are given and tentative values at x_s are adjusted to satisfy the boundary conditions at x_c and 1. In the event that uniqueness does not hold in this case the numerical results below will only be representative for solutions approaching Shtern & Hussain's asymptotic solution.)

For any given values of Γ_c, x_c and $\psi''(x_c)$ for which a two-cell solution exists, we can in principle calculate x_s numerically. If conversely, Γ_c, x_c and x_s are given we can find at least one value of $\psi''(x_c)$, which will give such a solution. However, as Γ_c increases the equations become increasingly singular, and thus we would like to have asymptotic formulae which express the relation between these quantities in the high-Γ_c limit. Fortunately, such formulae can be obtained from the asymptotic analysis of Shtern & Hussain [17], by substituting their asymptotic expression for F into (11) to obtain

$$\psi''(x_c) = \psi''(x_c)^{a1} + o\left(\Gamma_c^2\right) \tag{17}$$

$$\psi''(x_c)^{a1} = -\frac{\Gamma_c^2 (x_s - x_c)}{(1-x_c)^2 (1+x_c)(1+x_s)} \,. \tag{18}$$

Suppose that $\psi''(x_c) \approx \psi''(x_c)^{a1}$. We can then expect x_s^*, obtained by inverting the formula for $\psi''(x_c)^{a1}$, to yield reasonable approximations for x_s in terms of $\psi''(x_c)$. Indeed, by simple algebra we have

$$x_s^* = \frac{x_c\Gamma_c^2 - (1-x_c)^2 (1+x_c)\psi''(x_c)}{\Gamma_c^2 + (1-x_c)^2 (1+x_c)\psi''(x_c)} \,. \tag{19}$$

Since $q(x_c) = -x_c\psi''(x_c)$ and $q \propto \nu^{-2}$ we have that both the nominator and the denominator in (19) are proportional to ν^{-2}. Consequently, the predicted value of x_s does not depend on ν as long as ν is small enough to secure the validity of the asymptotic approximation. Hence this formula exhibits Reynolds number invariance for large Reynolds numbers.

To study the speed of convergence towards these asymptotic formulae numerical computations were performed for several values of x_s and Γ_c, and some of these results are presented in figure 4. From this figure we see that even when $\Gamma_c = 300$ the approximations differ from the true values by more than 15%, and thus it is evident that in this case much higher values of Γ_c than 300 are needed to obtain reasonable accuracy for the asymptotic formulae. Yet when $\Gamma_c = 300$, $x_c = -0.15$ and x_s is not too small, say larger than 0.3, the conically self-similar solutions are close to the asymptotic ones for most of the domain $[x_c, 1]$. Specifically, Γ is very close to the asymptotic function

$$\Gamma = \begin{cases} \Gamma_c\,, & \text{for } x_c \le x \le x_s \\ 0\,, & \text{for } x_s \le x \le 1 \end{cases} \tag{20}$$

which makes numerical computations rather difficult for larger values of Γ_c.

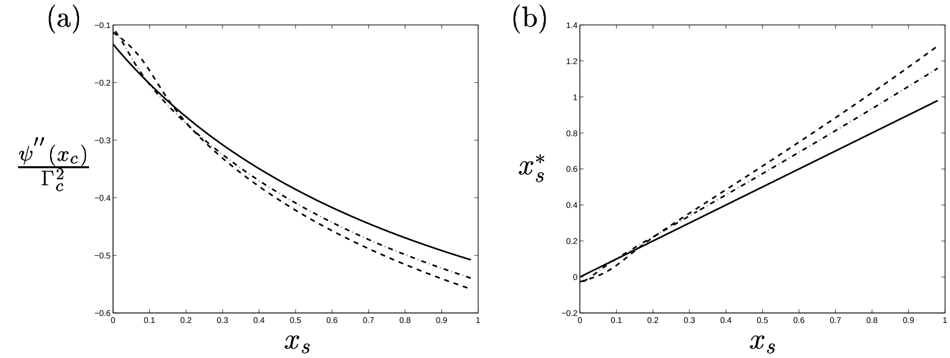

Figure 4: A comparison between (a) $\frac{\psi''(x_c)^{a1}}{\Gamma_c^2}$ and (b) x_s^* and their respective true values given by (a) $\frac{\psi''(x_c)}{\Gamma_c^2}$ and (b) x_s. In (a) the solid line shows $\frac{\psi''(x_c)^{a1}}{\Gamma_c^2}$ and the dashed lines represents the, numerically computed, true values of $\frac{\psi''(x_c)}{\Gamma_c^2}$. In (b) the solid line shows a line through the origin with slope 1, *i.e.* an ideal prediction curve, and the dashed lines show the values of x_s^* obtained for the numerically computed values of $\psi''(x_c)$. In all numerical computations $x_c = -0.15$ and $\Gamma_c = 150$ (dashed curves) or $\Gamma_c = 300$ (dot-dashed curves).

In order to derive more accurate formulae, we must understand where the discrepancies in figure 4 arise. Essentially, Shtern & Hussain's asymptotic analysis starts by assuming that Γ is given by the asymptotic function above. This function is then used to calculate F, to which we can add an arbitrary term of the form $C(1-x)^2$ for any C. This term gives rise to a corresponding arbitrary term for $\psi''(x_c)$ given by $-2C/(1+x_c)$, and hence the determination of this value is very important for the present analysis. No matter how C is chosen though, F is given by second degree polynomials in each of the domains $[x_c, x_s]$ and $[x_s, 1]$. In order to obtain the solutions of highest order in Γ_c Shtern & Hussain now removed the linear terms in (2) and calculated ψ from F by simple algebraic operations, and they then chose C in such a way that the boundary condition $\psi(x_c) = 0$ could be satisfied. It seems that the lack of accuracy of this choice is the main cause of the shortcomings of the formulae we presented above.

As a consequence of their choice of C, Shtern & Hussain could not satisfy the condition $\psi(x_s) = 0$, even though it was assumed to hold in the beginning of the analysis. This and other shortcomings were then remedied by using inner solutions around x_s and x_c. However, the inner solutions in [17] are not directly applicable in our case since they assume that F is constant in the boundary layer around $x = x_c$, which yields no correction to $F'(x_c)$ or $\psi''(x_c)$.

Instead of introducing more complex inner solutions, a different approach will be used to obtain formulae valid in the entire domain. To this end, we will assume that Γ is given by the asymptotic function above, but apart from that no other assumption will be made, and all boundary conditions will be satisfied. Just like in Shtern & Hussain's analysis we find that F is given by second order polynomials in each of the domains $[x_c, x_s]$ and $[x_s, 1]$, both of which have an arbitrary term $C\,(1-x)^2$ where for convenience we define C in such a way that for $C = 0$ we obtain the same functions that Shtern & Hussain used. For given values of x_c, Γ_c and x_s, our problem is now to find a C such that both of the boundary value problems

$$\begin{cases} \left(1-x^2\right)\psi_1' + 2x\psi_1 - \dfrac{1}{2}\psi_1^2 = F_1 \\ F_1 = \Gamma_c^2\left\{-\left(x-x_c\right)p_1\left(x\right) + C\left(1-x\right)^2\right\} \\ p_1\left(x\right) = \dfrac{2x_s - \left(1+x_s\right)x_c - \left(1+x_s-2x_c\right)x}{2\left(1+x_s\right)\left(1-x_c\right)^2} \\ \psi_1\left(x_c\right) = 0, \quad \psi_1\left(x_s\right) = 0\ . \end{cases} \tag{21}$$

$$\begin{cases} \left(1-x^2\right)\psi_2' + 2x\psi_2 - \dfrac{1}{2}\psi_2^2 = F_2 \\ F_2 = \Gamma_c^2\left(1-x\right)^2\left\{-\dfrac{\left(x_s-x_c\right)^2}{2\left(1-x_s^2\right)\left(1-x_c\right)^2} + C\right\} \\ \psi_2\left(x_s\right) = 0, \quad \psi_2\left(1\right) = 0\ . \end{cases} \tag{22}$$

have bounded solutions, *i.e.* we have an eigenvalue problem. It can be seen that there exists some C^* such that the second of the boundary value problems can be solved for any $C < C^*$. This follows from the fact that it is solved by a Squire-Potsch solution (see e.g. [13] or [21]). Hence, we must only solve the first eigenvalue problem. In fact we can find an exact solution to the first problem as well, but this solution is given by a complicated expression involving hypergeometric functions (see e.g. [23] or [12]) and hence C will be defined implicitly in terms of these functions. This, however, offers little insight into the problem, and numerically it is not easier to calculate the hypergeometric functions than to solve our original system. An alternative approach is to make the substitution $\psi = -2\left(1-x^2\right)U'/U$ which yields

$$-U'' + \frac{\Gamma_c^2\left(x-x_c\right)p_1\left(x\right)}{2\left(1-x^2\right)^2}U = \frac{C\Gamma_c^2}{2\left(1+x\right)^2}U\ , \tag{23}$$

with the boundary conditions $U'\left(x_c\right) = U'\left(x_s\right) = 0$, and the additional condition that $U \neq 0$ for $x \in [x_c, x_s]$. This is a Sturm-Liouville problem and hence we know that there exists an increasing sequence of non-negative eigenvalues, C_i, i=0,1,2,... such that for $C = C_i$ (23) has an eigenfunction with i zeros in $[x_c, x_s]$. The solution we are after is clearly the one with no zeros, which corresponds to the lowest eigenvalue $C = C_0$. For Sturm-Liouville problems there are several algorithms to facilitate numerical computation of the eigenvalues and eigenfunctions. Consequently, we can for given values of Γ_c, x_c and x_s compute a correction term to $\psi''(x_c)^{a1}$, which would greatly improve the accuracy of this estimate. However, with no analytical expression at hand we are at a loss in trying to invert the formula to obtain a more accurate formula for x_s^* in terms of Γ_c, x_c and $\psi''(x_c)$.

We may easily obtain an upper bound for C_0. To do this notice that if a bounded function U satisfies the boundary conditions then elementary calculus tells us that U'' must change sign somewhere strictly between x_c and x_s. Hence (23) implies that at this point either $U = 0$ or $w = 0$ where

$$w(x) = \frac{C}{2(1+x)^2} - \frac{(x - x_c)\, p_1(x)}{2(1-x^2)^2} \ . \tag{24}$$

However, we have required that $U \neq 0$ and thus there must be a point strictly between x_c and x_s where $w = 0$. It is easily seen that this requires that $0 \leq C < C^\dagger$ where

$$C^\dagger = \frac{(x_s - x_c)^2}{2(1-x_s^2)(1-x_c)^2} \ . \tag{25}$$

Since the zeros of w are zeros of a second degree polynomial, it is easily seen by studying the signs of w at x_c and x_s that it must have precisely one first order zero, x_t in the domain $[x_c, x_s]$. Hence we want to solve

$$U'' + \Gamma_c^2 w(x)\, U = 0 \ , \tag{26}$$

in $[x_c, x_s]$ where w has a first order zero at $x = x_t$, and is non-zero elsewhere. This problem is known as a second order ordinary differential equation with a (first-order) turning (transition) point.

Fortunately, the asymptotic solution for this problem is well known (see e.g. [1, p.451]) and is given by

$$U^a(x) = aAi\left(-\Gamma_c^{\frac{2}{3}}\xi(x)\right) + bBi\left(-\Gamma_c^{\frac{2}{3}}\xi(x)\right) \tag{27}$$

where Ai and Bi are the Airy functions of the first and second kind respectively and ξ satisfies

$$\xi\left(\frac{d\xi}{dx}\right)^2 = w(x) \ . \tag{28}$$

A straightforward calculation yields the following expression for ξ

$$\xi(x) = \begin{cases} \left[\frac{3}{2}\int_x^{x_t} w(z)^{\frac{1}{2}}\, dz\right]^{\frac{2}{3}} \ , & \text{for } x_c \leq x \leq x_t \ . \\ -\left[\frac{3}{2}\int_{x_t}^{x} (-w(z))^{\frac{1}{2}}\, dz\right]^{\frac{2}{3}} \ , & \text{for } x_t \leq x \leq x_s \ . \end{cases} \tag{29}$$

Let $\xi_c = \xi(x_c)$ and $\xi_s = \xi(x_s)$. The function

$$V(\xi) = aAi\left(-\Gamma_c^{\frac{2}{3}}\xi\right) + bBi\left(-\Gamma_c^{\frac{2}{3}}\xi\right) \tag{30}$$

has an infinite number of positive zeros, and hence, since $\xi_c > 0$, it will have zeros for $0 \le \xi \le \xi_c$ when Γ_c is large enough unless we make sure that

$$-\Gamma_c^{\frac{2}{3}}\xi_c > \alpha_1 \tag{31}$$

holds for all values of Γ_c, where α_1 denotes the first zero of V.

However, if we satisfy (31) there must be a negative ξ^* such that $\xi_s \le \xi^*$ for all values of Γ_c. Consequently, the asymptotic formulae for the Airy functions (see e.g. [1]) yield that

$$\frac{Bi'\left(-\Gamma_c^{\frac{2}{3}}\xi_s\right)}{Ai'\left(-\Gamma_c^{\frac{2}{3}}\xi_s\right)} = O\left(\exp\left(2\left(-\xi_s\right)^{\frac{3}{2}}\Gamma_c\right)\right) . \tag{32}$$

Hence, the condition $U'(x_s) = 0$ implies that $\frac{b}{a} = O\left(\exp\left(-2\left(-\xi_s\right)^{\frac{3}{2}}\Gamma_c\right)\right)$. However, for negative x, $Ai(x)$ and $Bi(x)$ are of the same order of magnitude, and hence the condition $U'(x_c) = 0$ is asymptotically satisfied if

$$\Gamma_c^{\frac{2}{3}}\xi_c = -\alpha_1' \tag{33}$$

where $\alpha_1' \approx -1.01879297$ is the first zero of Ai'.

To calculate ξ_c we make a change of variables in (29) to obtain

$$\xi_c = \left[\frac{3}{2}\left(x_t - x_c\right)\int_0^1 w\left(x_c + \left(x_t - x_c\right)y\right)^{\frac{1}{2}} dy\right]^{\frac{2}{3}} . \tag{34}$$

Suppose that $\lim_{\Gamma_c\to\infty} x_t = x^* \ne x_c$. In this case (34) implies that $\xi_c \to \xi^* \ne 0$, but according to (33) $\xi_c = O\left(\Gamma_c^{-\frac{2}{3}}\right)$ which is a contradiction. We have thus established that $\lim_{\Gamma_c\to\infty} x_t = x_c$. To obtain an asymptotic expression for xi_c we now make asymptotic expansions in $\Gamma_c^{-\frac{2}{3}}$ of x_t and C:

$$x_t = x_c + x_{t1}\Gamma_c^{-\frac{2}{3}} + \cdots \tag{35}$$

$$C = C_0 + C_1\Gamma_c^{-\frac{2}{3}} + \cdots \tag{36}$$

where the dots denote terms of higher order. First, recall that x_t was the zero of w in $[x_c, x_s]$ which is clearly determined by C. By substituting (36) into $w(x) = 0$ and identifying terms we obtain the relations

$$C_0 = 0 \tag{37}$$

$$x_{t1} = \frac{C_1\left(1 - x_c\right)^2}{p_1\left(x_c\right)} . \tag{38}$$

If we use (38) and Taylor's formula we obtain the following asymptotic formula

$$w\left(x_c+\left(x_t-x_c\right)y\right)=\frac{C_1}{2\left(1+x_c\right)^2}\Gamma_c^{-\frac{2}{3}}\left(1-y\right)+O\left(\Gamma_c^{-\frac{4}{3}}\right)\ . \tag{39}$$

After an additional application of Taylor's formula and integration this yields

$$\xi_c^{\frac{3}{2}}=\frac{C_1^{\frac{3}{2}}\left(1-x_c\right)^2}{2^{\frac{1}{2}}p_1\left(x_c\right)\left(1+x_c\right)}\Gamma_c^{-1}\left(1+O\left(\Gamma_c^{-\frac{2}{3}}\right)\right)\ . \tag{40}$$

Finally, when this is compared with (33) we obtain the following condition for C_1

$$C_1 = 2^{\frac{1}{3}}\left(-\alpha_1^{'}\right)\left(\frac{p_1\left(x_c\right)\left(1+x_c\right)}{\left(1-x_c\right)^2}\right)^{\frac{2}{3}} \tag{41}$$

$$= \frac{2^{\frac{1}{3}}\left(-\alpha_1^{'}\right)}{\left(1-x_c\right)^2}\left(\frac{\left(x_s-x_c\right)\left(1+x_c\right)}{1+x_s}\right)^{\frac{2}{3}}\ . \tag{42}$$

For our purposes, *i.e.* calculation of $\psi^{''}\left(x_c\right)$, knowledge of C_1 suffices, but it should be remarked that a somewhat more careful analysis is needed to obtain the eigenfunction U with such accuracy that we obtain a good approximation for ψ. However, if this is done this solution together with the Squire-Potsch solution in $[x_s,1]$ yields an asymptotic solution which contains both the inner solutions and the outer solutions of the asymptotic analysis in [17]. The inner solutions are a bit different, however, since our analysis allows F to vary in the boundary layers, which is not allowed in Shtern & Hussain's analysis. Finally, it should be remarked that neither of the two approaches take into account any effects of the variations of the circulation near $x=x_s$.

We now have the following refined asymptotic approximation for $\psi^{''}\left(x_c\right)$

$$\psi^{''}\left(x_c\right)^{a2} = -\frac{f\left(\frac{\Gamma_c^2\left(x_s-x_c\right)}{1+x_s},2^{\frac{4}{3}}\left(-\alpha_1^{'}\right)\left(1+x_c\right)^{\frac{2}{3}}\right)}{\left(1-x_c\right)^2\left(1+x_c\right)} \tag{43}$$

$$f\left(x,a\right) = x+ax^{\frac{2}{3}}\ . \tag{44}$$

In fact, the formula (43) can be inverted explicitly. To that end, let $z=\left(1-x_c\right)^2\left(1+x_c\right)\psi^{''}\left(x_c\right)$ and let $f^{-1}\left(x,a\right)$ denote inverse function of $f\left(x,a\right)$ with respect to x. We then have that

$$x_s^{\dagger}=\frac{x_c\Gamma_c^2-f^{-1}\left(z,a\right)}{\Gamma_c^2+f^{-1}\left(z,a\right)}\ , \tag{45}$$

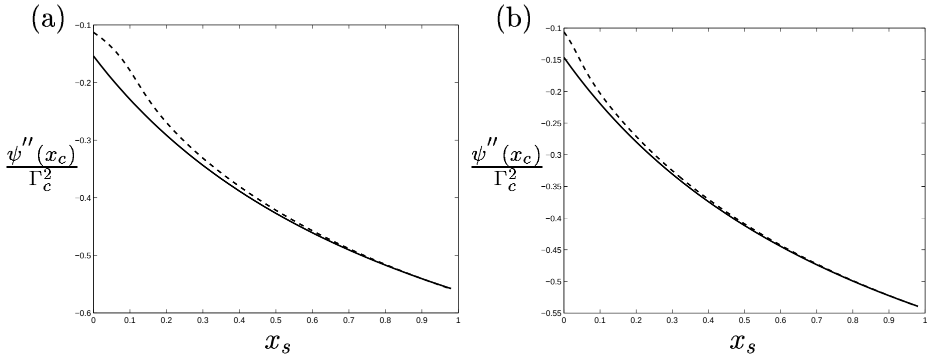

Figure 5: A comparison between $\frac{\psi''(x_c)^{a2}}{\Gamma_c^2}$ (the solid lines) and true values of $\frac{\psi''(x_c)}{\Gamma_c^2}$ (the dashed lines) obtained numerically. In all numerical computations $x_c = -0.15$, in (a) $\Gamma_c = 150$ and in (b) $\Gamma_c = 300$.

but using Cardano's formula we can calculate $f^{-1}(z, a)$ explicitly to obtain

$$f^{-1}(z,a) = \left\{ -\frac{a}{3} + \left(\frac{z}{2} - \frac{a^3}{27} + D \right)^{\frac{1}{3}} + \left(\frac{z}{2} - \frac{a^3}{27} - D \right)^{\frac{1}{3}} \right\}^3 \quad (46)$$

where

$$D = \left(\frac{z^2}{4} - \frac{a^3 z}{27} \right)^{\frac{1}{2}} . \quad (47)$$

(There are two other inverse formulae, but these are complex and hence of no interest here.)

In figures 5 and 6, the corrected asymptotic formulae (43) and (45) are compared with numerically obtained true values. These figures show that unless x_s is small then the refined asymptotic formulae are accurate within a few per cent even for $\Gamma_c = 150$. This is clearly a significant improvement compared to the situation presented in figure 4. Nevertheless, it is still true that for sufficiently large Γ_c the highest order formulae, which were presented in the beginning of this subsection, are still valid, as is the conclusion of the Reynolds number invariance of x_s^* and $x_s^\dagger$ for sufficiently large Reynolds numbers.

When x_s is small the refined asymptotic formulae are no longer as accurate, but this is due to the fact that Γ is no longer close to the asymptotic function as we have assumed it to be. This, however, means that the system is fairly simple to solve numerically, so we may obtain relations between x_s and $\psi''(x_c)$ that way. Indeed, it seems to be a general fact that the refined asymptotic formulae are accurate whenever numerical solutions are not readily available. Hence, if the approaches are combined we can for any

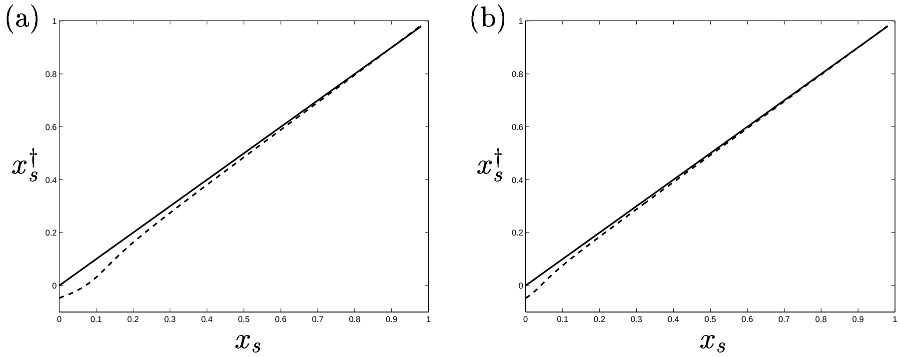

Figure 6: A comparison between $x_s^\dagger$ (the solid lines) calculated from numerically obtained values of $\psi''(x_c)$ and the ideal curve given by a straight line through the origin with slope 1 (the dashed lines). In all numerical computations $x_c = -0.15$, in (a) $\Gamma_c = 150$ and in (b) $\Gamma_c = 300$.

Γ_c and x_c obtain the relations between x_s and $\psi''(x_c)$, which are accurate to within a few per cent. Hence, we can with reasonable accuracy determine the angle of the separating cone in a two-cell flow solution, exclusively from knowledge of the flow at the bounding streamsurface. This means that we have solved our control problem.

Before concluding this section it should be said that we could equally well have derived formulae relating $\psi'(x_c)$ and x_s to one another. Indeed, from the calculations above we easily find that

$$\psi'(x_c) = \frac{2^{\frac{1}{3}}\left(-\alpha_1'\right)}{1-x_c^2}\left(\frac{(x_s-x_c)(1+x_c)}{1+x_s}\right)^{\frac{2}{3}}\Gamma_c^{\frac{4}{3}} + o\left(\Gamma_c^{\frac{4}{3}}\right) , \tag{48}$$

which we can easily invert. This is exactly the same solution as the one obtained in [17](except for a slight inaccuracy in their formula), but with the additional bonus that we have shown that their constant 1.2836 obtained numerically, is nothing but an approximation of $2^{\frac{1}{3}}\left(-\alpha_1'\right)$, which shows the mathematical origin of the constant.

There is, however, an important difference between the formulae obtained when using $\psi'(x_c)$ and those obtained using $\psi''(x_c)$, namely that in the former case the asymptotically leading terms come from the boundary layer around $x = x_c$ whereas in the latter case they come from the outer solution. Hence we can expect that formulae using $\psi'(x_c)$ are more sensitive to exact nature of the boundary layer.

4 Proof of Theorem 1

This section will be entirely devoted to proving Theorem 1, and we will begin by stating some known auxiliary results.

4.1 Some auxiliary results

The auxiliary function F in (3) was originally introduced by Goldshtik [6]. It was integrated once by Sozou [19] to yield

$$\left(1-x^2\right)F'' + 2xF' - 2F = \Gamma^2 + K \ , \tag{49}$$

where K is a constant. From the conditions (5) and (7) we have that

$$\lim_{x\to 1^-}\left(1-x^2\right)F'' = K \ . \tag{50}$$

If K is positive there must be a positive number ϵ such that in some neighbourhood of $x=1$ we have that

$$F'' > \frac{\epsilon}{1-x} \ , \tag{51}$$

which contradicts the condition $F'(1) = 0$. If K is negative, an analogous argument holds, and hence $K = 0$, which shows that in this case we may replace (3) with

$$\left(1-x^2\right)F'' + 2xF' - 2F = \Gamma^2 \ . \tag{52}$$

The major technical tool needed to prove Theorem 1 is the following result proved in [22]

Theorem 2 *(Real Analyticity)*
For any solution to (2),(4) and (52) with $x_c \neq -1$ such that $\psi \in C^1([x_c,1])$, $F \in C^2([x_c,1])$ and $\Gamma \in C^2[(x_c,1])$, and which satisfies the boundary conditions (5)-(9), there exists a domain $\Omega \subset \mathbf{C}$ which contains $[x_c,1)$ as well as a ball $B(1,r)$ for some $r>0$, such that ψ, F and Γ in fact belong to $A(\Omega)$. Here $A(\Phi)$ denotes the class of analytic (holomorphic) functions in Φ, and $B(a,r)$ is the disc in the complex plane with centre at a and radius r.

In addition we will on some occasions use the simple observation that since ψ is real analytic and satisfies (5) and (6) we may integrate (4) to obtain

$$\Gamma'(x) = \Gamma'(x_c)\exp\left(\int_{x_c}^{x}\frac{\psi(z)}{1-z^2}dz\right) \ . \tag{53}$$

From this it is evident that Γ' is monotonous in $[x_c,1]$ and thus if $\Gamma(x_c) = \Gamma_c > 0$ and $\Gamma(1) = 0$ it is clear that Γ' is negative in $[x_c,1]$ and hence that Γ is positive in $[x_c,1]$.

4.2 The proof of the theorem

To begin with, let us transform our equations to the form used by for example Serrin [15] and Yih *et al.* [24]. To this end let us make the substitutions

$$f(x) = -\frac{\psi(x)}{2(1-x^2)}, \quad \Omega(x) = \frac{\Gamma(x)}{\Gamma_c}, \quad G(x) = -\frac{2}{\Gamma_c^2}F(x) \ , \tag{54}$$

to obtain the system

$$\begin{aligned} f' + f^2 &= \left(\frac{\Gamma_c}{2}\right)^2 \frac{G(x)}{(1-x^2)^2} \ , & (55)\\ \left(1-x^2\right)G'' + 2xG' - 2G &= -2\Omega^2 \ , & (56)\\ \Omega'' + 2f\Omega' &= 0 \ . & (57) \end{aligned}$$

We remark that f is real analytic, because of (5)-(6) and the real analyticity of ψ. The equation (56) can be integrated to obtain

$$G(x) = 2(1-x)^2 \int_{x_c}^{x} \frac{t\Omega^2 \mathrm{d}t}{(1-t^2)^2} + 2x \int_x^1 \frac{\Omega^2 \mathrm{d}t}{(1+t)^2} + T(1-x)^2 \ . \tag{58}$$

If a solution to (55)-(57) satisfies the boundary conditions

$$\begin{aligned} f(x_c) &= 0 \ , & (59)\\ \lim_{x\to 1^-} |f(x)| &< \infty \ , \quad \lim_{x\to 1^-} \left|(1-x) f'(x)\right| < \infty \ , & (60)\\ \Omega(x_c) &= 1 \ , \quad \Omega(1) = 0 \ . & (61) \end{aligned}$$

then the boundary conditions for our original problem (5)-(9) are satisfied. Furthermore, we must transform our additional conditions to this form. In case (a) we specified $\psi'(x_c)$, but the definition of f tells us that once x_c is fixed this is equivalent to specifying $f'(x_c)$. However, from (55) it follows that if in addition Γ_c is given, this in turn implies that we have fixed $G(x_c)$ in case (a). In case (b) where we fix $\psi''(x_c)$ we must proceed differently. Now we differentiate (2) to obtain

$$\left(1-x^2\right)\psi'' + 2\psi - \psi\psi' = F' \ . \tag{62}$$

Thus, if we fix $\psi''(x_c)$ and x_c we have also fixed $F'(x_c)$, which means that if, in addition, we are given the value of Γ_c we have specified $G'(x_c)$. (It is not true that we have fixed $f''(x_c)$ though.) For the case (c) we specify T in (58), which can be interpreted as a boundary condition, by differentiating this equation twice to obtain

$$G''(x) = 2T - 2\frac{\Omega^2}{1-x^2} + 4\int_{x_c}^{x} \frac{t\Omega^2}{(1-t^2)^2}dt \tag{63}$$

which implies that $G''(x_c)\ (= 2T - 2/(1-x_c^2))$ is fixed if T and x_c is. From (56) we find that this in turn implies that the quantity $x_c G'(x_c) - G(x_c)$ is given.

Proof of Theorem 1

To prove Theorem 1 we will assume that there are two conically self-similar free-vortex solutions having the same values of x_c, Γ_c and either (a) $G(x_c)$ or (b) $G'(x_c)$ or (c) T, and form their differences. We will then show that the difference of the two Ω's must have an infinite number of zeros in the compact interval $[x_c, 1]$, and hence that the set of zeros must have a limit point. However, Theorem 2 tells us that a conically self-similar free-vortex solution to the Navier–Stokes equations is real analytic, and consequently the difference of two such solutions is also real analytic. Therefore the uniqueness theorem for real analytic functions implies that the two Ω's must me the same, which in turn implies that the two G's and the two f's must coincide.

To begin our analysis, let us fix the values of x_c, Γ_c and (a) $G(x_c)$ or (b) $G'(x_c)$ or (c) T and assume that we have two solutions (f_1, G_1, Ω_1) and (f_2, G_2, Ω_2) to our problem. Let us define

$$\mathcal{F} = f_1 - f_2\ , \quad \mathcal{G} = G_1 - G_2\ , \quad \mathcal{W} = \Omega_1 - \Omega_2, \quad \mathcal{T} = T_1 - T_2 \tag{64}$$

where T_1 and T_2 are the values of T of the two different solutions, which are the same in case (c) but not necessarily in the other two cases. These quantities clearly satisfy

$$\mathcal{F}' + (f_1 + f_2)\mathcal{F} = \left(\frac{\Gamma_c}{2}\right)^2 \frac{\mathcal{G}}{(1-x^2)^2} \tag{65}$$

$$\begin{aligned}\mathcal{G} &= 2(1-x)^2 \int_{x_c}^{x} \frac{t(\Omega_1+\Omega_2)\mathcal{W}}{(1-t^2)^2} dt + \\ &+ 2x \int_x^1 \frac{(\Omega_1+\Omega_2)\mathcal{W}}{(1+t)^2} dt + \mathcal{T}(1-x)^2\end{aligned} \tag{66}$$

$$\mathcal{W}'' + (f_1+f_2)\mathcal{W}' + \left(\Omega_1' + \Omega_2'\right)\mathcal{F} = 0 \tag{67}$$

as well as the boundary conditions

$$\mathcal{F}(x_c) = 0 \tag{68}$$

$$\mathcal{W}(x_c) = \mathcal{W}(1) = 0 \tag{69}$$

$$\mathcal{G}(1) = \mathcal{G}'(1) = 0\ . \tag{70}$$

In addition to these boundary conditions we have that

$$\mathcal{G}(x_c) = 0, \quad \text{for case (a)}, \tag{71}$$

$$\mathcal{G}'(x_c) = 0, \quad \text{for case (b)}, \tag{72}$$

$$\mathcal{G}''(x_c) = 0, \quad \text{for case (c)}. \tag{73}$$

For case (c) we could equivalently have expressed this as

$$x_c\mathcal{G}'(x_c) = \mathcal{G}(x_c) \tag{74}$$

and for case (a) we could equivalently have used the condition

$$\mathcal{F}'(x_c) = 0 \ . \tag{75}$$

Now let $Z(\mathcal{W})$ denote the number of zeros of $\mathcal{W}$ in the open interval $(x_c, 1)$, and let us analogously define $Z(\mathcal{G})$ and $Z(\mathcal{F})$. The key element of this proof is the establishment of the following proposition.

Proposition 1 *For cases (a) and (b), and if $x_c \geq 0$ for case (c) as well, $Z(\mathcal{W})$ cannot be finite.*

Let us for a moment assume that this proposition has been proven. This implies that the set of zeros for $\mathcal{W}$ must have a limit point in $[x_c, 1]$. Let A denote the (thus non-empty) set of limit points of this set. Trivially A must be closed in $[x_c, 1]$. On the other hand, Theorem 2 establishes that both Ω_1 and Ω_2 and hence $\mathcal{W}$ are real analytic in $[x_c, 1]$. Since a real analytic function is given by its Taylor series the function is either identically zero on the component or there exists a punctured neighbourhood around each zero where the function is non-zero. Thus every point $a \in A$ is in the interior of A and hence A is open. Since $[x_c, 1]$ is connected and A is nonempty we must thus have that $A = [x_c, 1]$, which is equivalent to saying that $\mathcal{W} \equiv 0$.

If $\mathcal{W} \equiv 0$ it immediately follows from (66) and (71)-(73) that $\mathcal{G} \equiv 0$. When this is substituted into (65) we obtain

$$\mathcal{F}(x) \exp\left(\int_{x_c}^{x} (f_1(t) + f_2(t))\, dt\right) = C \ . \tag{76}$$

However, the condition $\mathcal{F}(x_c) = 0$ requires that $C = 0$ and thus that $\mathcal{F} \equiv 0$. This concludes the proof of the theorem, provided that we can prove Proposition 1. **qed**

The basic idea in the proof of Proposition 1 is to assume that $Z(\mathcal{W}) = n$, where n is an arbitrary non-negative integer, and then to use properties of our system to show that this implies that $Z(\mathcal{W}) \geq n + 1$, which establishes the proposition by contradiction. However, zeros of even multiplicity where $\mathcal{W}$ does not change sign will cause difficulties for our argument, and therefore we will only consider zeros of odd multiplicity where $\mathcal{W}$ does change sign. We will denote the number of zeros of odd multiplicity of $\mathcal{W}$ in the open interval $(x_c, 1)$ by $O_Z(\mathcal{W})$, and the same notation will be used for other functions than $\mathcal{W}$. Of course $Z(\mathcal{W}) \geq O_Z(\mathcal{W})$ and thus proving that $O_Z(\mathcal{W})$ cannot be finite establishes the same conclusion for $Z(\mathcal{W})$.

The tool we will use most frequently is a simple fact in calculus, which is often termed Rolle's theorem, and which tells us that if a continuously differentiable function h is zero at two points a and b (the multiplicity of the zeros is immaterial) then either its derivative changes sign at some zero of odd multiplicity, strictly between a and b, or $h \equiv 0$. If the second alternative would hold anywhere in our subsequent argument we could immediately conclude that the concerned function would be identically zero, by invoking the uniqueness theorem for real analytic functions. This would immediately establish our theorem, and therefore for reasons of brevity this possibility will not be mentioned when Rolle's theorem is invoked below.

It turns out that one of the most difficult tasks of the proof is to get a lower bound of $O_Z(\mathcal{W})$ in terms of $O_Z(\mathcal{G})$. To accomplish this we introduce the auxiliary function

$$\mathcal{H} = \frac{\mathcal{G}}{(1-x)^2} , \tag{77}$$

which is similar to the one used by Yih *et al.*[24]. However, we will only be concerned with its derivative

$$\mathcal{H}' = \frac{(1-x)\,\mathcal{G}' + 2\mathcal{G}}{(1-x)^3} . \tag{78}$$

In fact we can now prove the following lemma.

Lemma 1 *If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{W}$ satisfy (65)-(70) and $\mathcal{H}'$ is as in (78) we have the following inequalities*

$$O_Z(\mathcal{G}) \geq O_Z(\mathcal{F}) \geq O_Z(\mathcal{W}) \geq O_Z(\mathcal{H}') \tag{79}$$

Proof The inequalities will be proved one at a time.

1. Since $\mathcal{F}(x_c) = 0$ we have that $\mathcal{F}$ has at least $O_Z(\mathcal{F}) + 1$ zeros in $[x_c, 1)$. Hence to prove the first inequality in Lemma 1 it suffices to prove that $\mathcal{G}$ changes sign at least once strictly between adjacent zeros of $\mathcal{F}$. Suppose to the contrary that there are two points x_1 and x_2 such that $\mathcal{F}(x_1) = \mathcal{F}(x_2) = 0$, and such that $\mathcal{G} \geq 0$ in (x_1, x_2) with equality at, at most, a finite number of points. (If $\mathcal{G} \leq 0$ we may of course interchange the role of the two solutions making it up.) If we now apply a Riccati transform to $f_1 = U_1'/U_1$ and to $f_2 = U_2'/U_2$ in (64) the equation (55) becomes

$$U_1'' - \frac{\Gamma_c^2}{2}\frac{G_1(x)}{(1-x^2)^2}U_1 = 0 \tag{80}$$

$$U_2'' - \frac{\Gamma_c^2}{2}\frac{G_2(x)}{(1-x^2)^2}U_2 = 0 \tag{81}$$

where $U_i^{'}(x_1) = 0$ for $i = 1, 2$. Since both f_1 and f_2 are real analytic, and hence non-singular, $U_i(x) \neq 0$ for $i = 1, 2$ and $x \in [x_1, 1]$. Therefore by Sturm's second comparison theorem, see for example [8, p. 229], we have that

$$f_1 = \frac{U_1^{'}}{U_1} > \frac{U_2^{'}}{U_2} = f_2 \ , \tag{82}$$

when $x \in (x_1, 1]$. This contradicts the assumption that $\mathcal{F}(x_2) = f_1(x_2) - f_2(x_2) = 0$. Hence $\mathcal{G}$ must change sign between adjacent zeros of $\mathcal{F}$, and thus we have that

$$O_Z(\mathcal{G}) \geq Z(\mathcal{F}) \geq O_Z(\mathcal{F}) \tag{83}$$

which establishes the first inequality in Lemma 1.

2. Since $\mathcal{W}(x_c) = \mathcal{W}(1) = 0$, Rolle's theorem implies that

$$O_Z\left(\mathcal{W}^{'}\right) \geq O_Z(\mathcal{W}) + 1 \ . \tag{84}$$

At zeros of multiplicity one of $\mathcal{W}^{'}$ equation (67) becomes

$$\mathcal{W}^{''} = -\left(\Omega_1^{'} + \Omega_2^{'}\right)\mathcal{F} \ . \tag{85}$$

In the last paragraph of the previous subsection we showed that $\Omega_1^{'} < 0$ and $\Omega_2^{'} < 0$ in $[x_c, 1]$. Hence the coefficient in front of $\mathcal{F}$ in (85) is positive at every single zero of $\mathcal{W}^{'}$, and hence $\mathcal{F}$ has the same sign as $\mathcal{W}^{''}$. More generally, at zeros of odd multiplicity p of $\mathcal{W}^{'}$ we have that $\mathcal{W}^{''}$ has a zero of even multiplicity $p-1$ at that point. Equation (67) and the above remark that the coefficient in front of $\mathcal{F}$ is positive together imply that $\mathcal{F}$ has a zero of even multiplicity $p-1$ at that point. Therefore $\mathcal{F}$ and $\mathcal{W}^{''}$ do not change sign at zeros of odd multiplicity of $\mathcal{W}^{'}$, and hence we may take a sufficiently small punctured neighbourhood around each of these zeros in which $\mathcal{F}$ has constant sign, which is the same as that of $\mathcal{W}^{''}$ in that neighbourhood. Now, from elementary calculus we know that $\mathcal{W}^{''}$ (and hence $\mathcal{F}$) must have different signs in sufficiently small punctured neighbourhoods around adjacent zeros of odd multiplicity of $\mathcal{W}^{'}$. Consequently, $\mathcal{F}$ must change sign at some point in between adjacent zeros of odd multiplicity of $\mathcal{W}^{'}$, and hence

$$\begin{aligned} O_Z(\mathcal{F}) &\geq O_Z\left(\mathcal{W}^{'}\right) - 1 \qquad (86)\\ &\geq O_Z(\mathcal{W}) \ , \qquad (87) \end{aligned}$$

which establishes the second inequality in Lemma 1.

3. If we use (66) we can obtain the following expression for $\mathcal{H}$

$$\begin{aligned} \mathcal{H} &= 2\int_{x_c}^{x} \frac{t(\Omega_1+\Omega_2)\mathcal{W}}{(1-t^2)^2}dt + \\ &+ 2\frac{x}{(1-x)^2}\int_x^1 \frac{(\Omega_1+\Omega_2)\mathcal{W}}{(1+t)^2}dt + \mathcal{T} \end{aligned} \tag{88}$$

and hence we have for $\mathcal{H}'$

$$\mathcal{H}'(x) = 2\frac{1+x}{(1-x)^3}K(x) \tag{89}$$

$$K(x) = \int_x^1 \frac{(\Omega_1+\Omega_2)\mathcal{W}}{(1+t)^2}dt\ . \tag{90}$$

Since the coefficient in front of K in (89) is positive for all $x \in (-1,1)$, we have that

$$O_Z(K) = O_Z\left(\mathcal{H}'\right). \tag{91}$$

Furthermore, $K(1) = 0$ and thus by Rolle's theorem

$$O_Z\left(K'\right) \geq O_Z(K). \tag{92}$$

However, we have that

$$K'(x) = -\frac{(\Omega_1+\Omega_2)\mathcal{W}}{(1+x)^2} \tag{93}$$

where the coefficient in front of $\mathcal{W}$ is negative in all of $[x_c, 1)$. Consequently,

$$\begin{aligned} O_Z(\mathcal{W}) &= O_Z\left(K'\right) && (94)\\ &\geq O_Z(K) && (95)\\ &= O_Z\left(\mathcal{H}'\right), && (96) \end{aligned}$$

which concludes the proof of the lemma.

qed

We now have the tools we need to prove Proposition 1.

Proof of Proposition 1

Assume that

$$O_Z(\mathcal{W}) = n \tag{97}$$

where n is an arbitrary positive integer. We will now establish that

$$O_Z\left(\mathcal{H}'\right) \geq n+1\ . \tag{98}$$

From the first two inequalities in Lemma 1 we obtain that

$$O_Z(\mathcal{G}) \geq O_Z(\mathcal{F}) \geq n \tag{99}$$

and in a right neighbourhood of each zero of odd multiplicity of $\mathcal{G}$, $\mathcal{G}$ and $\mathcal{G}'$ have the same sign, which is the opposite of the sign they have at the adjacent zeros of odd multiplicity of $\mathcal{G}$. Hence, (78) implies that $\mathcal{H}'$ changes sign at least once between adjacent zeros of odd multiplicity of $\mathcal{G}$. This establishes that

$$O_Z\left(\mathcal{H}'\right) \geq n-1 , \tag{100}$$

and thus we must find two more zeros. To this end let x_l and x_r denote the left- and the rightmost of the zeros of odd multiplicity of $\mathcal{G}$ in the open interval $(x_c, 1)$. Remember that the zeros found so far are located strictly between x_l and x_r. Our next task will be to prove that $\mathcal{H}'$ has at least n zeros of odd multiplicity in the semi-open interval $(x_l, 1)$.

If we do not have equality in both of the first two inequalities in Lemma 1 we know that $\mathcal{G}$ has at least $n+1$ zeros of odd multiplicity in $(x_c, 1)$, and consequently $\mathcal{H}'$ has at least n zeros of odd multiplicity in the open interval $(x_l, 1)$. Therefore, we may assume that equality holds in two first inequalities in Lemma 1, *i.e.* that

$$O_Z(\mathcal{G}) = O_Z(\mathcal{F}) = n . \tag{101}$$

Let q denote the sign of $\mathcal{W}$ in a right neighbourhood of x_c. From the proof of the second inequality in Lemma 1 we find that for this inequality to be an equality it is required that in a sufficiently small punctured neighbourhood of the leftmost zero of odd multiplicity of $\mathcal{W}'$ we must have that $sign(\mathcal{F}) = -q$. Furthermore, the proof of the second inequality in Lemma 1 also tells us that equality can only occur if $\mathcal{F}$ has constant sign to the left of the leftmost zero of odd multiplicity of $\mathcal{W}'$. This in turn implies that $sign(\mathcal{F}) = -q$ in a right neighbourhood of x_c. Together with Sturm's second comparison theorem this implies that $sign(\mathcal{G}) = -q$ in a right neighbourhood of x_c. Thus (101) tell us that $sign(\mathcal{G}(x)) = (-1)^{n+1} q$ for all $x \in (x_r, 1)$. Hence

$$sign\left(\mathcal{H}'(x)\right) = (-1)^{n+1} q \tag{102}$$

for x in some right neighbourhood of x_r, since $\mathcal{G}$ and $\mathcal{G}'$ have the same sign in a sufficiently small right neighbourhood of a zero of odd multiplicity of $\mathcal{G}$.

On the other hand, successive applications of l'Hospital's rule yield

$$\lim_{x\to 1^-} \mathcal{H}'(x) = \lim_{x\to 1^-} \frac{(1-x)\,\mathcal{G}' + 2\mathcal{G}}{(1-x)^3} \tag{103}$$

$$= \frac{G'''(1)}{6} . \tag{104}$$

From (3) we find that

$$\mathcal{G}''' = -\frac{2}{1-x^2}\left[\left(\Omega_1^2\right)' - \left(\Omega_2^2\right)'\right] . \tag{105}$$

However, from the discussion in the last paragraph of the previous subsection we know that $\Omega_1 > 0$ and $\Omega_2 > 0$ which implies that the sign of $(\Omega_1^2) - (\Omega_2^2)$ is the same as the sign of $\mathcal{W}$. Furthermore, both $(\Omega_1^2) - (\Omega_2^2)$ and $\mathcal{W}$ are zero at 1, and hence

$$sign\left[\left(\Omega_1^2\right)'(x) - \left(\Omega_2^2\right)'(x)\right] = -sign\left(\Omega_1^2(x) - \Omega_2^2(x)\right) = -sign\left(\mathcal{W}(x)\right) \tag{106}$$

in some left neighbourhood of 1. However, by definition we have that

$$sign\left(\mathcal{W}(x)\right) = (-1)^n q . \tag{107}$$

in this neighbourhood. To summarize, for x in some left neighbourhood of 1 we have that

$$\begin{aligned} sign\left(\mathcal{H}'(x)\right) &= sign\left(\mathcal{G}'''(x)\right) & (108)\\ &= -sign\left[\left(\Omega_1^2\right)'(x) - \left(\Omega_2^2\right)'(x)\right] & (109)\\ &= sign\left(\mathcal{W}(x)\right) & (110)\\ &= (-1)^n q & (111)\end{aligned}$$

A comparison between (102) and (111) tells us that $\mathcal{H}'$ changes sign somewhere between x_r and 1. Hence we have our additional zero, and we have proved that $\mathcal{H}'$ has at least n zeros of odd multiplicity in the open interval $(x_l, 1)$.

So far our analysis has been the same for all three of our cases, but now will use different methods to establish that for each of the three cases there is a zero of odd multiplicity of $\mathcal{H}'$ in the open interval (x_c, x_l). We will establish this for the three cases one at a time.

1. Case (a). We now know that $\mathcal{G}(x_c) = \mathcal{G}(x_l) = 0$, and that $\mathcal{G}$ does not change sign in (x_c, x_l) (though we cannot rule out the possibility of there being zeros of even multiplicity in the interval). Hence since $\mathcal{G}$ is real analytic elementary calculus tells us that $\mathcal{G}'$ has different signs in a right neighbourhood ofx_c and in a left neighbourhood of x_l. Furthermore, the multiplicity of a zero of $\mathcal{G}'$ at x_c or x_l is one order lower than the corresponding multiplicity of the zeros of $\mathcal{G}$ at these points. Hence (78) implies that $\mathcal{H}'$ has the same sign as $\mathcal{G}'$ in these neighbourhoods. Thus if x belongs to a right neighbourhood of x_c and y to a left neighbourhood of x_l we have that

$$sign\left(\mathcal{H}'(x)\right) = sign\left(\mathcal{G}'(x)\right) \tag{112}$$

$$= -sign\left(\mathcal{G}^{'}(y)\right) \tag{113}$$

$$= -sign\left(\mathcal{H}^{'}(y)\right) . \tag{114}$$

Consequently, $\mathcal{H}^{'}$ changes sign in (x_c, x_l). This concludes our task in case (a).

2. Case (b). We now know that $\mathcal{G}^{'}(x_c) = \mathcal{G}(x_l) = 0$. If in addition, $\mathcal{G}(x_c) = 0$ then we can apply the argument in Case (a), so we may assume that $\mathcal{G}(x_c) \neq 0$. Hence, $sign\left(\mathcal{H}^{'}(x_c)\right) = sign(\mathcal{G}(x_c))$. In addition, we know that the sign of $\mathcal{G}$ remains unchanged until x_l. Furthermore, in a left neighbourhood of x_l we know that the sign of $\mathcal{G}^{'}$ is the opposite to that of $\mathcal{G}$. Since the multiplicity of a zero of $\mathcal{G}^{'}$ at x_l is one order lower than the corresponding multiplicity of the zero of $\mathcal{G}$ at this point, it is clear from (78) that $\mathcal{H}^{'}$ has the same sign as $\mathcal{G}^{'}$ in this neighbourhood. Let therefore y be a point in a left neighbourhood of x_l, we then have

$$sign\left(\mathcal{H}^{'}(x_c)\right) = sign(\mathcal{G}(x_c)) \tag{115}$$

$$= sign(\mathcal{G}(y)) \tag{116}$$

$$= -sign\left(\mathcal{G}^{'}(y)\right) \tag{117}$$

$$= -sign\left(\mathcal{H}^{'}(y)\right) . \tag{118}$$

Consequently, $\mathcal{H}^{'}$ must change sign in the interval (x_c, x_l) in this case as well.

3. Case (c). In this case we will use the fact that

$$x_c \mathcal{G}^{'}(x_c) = \mathcal{G}(x_c) \tag{119}$$

If $x_c = 0$ then this implies that $\mathcal{G}(x_c) = 0$ which means that for this value of x_c, Case (c) coincides with the already treated Case (a). Let us therefore assume that $x_c > 0$, and substitution (119) into (78) in order to show that in this case

$$\mathcal{H}^{'}(x_c) = \frac{(1 + x_c)\,\mathcal{G}^{'}(x_c)}{(1 - x_c)^3} . \tag{120}$$

If $\mathcal{G}^{'} = 0$ we have Case (b) above, and we may therefore assume that this is not the case. Hence we have that

$$sign\left(\mathcal{H}^{'}(x_c)\right) = sign\left(\mathcal{G}^{'}(x_c)\right) . \tag{121}$$

By an argument entirely analogous to that in the previous two cases we find that for y in a left neighbourhood of x_l

$$sign\left(\mathcal{H}^{'}(y)\right) = sign\left(\mathcal{G}^{'}(y)\right) . \tag{122}$$

Now since $x_c > 0$, (119) implies that

$$sign\left(\mathcal{G}'(x_c)\right) = sign\left(\mathcal{G}(x_c)\right) \ . \tag{123}$$

Therefore, since $\mathcal{G}(x_l) = 0$ to be zero, the sign of $\mathcal{G}'$ in a left neighbourhood of x_l must be the opposite of that of $\mathcal{G}(x_c)$ and consequently to that of $\mathcal{G}'(x_c)$. This implies that for y in a left neighbourhood of x_l

$$\begin{aligned}
sign\left(\mathcal{H}'(x_c)\right) &= sign\left(\mathcal{G}'(x_c)\right) && (124)\\
&= sign\left(\mathcal{G}(x_c)\right) && (125)\\
&= sign\left(\mathcal{G}(y)\right) && (126)\\
&= -sign\left(\mathcal{G}'(y)\right) && (127)\\
&= -sign\left(\mathcal{H}'(y)\right) \ , && (128)
\end{aligned}$$

which proves that $\mathcal{H}'$ must have a zero in (x_c, x_l). (Note that if $x_c < 0$ then the second and all subsequent inequalities above are inverted and hence we cannot establish that $\mathcal{H}'$ must have a zero of odd multiplicity in (x_c, x_l). This is the only step of the proof which fails in this case.)

We have thus established that for all our three cases $\mathcal{H}'$ has at least one zero of odd multiplicity in (x_c, x_l), in addition to the at least n such zeros we had already shown that it had in the disjoint interval $(x_l, 1)$. To summarize we proved that

$$O_Z\left(\mathcal{H}'\right) \geq n + 1. \tag{129}$$

However, if we use the third inequality in Lemma 1 this tells us that

$$O_Z\left(\mathcal{W}\right) \geq O_Z\left(\mathcal{H}'\right) \geq n + 1 \tag{130}$$

which clearly contradicts (97) for any finite n. This concludes the proof of the proposition for $n \geq 1$.

When $n = 0$ is almost the same. Indeed, in this case we know that $\mathrm{sign}\left(\mathcal{W}\right) = q$ in the entire interval $(x_c, 1)$ except at a finite number of points where $\mathcal{W}$ have zeros of even multiplicity. By exactly the same argument as for $n > 1$ we have that for x in some left neighbourhood of 1 we have that

$$\begin{aligned}
sign\left(\mathcal{H}'(x)\right) &= sign\left(\mathcal{G}'''(x)\right) && (131)\\
&= -sign\left[\left(\Omega_1^2\right)'(x) - \left(\Omega_2^2\right)'(x)\right] && (132)\\
&= sign\left(\mathcal{W}(x)\right) && (133)\\
&= (-1)^n q \ . && (134)
\end{aligned}$$

In addition, we know that $\mathcal{W}'$ must have at least one zero, and just like above we know that either $\mathcal{F} = -q$ in a punctured neighbourhood of this point or $\mathcal{F}$ must have a zero. In the latter case $\mathcal{G}$ must have a zero in $[x_c, 1]$ and in the former case either $\mathcal{F}$ has constant sign throughout the interval, or $\mathcal{G}$ has a zero. If $\mathcal{G}$ has a zero, x_l the same arguments as for $n > 1$ may be used for each of the cases (a)-(c) to assure that $\mathcal{H}'$ has a zero in (x_c, x_l).On the other hand, if $\mathcal{F}$ has constant sign, then for y in a right neighbourhood of x_c we must have that $\text{sign}(\mathcal{F}(y)) = -q$, and hence by Sturm's second comparison theorem it follows that $\text{sign}(\mathcal{G}(y)) = -q$. In Case (b), this immediately implies that

$$\text{sign}\left(\mathcal{H}'(y)\right) = -q \ . \tag{135}$$

For Cases (a) and (c) the same conclusion holds, since in both cases $\mathcal{G}$ and $\mathcal{G}'$ have the same sign in a right neighbourhood of x_c. For Case (a) this is due to the fact that x_c is a zero of $\mathcal{G}$ and that any continuously differentiable function has the same sign as its derivative in a sufficiently small right neighbourhood of a zero. For Case (c) this is a direct consequence of (119) when $x_c > 0$, and when $x_c = 0$ Cases (c) and (a) coincide.

Hence, we have shown that either $\mathcal{G}$ and hence $\mathcal{H}'$ has a zero in $(x_c, 1)$ or both of (134) and (135) must hold. From this it is evident that $\mathcal{H}'$ must have at least one zero in $(x_c, 1)$. Now the final inequality of Lemma 1 implies that $O_Z(\mathcal{W}) \geq 1$. This concludes the proof of the proposition.

qed

5 Conclusion

The uniqueness result proved in this article tells us that within the class of conically self-similar free-vortex solutions a solution is uniquely determined by the opening angle of the bounding conical streamsurface, as well as the circulation and the radial velocity thereon. It is also shown that instead of the radial velocity we may take the surface radial tangential stress or the surface pressure as a parameter. These combinations of parameters can thus be used to control a conically self-similar free-vortex solution in terms of its properties at the bounding streamsurface only. Specifically, it is possible to control the opening angle of the separating cone in a two-cell flow in terms of these parameters.

In this paper, explicit formulae have been derived in the high Γ_c-limit, which interrelate the opening angle of the separating cone in a two-cell flow and either of the surface radial tangential stress or the surface pressure for given values of the opening angle of the bounding streamsurface and the circulation thereon. One striking feature of these formulae is that they show that the value of the opening angle of the separating cone in a two-cell flow is independent of the value of the viscosity when it is low enough, *i.e.* we have Reynolds number invariance for high Reynolds numbers. Some numerical checks have been performed, which show that whereas the lowest order formulae required high values of Γ_c to reach the asymptotic regime, the refined versions of the formulae are valid to within a few per cent even for moderate values of Γ_c.

The uniqueness question for the problem of Yih *et al.* [24] has been resolved with surprising results. For flows within a cone ($x_c \geq 0$) a uniqueness result is proven, which assures that no more than one solution can occur. For external flows ($x_c < 0$) this situation is different. Indeed, a specific case has been found numerically where at least two solutions exist. This striking property has been given a physical explanation based on a recent deciphering of the exact physical meaning of the problem itself.

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