On Stationary Solutions to the Linear Boltzmann Equation

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Abstract

This paper considers the stationary linear, space-dependent Boltzmann equation in the case of an interior source term together with an absorption term and general boundary reflections. First, mild L^1-solutions are constructed as limits of iterate functions. Then boundedness of all higher velocity moments are obtained in two physically interesting cases.

1 Introduction

The linear Boltzmann equation is frequently used for mathematical modelling in physics, (e.g. for discribing the neutron distribution in reactor physics, cf. [1]-[4]).

One fundamental question concerns the large time behavior of the function $f(\mathbf{x}, \mathbf{v}, t)$, representing the distribution of particles; in particular, the problem of convergence to a stationary equilibrium solution, when time goes to infinity. In our earlier papers [5]-[8] we have studied such convergence to equilibrium for the space-dependent linear Boltzmann equation with general boundary conditions and general initial data, under the assumption of existence of a corresponding stationary solution. For the proofs we use iterate functions, defined by an exponential form of the equation together with the boundary conditions, and we also use a general relative entropy functional for the quotient of the time dependent and the stationary solutions.

Then a fundamental question in kinetics concerns the existence and uniqueness of stationary solutions to the space-dependent transport equation, with general collision mechanism (including the case of inverse power forces), together with general boundary conditions, (including the periodic, specular and diffuse cases). We will study this problem in the angular cut-off case, using our earlier methods with iterate functions $F^n(\mathbf{x},\mathbf{v})$, (representing the distribution of particles having undergone at most n collisions, inside the body or at the boundary).

This will be done in the case of an interior source term $\alpha G(\mathbf{x},\mathbf{v})$, where $\alpha > 0$ is a constant and G is a given function, together with an absorption term $\alpha F(\mathbf{x},\mathbf{v})$ and general boundary reflections, (see Section 3). We will also (in Section 4) prove boundedness of all higher velocity moments for our solution $F(\mathbf{x},\mathbf{v}) = \lim_{n\to\infty} F^n(\mathbf{x},\mathbf{v})$, both in the specular and the diffuse boundary case, using our earlier estimation for the velocities in a binary collision. Finally, we study the problem of existence in a more general case (Section 5).

2 Preliminaries

We consider the stationary transport equation for a distribution function $F(\mathbf{x},\mathbf{v})$, depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$. The stationary linear Boltzmann equation in the case of given interior source $\alpha G(\mathbf{x},\mathbf{v})$, where $\alpha > 0$ is a constant and $G \geq 0$ is a given (measurable) function, together with an absorption term $\alpha F(\mathbf{x},\mathbf{v})$, is in the strong form

$$\alpha F(\mathbf{x},\mathbf{v}) + \mathbf{v}\nabla_{\mathbf{x}} F(\mathbf{x},\mathbf{v}) = \alpha G(\mathbf{x},\mathbf{v}) + (QF)(\mathbf{x},\mathbf{v}). \tag{2.1}$$

The collision term can be written

$$(QF)(\mathbf{x},\mathbf{v}) = \iint_{V\Omega} [Y(\mathbf{x},\mathbf{v}'_*)F(\mathbf{x},\mathbf{v}') - Y(\mathbf{x},\mathbf{v}_*)F(\mathbf{x},\mathbf{v})] \cdot B(\theta, |\mathbf{v}-\mathbf{v}_*|) d\theta d\zeta d\mathbf{v}_*, \tag{2.2}$$

where $Y \geq 0$ is a known distribution function, and $B \geq 0$ is given by the collision process. Here $\mathbf{v}, \mathbf{v}_*$ are the velocities before and $\mathbf{v}', \mathbf{v}'_*$ the velocities after a binary collision, and $\Omega = \{(\theta,\zeta) : 0 \leq \theta \leq \hat{\theta}, 0 \leq \zeta < 2\pi\}$ is the impact plane. In the angular cut-off case with $\hat{\theta} < \frac{\pi}{2}$ the gain and the loss term can be separated

$$(QF)(\mathbf{x},\mathbf{v}) = \int_V K(\mathbf{x},\mathbf{v}' \to \mathbf{v}) F(\mathbf{x},\mathbf{v}') d\mathbf{v}' - L(\mathbf{x},\mathbf{v}) F(\mathbf{x},\mathbf{v}), \tag{2.3}$$

where L is the collision frequency

$$L(\mathbf{x},\mathbf{v}) = \iint_{V\Omega} B(\theta, w) Y(\mathbf{x},\mathbf{v}_*) d\theta d\zeta d\mathbf{v}_*, w = |\mathbf{v}-\mathbf{v}_*|. \tag{2.4}$$

In the case of nonabsorbing body we have

$$L(\mathbf{x},\mathbf{v}) = \int_V K(\mathbf{x},\mathbf{v}\to\mathbf{v}'')d\mathbf{v}''. \tag{2.5}$$

One physically interesting case is that with inverse k-th power collision forces

$$B(\theta,w) = b(\theta)w^\gamma,\ \gamma = \frac{k-5}{k-1}, \tag{2.6}$$

with hard forces for $k>5$, maxwellian for $k=5$, and soft forces for $3<k<5$.

The equation (2.1) is supplemented with (general) boundary conditions

$$F_-(\mathbf{x},\mathbf{v}) = (1-\beta)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|}R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})F_+(\mathbf{x},\tilde{\mathbf{v}})d\tilde{\mathbf{v}}, \tag{2.7}$$

where β is a constant, $0\le\beta\le 1$. The function $R\ge 0$ satisfies

$$\int_V R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})d\mathbf{v}\equiv 1, \tag{2.8}$$

and $\mathbf{n}=\mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x}\in\Gamma$. The functions F_- and F_+ represent the ingoing and outgoing trace functions corresponding to F. Furthermore, in the specular reflection case, the function R is represented by Dirac measure $R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) = \delta(\mathbf{v}-\tilde{\mathbf{v}}+2\mathbf{n}(\mathbf{n}\mathbf{v}))$, and in the diffuse reflection case $R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) = |\mathbf{n}\mathbf{v}|W(\mathbf{x},\mathbf{v})$ with some given function $W\ge 0$ (e.g. maxwellian function).

Now using differentiation along the characteristics, the equation (2.1) can formally be written

$$\begin{aligned}&\frac{d}{dt}(F(\mathbf{x}+t\mathbf{v},\mathbf{v})) = \alpha G(\mathbf{x}+t\mathbf{v},\mathbf{v}) + \\ &+\int_V K(\mathbf{x}+t\mathbf{v},\mathbf{v}'\to\mathbf{v})F(\mathbf{x}+t\mathbf{v},\mathbf{v}')d\mathbf{v}' - [\alpha+L(\mathbf{x}+t\mathbf{v},\mathbf{v})]F(\mathbf{x}+t\mathbf{v},\mathbf{v}).\end{aligned} \tag{2.9}$$

Let

$$t_b\equiv t_b(\mathbf{x},\mathbf{v}) = \inf_{\tau\in\mathbb{R}_+}\{\tau : \mathbf{x}-\tau\mathbf{v}\notin D\}$$

and $\mathbf{x}_b\equiv\mathbf{x}_b(\mathbf{x},\mathbf{v}) = \mathbf{x}-t_b\mathbf{v}$. Here t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b=\mathbf{x}-t_b\mathbf{v}$ to the point $\mathbf{x}$.

Then we have the following *mild form* of the stationary linear Boltzmann equation

$$F(\mathbf{x},\mathbf{v}) = F_-(\mathbf{x}_b,\mathbf{v}) + \int_0^{t_b}[(QF)(\mathbf{x}-\tau\mathbf{v},\mathbf{v}) + \alpha G(\mathbf{x}-\tau\mathbf{v},\mathbf{v})]d\tau \tag{2.10}$$

and the *exponential form*

$$
\begin{aligned}
&F(\mathbf{x},\mathbf{v}) = F_-(\mathbf{x}_b,\mathbf{v})e^{-\int_0^{t_b}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds} + \\
&+\int_0^{t_b} e^{-\int_0^{\tau}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds}[\alpha G(\mathbf{x}-\tau\mathbf{v},\mathbf{v}) + \\
&+\int_V K(\mathbf{x}-\tau\mathbf{v},\mathbf{v}'\to\mathbf{v})F(\mathbf{x}-\tau\mathbf{v},\mathbf{v}')d\mathbf{v}']d\tau.
\end{aligned}
\tag{2.11}
$$

3 Construction of stationary solutions

We intend to construct mild L^1-solutions to our problem as limits of iterate functions F^n, when $n\to\infty$. Let first $F^{-1}(\mathbf{x},\mathbf{v})\equiv 0$ for all $\mathbf{x},\mathbf{v}\in\mathbb{R}^3$. Then define, for given function F^{n-1} the next iterate F^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation);

$$
F_-^n(\mathbf{x},\mathbf{v}) = (1-\beta)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|}R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})F_+^{n-1}(\mathbf{x},\tilde{\mathbf{v}})d\tilde{\mathbf{v}}, \tag{3.1}
$$

$$
\begin{aligned}
&\mathbf{n}\mathbf{v}<0, \mathbf{x}\in\Gamma=\partial D,\ \mathbf{v}\in V=\mathbb{R}^3; \\
&F^n(\mathbf{x},\mathbf{v}) = F_-^n(\mathbf{x}-t_b\mathbf{v},\mathbf{v})e^{-\int_0^{t_b}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds} + \\
&+\int_0^{t_b} e^{-\int_0^{\tau}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds}[\alpha G(\mathbf{x}-\tau\mathbf{v},\mathbf{v}) + \\
&+\int_V K(\mathbf{x}-\tau\mathbf{v},\mathbf{v}'\to\mathbf{v})F^{n-1}(\mathbf{x}-\tau\mathbf{v},\mathbf{v}')d\mathbf{v}']d\tau, \qquad (3.2) \\
&\mathbf{x}\in D\setminus\Gamma_-(\mathbf{v}),\ \mathbf{v}\in V=\mathbb{R}^3.
\end{aligned}
$$

Let also $F^n(\mathbf{x},\mathbf{v})\equiv 0$ for $\mathbf{x}\in\mathbb{R}^3\setminus D$.

Now we get a monotonicity lemma, which is essential in the following, and which can be proved by induction.

Lemma 3.1 $F^n(\mathbf{x},\mathbf{v})\geq F^{n-1}(\mathbf{x},\mathbf{v}), \mathbf{x}\in D, \mathbf{v}\in V, \mathbf{n}\in\mathbb{N}$.

Using differentiation along the characteristics, we get by (3.2) that

$$
\begin{aligned}
&\frac{d}{dt}(F^n(\mathbf{x}+t\mathbf{v},\mathbf{v})) = \alpha[G(\mathbf{x}+t\mathbf{v},\mathbf{v}) - F^n(\mathbf{x}+t\mathbf{v},\mathbf{v})] + \\
&+\int_V K(\mathbf{x}+t\mathbf{v},\mathbf{v}'\to\mathbf{v})F^{n-1}(\mathbf{x}+t\mathbf{v},\mathbf{v}')d\mathbf{v}' - L(\mathbf{x}+t\mathbf{v},\mathbf{v})F^n(\mathbf{x}+t\mathbf{v},\mathbf{v}).
\end{aligned}
\tag{3.3}
$$

Then integrating (3.3), it follows by Green's formula that

$$\begin{aligned}
&\alpha \iint_{DV} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} F^n_+(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma = \\
&= \alpha \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} F^n_-(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma + \\
&+ \iint_{DV} L(\mathbf{x},\mathbf{v}) [F^{n-1}(\mathbf{x},\mathbf{v}) - F^n(\mathbf{x},\mathbf{v})] d\mathbf{x} d\mathbf{v},
\end{aligned} \tag{3.4}$$

where by (2.8) and (3.1)

$$\iint_{\Gamma V} F^n_-(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma = (1-\beta) \iint_{\Gamma V} F^{n-1}_+(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma \tag{3.5}$$

Now by Lemma 3.1 it follows that

$$\alpha \iint_{DV} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \beta \iint_{\Gamma V} F^n_+(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma \le \alpha \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} \tag{3.6}$$

So, if $G \in L^1(D \times V)$, then we have for all $\alpha > 0$ that

$$\iint_{DV} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} \le \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} < \infty. \tag{3.7}$$

Then Levi's theorem gives existence of a mild L^1-solution $F(\mathbf{x},\mathbf{v}) = \lim_{n\to\infty} F^n(\mathbf{x},\mathbf{v})$ to the stationary linear Boltzmann equation (2.1) with (2.3), (2.7), and $F \equiv F_{\alpha,\beta}$ satisfies for all $\alpha > 0, 0 \le \beta \le 1$, the inequality

$$\begin{aligned}
&\alpha \iint_{DV} F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \beta \iint_{\Gamma V} F_+(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma \le \\
&\le \alpha \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}.
\end{aligned} \tag{3.8}$$

Furthermore, if $L(\mathbf{x},\mathbf{v})F(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$, then we get equality in (3.8) together with uniqueness in the relevant function space, cf. [6] and also [3].

So, for instance, if $\beta = \delta \cdot \alpha, \alpha > 0, \delta \ge 0$, then

$$\iint_{DV} F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \delta \iint_{\Gamma V} F_+(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma = \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}. \tag{3.9}$$

In summary, we have the following existence theorem for solutions to our stationary linear Boltzmann equation with general boundary reflections.

Theorem 3.2 *Assume that K, L and R are nonnegative, measurable functions, such that (2.5) and (2.8) hold, and $L(\mathbf{x},\mathbf{v}) \in L^1_{\mathrm{loc}}(D \times V)$. Let $\alpha > 0$ and $0 \le \beta \le 1$ be constants, and $G(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$ with $\iint G d\mathbf{x} d\mathbf{v} > 0$.*

a) Then there exists a mild L^1-solution $F(\mathbf{x}, \mathbf{v})$ to the problem (2.1)-(2.4) with (2.7). This solution, depending on α and β, satisfies the inequality (3.8).

b) Moreover, if $L(\mathbf{x}, \mathbf{v})F(\mathbf{x}, \mathbf{v}) \in L^1(D \times V)$, then the trace of the solution F satisfies the boundary condition (2.7) for a.e. $(\mathbf{x}, \mathbf{v}) \in \Gamma \times V$. Furthermore, mass conservation, giving equality in (3.8), holds, together with uniqueness in the relevant L^1-space.

Remark: For the case $\alpha = 0, \beta = 0$, we have in an earlier paper obtained uniqueness of mild L^1-solutions to the stationary linear Boltzmann equation, using a general entropy functional cf. [8].

4 On higher velocity moments

In this section we will prove boundedness of all higher velocity moments for our stationary solution (cf. Section 3), both for hard and soft collision forces in the case of inverse k-th power potentials. For that we will use our earlier results, cf. [5], giving an estimate for the velocities in a binary collision

$$\begin{aligned} &(1+(v')^2)^{\sigma/2} - (1+v^2)^{\sigma/2} \leq \\ &\leq C_1 w \cos\theta (1+v_*)^{\max(1,\sigma-1)}(1+v^2)^{\frac{\sigma-2}{2}} - \\ &- C_2 w \cos^2\theta (1+v^2)^{\frac{\sigma-1}{2}} \end{aligned} \tag{4.1}$$

with positive constants $C_1, C_2 > 0$ for all $\sigma > 0$.

To get higher moments estimates, we first multiply the equation (3.3) for the interate F^n by $(1+v^2)^{\sigma/2}$ and integrate, using Green's formula,

$$\begin{aligned} &\alpha \iint_{DV} (1+v^2)^{\sigma/2} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} (1+v^2)^{\sigma/2} F^n_+(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma = \\ &= \alpha \iint_{DV} (1+v^2)^{\sigma/2} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} (1+v^2)^{\sigma/2} F^n_-(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma \\ &+ \iiint_{DVV} [K(\mathbf{x}, \mathbf{v}' \to \mathbf{v})(1+v^2)^{\sigma/2} F^{n-1}(\mathbf{x},\mathbf{v}') - \\ &- K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')(1+v^2)^{\sigma/2} F^n(\mathbf{x},\mathbf{v})] d\mathbf{x} d\mathbf{v} d\mathbf{v}'. \end{aligned} \tag{4.2}$$

Using the collision estimate (4.1) together with the assumption on inverse power forces (2.6) and some elementary estimate, cf. [5], $w = |\mathbf{v} - \mathbf{v}_*|, -1 < \gamma \leq 1$,

$$-w^{\gamma+1} \leq (1+v_*)^{\gamma+1} - 2^{-1}(1+v^2)^{\frac{\gamma+1}{2}},$$

we find that the interior colision term in (4.2) is bounded from above by

$$
\begin{aligned}
&+\tilde{C}_1 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\
&+\tilde{C}_2 \iint_{DV} (1+v^2)^{\frac{\sigma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} - \\
&-\tilde{C}_0 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}
\end{aligned}
$$

with positive constants $\tilde{C}_0, \tilde{C}_1, \tilde{C}_2 > 0$. Here we have assumed that the function Y in (2.2) satisfies

$$\int_V (1+v_*)^{\gamma+\max(2,\sigma)} \sup_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*)) d\mathbf{v}_* < \infty, \int_V \inf_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*)) d\mathbf{v}_* > 0. \quad (4.3)$$

To handle the ingoing boundary term, $I_b^-(\sigma)$, in (4.2), we specialize in two physically interesting cases:

a) "Non-heating boundary" (e.g. specular reflection):

$$R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) \equiv 0 \quad \text{for } |\mathbf{v}| > |\tilde{\mathbf{v}}|. \quad (4.4)$$

Here we find that

$$I_b^-(\sigma) \leq (1-\beta) \iint_{\Gamma V} (1+v^2)^{\sigma/2} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma.$$

b) Diffuse reflection boundary:

$$R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = |\mathbf{nv}| W(\mathbf{x},\mathbf{v}). \quad (4.5)$$

Here we get

$$I_b^-(\sigma) \leq (1-\beta) C_{W,\sigma} \iint_{\Gamma V} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma$$

with a constant

$$C_{W,\sigma} = \int_V (1+v^2)^{\sigma/2} \sup_{\mathbf{x}\in\Gamma}(W(\mathbf{x},\mathbf{v})) d\mathbf{v} < \infty. \quad (4.6)$$

Then, the higher moment estimations follow in both the boundary cases, respectively

a) "non-heating boundary":

$$\alpha \iint_{DV} (1+v^2)^{\sigma/2} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \beta \iint_{\Gamma V} (1+v^2)^{\sigma/2} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma$$
$$+ \tilde{C}_0 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} \leq \alpha \iint_{DV} (1+v^2)^{\sigma/2} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} +$$
$$+ \tilde{C}_1 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \tilde{C}_2 \iint_{DV} (1+v^2)^{\frac{\sigma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}, \tag{4.7}$$

b) diffuse boundary:

$$\alpha \iint_{DV} (1+v^2)^{\sigma/2} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} (1+v^2)^{\sigma/2} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma +$$
$$+ \tilde{C}_0 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} \leq \alpha \iint_{DV} (1+v^2)^{\sigma/2} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} +$$
$$+ \tilde{C}_1 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \tilde{C}_2 \iint_{DV} (1+v^2)^{\frac{\sigma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} +$$
$$+ (1-\beta) C_{W,\sigma} \iint_{\Gamma V} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma, \tag{4.8}$$

where by (3.6)

$$\beta \iint_{\Gamma V} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma \leq \alpha \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}.$$

Letting $n \to \infty$ and using that $F^n(\mathbf{x},\mathbf{v}) \nearrow F(\mathbf{x},\mathbf{v})$, then the estimates (4.7) and (4.8) hold also for $F(\mathbf{x},\mathbf{v})$. Now by (3.8), the moment for $\sigma = 0$, i.e., the mass $\int\int F d\mathbf{x} d\mathbf{v}$, is bounded. So, successively, by (4.7), (4.8), we get boundedness of all higher velocity moments, $\sigma > 0$, for both soft and hard collision forces, $-1 < \gamma \leq 1$, i.e., $-3 < k \leq \infty$, and for both "non-heating" and diffuse boundaries.

Theorem 4.1 *Assume that the collision function $B(\theta, w)$ is given for inverse k-th power forces by equation (2.6) with $3 < k \leq \infty$, i.e., $-1 < \gamma \leq 1$, and suppose that the function $Y(\mathbf{x}, \mathbf{v}_*)$ satisfies (4.3). Let the boundary relation (3.7) be given by a "non-heating" boundary (4.4), (e.g. specular reflection), or by diffuse reflections (4.5) with (4.6).*

Then the higher velocity moments belonging to the mild solution F in Theorem 3.2, are all bounded,

$$\iint_{DV} (1+v^2)^{\sigma/2} F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} < \infty;$$

i.e., for all $\sigma > 0$, $\alpha > 0, 0 < \beta \leq 1$, *and* $-1 < \gamma \leq 1$, *if* $(1+v^2)^{\sigma/2}G(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$.

5 A more general case

In this section we will shortly handle a little more general case, assuming also a boundary source term $\beta_0 S_b(\mathbf{x},\mathbf{v})$ with a given function $S_b \geq 0$ together with the interior source $\alpha_0 G(\mathbf{x},\mathbf{v})$, and with interior and boundary absorption coefficients α, ϵ and $\beta \geq 0$.

Then define iterate functions, cf. (3.1), (3.2), by

$$F_-^n(\mathbf{x},\mathbf{v}) = \beta_0 S_b(\mathbf{x},\mathbf{v}) + (1-\beta)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}} \to \mathbf{v}) F_+^{n-1}(\mathbf{x},\tilde{\mathbf{v}})d\tilde{\mathbf{v}}, \qquad (5.1)$$

and

$$\begin{aligned} F^n(\mathbf{x},\mathbf{v}) &= F_-^n(\mathbf{x}_b,\mathbf{v}) e^{-\int_0^{t_b}[\alpha+(1+\epsilon)L(\mathbf{x}-s\mathbf{v},\mathbf{v})]ds} + \\ &+ \int_0^{t_b} e^{-\int_0^{\tau}[\alpha+(1+\epsilon)L(\mathbf{x}-s\mathbf{v},\mathbf{v})]ds}[\alpha_0 G(\mathbf{x}-\tau\mathbf{v},\mathbf{v}) + \\ &+ \int_V K(\mathbf{x}-\tau\mathbf{v},\mathbf{v}' \to \mathbf{v}) F^{n-1}(\mathbf{x}-\tau\mathbf{v},\mathbf{v}')d\mathbf{v}']d\tau \end{aligned} \qquad (5.2)$$

with constants $\alpha, \alpha_0, \beta_0, \epsilon \geq 0$, $0 \leq \beta \leq 1$. Here we also get a useful monotonicity lemma:

$$F^n(\mathbf{x},\mathbf{v}) \geq F^{n-1}(\mathbf{x},\mathbf{v}),\ \mathbf{x} \in D, \mathbf{v} \in V,\ n \in \mathbb{N}.$$

Then differentiating (5.2) along the characteristics, and integrating with Green's formula, it follows, as in Section 3, that

$$\begin{aligned} &\alpha \iint_{DV} F^n(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \beta \iint_{\Gamma V} F_+^{n-1}(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma + \\ &+ \epsilon \iint_{DV} L(\mathbf{x},\mathbf{v})F^n(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} = \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \\ &+ \beta_0 \iint_{\Gamma V} S_b(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma - \iint_{DV} L(\mathbf{x},\mathbf{v})[F^n - F^{n-1}]d\mathbf{x}d\mathbf{v} \\ &- \iint_{\Gamma V} [F_+^n - F_+^{n-1}]|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma. \end{aligned}$$

So, Levi's theorem gives (for $\alpha > 0$) existence of L^1-solutions

$$F(\mathbf{x},\mathbf{v}) = \lim_{n\to\infty} F^n(\mathbf{x},\mathbf{v}),$$

where F depends on $\alpha, \alpha_0, \beta, \beta_0, \epsilon$ and G, S_b.

Using the technique from Section 4, we can also get analogous results for higher velocity moments, cf. Theorem 4.1.

Remark: One remaining problem concerns the question: what happens when the coefficients $\alpha, \alpha_0, \beta, \beta_0$ and ϵ go to zero? We hope to come back to this question in a forthcoming paper.

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