Numerical experiment for unsteady motions of a rarefied gas caused by initially non-equilibrium states and nonuniform boundary conditions.

A.I. Khisamutdinov
Sobolev institute of mathematics, Acad. Sci. of Russia
630090, Akad. Koptyg pr. Novosibirsk, Russia.

A. Heintz
Department of Mathematics, University of Göteborg and
Chalmers University of Technology,
41296 Göteborg, Sweden

Abstract

A numerical experiment by means a Monte Carlo method with many particle system and without discretisation in time is used for the investigation of the evolution of a rarefied gas in a slab with initial distribution function having singular components and for non-uniform boundary conditions. A short description for the method and the algorithm is given. The qualitative dependence of the solution on the initial data and boundary conditions is discussed.

1 Introduction

Consider a rarefied gas in a slab with diffuse reflection on boundary planes that are kept at different temperatures T_1, T_2. Starting from a plane symmetric non-equilibrium initial state and under the influence of the combination of the initial disturbance and the disturbance from the walls the gas begins to evolve in time. The mathematical description of the problem is given by the Boltzmann equation with reflection boundary conditions. One suspects by physical reasons that with time tending to infinity the distribution of particles in the gas should tend to a stationary state corresponding to a given mean

density and the temperatures on the walls. From the mathematical point of view this question is not trivial.

The asymptotics of initial boundary value problems for the Boltzmann equation and the stability of the stationary solutions for small temperature changes on the boundary when solutions are close to a global equilibrium were investigated in detail in: [19], [4], [20].

The existence of a stationary solution to the Boltzmann equation for large difference between wall temperatures T_1, T_2 is proved only in a weak L_1 sense [3]. There are at present no uniqueness results and no rigorous mathematical results proving the stability of stationary solutions in the case when a stationary solution is strongly non-equilibrium. The lack of such rigorous stability results for the Boltzmann equation makes it interesting to investigate such an evolution and the stability of stationary solutions in a numerical experiment.

A systematical study of the development of the strongly non-equilibrium distributions based on BGK equations was made by the group of Y. Sone in Kyoto [1], [2]. The essential result of these papers is not only the description of the evolution of macro parameters: density, velocity, temperature. There are also detailed results on the development of the distribution function which as it is shown there has due to boundary conditions an interesting structure, including jumps in velocity variables.

There are few numerical results for the Boltzmann equation where a detailed analysis of the development of the distribution function is given. It is because of relatively strong requirements to computer memory and computational resources to reproduce details of such a multidimensional evolution problem for the full Boltzmann equation.

The objective of the present paper is to investigate by means of numerical experiments the character of relaxation of a rarefied gas for a family of space inhomogeneous problems having a non-equilibrium stationary states. In other words the paper concerns a natural from the physical point of view question how the stabilisation of different non-equilibrium initial states to a nonuniform stationary state depends on the geometrical structure of initial data and on boundary conditions. We are essentially concerned with a detailed analysis of the evolution of the distribution function. Boundary conditions are of diffuse reflection type with non-uniform temperature at the boundary. This guarantees that the stationary solution is thermodynamically non-equilibrium.

We use in the present work a Monte Carlo method by A. Khissamutdinov called in his publications Continuous Time Monte Carlo (CTMC) method and a code written by him and L. Sidorenko in Novosibirsk. The Continuous Time Monte Carlo methods are based on a special Markov process. They were introduced in [6]-[9] and developed further in [11]-[14] and [15]. Physically a system of moderately large number N of particles is modelled.

Mathematically CTMC methods belong to Monte Carlo methods for computation of iterations of linear operators [14], [7], [5]. By means of these methods linear master equations for the many particles system are solved.

When $n_s \to \infty$ where n_s is the mean initial number of particles one believes that the solution of the master equation also gives a solution to a smeared in space Boltzmann equation. Rigorous proofs for this fact exist in some particular situations, but not in full generality.

Different algorithms for realisation of CTMC methods are described and investigated in [16], [12], [11],[15]. Three different models for interaction between particles were introduced in [9],[8], [6], impoved in [13] and are used in these algorithms. In particular a space-inhomogeneous variant [16], [12], of the null-collision technique [17], [18] is used here.

Well known direct simulation Monte Carlo (DSMC) methods can be interpreted as approximations to CTMC methods because as it was shown in [11], [9], [5], DSMC methods can be deduced from CTMC methods by discretising the time variable and splitting the master equation into collisionless and space homogeneous relaxation steps.

In more details the method and the algorithm are described in Section 2.

The evolution of several types of initial states is investigated. These are:

1) A jump of density for two Maxwellian initial distributions with the same temperature.

2) In all space points a sum of a Maxwellian distribution with a delta function with velocity V_x directed along the planes. Qualitatively different effects are observed for different velocities V_x.

3) The same distribution as before but with delta function added only in a thin sub-domain of slab shape between planes.

These initial data are used for different, but quite small Knudsen numbers $Kn = 0.1$, $Kn = 0.2$ and different temperatures on the boundary, which correspond to different stationary solutions.

The main observation we have made based on numerical results is that after a few mean collision times and independently of the initial data the distribution function loses initial individual structure and becomes smooth and Maxwellian-like. After this it takes a longer time for through a diffusion like process to get to the stationary state. We make a remark that we investigate this dynamics in the case when the stationary state is space nonuniform and non-equilibrium.

It gives a numerical evidence to the property which is not mathematically proved that an arbitrary initial distribution function of a rarefied gas relaxates after several mean collisionless times to a smoother local "equilibrium" which then much longer goes to a stable stationary distribution.

2 Continuous time Monte Carlo methods and the algorithm used in the modelling.

2.1

We give here a short description of the method and the algorithm based on ideas and constructions from [9], [15], [14]. The motion of a rarefied gas in the slab is modelled by a system of a moderately large number of particles in a parallelepiped V with the boundary surface S consisting of sides S_1-S_6. Sides S_1, S_2 are orthogonal to z -axis. Other sides S_3 - S_6 are orthogonal to the x and y axises. Later we omit the domain of integration in integrals.

We use the following notations and assumptions:

$\bar{V} = V \cup S$, $\mathcal{G} = V \times R^3$, $\bar{\mathcal{G}} = \bar{V} \times R^3$;

N, $1 < N < \infty$ is the number of particles in V, fixed in the problem;

(r^i, v^i), $i = 1, ...N$, are coordinates and velocities of the particles, $x^i = (r^i, v^i)$;

$\mathcal{R} = (r^1, ...r^N)$; $\mathcal{V} = (v^1, ...v^N)$, $\mathcal{X} = (\mathcal{R}, \mathcal{V})$, $\mathcal{X} = (\mathcal{R}\ , \mathcal{V}) = (x^1, ..., x^N)$;

$\Gamma(x' \to x)$ is a density of probability for a particle with phase coordinates x' to change its coordinates to x when the particle reaches the boundary S;

the initial state of the gas in the slab we are interested in is uniform with respect to x and y coordinates. This lets us model the flow in the slab by the flow in the parallelepiped V with periodical boundary conditions on sides S_3 - S_6;

on sides S_1, S_2 we use the diffuse reflection conditions corresponding to temperatures T_1, T_2 on S_1, S_2. $\Gamma(x' \to x) = \Gamma_{diff}(v' \to v)\delta(r - r')$. On sides S_3 - S_6 periodical boundary conditions with respect to space variables are used. In the case when initial data are uniform with respect to x and y it gives a natural normalisation of mass and useful modelling of the volume of collision for the flow in the slab;

$w^{(1)}('v^l, 'v^l \to v^l, v^l | r^l, r^m)$ is a given differential frequency of elastic scattering of a pair (l, m) of particles with indexes l, m.

$$w^{(1)}(x^l, x^m) \equiv w_{lm}(\mathcal{X}) = \int w^{(1)}('v^l, 'v^l \to v^l, v^l | r^l, r^m) d'v^l d'v^m$$

is the integral frequency for the pair (l, m). We suppose that scattering is isotropic with respect to the center of mass;

$w(\mathcal{X}) = \sum_{(l,m)} w_{lm}(\mathcal{X})$ is the total frequency of interactions of N particles;

$\mathcal{P}_0 = \{\mathcal{X}_t\}_{t=0}^{\infty}$ is a Markov process with states in $(\bar{\mathcal{G}})^N$;

$\mu_t(\mathcal{X})$ is a phase density of the gas at the time t and $\mu_0(\mathcal{X}) = \mu_t(\mathcal{X})|_{t=0}$ is a given initial phase density such that such that it is a product of one particle densities; $\mu_0(\mathcal{X}) = \prod_{i=1}^{N} \mu_0^{(1)}(x^i)$; $\quad \mu_t^{(1)}(x^1) = \int \mu_t(\mathcal{X}) dx^2 ... dx^N$;

$f(t, x^1) = N \cdot \mu_t^{(1)}(x^1)$ is considered when $N \to \infty$ as a good approximation for the solution to the smeared with respect to space coordinates Boltz-

mann equation corresponding to the scattering model given by $w^{(1)}('v^l,' v^l \to v^l, v^l|r^l, r^m)$;

$\tilde{\omega}$, $\mathcal{M}(\cdot)$ and $Var(\cdot)$ denote the stochastic trajectory, the mean value and the dispersion for the process $\mathcal{P}_0$;

$\mathcal{E}(x|t^*)$ is a given measurable function on $\bar{\mathcal{G}}$ with t^* being a parameter which is a fixed time instant here $0 < t^* < \infty$.

The functional $\mathcal{J}_{t^*} = \int f(t^*, x)\mathcal{E}(x|t^*)dx$ represents physical variables we are going to compute.

2.2

Taking into account that the initial data are taken uniform with respect to r_x, r_y the process $\mathcal{P}$ is a process with jumps with respect to the velocities and continuous with respect to the space coordinate r_z of particles. Jumps take place instantly and are of two types:

a) the change of velocities of particle colliding in the volume

b) the change of velocity of the particle colliding with S_1 or S_2.

Between collisions particles are moving along straight lines. The velocities are continuous from right with respect to the time variable. The initial state $\mathcal{P}_0$ given by $\mu_0^{(1)}(x)$ is considered as a state after a jump.

Let $\mathcal{X}_t$ be a state of the process at time t and $t_{sc} = t_{sc}(t)$ and $t_{ex} = t_{ex}(t)$ be dependent on t random times of the next jump. Namely $t_{sc} > t$ is the minimal time of the next collision between two particles and $t_{ex} > t$ is the minimal time when a particle reaches the boundary S. The time $t'(t)$ of the next jump is distributed as $\min\{t_{sc}, tex\}$. The modelling of these random times is described in detail in [10]-[12],[14]-[16] .

We give here the master equation for the process $\mathcal{P}_0$:

$$\frac{\partial}{\partial t}\mu_t(\mathcal{X}) + (\mathcal{V}, \nabla_{\mathcal{R}}\mu_t(\mathcal{X}) + w(\mathcal{X})\mu_t(\mathcal{X}) = \left[\hat{S}^+\mu_t\right](\mathcal{X}) \tag{2.1}$$

where

$$(\mathcal{V}, \nabla_{\mathcal{R}}\mu_t(\mathcal{X}) \equiv \sum_{i=1}^{N}(v^i, \nabla_{r^i}\mu_t(\cdot)) \tag{2.2}$$

$$\left[\hat{S}^+\mu_t\right](\mathcal{X}) = \int \mu_t(\mathcal{R}, \mathcal{X}) \times S('\mathcal{V} \to \mathcal{V}, \mathcal{R})d'\mathcal{V} \tag{2.3}$$

$$S('\mathcal{V} \to \mathcal{V}, \mathcal{R}) = \sum_{(l,m)} w^{(1)}('v^l,' v^m \to v^l, v^m|r^l, r^m) \prod_{i=1,\ i\neq l,m}^{N} \delta(v^i -' v^i) \tag{2.4}$$

The discussed above initial and boundary conditions should be used together with this equation.

A random trajectory of the process $\mathcal{P}_0$ can be written down as $\tilde{\omega} = (t_0, \mathcal{X}_{t_0}), (t_1, \mathcal{X}_{t_1}), ..., (t_j, \mathcal{X}_{t_j})$ namely in the form of the trajectory of a Markov chain. Here each state with $j \geq 1$ is a state after the corresponding jump. For brevity we write $\tilde{\omega} = (t_0, \mathcal{X}_0), (t_1, \mathcal{X}_1), ..., (t_j, \mathcal{X}_j)$

With the constructed Markov chain the following "imitational" estimator for the functional $\mathcal{J}_{t^*}$ of the process will be used:

$$\theta_0(\tilde{\omega}) = \mathcal{E}^N(t^*, \mathcal{X}_{t^*}), \tag{2.5}$$

where $\mathcal{E}^N(t, \mathcal{X}_t) = \sum_{i=1}^N \mathcal{E}(x_t^i|t)$.

The algorithm of modelling consists of the following steps.

1. Modelling the initial velocities and the coordinates of N particles. For each particle we choose coordinates randomly with density corresponding to the initial density profile we want to reproduce. After that for each particle an initial velocity corresponding to a given initial distribution with respect to velocities is generated. In our case it is typically a linear combination of Maxwellian distributions and delta functions. We start at time $t = 0$ and follow the jumps of the system corresponding to collisions of particles with each other and with the boundary and stop the modelling at time $t = T$ which we are interested in. We keep for each particle the time and the coordinate of its last collision with the boundary or with another particle. It lets us to avoid numerous recalculation of the coordinates of the particles.
2. Simulation for the time of collision with the density $W \exp(W(t - t'))$. The constant frequency W here is an appropriate upper bound for the frequency of binary collisions of particles with each other in the volume. We use in the present paper an estimate in the form $W = (W^{(1)} N_{1M} N_t)/2$ with $W^{(1)} = 2 \max |v| \sigma_0$, $N_{1M} = \bar{N}_M + 3\sqrt{\bar{N}_M}$, where $\bar{N}_M = \max\{\bar{N}_t\}$ and is the maximal mean number of particles that can collide with a given one in its interaction volume. The interaction volume is taken enough small - 1/5-th of the mean free path along the coordinate across the slab times a unit area in the orthogonal plane.
3. We compare the time t' of collision in the volume with the minimal over all particles time t_b for crossing the boundary. If $t' > t_b$ we calculate for the particle closest to the boundary the coordinate and the time when it reaches the boundary and according to the diffuse reflection model compute the random velocity after collision. After that the new minimal time t_b for reaching the boundary is recalculated.
4. If the time t' of collision in the volume is smaller then t_b we modell the collision between two molecules. It consists of several steps.
   (a) We choose first a uniformly distributed random number of a molecule and a random number of the second molecule.

(b) We check if they are at the preliminarily declared small distance ρ in z that lets them collide. If it is the case, a vector uniformly distributed on a unit sphere is chosen and new velocities for these two particles are calculated according to the hard sphere model.

(c) If a chosen partner is able to collide with the chosen particle we choose one more random number of the partner and repeat the procedure. This algorithm for modelling a collision is repeated up to the step when the number of already checked potential partners reaches an upper bound $(N-1)/N_{1M}$. Then we call the collision "fictions" and do not change velocities of particles at the time instance t'.

5. After a change of the velocity of at least one particle the minimal time t_b is recalculated.

6. The modelling continues up to the time T. When the minimal time of the next collision in the volume or with the boundary reaches T all coordinates of the particles at time T are calculated and the input of particles to the distribution function and to macro parameters - density, temperature, velocity are calculated.

7. The same algorithm is repeated enough many times to collect a large statistics that lets us get an approximation of mean parameters and the distribution function with small dispersion.

3 Results and discussion

We describe in this section results of the numerical experiments. Several families of essentially different initial data and boundary conditions were investigated. The coordinate orthogonal to planes is denoted by z, one coordinate parallel to planes is denoted by x. The length unit is equal to mean free path in the volume. It takes only several mean free times between collisions to reach a distribution close to local Maxwellian out of the Knudsen layer.

Results on the classical problem with initial data when the gas has Maxwellian distribution with zero mean velocity and constant temperature equal to one. In two halfs of the slab density has two different constant values. The interaction of smeared shock waves and expansion waves from this initial jump and from the walls having different temperatures is observed during several mean free times. After that distribution function of the gas becomes smooth and Maxwellian-like a slow process of energy exchange between the gas and the walls turns the gas to the stationary state. Corresponding stationary profiles of density and temperature are almost linear in our case with relatively small Knudsen number $Kn = 0.1$. These results are shown in Fig.1 - Fig.2.

Mathematical reasons from the analysis of the stationary Boltzmann equation [3] show that the concentration of particles having velocities almost parallel to the boundary planes of the slab is dangerous for regular behaviour of

the stationary solution. It means that the relaxation to a stationary state for distributions having such concentrations of particles can be much slower than those that do not include such singular parts. We investigated two different situations when the initial distribution has constant density but the distribution with respect to velocities is very non-equilibrium. It includes here a singular component consisting of molecules with velocities parallel to the slab. In one case the initial distribution in the whole slab is a sum of a Maxwellian distribution with temperature 1 and a delta function corresponding to particles with velocity 0.0, 0.5, 3.0. In another case this beam - like monochromatic component is present only in a part of the slab: in a relatively thin slab the center or close to one of the boundaries.

In these cases several interesting effects were observed. The interaction of the cold monochromatic beam of fast or slow molecules with the smoother part of the distribution with finite temperature depends on the velocity of the beam and its location with respect to the boundary.

One typical case is when the velocity of the beam is high, larger than 2. During the time of several mean free times between collisions fast particles from the beam are attracting particles around it. The flow directed to the center of the slab from the neighbourhood of the beam is clearly visible. The density drops around the beam and by this way a kind of ejection effect is observed at the first stage of the evolution of this type of flow. At the same time the temperature in the beam and around it increases because the kinetic energy of the beam passes through collisions with thermalised particles to the kinetic energy of chaotic motion. Despite of that the temperature has a small local minimum at the center of the beam even after several mean collision times. This ejection effect and the effect of smearing of the beam depends strongly on parameters of the beam. If the velocity of particles in the beam is large enough $V_x > 1.5$, the thermalisation of particles hard spheres is fast. The delta function component in the distribution function disappears after a very short time less than mean free time between collisions. The z - component of velocity is directed to the center of the beam and is equal to zero on the boundaries. The x- velocity has a maximum at the center of the beam. These results are shown for different times in Fig.7 - Fig.9, Fig.12, Fig.13.

If the velocity of particles in the beam is small $V_x < 1$ the thermalisation of hard spheres is relatively slow, because the cross section is proportional to the relative velocity. The delta function from the initial distribution is present after several mean free times between collisions. The profiles of macroscopic parameters evolve also in a different way here. The behaviour of temperature has a qualitatively another character. In contrast to the previous case temperature attains in the beam a global minimum which is still present in time $t = 4$ and has no other local minima. The density has the behaviour similar to one for a fast beam, but with an additional local minimum in the center of the beam and a zone of low density almost approaching zero around the beam generating a specific non-monotone profile of the density.

The z - component of velocity is directed to the center of the beam and is equal to zero on the boundaries. The x- component of velocity has the only maximum at the center of the beam. These results are shown in Fig.3 - Fig.6, Fig.10, Fig.11.

Conclusions.

The main result of the present paper is that for a wide class of non-equilibrium initial data with and for non-uniform boundary conditions it is shown that it takes only several mean free times for the distribution function of the gas to become rather smooth and to lose its individual properties including possible singular behaviour at the beginning.

Another result is a useful testing of the continuous time Monte Carlo method in quite different situations. In particular it is shown that valid computations for the evolution of the distribution function of a rarefied gas can be done for large intervals in time and for strongly nonequilibrium initial data.

Acknowledgements The work was supported in part by the grant from for joint research with states from the former Soviet Union from the Royal Swedish academy of Sciences (No 12505) and by TFR grant (No 222 - 97 - 370). The first author is thankful to Leif Arkeryd for useful discussions and hospitality during the period of his work at Chalmers. The authors wish to express their gratitude to L.L. Sidorenko for his help while carrying out this research.

References

[1] K. Aoki, Y. Sone, K. Nishino, and H. Sugimoto, *Numerical analysis of unsteady motion of a rarefied gas caused by sudden chaneges of wall temperature with special interest in the propagation of a discontinuity in the velocity distribution function*, in 17-th International symposium on RGD, (Aahen, 1990), VCH, Weinheim, 1991, pp. 222–231.

[2] K. Aoki, I. Yamashita, and Y. Sone, *Numerical analysis of steady flows of a gas condensing on or evaporating from its plane condensed phase on the basis of kinetic theory: effect of gas motion along the condensed phase*, in 15-th International symposium on RGD 2, 1991 (Grado, 1986), Teubner, Stuttgart, 1986, pp. 323–333.

[3] L. Arkeryd and A. Nouri, *The stationary Boltzmann equation in the slab with given weighted mass for hard and soft forces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 27 (1998), pp. 533–556.

[4] A. G. Heintz, *Solvability of the initial-boundary value problem for a nonlinear Boltzmann equation in a bounded domain*, in Aerodynamics of rarefied gases, No. 11, Leningrad. Univ., Leningrad, 1983, pp. 166–174.

[5] A. I. Khisamutdinov, *Kinetic equation for rarefied gases and Monte Carlo method for iterations of linear operators.*

[6] ——, *A simulation Monte Carlo method for modeling the dynamics of rarefied gases* , Preprint N 599, Akad. Nauk SSSR Sibirsk. Otdel. Vychisl. Tsentr, Novosibirsk, 1985.

[7] ——, *Justification of simulation and nonsimulation methods for statistical modeling of rarefied gas kinetics* , Preprint N 565, Akad. Nauk SSSR Sibirsk. Otdel. Vychisl. Tsentr, Novosibirsk, 1985.

[8] ——, *A simulation method for statistical modeling of rarefied gases*, Dokl. Akad. Nauk SSSR, 291 (1986), pp. 1300–1304.

[9] ——, *Simulation statistical modeling of the kinetic equation of rarefied gases*, Dokl. Akad. Nauk SSSR, 302 (1988), pp. 75–79.

[10] ——, *On Monte Carlo algorithms with continuous time for a kinetic equation of rarefied gases* , Preprint N 35, Akad. Nauk SSSR Inst. of Mathematics, Novosibirsk, 1989.

[11] ——, *Algorithms with "different time coordinates" of Monte Carlo methods for a nonlinear "smoothed" Boltzmann equation*, Dokl. Akad. Nauk SSSR, 316 (1991), pp. 829–833.

[12] ——, *Algorithms with fictitious collisions of the Monte Carlo method with continuous time for the Boltzmann equation*, Dokl. Akad. Nauk, 328 (1993), pp. 662–665.

[13] A. I. Khisamutdinov and N. F. Kobeleva, *"Nonsymmetric" interactions in Monte Carlo methods with continuous time and the approximation of the Boltzmann equation*, Mat. Model., 4 (1992), pp. 110–119.

[14] A. I. Khisamutdinov and S. O. Shilova, *The nonlinear Boltzmann equation, methods with "continuous time", and some general constructions of Monte Carlo methods*, Sibirsk. Mat. Zh., 39 (1998), pp. 456–473, iv.

[15] A. I. Khisamutdinov and L. L. Sidorenko, *Algorithms of the continuous-time Monte Carlo method for the kinetic equation of rarefied gases*, Mat. Model., 6 (1994), pp. 47–60.

[16] ——, *Monte Carlo fictitious collision algorithms for nonlinear Boltzmann equation*, Monte Carlo Methods Appl., 1 (1995), pp. 221–240.

[17] K. Koura, *Null-collision technique in direct simulation Monte Carlo method*, Phys. Fluids, 2 (1990), pp. 3509–3511.

[18] ——, *A sensitive test for accuracy in evaluation of molecular collision number in the direct simulation method*, Phys. Fluids, 2 (1990), pp. 1287–1289.

[19] N. B. Maslova, *Nonlinear evolution equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1993. Kinetic approach.

[20] S. Ukai, *Solutions of the Boltzmann equation*, in Patterns and waves, North-Holland, Amsterdam, 1986, pp. 37–96.

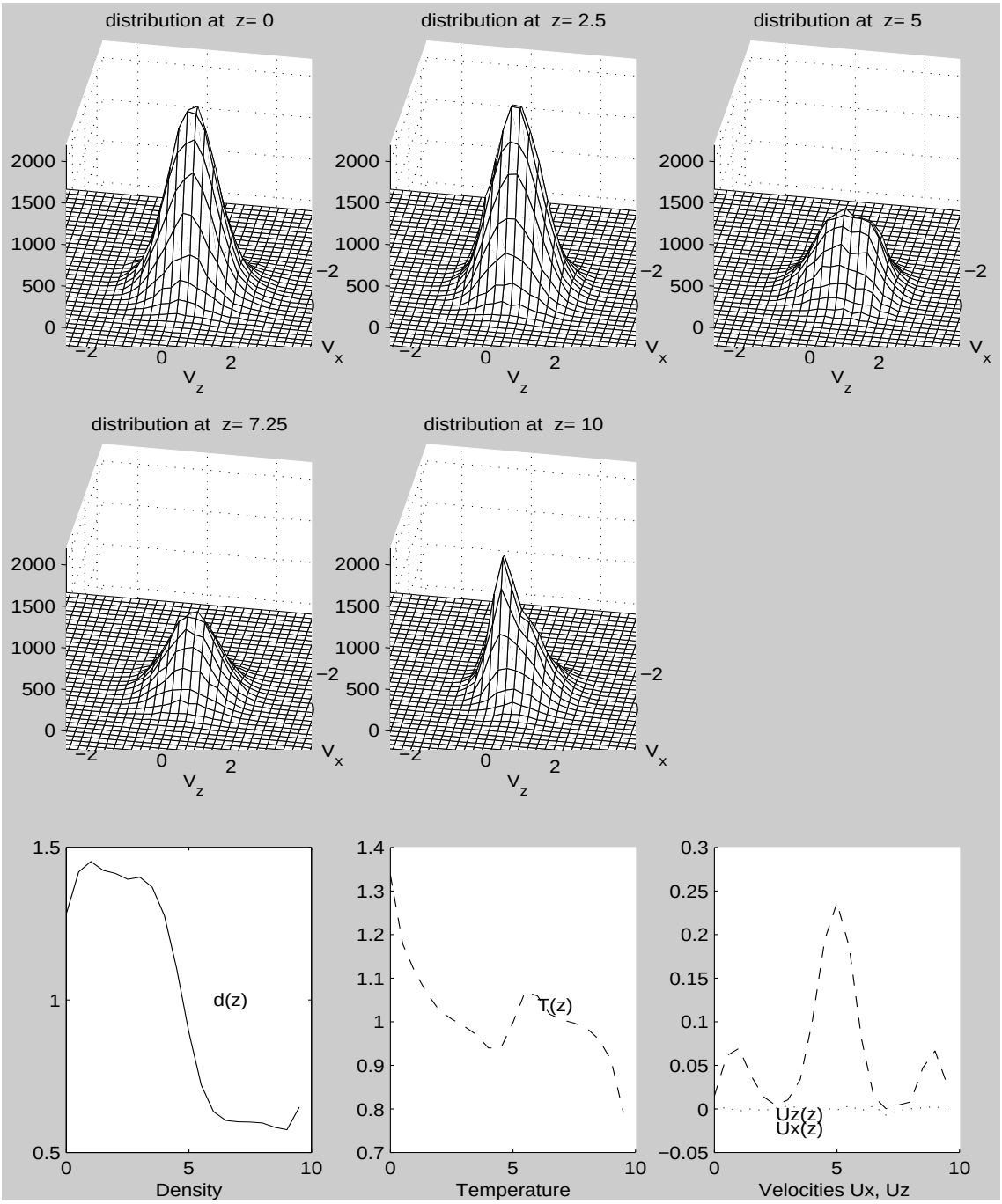

Figure 1: Development of the initial state which is a maxwellian distribution with densities in two halfs of the slab corresponding to 14 and 6 particles on the mean free path. $T_1 = 2$, $T_2 = 0.5$, Time $t = 1$, $Kn = 0.1$.

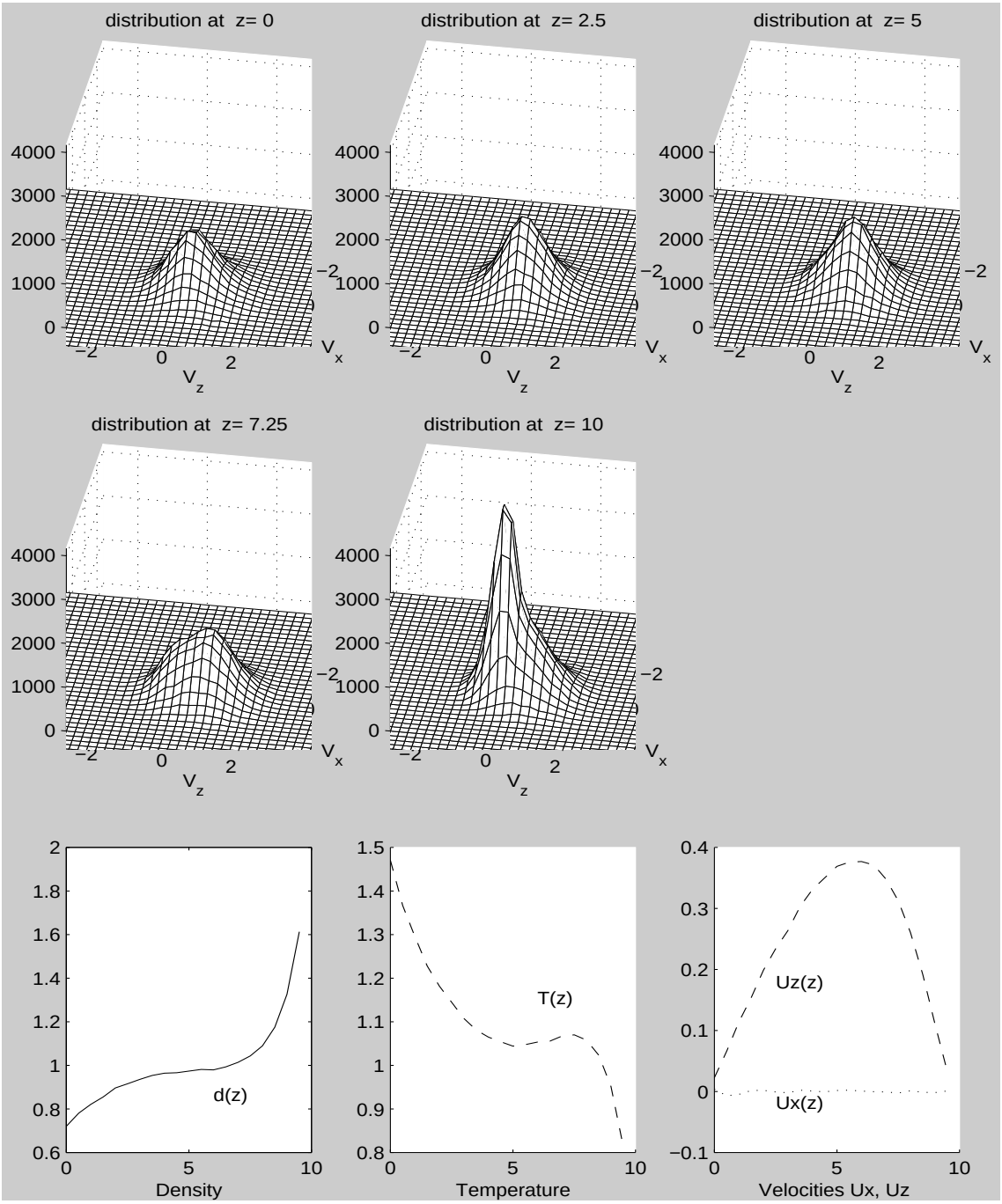

Figure 2: Development of the initial state which is a maxwellian distribution with densities in two halfs of the slab corresponding to 14 and 6 particles on the mean free path. $T_1 = 2$, $T_2 = 0.5$, Time $t = 8$, $Kn = 0.1$.

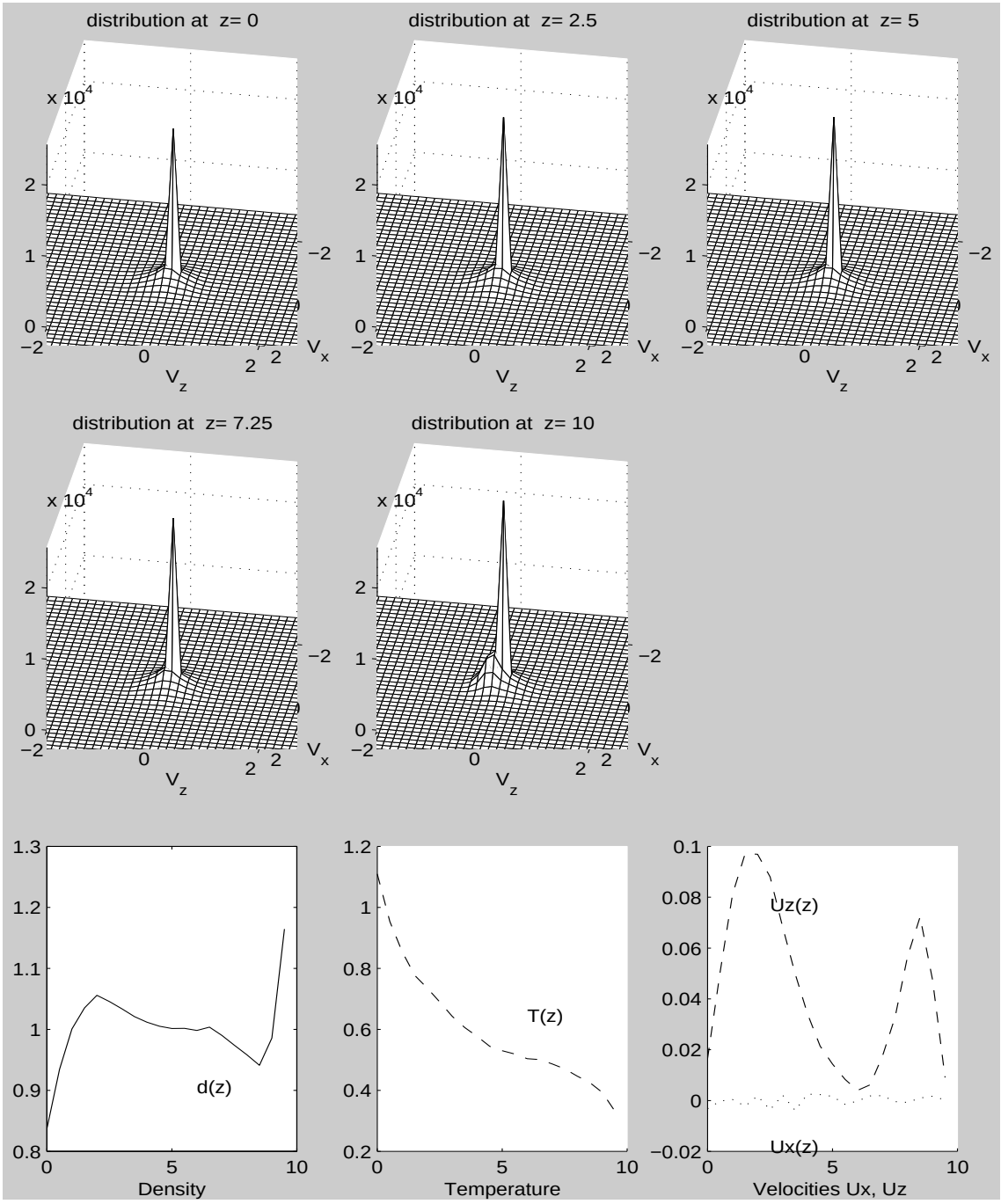

Figure 3: Development of the initial state which is a uniform mixture of a 50% maxwellian distribution and 50% particles at rest. $T_1 = 4$, $T_2 = 0.5$, Time $t = 2$, $Kn = 0.1$.

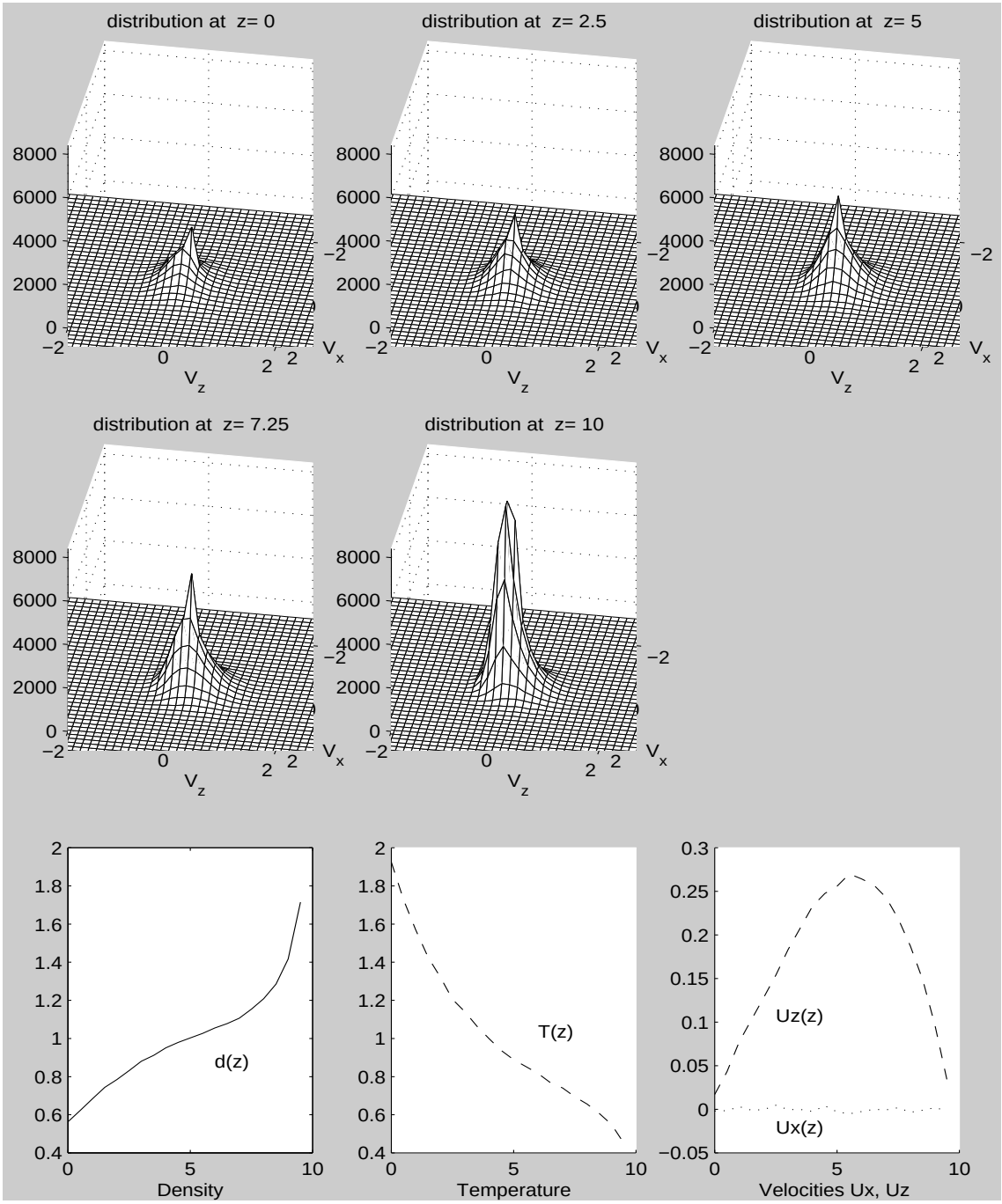

Figure 4: Development of the initial state which is a uniform mixture of a 50% Maxwellian distribution and 50% particles at rest. $T_1 = 4$, $T_2 = 0.5$, Time $t = 8$, $Kn = 0.1$.

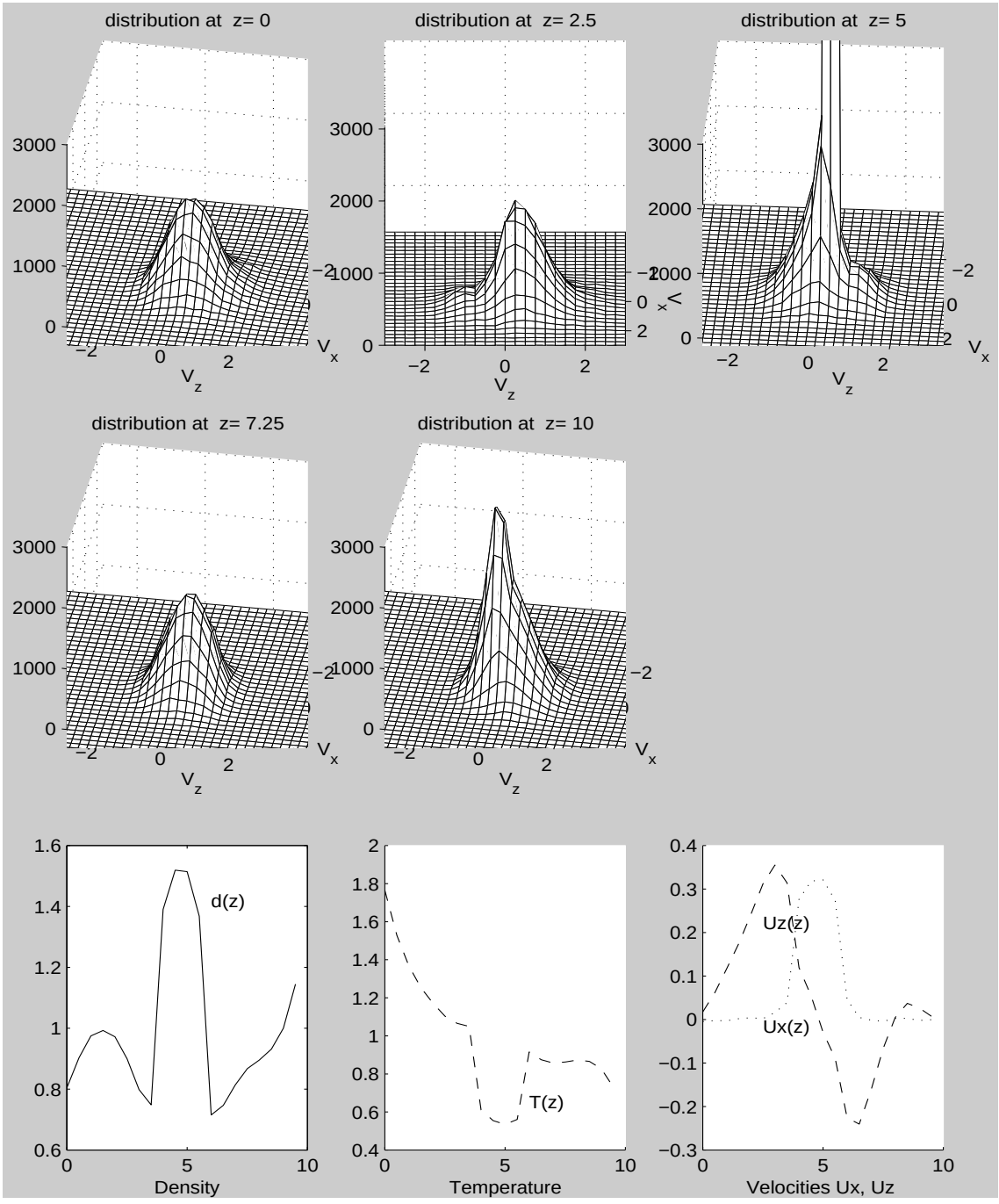

Figure 5: Development of the initial state as a Maxwellian distribution for $0 < 4 < 0$, $6 < z < 10$ and is mixture of 10 % of Maxwellian distribution and 90% of particles with velocity $V_x = 0.5$, $V_y = 0.0$, $V_z = 0.0$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 2$, $Kn = 0.1$.

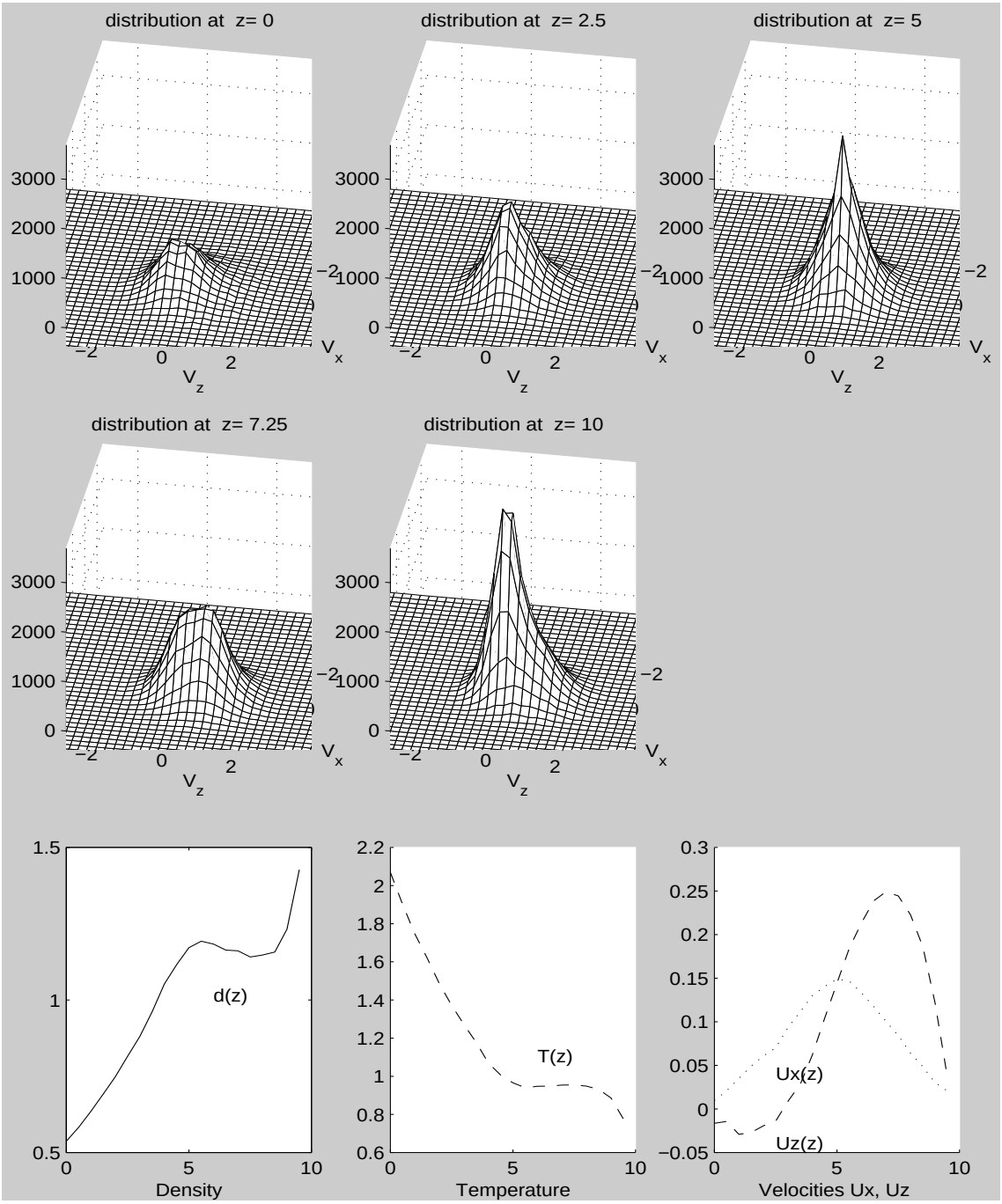

Figure 6: Development of the initial state which is a Maxwellian distribution for $0 < z < 4$, $6 < z < 10$ and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 0.5$, $V_y = 0.0$, $V_z = 0.0$ for $4 \leq z \leq 6$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 8$, $Kn = 0.1$.

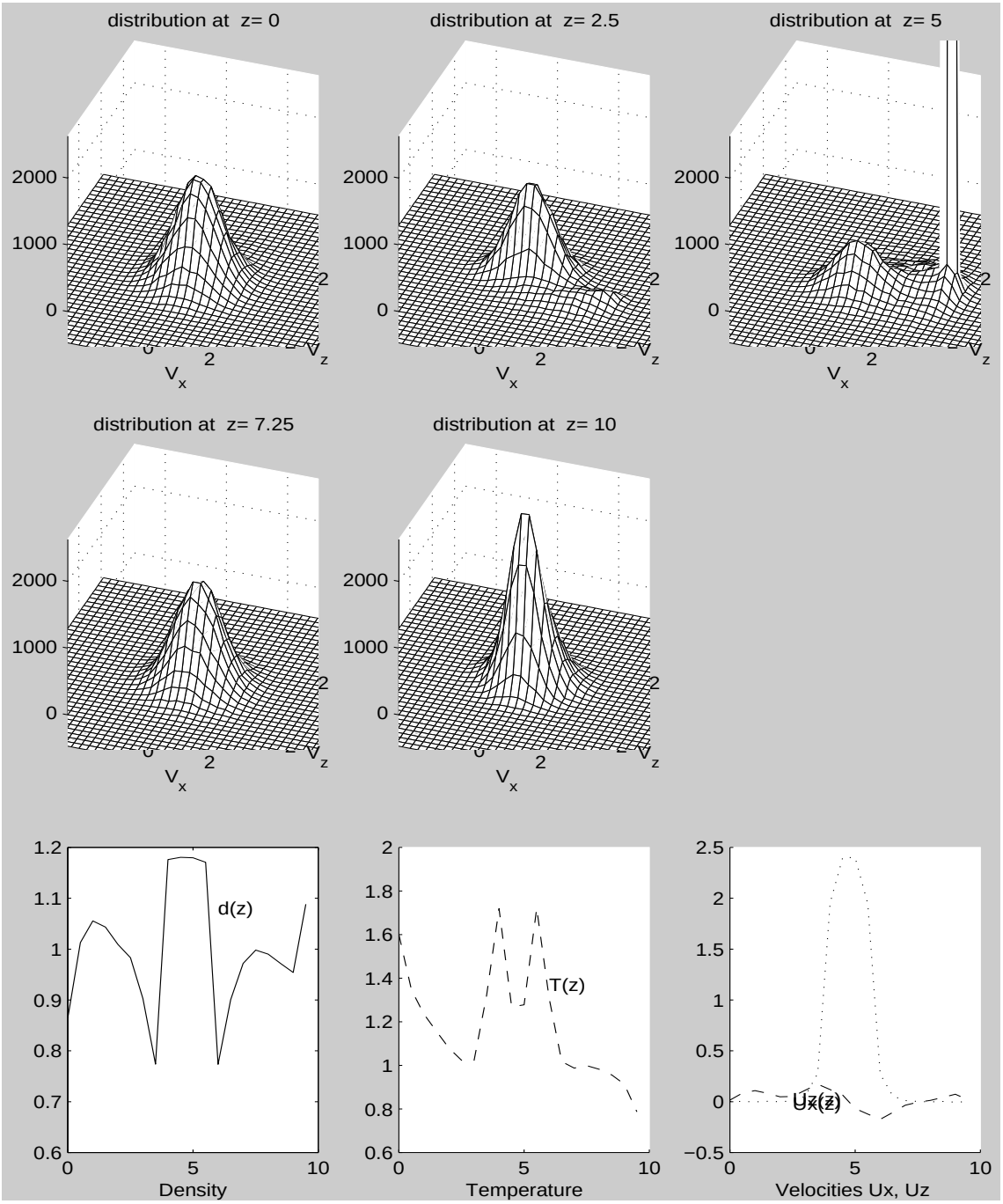

Figure 7: Development of the initial state which is a Maxwellian distribution for $0 < z < 4$, $6 < z < 10$ and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 3$, $V_y = 0.0$, $V_z = 0.0$ for $4 \leq z \leq 6$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 1$, $Kn = 0.1$.

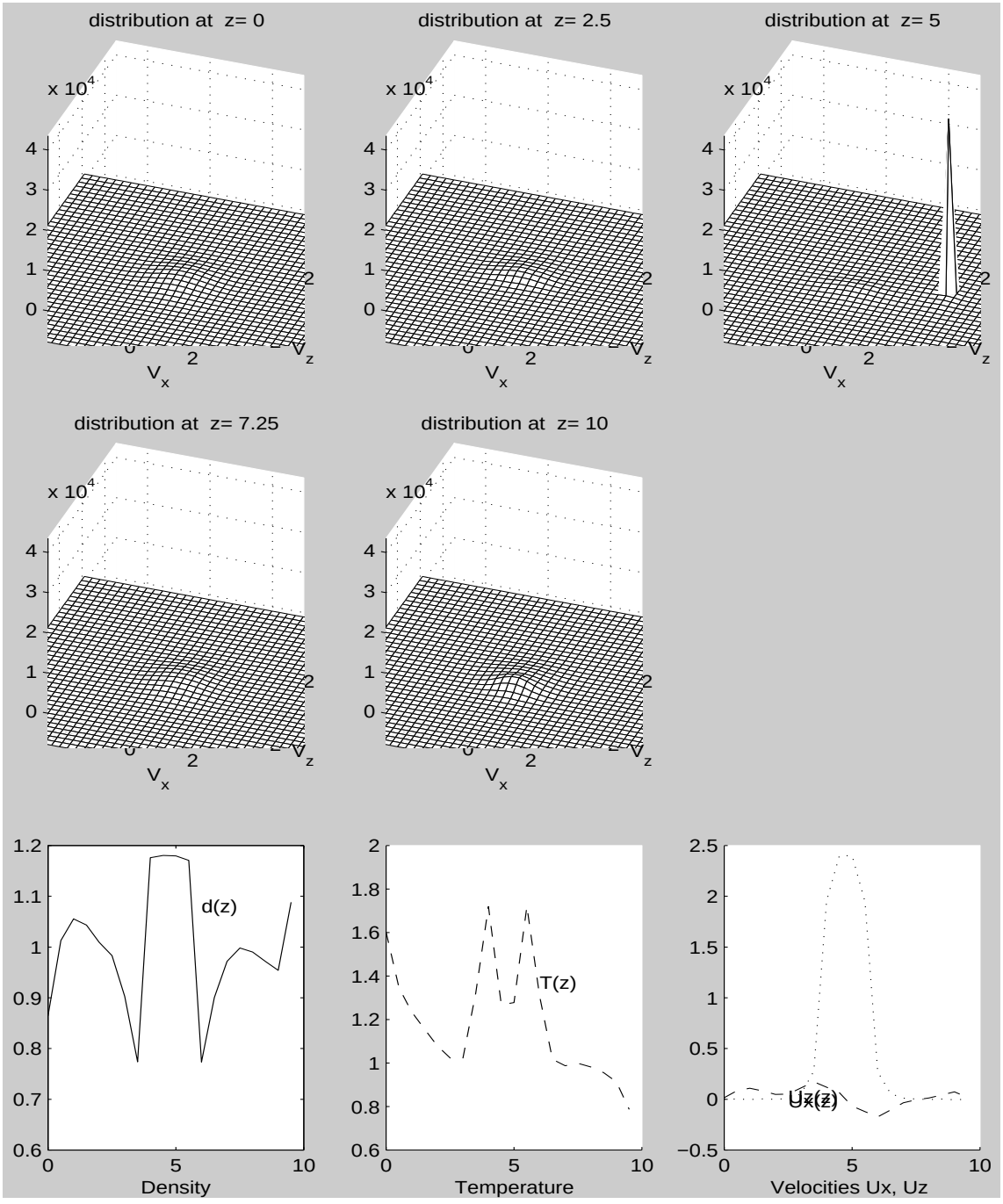

Figure 8: Development of the initial state which is a Maxwellian distribution for $0 < z < 4$, $6 < z < 10$ and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 3$, $V_y = 0.0$, $V_z = 0.0$ for $4 \leq z \leq 6$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 1$, $Kn = 0.1$. Units with respect to the maximum of the distribution function.

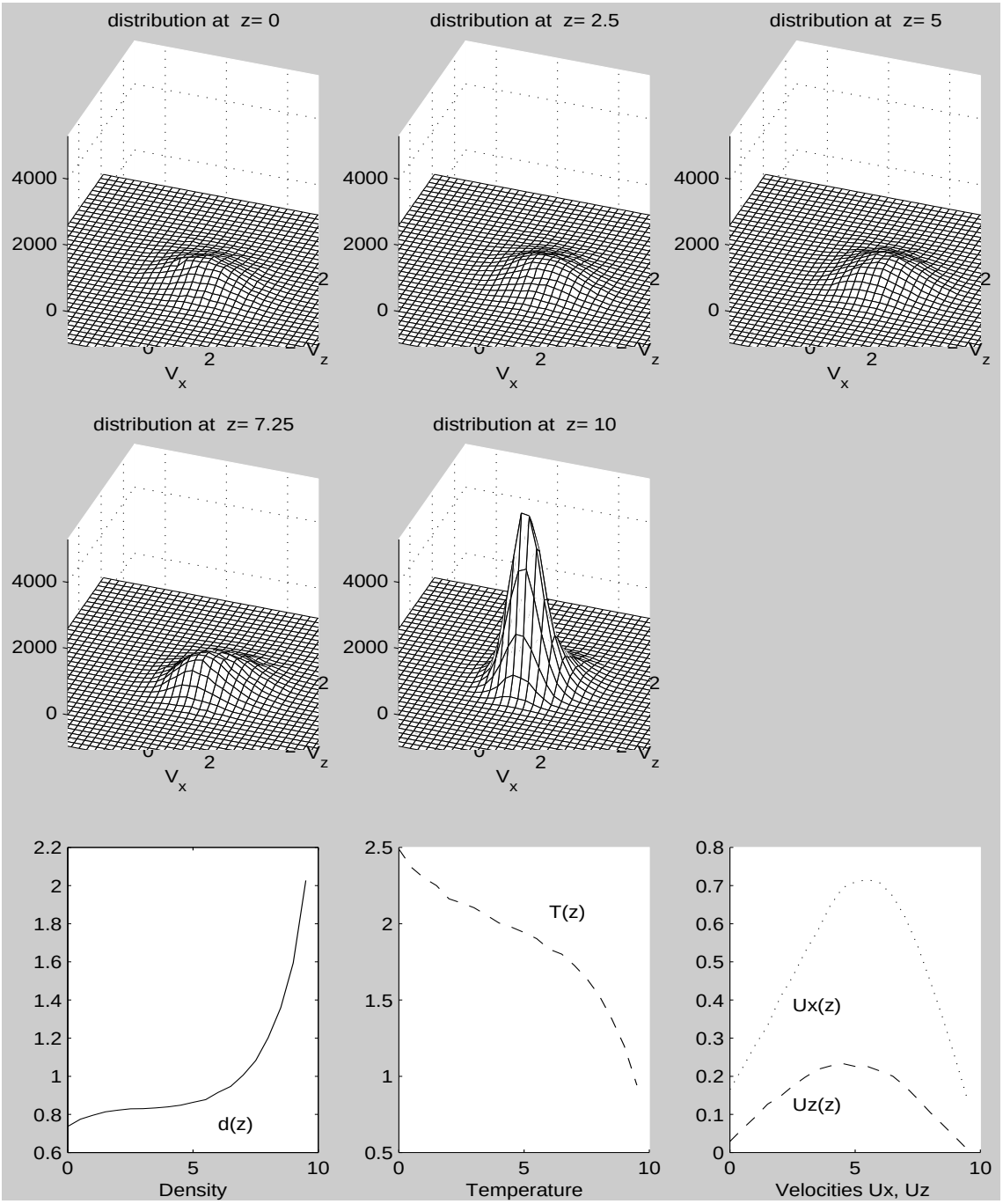

Figure 9: Development of the initial state which is a Maxwellian distribution for $0 < z < 4$, $6 < z < 10$ and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 3$, $V_y = 0.0$, $V_z = 0.0$ for $4 \leq z \leq 6$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 8$, $Kn = 0.1$.

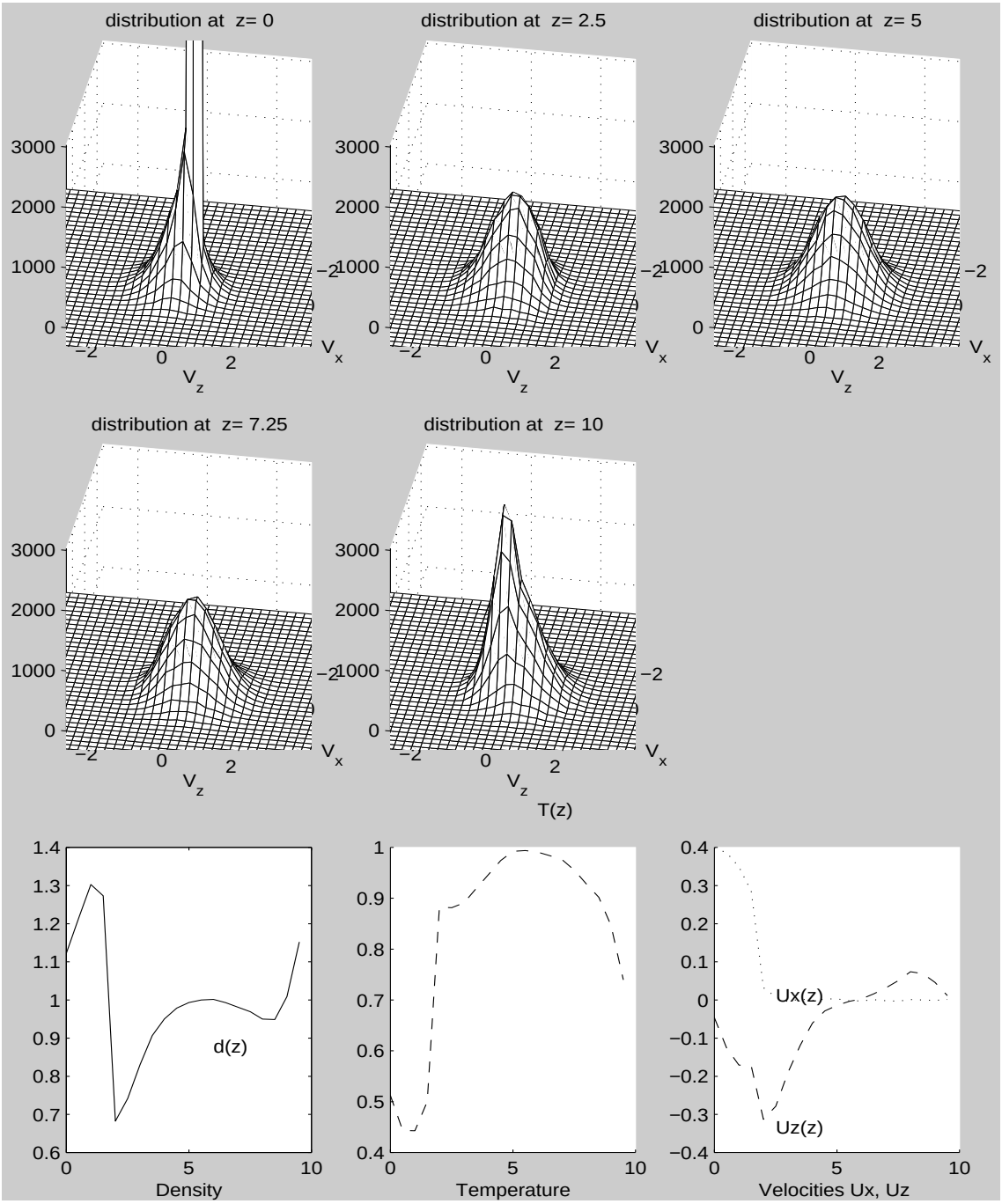

Figure 10: Development of the initial state which is a Maxwellian distribution for $2 < z < 10$, and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 0.5$, $V_y = 0.0$, $V_z = 0.0$ for $0 < z \leq 2$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 2$, $Kn = 0.1$.

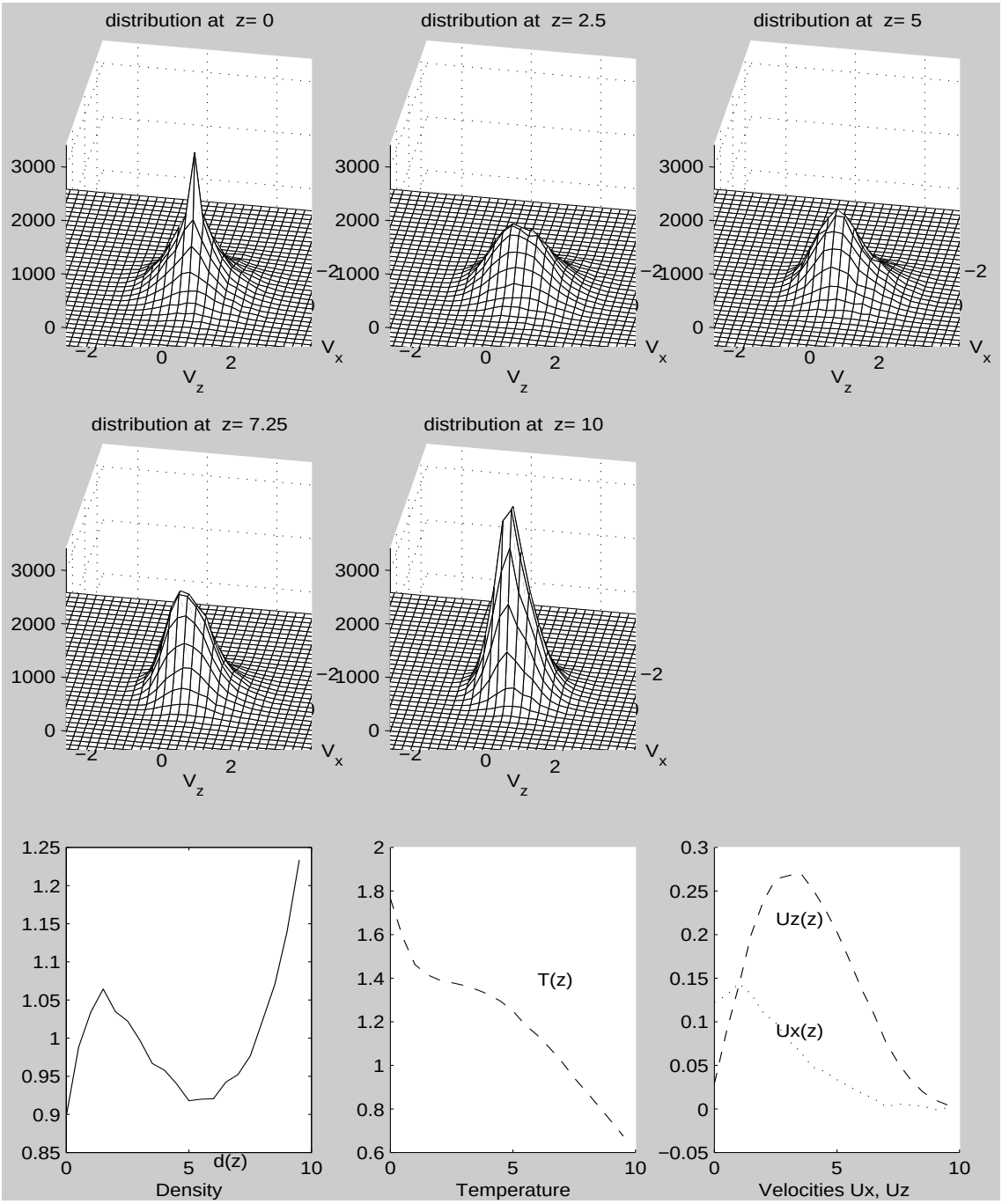

Figure 11: Development of the initial state which is a Maxwellian distribution for $2 < z < 10$, and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 0.5$, $V_y = 0.0$, $V_z = 0.0$ for $0 < z \leq 2$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 8$, $Kn = 0.1$.

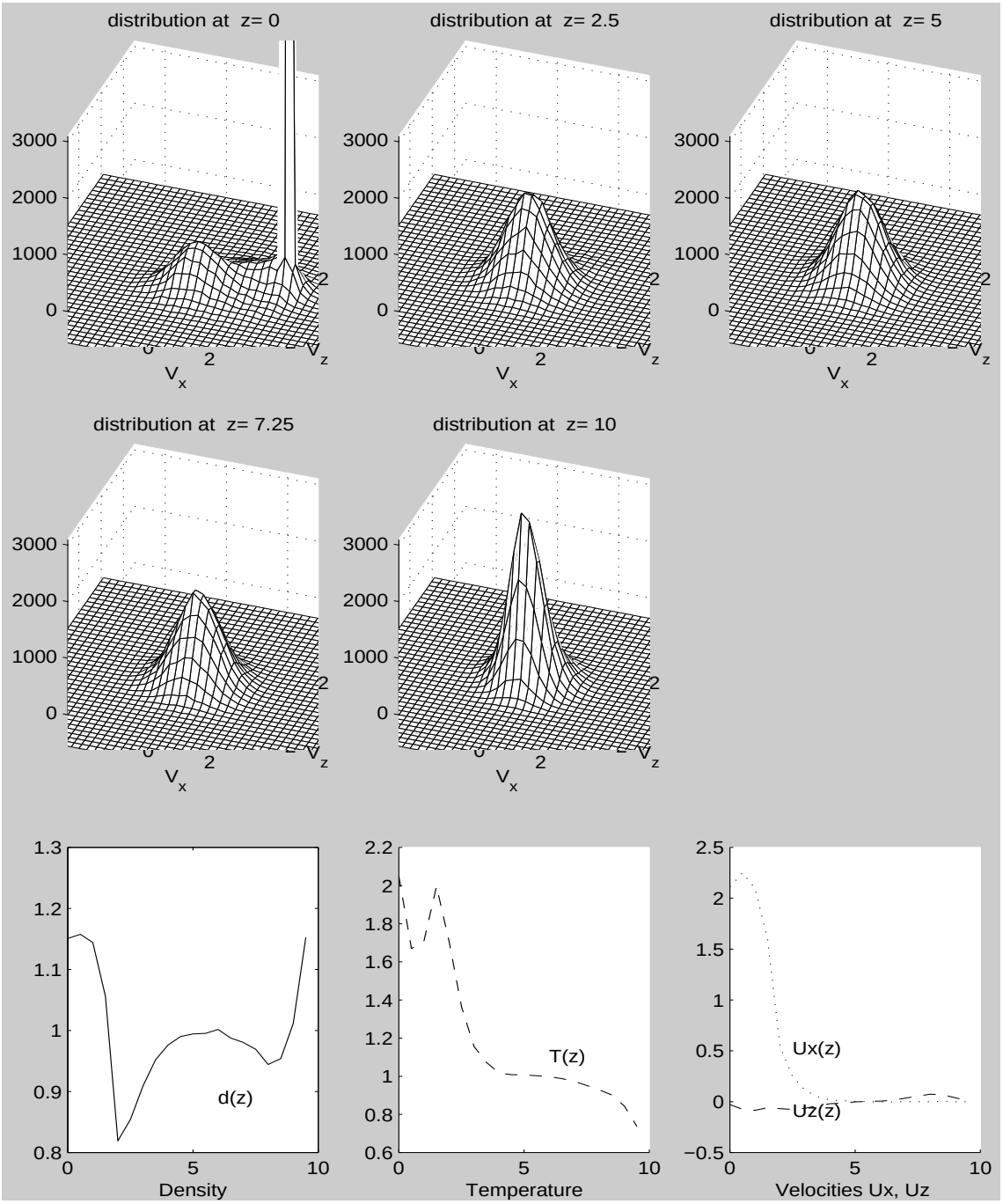

Figure 12: Development of the initial state which is a Maxwellian distribution for $2 < z < 10$, and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 3.0$, $V_y = 0.0$, $V_z = 0.0$ for $0 < z \leq 2$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 2$, $Kn = 0.1$.

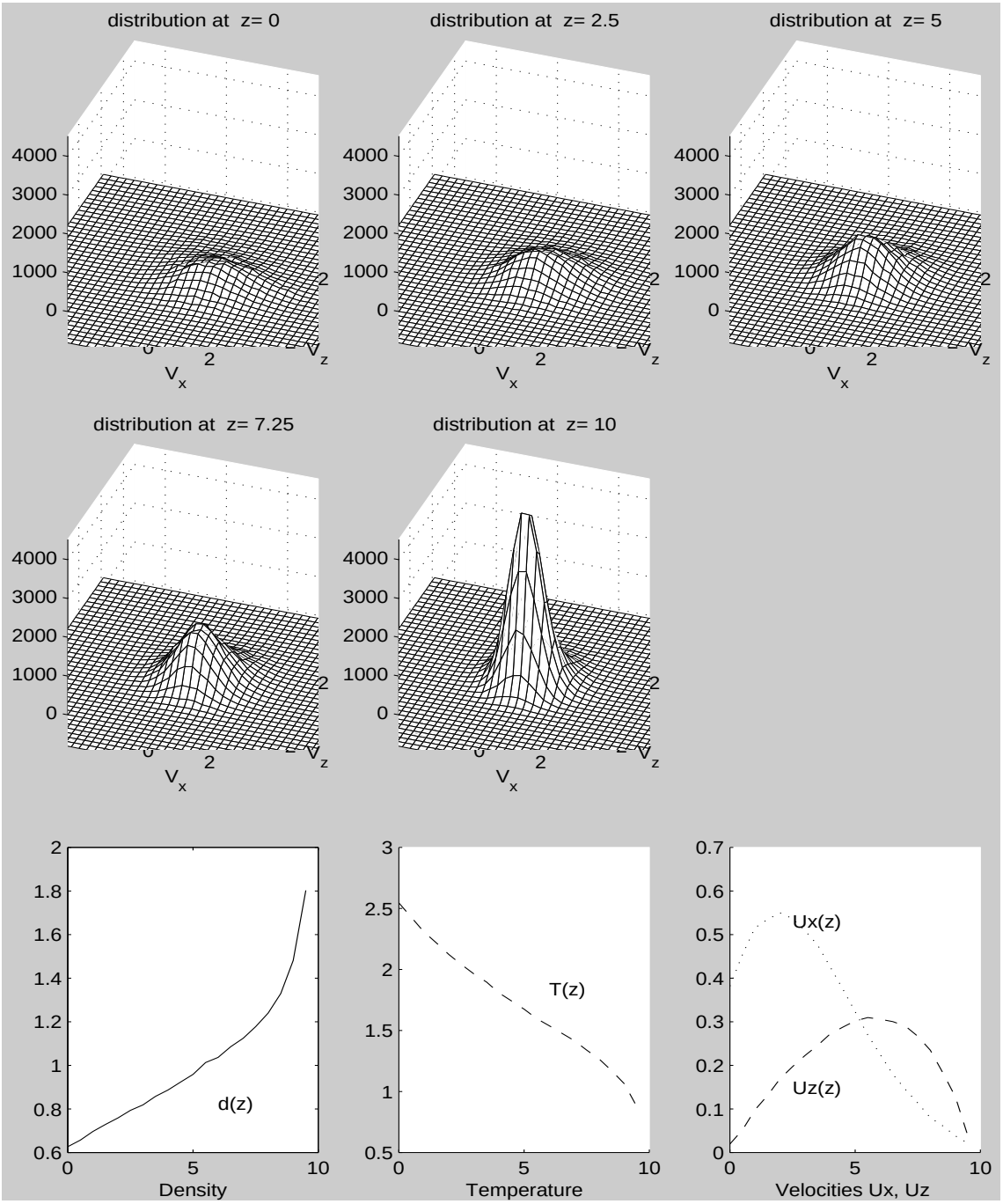

Figure 13: Development of the initial state which is a Maxwellian distribution for $2 < z < 10$, and is mixture of 10% of Maxwellian distribution and 90% of particles with velocity $V_x = 3.0$, $V_y = 0.0$, $V_z = 0.0$ for $0 < z \leq 2$. $T_1 = 3$, $T_2 = 0.5$, Time $t = 8$, $Kn = 0.1$.