# CONVERGENCE OF A DISCONTINUOUS GALERKIN SCHEME FOR THE NEUTRON TRANSPORT

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Abstract. We study the spatial discretization for the numerical solution of a model problem for the neutron transport equation in an infinite cylindrical media. Based on using an interpolation technique in the discontinuous Galerkin finite element procedure, and regularizing properties of the solution operator, we derive an *optimal* error estimate in $L_2-$norm for the scalar flux. This result, combined with a duality argument and previously known semidiscrete error estimates for the velocity discretizations, gives *globally optimal* error bounds for the critical eigenvalue.

**1. Introduction.** We consider a fully discrete scheme for the numerical solution of the stationary, isotropic, one-velocity neutron transport equation in an infinite cylindrical domain in $\mathbb{R}^3$ with a polygonal cross section $\Omega$. The cylindrical symmetry reduces the problem to $\mathbb{R}^2$ by projecting along the axis of the cylinder. Thus we study the neutron transport equation in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with the velocity space being the unit disc $\mathbf{D} \subset \mathbb{R}^2$.

We analyze the discontinuous Galerkin finite element method, with piecewise linear trial functions, for the space discretization, by means of a quasiuniform triangulation of the space domain $\Omega$ with the mesh size $h$. In order to obtain *sharp* error bounds, we use a *K-method* of interpolation based on a splitting with respect to the maximal available regularity of the partial derivatives of the exact solution. For this method we give an $L_2$ error estimate for the scalar flux of order $h^{1-\varepsilon}$, $\varepsilon \geq 0$ small, resulting a *globally optimal* error bound for the largest (critical) eigenvalue of order $h^{3-\varepsilon}$.

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In approximating the hyperbolic problems with the discontinuous Galerkin an $L_2$-error estimate of the form

$$\|\varphi - \varphi_h\|_{L_2(\Omega)} \leq h^{s-1/2}\|\varphi\|_{H^s(\Omega)}, \quad \varphi \in H^s(\Omega),$$

is optimal, where $H^s(\Omega)$, $s > 0$ is the usual Sobolev space containing all functions with $s$ derivatives in $L_2$ and for non-integer $s$, $H^s(\Omega)$ is defined by the interpolation, see [7]. Since for the neutron transport equation the exact scalar flux is at most in $H^{3/2-\varepsilon}(\Omega)$ see [5], therefore the convergence rate $\mathcal{O}(h^{1-\varepsilon})$ is sharp. The usual way of deriving convergence rates for functions of fractional regularity is through embeddings between Sobolev and Besov spaces. However, the embeddings (Sobolev-Besov-Sobolev) cause twice $\mathcal{O}(h^{-\delta})$, $\delta \sim \varepsilon$, ($\varepsilon$ small), factor affecting the convergence rate, see [3]. Further, as an ingredience of an approximation procedure, the emdedding process is too technical. To avoid these undesired features, the present study is based on $K$-method of interpolation.

The most related studies of this type (same geometry) can be found in [3], [5] and [6]. In [5], $L_2$ error estimates are proved for both semidiscrete and fully discrete problems. In [6] the semidiscrete problem is studied in the $L_1$-norm, which is the most relevant norm from a physical point of view, since the scalar flux represents a particle density. Also, because of the limited regularity of the exact solution, error estimates in the $L_1$ norm for eigenfunctions yields the sharpest error bound for the eigenvalues. However, in [3] (as in the present studies), i.e., for the spatial discretization based on the finite element method, the $L_2$ norm is more suitable, since in estimations in $L_2$, the duality concept can be used more efficiently giving the sharpest error estimate for the critical eigenvalue (even in here in the weaker norm $L_1$). This improves the convergence rate for the eigenvalue *more than* three times ($\mathcal{O}(h^{3-\varepsilon})$) as that we obtained for the pointwise scalar flux ($\mathcal{O}(h^{1-\varepsilon})$). Therefore, despite the *desirability* of studying both spatial and angular discretizations, combined, in the same underlying function space; there are significant advantages in considering different norms.

Some relevant classical approaches are, e.g., the discrete ordinates studies in [12] for the angular discretizations, the finite element studies in [9] for the spatial discretizations, and finally a Ritz-Galerkin approach in [8] for both angular and spatial discretizations in a general setting. Some recent numerical approximations for the transport equation by discontinuous finite elements can be found, e.g., in [1], and [2]. For mathematical analysis and theory of the neutron transport we refer to [11].

An outline of this paper is as follows: In Section 2 we introduce the model problem, drive the governing integral equation and recall some previous estimates for the angular discretizations which are relevant to our purpose. In Section 3 we give error estimates for the space discretization and prove the main result: Theorem 3.1. Our concluding Section 4 is devoted to a discussion on duality approach for the eigenvalue estimates and some compatibility conditions between the number of odrinates and the spatial mesh size in a fully discretized scheme.

Throughout this paper $C$ will denote a positive constant not necessarily the same at each occurrence and independent of all the parameters involved, unless otherwise explicitly stated.

## 2. Model problems.

**2.1. The continuous model problem.** We consider a problem of mono-energetic transport of neutrons in an infinite cylindrical media $\tilde{\Omega} \subset \mathbb{R}^3$, with the isotropic source and scattering. We assume that the cross-section $\Omega$ of the cylinder $\tilde{\Omega}$ is a bounded convex polygonal domain in $\mathbb{R}^2$ with the boundary $\Gamma$. Assuming also that the source term $f$ is constant along the axial direction of the cylinder we may project the equation, see [5], on the cross section $\Omega$ to obtain

$$\begin{aligned} &\mu \cdot \nabla_x u(x,\mu) + u(x,\mu) = \lambda \int_{\mathbf{D}} u(x,\eta)(1-|\eta|^2)^{-1/2} d\eta + f(x), \quad (x,\mu) \in \Omega \times \mathbf{D}, \\ &u(x,\mu) = 0, \quad \text{for } (x,\mu) \in \Gamma_\mu^- \times \mathbf{D}, \end{aligned} \tag{2.1}$$

where, $\mu \cdot \nabla_x = \sum_{i=1}^2 \mu_i(\partial/\partial x_i)$, $\lambda$ is a real parameter, $u(x,\mu)$ is the density of neutrons at the point $x \in \Omega$ moving in the direction $\mu \in \mathbf{D} = \{\mu \in \mathbb{R}^2 : |\mu| \le 1\}$ (the projection of the unit sphere), and $\Gamma_\mu^-$ is the inflow boundary of $\Omega$ with respect to $\mu$ defined by

$$\Gamma_\mu - = \{x \in \Gamma : \mu \cdot n(x) < 0\}, \tag{2.2}$$

where $n(x)$ is the outward unit normal to $\Gamma$ at $x \in \Gamma$. Let us also introduce the *scalar flux* $U$ defined by

$$U(x) = \int_{\mathbf{D}} u(x,\mu)(1-|\mu|^2)^{-1/2} d\mu. \tag{2.3}$$

Now consider the following hyperbolic partial differential equation: given $g \in L_p(\Omega), 1 \le p \le \infty$, find $w(x,\mu)$ such that for $\mu \in \mathbf{D} \setminus \{0\}$,

$$\mu \cdot \nabla w + w = g \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \Gamma_\mu^-, \tag{2.4}$$

the solution of this problem is given by

$$w(x,\mu) = T_\mu g(x) = \int_0^{d(x,\mu)/|\mu|} e^{-s} g(x - s\mu) ds, \tag{2.5}$$

where $T_\mu$ is the solution operator and $d(x,\mu)$ is the distance from $x \in \Omega$ to the inflow boundary in the direction $-\mu$ :

$$d(x,\mu) = \inf\{s > 0 : (x - s\mu/|\mu|) \notin \Omega\}.$$

Let $g = \lambda U + f$, then using equations (2.4) and (2.5) our model problem (2.1) has a solution of the form

$$u(x,\mu) = T_\mu(\lambda U + f)(x), \qquad (x,\mu) \in \Omega \times \mathbf{D}, \tag{2.6}$$

consequently we have the following integral equation, for the scalar flux $U$:

$$(I - \lambda T)U = Tf, \tag{2.7}$$

where

$$Tg(x) = \int_{\mathbf{D}} T_\mu g(x)(1-|\mu|^2)^{-1/2} d\mu.$$

$T$ is an integral operator with weakly singular kernel i.e. $T : L_p(\Omega) \to W_p^1(\Omega), 1 \le p \le \infty$, see [5, Lemma 1.1], specially $T : L_p(\Omega) \to L_p(\Omega), 1 \le p \le \infty$, is compact. Thus (2.7) is a Fredholm integral equation of the second kind, hence if $\lambda^{-1} \notin \sigma(T)$, where $\sigma$ is the spectrum of the operator $T$, then there is a unique $U \in L_p(\Omega), 1 \le p \le \infty$, satisfying (2.7).

**Remark 2.1.** We know that the scalar flux $U$, no matter how smooth the given data $f$ is, has a limited regularity, in fact we have $U \in H^{3/2-\varepsilon}(\Omega); 0 < \varepsilon << 1$, see the splitting argument below. Moreover the solution operator $T_\mu$ is more regularizing in the streamline direction (direction of $\mu$) in the sense that, for the angular flux $u(x,\mu)$, we have more derivatives in one direction than the others and hence the partial derivatives of the angular flux $u$ do not have the same regularity, (for the scalarf flux $U$ this phenomena is integrated away ). There are also singularities arising from the *closeness* of the directions of the velocity variable $\mu$ to the directions of the sides of the polygonal domain $\Omega$. Therefore we shall split the velocity directions into the so called *"good ones"* and *"bad ones"*. We split the less regular partial derivative of the angular flux, loosely speaking, in such a way that in the final step the terms are balanced in the sense that each split part contributes to the same order of convergence. For further studies of such singularities see [13].

**2.2 The semidiscrete model problem.** We introduce the semidiscrete analogue of the model problem (2.7): given a function $f$, find $u_n(x,\mu)$ such that

$$u_n(x,\mu) = T_\mu(\lambda U_n + f)(x), \qquad (x,\mu) \in \Omega \times \mathbf{D}, \tag{2.8}$$

where $U_n$ is the quadrature approximation of the scalar flux $U$ constructed as in [5], i.e.

$$U_n(x) = \sum_{\mu\in\Delta} u_n(x,\mu)\omega_\mu \cong \int_{\mathbf{D}} u(x,\mu)(1-|\mu|^2)^{-1/2}\,d\mu,$$

with $\Delta = \{\mu^1, \mu^2, ..., \mu^n\}$ being a discrete set of quadrature points $\mu^i \in \mathbf{D}$, $i = 1, ..., n$, with the corresponding positive weights $\omega_\mu, \mu \in \Delta$. We have $n = MN$, where $M$ is the number of, equidistributed, discrete points on the unit circle and $N$ is the number of Gauss points on $[0,1]$, chosen according to a special quadrature structure. Multiplying (2.8) by $\omega_\mu$ and summing over $\Delta$ we obtain the following semidiscrete analogue of (2.7): find $U_n$, such that

$$(I - \lambda T_n)U_n = T_n f, \qquad \text{with} \quad T_n g(x) = \sum_{\mu\in\Delta} \omega_\mu T_\mu g(x). \tag{2.9}$$

Then for the semidiscrete error $U - U_n$ we have the following estimates:

**Proposition 2.1.** *Assume that $\lambda^{-1} \notin \sigma(T)$. For each $\varepsilon > 0$ there is a constant $C$ such that, for sufficiently large $N$ and $M$,*

$$\|U - U_n\|_{L_1(\Omega)} \le C\left[\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon}}\right]\left(\|U\|_{L_1(\Omega)} + \|f\|_{W_1^1(\Omega)}\right), \tag{2.10}$$

$$\|(T - T_n)(\lambda U + f)\|_{L_2(\Omega)} \le Cn^{-1/2}\|\lambda U + f\|_{H^1(\Omega)}. \tag{2.11}$$

The estimate (2.10) is the main result of [6] and (2.11) is given in Lemma 4.3 of [5].

**3. The discontinuous Galerkin and fully discrete problem.** We shall denote by $\{\mathcal{C}_h\}$ a family of quasiuniform triangulation $\mathcal{C}_h = \{K\}$ of $\Omega$ indexed by the parameter $h$, the maximum diameter of triangles $K \in \mathcal{C}_h$. We introduce the finite element space:

$$V_h = \{v \in L_2(\Omega) : v|_K \text{ is linear}, K \in \mathcal{C}_h\}$$

and define a discrete solution operator $T_\mu^h : L_2(\Omega) \to V_h$ approximating $T_\mu$ by the following discontinuous Galerkin finite element method for (2.4):

$$\sum_{K\in\mathcal{C}_h}\left[(\mu\cdot\nabla u^h + u^h, v)_K + \int_{\partial K_-}[u^h]v_+\,|\mu\cdot\hat{n}|d\sigma\right] = \int_\Omega gv dx, \qquad \forall v \in L_2(\Omega), \tag{3.1}$$

where

$$(u,v)_K = \int_K uv dx, \qquad \partial K^- = \{x \in \partial K : \mu \cdot \hat{n}(x) < 0\},$$
$$[v] = v_+ - v_-, \qquad v_\pm(x) = \lim_{s\to 0\pm} v(x+s\mu), \quad x \in \partial K,$$

$\hat{n} = \hat{n}(x)$ is the outward unit normal to $\partial K$ at $x \in \partial K$ and $u^h_- = 0$ on $\Gamma^-_\mu$.

Now, let us formulate the following fully discrete analogue of (2.6): given $f$ find $u^h_n(\cdot,\mu) \in V_h$ such that

$$u^h_n(\cdot,\mu) = T^h_\mu(\lambda U^h_n + f), \qquad \mu \in \Delta, \tag{3.2}$$

where $U^h_n$ is a totally discretized approximation of the scalar flux $U$, i.e.

$$U^h_n = \sum_{\mu\in\Delta} u^h_n(\cdot,\mu)\omega_\mu.$$

Equation (3.2) yields to the problem of finding $U^h_n \in V_h$ such that

$$(I - \lambda T^h_n)U^h_n = T^h_n f, \qquad \text{with} \quad T^h_n = \sum_{\mu\in\Delta} T^h_\mu \omega_\mu. \tag{3.3}$$

For $\lambda^{-1} \notin \sigma(T)$ and $\max(h, 1/n)$ sufficiently small, (3.3) has a unique solution $U^h_n \in V_h$; see [4, section 5]. Our main concern is the estimates of the error in the scalar flux for the fully discrete problem (3.3) i.e. $U - U^h_n$. The parameters $h$ and $n$ will be related according to the following compatibility conditions:

$$h^{-1}(n) \sim \sqrt{n} = \sqrt{MN}, \qquad \text{and} \quad M \sim N. \tag{3.4}$$

Optimal semidiscrete error estimates, (according to Proposition 2.1), and sharper estimates for the fully discrete eigenvalues, require a slight modification of the second relation in (3.4) which, does not effect the results in here (see also Remark 4.1).

We shall use the following notation: for $s \geq 0$, $\|\cdot\|_s$ will denote the norm in the sobolev space $H^s(\Omega)$ and $|\cdot|_s$ the corresponding seminorm of highest order derivatives. For non-integer $s$, these norms are defined by interpolation, see [7]. For $s = 0$ we omit the index 0 and let $\|\cdot\|$ denote the $L_2(\Omega)$ norm. Now let $\mathcal{R} := I_{x_1} \times I_{x_2}$ be an axi-parallel rectangular domain containing $\Omega$ and the union of supports of all functions under consideration. For $r, s \geq 0$, we shall use the space $H^{r,s}(\mathcal{R})$ with the norm

$$\|v\|_{r,s} = \Big[\int_{I_{x_2}} \|v(\cdot,x_2)\|_{H^r(I_{x_1})}\, dx_2 + \int_{I_{x_1}} \|v(x_1,\cdot)\|_{H^s(I_{x_2})}\, dx_1\Big]^{1/2},$$

and the corresponding semi-norm obtained by replacing the norm $\|\cdot\|_{H^t}$ by the semi-norm $|\cdot|_{H^t}$. Since all fuctions are vanishing outside $\mathcal{R}$, thus we may in the norms $\|v\|_{r,s}$ and $|v|_{r,s}$, replace both $I_{x_1}$ and $I_{x_2}$ by $\mathbb{R}$, see Lions and Magenes [10] for the details.

We shall also use the following splitting of $\Delta$ in two sets:

$$J'_\delta = \{\mu \in \Delta : \theta = \min(|\sin(\mu, d_k)|) \geq \delta \sim \frac{1}{M},\ k = 1, 2, \ldots, P_0\},$$
$$J_\delta = \{\mu \in \Delta : \mu \notin J'_\delta\},$$

where $d_k$'s are the sides of $\Omega$ and $P_0$ is the number of sides of $\Omega$. Our main result is:

**Theorem 3.1.** *Assume (3.4) and that $\lambda^{-1} \notin \sigma(T)$. Let $U$ and $U_n^h$ satisfy (2.7) and (3.3) respectively. Then there is a constant $C$ such that for sufficiently small $h$ (large $n$) and for $g \in H^{3/2-\varepsilon}(\Omega)$,*

$$||U - U_n^h|| \leq Ch^{1-\varepsilon}|\log h| ||g||_{H^{3/2-\varepsilon}(\Omega)},$$

*where $g = \lambda U + f$ and $\varepsilon > 0$ is small.*

**Remark 3.1.** Note that by the trace Theorem, see [7], for a Lipschitz domain $\Omega$, $g \in H^s(\Omega)$, implies that $g \in H^{s-1/2}(\Gamma)$, where $\Gamma = \partial\Omega$, $s > 1/2$. For the polygonal domain $\Omega$ and with $s = 3/2 - \varepsilon$, as in Theorem 3.1 above, we get the right regularity at the boundary. The crucial part that remains is to make sharp approximations so that, due to this regularity at the boundary, the estimates give optimal convergence rate.

In the proof of Theorem 3.1 we use the following two results:

**Proposition 3.1.** *[Stability]; cf.[5]: For $g \in L_2(\Omega)$ we have*

$$||\mu \cdot \nabla T_\mu g|| + ||T_\mu g|| + \Big(\int_\Gamma (T_\mu g)^2 |\mu \cdot \hat{n}| d\sigma\Big)^{1/2} \leq C||g||. \tag{3.5}$$

**Proposition 3.2.** *[Convergence]; cf. [5]. Given $g \in L_2(\Omega)$, there is a unique $u^h(\cdot, \mu) = T_\mu^h g(\cdot) \in V_h$ satisfying (3.1). Moreover, there is a constant $C$ independent of $g, \mu, h$ and $\Omega$ such that*

$$|||(T_\mu - T_\mu^h)g||| \leq Ch^{s-1/2}|T_\mu g|_s, \qquad s = 1, 2, \tag{3.6}$$

$$|||T_\mu^h g||| \leq C||g||, \tag{3.7}$$

$$\sum_{\mu \in J_\delta} \omega_\mu ||(T_\mu - T_\mu^h)g|| \leq Ch||g||, \tag{3.8}$$

*where $J_\delta$ is defined above and*

$$|||v||| = \Big[||v||^2 + h\sum_K ||\mu \cdot \nabla v||_K^2 + \sum_K \int_{\partial K} |[v]|^2 |\mu \cdot \hat{n}| ds\Big]^{1/2}, \tag{3.9}$$

$$||v||_K = (v, v)_K^{1/2}.$$

*Recall that $|\cdot|_s$ is the seminorm with the corresponding maximal number of derivatives.*

We prove theorem 3.1 using the following Lemma based on the above splitting:

**Lemma 3.1.** *There is a constant $C$ such that for $g \in H^{3/2-\varepsilon}(\Omega)$,*

$$\sum_{\mu \in J_\delta'} \omega_\mu |T_\mu g|_{3/2-\varepsilon} \leq C|\log \delta| ||g||_{H^{3/2-\varepsilon}(\Omega)}.$$

We postpone the proof of Lemma 3.1 and first show that Theorem 3.1 follows from that.

**Proof of Theorem 3.1.** We have using (2.7) and (3.3) that

$$(U - U_n^h)(T - T_n^h)(\lambda U + f) + \lambda T_n^h(U - U_n^h),$$

i.e.,

$$(I - \lambda T_n^h)(U - U_n^h) = (T - T_n)(\lambda U + f) + (T_n - T_n^h)(\lambda U + f) := e_n + e_n^h.$$

According to a stability estimate, see [5.,Theorem 5.1], if $\lambda^{-1} \notin \sigma(T)$, then for sufficiently large $n$, $(I-\lambda T_n^h)^{-1} : L_2(\Omega) \to L_2(\Omega)$ exists and is uniformly bounded. Thus we have

$$\|U - U_n^h\| \le C_\lambda \left(\|e_n\| + \|e_n^h\|\right). \tag{3.10}$$

Using Proposition 3.1, an interpolation in (3.6) together with (3.8) and Lemma 3.1 we get

$$\begin{aligned}\|e_n^h\| =& \|(T_n - T_n^h)g\| = \| \left( \sum_{\mu\in J_\delta} + \sum_{\mu\in J'_\delta} \right) \omega_\mu (T_\mu - T_\mu^h) g\| \\ \le& \sum_{\mu\in J_\delta} \omega_\mu \|(T_\mu - T_\mu^h)g\| + \sum_{\mu\in J'_\delta} \omega_\mu \|(T_\mu - T_\mu^h)g\| \\ \le& Ch\|g\| + Ch^{1-\varepsilon} \sum_{\mu\in J'_\delta} \omega_\mu |T_\mu g|_{3/2-\varepsilon} \\ \le& Ch^{1-\varepsilon} |\log\delta| \|g\|_{H^{3/2-\varepsilon}(\Omega)}.\end{aligned} \tag{3.11}$$

Thus (2.11), (3.4) and the relation $\log\delta \sim \log h$ give the desired result. ■

Lemma 3.1 follows combining the following two results:

**Lemma 3.2.** *There is a constant $C$ such that for $g \in H^s(\Omega)$ and $\mu \in J'_\delta$,*

$$\|T_\mu g\|_{s,0} \le \frac{C}{|\mu|^{s-1/2}} \|g\|_{H^s(\Omega)}, \tag{3.12}$$

$$\|T_\mu g\|_{\alpha,\beta} \le \frac{C}{|\mu|^{s-1/2}} \cdot \frac{1}{(\min_k |\sin(\mu, d_k)|)^{1/2}} \|g\|_{H^s(\Omega)}, \tag{3.13}$$

*with $\alpha + \beta = s \le 2$ and $\beta \le 1$.*

**Lemma 3.3.** *There is a constant $C$ such that for $g \in H^s(\Omega), s = 3/2 - \varepsilon$, we have*

$$\|T_\mu g\|_{0,s} \le \frac{C}{|\mu|} \cdot \frac{1}{\min_k |\sin(\mu, d_k)|} \|g\|_{H^s(\Omega)}. \tag{3.14}$$

**Proof of Lemma 3.1.** By lemma 3.3 we have the estimate for (3.13) when $\beta$ is beyond 1, actually when $\beta \le s = 3/2 - \varepsilon$. Now since we have

$$\begin{aligned}\sum_{\mu\in J_{\delta'}} \frac{\omega_\mu}{|\mu|} \cdot \frac{1}{\min_k |\sin(\mu, d_k)|} &\cong C \left( \sum_{i=1}^N \frac{A_i}{|\mu_i|} \right) \left( \frac{2\pi}{M} \sum_{\theta_j \ge \delta} \frac{1}{|\sin(\mu, d_j)| = \theta_j} \right) \\ &\cong C \left( \int_\delta^{2\pi} \frac{d\theta}{\theta} \right) \sim C \left( |\log\delta| \right),\end{aligned}$$

the proof follows from Lemmas 3.2 and 3.3. ■

Below in the proofs for Lemmas 3.2 and 3.3 we shall carry out the basic estimates and leave the task of after-hand arithmetics to interested reader. See also [7] and [13].

**Proof of Lemma 3.2.** By an orthogonal coordinate transformation we may assume that $\mu = (\mu_1, 0), |\mu| = r$ and let $\nu = (0, \nu_2)$ with $|\nu| = 1$ be perpendicular to $\mu$. Recall that

$$T_\mu g(x) = \int_0^{d(x,\mu)/r} e^{-s} g(x - s\mu) ds, \qquad \frac{\partial d}{\partial \nu} = r\frac{\nu\cdot\hat{n}}{\mu\cdot\hat{n}} \qquad \text{and} \qquad \frac{\partial d}{\partial \mu} = r, \tag{3.15}$$

see [5]. For a sufficiently regular function $\varphi$ we have

$$\frac{\partial}{\partial x_1}(\varphi(x))=\frac{1}{\mu_1}\mu_1\frac{\partial}{\partial x_1}(\varphi(x))=\frac{1}{\mu_1}[\mu\cdot\nabla\varphi(x)]=\frac{1}{\mu_1}\left(\frac{\partial}{\partial\mu}\varphi(x)\right). \tag{3.16}$$

Assume for a moment that $g$ and consequently $T_\mu g$ are regular enough, then by the definition of $T_\mu g$ and using (3.16) repeatedly, with $\varphi=T_\mu g$, we get

$$\begin{aligned}\frac{\partial}{\partial x_1}(T_\mu g(x)) &= \frac{1}{\mu_1}\frac{1}{r}\frac{\partial d}{\partial\mu}e^{-d/r}g(\bar{x})+\frac{1}{\mu_1}\int_0^{d/r}e^{-s}\frac{\partial}{\partial\mu}g(x-s\mu)ds\\ &= \frac{1}{\mu_1}e^{-d/r}g(\bar{x})+\int_0^{d/r}e^{-s}\frac{\partial}{\partial x_1}g(x-s\mu)ds,\end{aligned} \tag{3.17}$$

where $\bar{x}=(x-d\frac{\mu}{r})\in\Gamma$. Squaring (3.17) and integrating over $\Omega$, since $dx_2=\frac{\mu\cdot\hat{n}}{|\mu|}d\sigma$, $dx=dx_1\,dx_2$, we find that

$$\begin{aligned}\|\frac{\partial}{\partial x_1}(T_\mu g)\|^2 &\le C\left(\int_\Omega\frac{1}{r^2}e^{-2d/r}g^2(\bar{x})dx+\int_\Omega\int_0^{d/r}e^{-2s}(\frac{\partial}{\partial x_1}g)^2dsdx\right)\\ &\le \frac{C}{r}\int_0^{\mathrm{diam}\Omega}\frac{1}{r}e^{-2d/r}dx_1\int_\Gamma|g|^2\frac{|\mu\cdot\hat{n}|}{|\mu|}\,d\sigma+C\|\nabla g\|^2\le\frac{C}{r}\|g\|^2_{H^1(\Omega)}.\end{aligned}$$

Thus

$$\|\frac{\partial}{\partial x_1}(T_\mu g)\|\le\frac{C}{r^{1/2}}\|g\|_{H^1(\Omega)}. \tag{3.18}$$

Similarly

$$|\frac{\partial^2}{\partial x_1^2}(T_\mu g(x))|=\frac{1}{|\mu|^2}|\frac{\partial^2}{\partial\mu^2}(T_\mu g(x))|, \tag{3.19}$$

and

$$\frac{\partial^2}{\partial\mu^2}(T_\mu g(x))=-e^{-d/r}g(\bar{x})+2e^{-d/r}\frac{\partial}{\partial\mu}g(\bar{x})+\int_0^{d/r}e^{-s}\frac{\partial^2}{\partial\mu^2}(g(x-s\mu))ds.$$

Thus by repeatedly using of (3.19) we have

$$\begin{aligned}|\frac{\partial^2}{\partial x_1^2}(T_\mu g(x))| &\le \frac{1}{r^2}e^{-d/r}|g(\bar{x})|+\frac{2}{r}e^{-d/r}|\frac{\partial}{\partial x_1}g(\bar{x})|\\ &\quad+|\int_0^{d/r}e^{-s}\frac{\partial^2}{\partial x_1^2}(g(x-s\mu))ds|.\end{aligned} \tag{3.20}$$

Squaring (3.20) and integrating over $\Omega$, using the same technique as above we find that

$$\|\frac{\partial^2}{\partial x_1^2}(T_\mu g)\|\le\frac{C}{r^{3/2}}\|g\|_{H^2(\Omega)}. \tag{3.21}$$

Now since $s=3/2-\varepsilon$, (3.12) is a result of interpolating between (3.18)and(3.21).

As for the derivative with respect to $x_2$ we have by (3.15) that

$$\frac{\partial}{\partial x_2}(T_\mu g(x)) = \frac{\partial}{\partial \nu}(T_\mu g(x)) = e^{-d/r} g(\bar{x}) \frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}} + \int_0^{d/r} e^{-s} \frac{\partial}{\partial \nu} g(x - s\mu) ds. \tag{3.22}$$

Thus, using $dx_2 = (\mu \cdot \hat{n})/|\mu|\, d\sigma$ once again, we get

$$\begin{aligned}
\|\frac{\partial}{\partial x_2}(T_\mu g)\|^2 &\leq C \left( \int_\Omega e^{-2d/r} g^2(\bar{x}) |\frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}|^2 dx + \int_\Omega \int_0^{d/r} e^{-2s} (\frac{\partial}{\partial \nu} g)^2 ds\, dx \right) \\
&\leq C \left( \int_0^{\operatorname{diam}\Omega} \frac{1}{r} e^{-2d/r} dx_1 \int_\Gamma |g|^2 |\frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}|^2 |\mu \cdot \hat{n}| d\sigma + \|\nabla g\|^2 \right) \\
&\leq \frac{C}{\min_k |\mu \cdot \hat{n}_k|} \left( \int_0^{\operatorname{diam}\Omega} \frac{1}{r} e^{-2d/r} dx_1 \right) \int_\Gamma |g|^2 \, d\sigma + C \|\nabla g\|^2 \\
&\leq \frac{C}{r \min_k |\sin(\mu, d_k)|} \|g\|^2_{H^1(\Omega)}.
\end{aligned}$$

This implies that

$$\|\frac{\partial}{\partial x_2}(T_\mu g)\| \leq \frac{1}{r^{1/2}} \frac{C}{(\min_k |\sin(\mu, d_k)|)^{1/2}} \|g\|_{H^1(\Omega)}. \tag{3.23}$$

Moreover since

$$|\frac{\partial^2}{\partial x_1 \partial x_2}(T_\mu g(x))| = \frac{1}{|\mu|} |\frac{\partial}{\partial \nu}(\frac{\partial}{\partial \mu} T_\mu g(x))|,$$

differentiating $\frac{\partial}{\partial \mu}(T_\mu g(x) = \mu_1 \frac{\partial}{\partial x_1}(T_\mu g(x))$ in (3.17) with respect to $\nu$ we get

$$\begin{aligned}
\frac{\partial^2}{\partial \nu \partial \mu}(T_\mu g(x)) = &- e^{-d/r} g(\bar{x}) (\frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}) + e^{-d/r} \frac{\partial}{\partial \nu} g(\bar{x}) + e^{-d/r} \frac{\partial}{\partial \mu}(g(\bar{x}))(\frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}) \\
&+ \int_0^{d/r} e^{-s} \frac{\partial^2}{\partial \nu \partial \mu}(g(x - s\mu)) ds,
\end{aligned}$$

so that using (3.16),

$$\begin{aligned}
|\frac{\partial^2}{\partial x_1 \partial x_2}(T_\mu g(x))| \leq &\frac{1}{r} e^{-d/r} |g(\bar{x})| |\frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}| + \frac{1}{r} e^{-d/r} |\frac{\partial}{\partial \nu} g(\bar{x})| \\
&+ e^{-d/r} |\frac{\partial}{\partial x_1}(g(\bar{x}))| |\frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}| + \int_0^{d/r} e^{-s} |\frac{\partial^2}{\partial x_1 \partial x_2}(g(x - s\mu))| ds.
\end{aligned}$$

This gives that

$$\|\frac{\partial^2}{\partial x_1 \partial x_2}(T_\mu g)\| \leq \frac{C}{r\, (\min_k |\sin(\mu, d_k)|)^{1/2}} \|g\|_{H^2(\Omega)}. \tag{3.24}$$

Now interpolating between (3.18), (3.21), (3.23) and (3.24) gives (3.13) and completes the proof of Lemma 3.2. ■

**Proof of Lemma 3.3.** Recall that by (3.22) we have

$$\frac{\partial}{\partial x_2}(T_\mu g(x)) = e^{-d/r} g(\bar{x})\psi(x_2) + \int_0^{d/r} e^{-s}\frac{\partial}{\partial \nu} g(x - s\mu)ds := F(x,\mu) + G(x,\mu),$$

where

$$\psi(x_2) = \frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}.$$

Evidently, for $g \in H^q(\Omega)$ we have $G \in H^{q-1}(\Omega), q = 1, 2$. Then by interpolation for $g \in H^s(\Omega)$, $1 < s < 2$, $G \in H^{s-1}(\Omega)$. It remains to show that $F \in H^{s-1}(\Omega)$. We split the piecewise constant function $\psi$ to a continuous part $\psi_1$ and a remaining part $\psi_2$, see (3.25) below, so that

$$\psi(x_2) = \psi_1(x_2) + \psi_2(x_2).$$

Observe that $\psi(x) \equiv \psi(x_2)$ has a well-defined (nonoverlaping) domain of definition:

$$D_\psi := \{x_2 : (x_1, x_2) \in \Gamma_\mu^-\},$$

we extend $\psi$ by zero outside of $D_\psi$ and define

$$\psi(x_2) = \begin{cases} \psi(x), & \text{for} \quad x \in \Gamma_\mu^- \quad \text{i.e. for} \quad x_2 \in D_\psi, \\ 0, & \text{for} \quad x_2 \in \mathbb{R} \setminus D_\psi. \end{cases}$$

Thus we have split $F$ as

$$F(x,\mu) = e^{-d/r} g(\bar{x})\psi_1(x_2) + e^{-d/r} g(\bar{x})\psi_2(x_2) := \varphi_1 + \varphi_2,$$

where $\varphi_1$ is continuous and differentiable except at a finite number of points, while $\varphi_2$ contains jump discontinuities. Below we construct a suitable function $\psi_1$ :

Let $B$ denote the projection of $\Gamma_\mu^-$ on $x_2$, then $B \subset \mathbb{R}$ is bounded and since $\Omega$ is convex we have that for a fixed $\mu$,

$$B = \bigoplus_{S_j \subset \Gamma_\mu^-} \text{Proj}_{x_2} S_j = \bigcup_{S_j \subset \Gamma_\mu^-} I_j.$$

We change notation $x_2 \to \xi$ and assume, without loss of generality that the $I_j$'s are ordered in a sense that: $\xi_i \in I_i$ and $\xi_k \in I_k$ with $i < k$ imply $\xi_i < \xi_k$. Moreover we assume that the first interval is $I_1$ and the last interval is $I_{M_\circ - 1}$ and $\xi_i$ is the left end point of $I_i$ and $\xi_{M_\circ}$ is the right end point of $I_{M_\circ-1}$. Now we fix a small $b > 0$ , such that for $\mu \in J'_\delta$, $|I_j| \geq 3b, \quad \forall j = 1, 2, ..., M_\circ - 1$. Observe that for $S_j \subset \Gamma_\mu^-$, $|\text{Proj}_{x_2} S_j|$ may be very small. This can happen when either we have a very small side or an arbitrary side has an arbitrary small angle with $\mu$. The recent case is included in $J_\delta$ and is treated separately. In the first case we can, either, have a jump over the smaller side to the adjacent sides, or choose a smaller $b$ in the construction of $\psi_1$. For a usual choice of $b$ we have $\sin\theta_j = \frac{|I_j|}{|S_j|} = \frac{|\text{Proj}_{x_2} S_j|}{|S_j|}$, and since for $\mu \in J'_\delta$, $\sin\theta_j \geq \delta$, we get

$$|I_j| = |\text{Proj}_{x_2} S_j| \geq \delta |S_j|.$$

Thus for $\mu \in J'_\delta$, we choose $b$ such that $\forall j$, $\delta |S_j| \geq 3b$, hence

$$b \leq \frac{1}{3}\delta \min_{S_j \subset \Gamma_\mu^-}(|S_j|).$$

Note that

$$\frac{\nu\cdot\hat n}{\mu\cdot\hat n}=\psi(x_2)=\frac{1}{r}\frac{\cos(\nu,\hat n)}{\cos(\mu,\hat n)}=\frac{1}{r}\frac{\sin(\mu,\hat n)}{\cos(\mu,\hat n)}=\frac{1}{r}\tan(\mu,\hat n).$$

Now we define

(3.25)

$$\psi_1(\xi)=\begin{cases}0, & \xi\le\xi_1 \quad\text{and}\quad \xi\ge\xi_{M_\circ},\\ \psi(\xi_j-b)^-+\frac{1}{2b}[\psi(\xi_j+b)^+-\psi(\xi_j-b)^-](\xi-\xi_j+b), & \\ & \xi_j-b\le\xi\le\xi_j+b,\quad j=2,...,M_\circ-1,\\ \frac{1}{b}\psi(\xi_1+b)^+(\xi-\xi_1), & \xi_1\le\xi\le\xi_1+b,\\ \frac{1}{b}\psi(\xi_{M_\circ}-b)^-(\xi_{M_\circ}-\xi), & \xi_{M_\circ}-b\le\xi\le\xi_{M_\circ},\\ \psi(\xi), & \text{else,}\end{cases}$$

this implies that

$$\psi_1'(\xi)=\begin{cases}0, & \xi\le\xi_1 \quad\text{and}\quad \xi\ge\xi_{M_\circ},\\ \frac{1}{2b}[\psi]_j, & \xi_j-b\le\xi\le\xi_j+b,\quad j=2,...,M_\circ-1,\\ \frac{1}{b}\psi(\xi_1+b)^+, & \xi_1\le\xi\le\xi_1+b,\\ -\frac{1}{b}\psi(\xi_{M_\circ}-b)^-, & \xi_{M_\circ}-b\le\xi\le\xi_{M_\circ},\\ 0, & \text{else.}\end{cases}$$

Here $M_\circ=P_\circ+1$, where $P_\circ$ is the number of sides of $\Omega$ included in $\Gamma_\mu^-$ and for $j=2,...,M_\circ-1$, $[\psi]_j$ is the gap $[\psi(\xi_j+b)^+-\psi(\xi_j-b)^-]$. Now differentiating $\varphi_1$ yields

$$\begin{aligned}\frac{\partial\varphi_1}{\partial x_2}&=-e^{-d/r}g(\bar x)\psi_1(x_2)\frac{\nu\cdot\hat n}{\mu\cdot\hat n}+e^{-d/r}\frac{\partial}{\partial\nu}(g(\bar x))\psi_1(x_2)+e^{-d/r}g(\bar x)\frac{\partial}{\partial x_2}(\psi_1(x_2))\\ &:=\chi_1(x,\mu)+\chi_2(x,\mu)+\chi_3(x,\mu),\end{aligned}$$

and hence

$$\|\frac{\partial\varphi_1}{\partial x_2}\|_{L_2(\Omega)}\le\sum_{j=1}^{3}\|\chi_j\|_{L_2(\Omega)}. \tag{3.26}$$

Below we estimate each norm in the right hand side of (3.26) separately:

$$\begin{aligned}\|\chi_1\|^2_{L_2(\Omega)}&=\int_\Omega e^{-2d/r}g^2(\bar x)\psi_1^2(x_2)(\frac{\nu\cdot\hat n}{\mu\cdot\hat n})^2\,dx\\ &\le C\left[\int_0^{\mathrm{diam}\Omega}e^{-2x_1/r}\,dx_1\int_{\mathbb R}g^2(\xi)\psi_1^2(\xi)(\frac{\nu\cdot\hat n}{\mu\cdot\hat n})^2\frac{|\mu\cdot\hat n|}{|\mu|}\,d\xi\right]\\ &\le C\frac{\sin^2(\mu,\hat n)}{r\,\cos(\mu,\hat n)}\int_0^{\mathrm{diam}\Omega}\frac{1}{r}e^{-2x_1/r}\,dx_1\int_{\mathbb R}g^2(\xi)\psi_1^2(\xi)\,d\xi\\ &\le\frac{C}{r\,\min_k|\sin(\mu,d_k)|}\sup_{\mathbb R}|\psi_1(\xi)|^2\|g\|^2_{L_2(\Gamma)}.\end{aligned}$$

Now using the trace estimate we obtain

$$\|\chi_1\|_{L_2(\Omega)} \le C_\psi \frac{1}{\sqrt{r\ \min_k |\sin(\mu, d_k)|}} \|g\|_{H^1(\Omega)}, \tag{3.27}$$

where $C_\psi = C(\sup_{\xi\in\mathbb{R}} |\psi_1(\xi)|^2)^{1/2} \sim C\sup_{\xi\in\mathbb{R}} |\psi_1(\xi)|$. Without loss of generality, we may extend $g$ to $\mathbb{R}^2$ so that $\nu\cdot\nabla g = 0$ in $\mathbb{R}^2\setminus\Omega$. Now with a similar argument as above we find that

$$\|\chi_2\|^2_{L_2(\Omega)} = \int_\Omega e^{-2d/r}(\frac{\partial}{\partial\nu} g(\bar{x}))^2(\psi_1(x_2))^2\, dx \le C_\psi^2 \int_{\mathbb{R}^2} |\nabla g|^2\, dx \le C_\psi^2 \|g\|^2_{H^1(\Omega)}$$

and hence

$$\|\chi_2\|_{L_2(\Omega)} \le C_\psi \|g\|_{H^1(\Omega)}. \tag{3.28}$$

As for the last norm we have

$$\begin{aligned}
\|\chi_3\|^2_{L_2(\Omega)} &= \left(\int_\Omega e^{-2d/r} g^2(\bar{x})[\frac{\partial}{\partial x_2}(\psi_1(x_2))]^2\, dx\right)\\
&\le C\left[\int_0^{\mathrm{diam}\Omega} e^{-2x_1/r}\, dx_1 \int_\Gamma g^2(\xi)[\frac{\partial}{\partial\xi}(\psi_1(\xi))]^2 \frac{|\mu\cdot\hat{n}|}{|\mu|}\, d\xi\right]\\
&\le C\left[\int_0^{\mathrm{diam}\Omega} \frac{1}{r} e^{-2x_1/r}\, dx_1\right] \int_\Gamma g^2(\xi)[\frac{\partial}{\partial\xi}(\psi_1(\xi))]^2\, d\xi\\
&\le \frac{C}{b^2}\left(\max_{1\le j\le M_\circ} |[\psi]_j|^2\right) \|g\|^2_{L_2(\Gamma)}.
\end{aligned}$$

Once again using the trace estimate we get

$$\|\chi_3\|_{L_2(\Omega)} \le Cb^{-1}(\max_j |[\psi]_j|)\|g\|_{H^1(\Omega)}. \tag{3.29}$$

Hence (3.26)-(3.29) imply that

$$\|\frac{\partial\varphi_1}{\partial x_2}\| \le C\left[\sup_{\xi\in\mathbb{R}} |\psi_1(\xi)| \left(1 + \frac{1}{\sqrt{r\ \min_k |\sin(\mu, d_k)|}}\right) + \frac{1}{b}\sup_j |[\psi]_j|\right] \|g\|_{H^1(\Omega)}. \tag{3.30}$$

Further recall that

$$\varphi_2(x,\mu) = e^{-d/r} g(\bar{x})\psi_2(x_2),$$

and by the construction of $\psi_1$, the function $\psi_2$ has a small support, $|\mathrm{supp}\psi_2| \le 2bM_\circ$. An application of the trace estimate, as before, easily yields to

$$\|\varphi_2\|_{L_2(\Omega)} \le CbM_\circ \sup_{\xi\in\mathbb{R}} |\psi_2(\xi)|\|g\|_{H^1(\Omega)}. \tag{3.31}$$

Recall that, we need to show that $F\in H^{s-1}(\Omega)$, $1 < s < 2$, to this approach set $A_2 = L_2(\Omega)$, define the function space $A_1$ by the following implicit relation: for a sufficiently smooth function $f$ defined in $\Omega$, let

$$\|f\|_{A_1} = \|\frac{\partial f}{\partial x_2}\|_{L_2(\Omega)}$$

and introduce the $K-$functional:

$$K(t,F) = \inf_{F=\varphi_1+\varphi_2} (t\|\varphi_1\|_{A_1} + \|\varphi_2\|_{A_2}). \tag{3.32}$$

Now the following interpolation result completes the proof of Lemma 3.3. See also embedding theorems in [7]. ■

**Lemma 3.4.** *For $g \in H^1(\Omega)$, $\mu \in J'_\delta$, $|\mu| = r$ and $0 < \varepsilon \ll 1/2$ we have*

$$\Phi_{1/2-\varepsilon,2}(K(t,F)) \leq \frac{C}{r\ \min_k |\sin(\mu, d_k)|} \|g\|_{H^1(\Omega)}, \tag{3.33}$$

*where $\Phi_{\rho,q}$ is the $K$-functional defined by*

$$\Phi_{\rho,q}(\varphi(t)) = \left( \int_0^\infty (t^{-\rho}\varphi(t))^q \frac{dt}{t} \right)^{1/q}, \qquad 1 \leq q \leq \infty, \quad 0 < \rho < 1.$$

**Proof.** It suffices to prove that

$$\|F\|_{B_2^{1/2,\infty}(\Omega)} \leq \frac{C}{r\ \min_k |\sin(\mu, d_k)|} \|g\|_{H^1(\Omega)}, \tag{3.34}$$

where $B_2^{1/2,\infty}(\Omega)$ is the usual Besov space, see [3] ($B_2^{1/2,\infty}$ is continuously embedded in $H^s$ for all $0 < s < 1/2$). To prove (3.34) is equivalent to showing that

$$\sup_{t>0} t^{-1/2} K(t,F) \leq \frac{C}{r\ \min_k |\sin(\mu, d_k)|} \|g\|_{H^1(\Omega)}. \tag{3.35}$$

We concentrate on (3.35) and first consider the simple case, i.e., $t$ large such hat

$$t \geq r \min_k |\cos(\mu, \hat{n}_k)| \sim r \min_k |\sin(\mu, d_k)|,$$

where $n_k$ is the outward unit normal to the $k$-th side $S_k$ of $\Omega$ and $\mu \in J'_\delta$ is fixed. Then for the partition $\psi_1 = 0$ and $\psi_2 = \psi$ we have using the same techniques as in the estimations of $\|\chi_i\|_{L_2(\Omega)}$'s that

$$\begin{aligned}
K(t,F)^2 \sim \|\varphi_2\|_{A_2}^2 &= \|e^{-d/r} g(\bar{x}) \frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}}\|_{L_2(\Omega)}^2 \\
&\leq C \int_0^{\text{diam}\Omega} e^{-2d/r}\, dx_1 \int_{\mathbb{R}} g^2(\xi) \left( \frac{\nu \cdot \hat{n}}{\mu \cdot \hat{n}} \right)^2 \frac{|\mu \cdot \hat{n}|}{|\mu|}\, d\xi \\
&\leq C \int_0^{\text{diam}\Omega} \frac{1}{r} e^{-2d/r}\, dx_1 \int_{\mathbb{R}} g^2(\xi) \frac{1}{|\mu \cdot \hat{n}|}\, d\xi \\
&\leq \frac{C}{r \min_k |\mu \cdot \hat{n}_k|} \|g\|_{L_2(\Gamma)}^2 \leq \frac{C}{r \min_k |\sin(\mu, d_k)|} \|g\|_{H^1(\Omega)}^2,
\end{aligned}$$

and thus

$$t^{-1/2} K(t,F) \leq \frac{C}{r\ \min_k |\sin(\mu, d_k)|} \|g\|_{H^1(\Omega)}. \tag{3.36}$$

Let now $t < r\ \min_k |\sin(\mu, d_k)|$ then with $\psi_1$ constructed as in (3.25) and $\psi_2 = \psi - \psi_1$ we have using (3.30)-(3.32) that

(3.37)

$$\begin{aligned}
&K(t,F) \leq (t\|\varphi_1\|_A + \|\varphi_2\|_{L_2(\Omega)}) \\
&\leq C \left\{ t \left[ \sup |\psi_1| \left( 1 + \frac{1}{\sqrt{r\ \min_k |\sin(\mu, d_k)|}} \right) + \frac{1}{b} \sup_j |[\psi]_j| \right] + 2bM_\circ \sup |\psi_2| \right\} \times \\
&\|g\|_{H^1(\Omega)}.
\end{aligned}$$

Now let us use $b^2 = t$, since $\sup|\psi_2| \sim \sup|\psi|$, $\sup|\psi_1| \le \sup|\psi|$ and also for the jumps $[\psi]_j$ at the points $\xi_j$, $j = 1, 2, ..., M_\circ$, $\sup_j |[\psi]_j| < 2\sup|\psi|$, hence (3.37) can be rewritten as

$$K(t,F) \le Ct^{1/2} \sup|\psi| \left[ t^{1/2}\left(1 + \frac{1}{\sqrt{r}\, \min_k |\sin(\mu, d_k)|}\right) + 1 + 4M_\circ \right] \|g\|_{H^1(\Omega)},$$

or equivalently as

$$t^{-1/2}K(t,F) \le C \sup|\psi| \left( 2\frac{t^{1/2}}{\sqrt{r}\, \min_k |\sin(\mu, d_k)|} + 1 + 4M_\circ \right) \|g\|_{H^1(\Omega)},$$

where we have used the fact that $\frac{t}{b} = t^{1/2}$. Further since $\frac{t^{1/2}}{\sqrt{r}\, \min_k |\sin(\mu, d_k)|} < 1$, hence

$$t^{-1/2}K(t,F) \le C(4M_\circ + 3)(\sup|\psi|)\|g\|_{H^1(\Omega)}. \tag{3.38}$$

Now using the fact that $\psi = \frac{1}{|\mu|} \cdot \frac{1}{\sin(\mu, d_k)}$ a combination of (3.36)-(3.38) complete the proof of Lemma 3.5. ∎

**4. Eigenvalue estimates.** Below we shall see by means of a *weak norm* estimate of the scalar flux that the largest eigenvalue $\lambda^{-1}$ of the transport operator $T$, (which makes $(I - \lambda T)^{-1}$ singular), can be found more accurately than the pointwise scalar flux. Observe that the kernel of the integral operator $T$ is symmetric and positive, see the representation of $T$ in [4]; (1.9). Hence $T$ is self-adjoint (on $L_2(\Omega)$), and thus has only real eigenvalues. Furthermore, by the Krien-Rutman theory, its largest eigenvalue is positive and simple. In this part we have the following result:

**Proposition 4.1.** *Let $\kappa$, $\kappa_n$ and $\kappa_n^h$ be the largest eigenvalues of the operators $T$, $T_n$ and $T_n^h$, respectively. Then for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there are constants $C = C(\varepsilon_1, \kappa)$ and $C(Q) = C(\varepsilon, \kappa, Q)$ such that for sufficiently large $N$ and $M$ (even) and sufficiently small $h$, we have*

$$|\kappa - \kappa_n| \le C\left(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\right), \tag{4.1}$$

$$|\kappa - \kappa_n^h| \le C\left(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon_1}}\right) + C(Q)h^{3-\varepsilon}, \tag{4.2}$$

*where $Q$ is an arbitrary quadrature set.*

These estimates follow from the semidiscrete and fully dscrete results of [6] and [3], respectively. The assumption on the number of angular quadrature points $M$ (even), makes the quadrature set $\Delta$ symmetric, so that, $\mu \in \Delta$ implies that $-\mu \in \Delta$. Then it follows that $T_n$ is self-adjoint (see e. g. [4] Lemma 2.1) and thus its eigenvalues are real, which is crucial in the proof of (4.1).

**Remark 4.1.** Similar results hold for a two-dimensional problem with the cylindrical domain $\tilde{\Omega}$ replaced by $\Omega \subset \mathbb{R}^2$ and the velocity space $\mathbf{D}$ replaced by $S = \{\mu \in \mathbb{R}^2 : |\mu| = 1\}$. Then, using a quadrature rule with $N$ discrete points on a quadrature set $Q \subset S$, and by the duality argument for spatial discretization, it is possible to show that for the largest eigenvalue $\kappa_N$, of the corresponding semidiscrete operator $T_N$, there exists an eigenvalue $\kappa_N^h$, for the fully discrete operator $T_N^h$ such that

$$|\kappa_N - \kappa_N^h| \le C(Q)h^{3-\varepsilon}. \tag{4.3}$$

However if we use the discrete ordinates method, with $N$ uniformly distributed points on the unit circle $S$, then the results in [3] give that

$$|\kappa_N - \kappa_N^h| \leq Ch^{1-\varepsilon}. \tag{4.4}$$

Now we may assume that $\kappa$ is the largest eigenvalue for the operator $T$ in the two-dimensional case, then recalling Theorem 5.1 in [4]; we have

$$|\kappa - \kappa_N| \leq CN^{-2+\theta}, \qquad 0 < \theta \ll 1. \tag{4.5}$$

Combining (4.3) and (4.5) we obtain

$$|\kappa - \kappa_N^h| \leq CN^{-2+\theta} + C(Q)h^{3-\varepsilon}, \tag{4.6}$$

while (4.4) together with (4.5) yields to

$$|\kappa - \kappa_N^h| \leq CN^{-2+\theta} + C\, h^{1-\varepsilon}. \tag{4.7}$$

Similarly, for our case in here, i.e. the case of infinite cylindrical domains, we have using Theorem 3.1 that

$$|\kappa_n - \kappa_n^h| \leq Ch^{1-\varepsilon}, \tag{4.8}$$

here $\kappa_n$ and $\kappa_n^h$ are the largest eigenvalues of $T_n$ and $T_n^h$ respectively, $n$ is the number of discrete points on the unit disc and the constant $C$ is independent of the quadrature set. Combination of (4.1) and (4.8) gives

$$|\kappa - \kappa_n^h| \leq C\left(\frac{1}{N^4} + \frac{1}{M^{2-\varepsilon}} + h^{1-\varepsilon}\right). \tag{4.9}$$

Comparing (4.6) with (4.7) and (4.2) with (4.9), we find that, if the quadrature set $Q \subset \mathbf{D}$ is properly chosen, so that $C(Q)$ does not cause a decay on efficiency of the scheme, then in order to obtain sharper fully discrete eigenvalue estimates, in both, two-dimensions and the case of cylindrical domains, a change of the compatibility concept; i.e. the condition $h \sim \frac{1}{N}$, is necessary.

The optimal relations $h = h(N)$, as well as $M = M(N)$, in the above estimates should be chosen in such a way that the contributions of the spatial and angular errors, to the global error, be of the same order of magnitude. Omitting logarithms and both $\varepsilon$ and $\theta$ powers of $h, M$ and $N$, respectively, we conclude that for the two-dimensional problem, with $N$ uniformly distributed quadrature points on the unit circle, if we use the duality, then as the spatial mesh size, the choice of $h \sim N^{-2/3}$ is optimal, while without the duality argument $h \sim N^{-2}$ is required. Similarly, in the case of the infinite cylindrical domains, with the duality argument $h \sim N^{-4/3} \sim M^{-2/3}$ is optimal whereas the corresponding condition without using the duality is $h \sim N^{-4} \sim M^{-2}$. Note specially that $M \sim N^2$ and the convergence rates, for the eigenvalues, are substantially improved. We point out that for the second condition in (3.4), i.e., $N \sim M$ a replacement of the form $M \sim N^2$ will not cause any serious restrictions in the proofs, since then, e.g., in the proof of (3.8) the use of $\frac{C}{\sqrt{M}} \sim h^\alpha$ with $h \sim n^{-1/2}$ implies that $\frac{1}{\sqrt{M}} \sim h^{2/3} \leq h^{1/2}$, $(h < 1)$, further recalling (4.2), $h \sim N^{-4/3} \sim M^{-2/3}$ gives $\frac{1}{\sqrt{M}} \sim h^{3/4} \leq h^{1/2}$, which in both cases our original estimates will be preserved. In conclusion: in connection with the duality concept, the compatibility conditions of the form $h \sim n^{-1/2}$, $n = MN$ and $M \sim N^2$(i.e., $h \sim N^{-3/2}$), containing both dual cases, are in *"good agreement"* with the theory and in comparison with an optimal choice of the form $h \sim N^{-4}$ in (5.13), we would only need $h \sim N^{-3/2}$ (with $M \sim N^2$).

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