NAKAYAMA AUTOMORPHISMS OF FINITE PROJECTIVE HOPF ALGEBRAS

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1. Introduction

The circle of ideas relating Frobenius algebras to Hopf algebras began with the example by Berkson [4] of the restricted universal enveloping algebra of a finite dimensional restricted Lie algebra. Together with the well-known Frobenius algebra examples of finite group algebras, this made a strong case for conjecturing that finite dimensional Hopf algebras are Frobenius, something which was established as a corollary of the work of Larson and Sweedler in [20]. Pareigis extended the theory in [20] to a finite projective Hopf algebra H over a commutative ring k in [30, 32]. His main result is that if k has trivial Picard group, then H is still a Frobenius algebra [30]. However, in general H is a somewhat modified Frobenius algebra depending on the element P in the Picard group of k represented by the dual of the space of integrals in the dual Hopf algebra H^* [32].

Schneider [37] made the first remarkable application of the Frobenius theory of a Hopf algebra to establishing a nice and important formula; namely, Radford's formula for the fourth power of the antipode S (Eq. 38) [33]. The authors independently used another Frobenius approach to proving the Radford formula for FH-algebras, or Hopf algebras over rings which are Frobenius algebras, in [12]. The idea in [12] and more clearly in the present paper is the following conceptually. First, from a complete set of Frobenius data called a Frobenius system for a Hopf algebra we obtain another Frobenius system by applying the antipodal anti-automorphism. Second, we obtain two Nakayama automorphisms with formulas involving $S^{\pm 2}$ acted on from the right and left, respectively, by the left modular function for H. Third, the Kasch-Pareigis principle that any two Frobenius systems are unique up to an invertible element, which we call the derivative, leads after a computation to the modular function for H^*, $b \in H$ as derivative. Finally, since the two Nakayama automorphisms are related by an inner automorphism determined by b, we easily derive the Radford formula for S^4. In principle, this technique might produce nice formulas or new proofs wherever one deals with examples of Frobenius algebras or extensions.

The aim of this paper is to give explicit reasons why a finite projective Hopf algebra H over a commutative ring k is very close to being a FH-algebra, a principle first formulated by Pareigis [31]. Our main example of this principle is to make a Frobenius proof of Radford's formula work for a general finite projective Hopf algebra H. The first part of our paper is organized as follows. In the first two sections, we review a chapter in the unpublished lecture notes of Pareigis [32], specializing to a theory of P-Frobenius algebras with Frobenius homomorphism, dual bases and Nakayama automorphisms, which we call here *a Frobenius system* for

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H. To this we add the conceptually useful *comparison theorem* and *transformation theorem* for P-Frobenius algebras. In Section 3 we follow [32] by showing that H is a P-Frobenius algebra with left integral-like Frobenius homomorphism ψ and dual bases determined by a left norm N. We emphasize that H *has a Nakayama automorphism*, as Pareigis has shown. In Section 4, we meet the problem that the space of integrals in H^* is not freely generated by a left norm, by defining a modular function as the Nakayama automorphism composed with the counit, a formula from the authors' [12]: the usual definition of modular function depends on the norm element being a free generator of the space of integrals, which is just not the case for general H. In Section 5, we extend the validity of the Fischman-Montgomery-Schneider formula [28, 8] to include the Nakayama automorphism ν of H. Then we transform the P-Frobenius system of Pareigis by the antipode S and prove that the derivative is proportional to the distinguished group-like $b \in H$. We finally apply the comparison theorem and obtain a clear proof of Radford's formula for S^4 in the general case. Using these formulas we extend results in [33, 8, 12] to the result (Corollary 5.5) that $\nu^{2N} = S^{4N} = \mathrm{Id_H}$ for a finite projective Hopf algebra H, where N is the least common multiple of the local ranks of H.

Oberst and Schneider showed in [28] that finite projective Hopf subalgebras are Frobenius extensions of the second kind defined by Nakayama-Tsuzuku [26] if the overalgebra is cocommutative and the subalgebra is a k-direct summand. These conditions were required because Nichols and Zoeller had not yet proved that a finite dimensional Hopf algebra is free over a Hopf subalgebra [27]. By localization-type arguments, the Nichols-Zoeller theorem can be extended to showing that finite projective Hopf subalgebras form a free extension if k is a local ring, and a projective extension in general: see Schneider [36] and the authors' [12]. One of the objectives of this paper is to show that the most general case of a finite projective Hopf subalgebra pair $K \subseteq H$ is a Frobenius extension of a *third kind*, depending not only on a relative Nakayama automorphism but also on the two Picard group elements of k represented by the space of integrals of K^* and H^*. It is implicit in the literature on Frobenius extensions and Frobenius algebras [15, 23, 29] that the induced representation theory of a subalgebra pair $H \supseteq K$ is substantially simplified once they are classified as a Frobenius extension of any kind. For example, Kasch [15] proves that a Frobenius extension of a self-injective ring is self-injective. In this respect, the only difference between Frobenius extensions of the first kind and Frobenius extensions of the second and third kinds is that the functors of co-induction and induction are naturally equivalent for the first kind and this differs only by a Morita auto-equivalence of the module category $\mathcal{M}_K$ for the second and third kinds (cf. Section 6).

The rest of the paper is organized as follows. In Section 6, we show that a finite projective Hopf algebra H is separable precisely when the counit of its norm is invertible in some generalized sense for modules. We show that H is moreover strongly separable if H is involutive and separable. In Section 7, we show that a finite projective Hopf subalgebra pair forms a Frobenius extension of a *third kind*, a notion due to Morita [23], [24] and Pareigis [32]. In Section 7, we return to the idea that a finite projective Hopf subalgebra H is close to being an FH-algebra by proving that H is a Hopf subalgebra of an FH-algebra in two ways. First, we prove that Drinfeld's quantum double $D(H)$ is an FH-algebra. Second, we find a ring extension $k \subset K$ such that $Pic(K) = 0$: therefore the FH-algebra $H \otimes_k K$ is a flat extension of H.

2. P-Frobenius Algebras

In this section we sketch a theory of P-Frobenius algebras based on Pareigis's P-Frobenius extensions [32, unpublished], which generalized P-Frobenius algebras and β-Frobenius extensions at once. This generality will not be central to this paper, but we will need all the results below on P-Frobenius algebras in the later sections (except Proposition 2.2). All the results in this section save Theorems 2.7, 2.8, Proposition 2.2, and Lemma 2.9 are based on [32].

Let k be a commutative ring throughout this paper. A tensor $\otimes$ without subscript will means $\otimes_k$ as will a homomorphism group $\mathrm{Hom} = \mathrm{Hom}_k$. The k-dual of a k-module V is denoted by V^*. If A is a k-algebra, its V-dual $\mathrm{Hom}(A, V)$ has a standard A-bimodule structure given by $(bfc)(a) := f(cab)$ for every $f \in \mathrm{Hom}(A, V), a, b, c \in A$.

Let P be an invertible k-module throughout, i.e. P is finite projective of constant rank 1 [38]. The functor represented by $P \otimes -$ is a Morita auto-equivalence of the category of k-modules, denoted by $\mathcal{M}_k$, and P represents an isomorphism class in the Picard group $Pic(k)$ of k [1, 38]. Let Q be its inverse as an element of $Pic(k)$, so $Q \cong P^*$, and both $P \otimes Q \cong k$ and $Q \otimes P \cong k$ are given by canonical isomorphisms ϕ_1 and ϕ_2, respectively, which we choose so that associativity holds

$$(qp)q' = q(pq') \tag{1}$$

for every $p \in P$ and $q, q' \in Q$, and a corresponding associativity equation on $P \otimes Q \otimes P$ [1], where the values of these isomorphisms are denoted simply by $p \otimes q \mapsto pq$ and $q \otimes p \mapsto qp$. Since $\phi_2 \circ \phi_1^{-1}$ is an automorphism of k, we have $\chi, \gamma \in k$ such that $\chi\gamma = 1_k$ and

$$\begin{aligned} pq &= \gamma qp \\ qp &= \chi pq \end{aligned} \tag{2}$$

for every $p \in P$, $q \in Q$.

Definition 2.1. A k-algebra A is said to be a P-Frobenius algebra if

1. A is finite projective as a k-module;
2. $A_A \cong \mathrm{Hom}_k(A, P)_A$.

If $P \cong P'$, then a P-Frobenius algebra is also P'-Frobenius. In particular, if $P \cong k$, then a P-Frobenius algebra is an ordinary Frobenius algebra. The following converse statement is *false*: if a P-Frobenius algebra is also P'-Frobenius, then $P \cong P'$. This may be somewhat surprising if one recalls that the corresponding statement is true for β-Frobenius extensions [26]. A counterexample is based on the Steinitz isomorphism theorem for ideals in a Dedekind domain R [21]:

Proposition 2.2. *Suppose R is a Dedekind domain and I is a non-principal ideal in R such that $I \cong I^{-1}$. Let $A := M_2(R)$. Then*

$${}_A\mathrm{Hom}_R(A, I) \cong {}_A A. \tag{3}$$

Proof. Let F denote the field of fraction of R, and e_{ij} the matrix units in A. We first note that $\mathrm{Hom}_R(A, I) \cong M_2(I)$, since

$$f \mapsto \begin{pmatrix} f(e_{11}) & f(e_{12}) \\ f(e_{21}) & f(e_{22}) \end{pmatrix}$$

is a left A-isomorphism if we define the left A-module structure on $M_2(I)$ by $X \cdot B := BX^t$ for every $B \in M_2(I), X \in A$.

By the Steinitz isomorphism theorem, $I \oplus I \cong R \oplus R$ as R-modules determined by a matrix $C \in M_2(F)$ as $(x\ y) \mapsto (x\ y)C^t$. Then the mapping $X \mapsto (CX)^t$ for every $X \in M_2(I)$ determines an R-isomorphism $\Psi : M_2(I) \to A$. But for every $Y \in A$ we have

$$\Psi(Y \cdot X) = (CXY^t)^t = YX^tC^t = Y\Psi(X)$$

whence Ψ is a left A-module isomorphism as desired. □

A is of course a well-known example of a Frobenius algebra over R. That it is also an I-Frobenius algebra where $I \not\cong R$ follows directly from Theorem 2.4 below. R is for example realized by the ring of integers of an algebraic number field with two element ideal class group.

Recall that an algebra A is QF (quasi-Frobenius) in the sense of Müller [25], if A is finite projective as a k-module, and A_A is isomorphic to a direct summand of the direct sum of n copies of A^*_A, for $n \geq 1$. Also recall that a QF ring A is Artinian and injective as a right or left module over itself [18]. It follows straightaway from Definition 2.1 that:

Proposition 2.3. *A P-Frobenius algebra A is a QF algebra. A is a QF ring if k itself is a QF ring.*

Proof. If $P \oplus N \cong k^n$, then

$$A_A \oplus \mathrm{Hom_k(A, N)} \cong \mathrm{nA^*}$$

It is shown in [25] that a QF extension of a QF ring is a QF ring. □

However, P-Frobenius algebras are much closer to being Frobenius algebras than QF algebras as we shall see below.

Theorem 2.4. *The following conditions on a k-algebra A are equivalent:*

1. *A is a P-Frobenius algebra;*
2. *A_k is finite projective and ${}_AA \cong {}_A\mathrm{Hom_k(A, P)}$;*
3. *there are $\phi \in \mathrm{Hom_k(A, P)}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ and $q_1, \ldots, q_n \in Q$ such that*

$$\sum_i \phi(ax_i)q_iy_i = a \tag{4}$$

or

$$\sum_i x_iq_i\phi(y_ia) = a \tag{5}$$

for every $a \in A$. (ϕ is referred to as a Frobenius homomorphism and $\{x_i\}, \{q_i\}, \{y_i\}$ as dual bases for ϕ.)

Proof. (1 ⇒ 2.) We compute using the Hom-tensor relation:

$$\begin{aligned} {}_A\mathrm{Hom_k(A, P)} &\cong \mathrm{Hom_k(Hom_k(A, P)_A, P)} \\ &\cong {}_A\mathrm{Hom_k(A^* \otimes P, P)} \\ &\cong {}_A\mathrm{Hom_k(A^*, k)} \cong {}_\mathrm{A}\mathrm{A}, \end{aligned}$$

since P is an invertible module.

(2 $\Rightarrow$ 3.) Given $\Psi : {}_AA \stackrel{\cong}{\to} {}_A\mathrm{Hom_k(A,P)}$ and $\phi := \Psi(1_A)$, then $\Psi(a) = a\phi$ for every $a \in A$. Then ${}_AA \otimes Q \cong {}_AA^*$ via $a \otimes q \mapsto a\phi q$. If $\{y_i \in A\}$, $\{f_i \in A^*\}$ is a finite projective base for A_k, one finds $x_{ij} \in A$, $q_{ij} \in Q$ such that $\sum_j x_{ij}\phi q_{ij} = f_i$. Setting $y_{ij} := y_i$ for each i and j, we have for every $a \in A$,

$$\begin{aligned} a &= \sum_i f_i(a) y_i \\ &= \sum_{i,j} (x_{ij}\phi)(a) q_{ij} y_{ij} = \sum_{i,j} \phi(a x_{ij}) q_{ij} y_{ij}. \end{aligned}$$

We merely reindex to get Equation 4. Equation 5 follows from a computation showing $\Psi(\sum_i x_i q_i \phi(y_i a))(x) = \Psi(a)(x)$ for $x, a \in A$, which is similar to [8, 1.3].

(3 $\Rightarrow$ 1.) Suppose $\sum_{i=1}^n x_i q_i(\phi y_i) = \mathrm{Id_A}$. Then A is finite projective. Define $\Psi : A_A \to \mathrm{Hom_k(A,P)_A}$ by $\Psi(a) := \phi a$ for every $a \in A$. Then Ψ is epi since for every $f \in \mathrm{Hom_k(A,P)}$ we have $\Psi(\sum_i f(x_i) q_i y_i)(a) = f(a)$ for every $a \in A$. Since $\Psi : A \to \mathrm{Hom_k(A,P)} \cong \mathrm{A^* \otimes P}$ is an epimorphism between finite projective modules of the same local rank, (i.e. $\mathcal{P}$-rank for every prime ideal $\mathcal{P}$ in k), Ψ is bijective [38, 30].

A similar argument shows that we may establish Condition 2 from Equation 4. □

Throughout this section, we continue our use of the notation ϕ and x_i, q_i, y_i for the Frobenius homomophism and dual base of a P-Frobenius algebra A. The automorphism on A in the next result measures the deviation of ϕ from satisfying the trace condition $\phi(ab) = \phi(ba)$ for every $a, b \in A$.

Corollary 2.5. *In a P-Frobenius algebra A there is an algebra automorphism $\nu : A \to A$ given by*

$$a\phi = \phi\nu(a) \tag{6}$$

for every $a \in A$. (Call ν the Nakayama automorphism.)

Proof. In the proof of the last theorem we established $3 \Rightarrow 1$ by showing $a \mapsto \phi a$, for every $a \in A$, is an isomorphism. As we noted, we may equally well establish $3 \Rightarrow 2$ in this proof by showing that $a \mapsto a\phi$ is an isomorphism ${}_AA \cong {}_A\mathrm{Hom_k(A,P)}$. Since $a\phi \in \mathrm{Hom_k(A,P)}$ for each $a \in A$, it follows that there is a unique $a' \in A$ such that $a\phi = \phi a'$. One defines $\nu(a) = a'$ and easily checks that ν is an automorphism. □

We fix the data $(\phi, x_i, q_i, y_i, \nu)$ for the rest of this section and refer to this as the *Frobenius system* of A in this paper.

Proposition 2.6. *Given a P-Frobenius algebra A, the dual base tensor $\sum_i x_i \otimes q_i \otimes y_i$ satisfies $\forall\, \mathrm{a} \in \mathrm{A}$:*

1. $\sum_i a x_i \otimes q_i \otimes y_i = \sum_i x_i \otimes q_i \otimes y_i a$, *and*
2. $\sum_i x_i a \otimes q_i \otimes y_i = \sum_i x_i \otimes q_i \otimes \nu(a) y_i$.

Proof. We give only the proof of the second equation, the first being similar. By Equations 5, 1, 6 and 4, we compute:

$$\begin{aligned}\sum_i x_i a \otimes q_i \otimes y_i &= \sum_{i,j} x_j q_j \phi(y_j x_i a) \otimes q_i \otimes y_i \\ &= \sum_{i,j} x_j \otimes q_j \otimes \phi(y_j x_i a) q_i y_i \\ &= \sum_{i,j} x_j \otimes q_j \otimes \phi(\nu(a) y_j x_i) q_i y_i \\ &= \sum_j x_j \otimes q_j \otimes \nu(a) y_j \quad \square\end{aligned}$$

We next prove that P-Frobenius systems for A are unique up to an invertible element in A, which we call the *comparison theorem.*

Theorem 2.7. *Suppose (ϕ, x_i, q_i, y_i) and $(\phi', x'_j, q'_j, y'_j)$ are two P-Frobenius systems for a P-Frobenius algebra A. Then there is $d \in A^\circ$ such that*

$$\phi' = \phi d \tag{7}$$

and

$$\sum_j x'_j \otimes q'_j \otimes y'_j = \sum_i x_i \otimes q_i \otimes d^{-1} y_i. \tag{8}$$

If ν, ν' are the Nakayama automorphisms of ϕ and ϕ', then $\forall\, \mathrm{a} \in \mathrm{A}$,

$$v'(a) = d^{-1} v(a) d. \tag{9}$$

Proof. Since ϕ and ϕ' freely generate $\mathrm{Hom_k(A,P)}$ as right A-modules, Equation 7 is clear with d an invertible in A.

To verify Equation 8, we note that

$$\sum_i x_i q_i \phi d(d^{-1} y_i a) = a \tag{10}$$

for every $a \in A$. There is an isomorphism

$$A \otimes Q \otimes A \cong \mathrm{End_k(A)}$$

given by $a \otimes q \otimes b \mapsto aq\phi' b$, for every $a, b \in A, q \in Q$, since $A \otimes A^* \cong \mathrm{End_k(A)}$ and $Q \otimes A \cong A^*$. Equation 8 follows from the injectivity of this mapping and Equation 10.

We note that for every $x, a \in A$

$$\phi'(xa) = \phi'(\nu'(a)x) \;\Leftrightarrow\; \phi(dxa) = \phi(\nu(a)dx) = \phi(d\nu'(a)x) \tag{11}$$

the last equation implying that for all $a \in A$,

$$\nu(a)d = d\nu'(a)$$

which is equivalent to Equation 9. $\square$

We also need to know the effect of an algebra anti-automorphism on a Frobenius system, as given in the following *transformation theorem.*

Theorem 2.8. *Let A be a P-Frobenius algebra with Frobenius system $(\phi, x_i, q_i, y_i, \nu)$. If α is a k-algebra anti-automorphism of A, then*

$$(\alpha\phi,\ \chi\overline{\alpha}(y_i),\ q_i,\ \overline{\alpha}(x_i),\ \overline{\alpha}\circ\overline{\nu}\circ\alpha) \tag{12}$$

is another Frobenius system for A, where $\overline{\alpha}$ and $\overline{\nu}$ denote the inverses of α and ν, and $\alpha\phi := \phi\circ\alpha$.

Proof. We compute using the identity $\alpha(ab) = \alpha(b)\alpha(a)$ for all $a, b \in A$:

$$a = \sum_i x_i q_i \phi(y_i a) = \sum_i \chi(\alpha\phi)(\overline{\alpha}(a)\overline{\alpha}(y_i))q_i x_i,$$

and by applying $\overline{\alpha}$ to both sides we obtain

$$\overline{\alpha}(a) = \sum_i \chi(\alpha\phi)(\overline{\alpha}(a)\overline{\alpha}(y_i))q_i\overline{\alpha}(x_i).$$

It follows from Theorem 2.4 that $\alpha\phi$ is a Frobenius homomorphism with dual bases $\{\chi\overline{\alpha}(y_i)\}$, $\{q_i\}$, $\{\overline{\alpha}(x_i)\}$.

We compute the Nakayama automorphism η for $\alpha\phi$ in terms of α and ν: for all $a, b \in A$,

$$\phi(\alpha(a)\alpha(b)) = (\alpha\phi)(ba) = (\alpha\phi)(\eta(a)b) = \phi(\alpha(b)\alpha\eta(a)) = \phi((\nu\alpha\eta)(a)\alpha(b))$$

by applying Equation 6 twice. Since ϕ freely generates A^*, it follows that $\nu\circ\alpha\circ\eta = \alpha$, whence

$$\eta = \overline{\alpha}\circ\overline{\nu}\circ\alpha. \quad \square \tag{13}$$

We will need the following lemma in our last section.

Lemma 2.9. *If A is a P-Frobenius algebra and B is a Q-Frobenius algebra, then the tensor product algebra $A\otimes B$ is a $P\otimes Q$-Frobenius algebra.*

Proof. First, $C := A\otimes B$ is finite projective as a k-module. Secondly,

$${}_C C \cong {}_A A \otimes {}_B B \cong {}_A\mathrm{Hom}(A, P)\otimes {}_B\mathrm{Hom}(B, Q) \cong {}_C\mathrm{Hom}(C, P\otimes Q),$$

since A, B, P and Q are finite projective k-modules. $\square$

3. Finite Projective Hopf Algebras are P-Frobenius Algebras

Let H be a Hopf algebra over a commutative ring k which is finite projective as a k-module. *A "Hopf algebra" refers to such a finite projective Hopf algebra for the rest of this paper, unless otherwise stated.* In this section, we review the results of Pareigis on the Hopf module structure on the dual Hopf algebra H^* and the P-Frobenius structure on H [30, 32]. For the convenience of the reader we offer complete proofs for results in the unpublished [32].

For the Hopf algebra H we denote its comultiplication by $\Delta : H \to H\otimes H$, its counit by ϵ, and its antipode by S. The values of Δ are denoted by $\Delta(x) = \sum x_1\otimes x_2$. If M is a right comodule over H the values of its coaction on an element $m\in M$ is denoted by $\sum m_0\otimes m_1$. The dual of H is itself a Hopf algebra H^* where its multiplication is the convolution product (dual to Δ), comultiplication is the dual of multiplication on H, the counit is $1\in H\cong H^{**}$ ($x\mapsto$ evaluation at x). We also denote its antipode by S where the context is clear.

Proposition 3.1. *If H is a Hopf algebra, then H^* is right Hopf module.*

Sketch of Proof. The proof in [30] notes that the natural left H^*-module structure on the dual algebra H^* induces a comodule structure mapping $\chi : H^* \to H^* \otimes H$, determined by

$$gh = \sum h_0 g(h_1) \tag{14}$$

for every $g, h \in H^*$.

The right H-module structure on H^* is given by $(h^* \cdot h)(x) = h^*(xS(h))$ for every $x, h \in H$ and $h^* \in H^*$. A long computation in [30] shows this compatible with the H^*-comodule structure in the sense of Hopf modules. □

Proposition 3.2. *Every right Hopf module M over a Hopf algebra H is isomorphic to the trivial Hopf module, $M \cong P(M) \otimes H$, where*

$$P(M) = \{m \in M \mid \chi(m) = m \otimes 1_H\}$$

is a k-direct summand of M and $\chi : M \to M \otimes H$ denotes the right H-comodule structure mapping.

Sketch of Proof in [30]. One shows that the map $M \to M$ given by $m \mapsto \sum S(m_0)m_1$ is a k-linear projection onto $P(M)$. Then the mapping $\beta : M \to P(M) \otimes H$ given by $\beta(m) = \sum m_0 S(m_1) \otimes m_2$ has inverse given by the Hopf module map $\alpha : P(M) \otimes H \to M$ given by $\alpha(m \otimes h) = mh$. □

Corollary 3.3. *The k-module $P(H^*)$ associated to a Hopf algebra H by Propositions 3.1 and 3.2 is an invertible k-direct summand in H^*.*

Proof. Since $P(H^*) \otimes H \cong H^*$ and H, H^* have the same local ranks, it follows that the finite projective k-module $P(H^*)$ has constant rank 1. Then $P(H^*) \otimes P(H^*)^* \cong k$ and $P(H^*)$ is invertible [38]. □

We note that $P(H^*)$ is the space of left integrals $\int_{H^*}^{\ell}$ in H^*:

$$P(H^*) = \{f \in H^* | gf = g(1)f\} \tag{15}$$

which follows from Equation 14 since $\sum f_0 \otimes f_1 = f \otimes 1$.

Proposition 3.4. *The antipode S of a Hopf algebra H is bijective.*

Sketch of Proof in [30]. Assuming that $S(x) = 0$, one then notes that multiplication from the right by x on $P(H^*) \otimes H$ is zero by the existence of the (H-module) isomorphism $\alpha : P(H^*) \otimes H \to H^*$ in Proposition 3.2. If k is field $P(H^*) \cong k$ and it is clear that x is then zero. The general case follows from a localization argument. Surjectivity for S is apparent if k is a field, and the general case follows again from a localization argument. □

Denote the composition-inverse of S by $\overline{S}$.

Theorem 3.5 (Pareigis)**.** *If H is a Hopf algebra and*

$$P := P(H^*)^*,$$

then H is a P-Frobenius algebra.

Proof. We set $\Phi : P(H^*) \otimes H \xrightarrow{\cong} H^*$, $f \otimes x \mapsto f \cdot x$, where we note that the right H-module structure is related to the standard left H-module structure on H^* via a twist by S: for every $g \in H^*, x, y \in$

$$(g \cdot x)(y) = g(yS(x)) = (S(x)g)(y).$$

Let $Q := P(H^*)$, which is canonically isomorphic to the dual of P, and satisfies $P \otimes Q \cong k$ by Corollary 3.3.

Define $\Psi' : H \to \mathrm{Hom_k(H, P)}$ as the composite of the right H-module isomorphisms

$$H \longrightarrow P \otimes Q \otimes H \stackrel{1 \otimes \Phi}{\longrightarrow} P \otimes H^* \longrightarrow \mathrm{Hom_k(H, P)}.$$

It is easy to check that

$$\Psi'(x)(y)(q) := \Phi(q \otimes x)(y) = q(yS(x)) \tag{16}$$

for all $x, y \in H$ and $q \in Q$.

Now let $\Psi := \Psi' \circ \overline{S}$. Ψ is a Frobenius isomorphism ${}_H H \cong {}_H\mathrm{Hom_k(H, P)}$, since $\overline{S}$ is an anti-automorphism of H and

$$\Psi(xy) = \Psi'(\overline{S}(y)\overline{S}(x)) = \Psi(y) \cdot \overline{S}(x) = x\Psi(y). \quad \square$$

Corollary 3.6. *The Frobenius homomorphism $\psi : H \to P$ defined by the theorem satisfies for every $a \in H$*

$$\sum a_1 \otimes \psi(a_2) = 1 \otimes \psi(a) \tag{17}$$

Proof. We note that the Frobenius homomorphism $\psi := \Psi(1) = \Psi'(1)$ satisfies by Equation 16, for every $q \in P(H^*), a \in H$,

$$\psi(a)(q) = q(a),$$

and

$$q(a)1_H = \sum a_1 q(a_2)$$

since $q \in \int^{\ell}_{H^*}$.

Since $P = Q^*$ and H is finite projective over k, we canonically identify $H \otimes P \cong \mathrm{Hom_k(Q, H)}$, and compute $\forall q \in Q, a \in H$

$$(\sum a_1 \otimes \psi(a_1))(q) = \sum a_1 \psi(a_2)(q) = \sum a_1 q(a_2) = 1_H q(a) = (1 \otimes \psi(a))(q)$$

whence Equation 17. $\square$

If $\int^{\ell}_{H^*} \cong k$, we see from the theorem and the corollary that H is an ordinary Frobenius algebra with Frobenius homomorphism a left integral in H^*: this is called an *FH-algebra* [31, 12]. Conversely, we have the following result, which does not seem to be explicitly noted in earlier literature.

Proposition 3.7. *If H is a Frobenius algebra and Hopf algebra, then H is an FH-algebra.*

Proof. We use the fact that the k-submodule of integrals of an augmented Frobenius algebra is free of rank 1 (cf. Lemma 4.3, [30, Theorem 3] or [12, Prop. 3.1]). Then $\int^{\ell}_{H} \cong k$. It follows from Pareigis's Theorem that the dual Hopf algebra H^* is a Frobenius algebra. Whence $\int^{\ell}_{H^*} \cong k$ and H is an FH-algebra. $\square$

Next we obtain as in [32] a left norm for the Frobenius homomorphism $\psi : H \to P$ and study its properties. Since $x \mapsto x\psi$ is an isomorphism ${}_H H \to {}_H\mathrm{Hom(H, P)}$ and

$\mathrm{Hom(H,P)} \otimes \mathrm{Q} \cong \mathrm{H}^*$ affords a canonical identification, it follows that there are elements $N_i \in H, q_i \in Q$ such that the counit of H,

$$\epsilon \stackrel{\cong}{\longmapsto} \sum_i N_i \psi \otimes q_i. \tag{18}$$

Call $N := \sum_i N_i \otimes q_i$ in $H \otimes Q$ the *left norm* of ψ, and note that $\sum_i \psi(aN_i)q_i = \epsilon(a)$ for every $a \in H$. In the natural left H-module ${}_H H \otimes Q$ we have

$$aN = \epsilon(a)N, \tag{19}$$

since both aN and $\epsilon(a)N$ map to $\epsilon(a)\epsilon$ under the composite isomorphism, $H \otimes Q \stackrel{\cong}{\to} \mathrm{Hom_k(H,P)} \otimes \mathrm{Q} \stackrel{\cong}{\to} \mathrm{H}^*$ given by $a \otimes q \mapsto a\psi q$.

For all $p \in P$, we note that

$$\sum_i N_i q_i(p) \in \int_H^\ell, \tag{20}$$

since this follows by applying Equation 19 to p.

Theorem 3.8 (Pareigis). *If H is a Hopf algebra with Frobenius homomorphism ψ given above and left norm $\sum_i N_i \otimes q_i$, then the dual bases for ψ is given by*

$$\{N_{i2}\},\ \{q_i\},\ \{\overline{S}(N_{i1})\} \tag{21}$$

Proof. We compute as in [32, Lemma 3.16], using Equation 17 at first and Equation 19 next (for every $a \in A$):

$$\begin{aligned} \sum \psi(aN_{i2})q_i\overline{S}(N_{i1}) &= \sum a_1 N_{i2}(\psi(a_2N_{i3})q_i)\overline{S}(N_{i1}) \\ &= \sum a_1\psi(a_2N_i)q_i \\ &= \sum a_1\epsilon(a_2)\psi(N_i)q_i = a\epsilon(1) = a. \end{aligned}$$

It follows from Theorem 2.4 that $\{N_{i2}\}$, $\{q_i\}$, $\{\overline{S}(N_{i1})\}$ are dual bases for ψ. □

4. Pinning down the Modular Functions

This and the remaining sections are essentially new. In this section we give a definition of modular function in Equation 25 based on [12], and find two formulas, Eqs.26 and 28 which will be used later. The rest of this section is somewhat technical and might be browsed on a first reading.

It follows from appying S to the equation in the last proof, and setting $a = 1$, that

$$\sum_i (\psi q_i) \rightharpoonup N_i = 1, \tag{22}$$

where $\psi q_i \in H^*$ is the mapping $a \mapsto \psi(a)q_i$ for each i and $a \in H$. Of course $1 \in H^{**} \cong H$ is the counit of H^*. It follows from Equations 18 and 22 that the antipode on the dual Hopf algebra H^* is given by

$$S(g) = \sum N_i(g(\psi q_i)_2)(\psi q_i)_1, \tag{23}$$

since one computes that $\sum g_1 S(g_2) = g(1)\epsilon$ for every $g \in H^*$.

Proposition 4.1. *H is a Hopf algebra and P-Frobenius algebra if and only if H^* is a Hopf algebra and P^*-Frobenius algebra.*

Proof. Let $Q = P^*$. It suffices to show the forward implication. Let $p_i \in P$ be such that $\sum_i q_i p_i = 1_k$. Then Equation 23 implies that

$$(N_i \otimes q_i, (\psi q_i)_2,\ p_i,\ \overline{S}(\psi q_i)_1) \tag{24}$$

is a Q-Frobenius system for H^*, where we identify $H \otimes Q \cong \mathrm{Hom(H^*, Q)}$ via the obvious isomorphism. □

We next define a *left modular function* for a Hopf algebra H. We continue the notation established in the previous section.

Definition 4.2. Define the *left modular function*, or *left distinguished group-like element*, $m : H \to k$ by

$$m := \epsilon \circ \nu \tag{25}$$

where ν is the Nakayama automorphism of H relative to ψ (cf. Corollary 2.5).

First note that m does not depend on the choice of Nakayama automorphism, since $\epsilon(d\nu(a)d^{-1}) = \epsilon(\nu(a))$ for every $a \in A$. Next note that m is an algebra homomorphism (an augmentation in fact), and therefore a group-like element in the dual Hopf algebra H^*. With respect to the natural right H-module $H_H \otimes_k Q$, we note that for all $a \in H$,

$$Na = Nm(a), \tag{26}$$

since Na is mapped into $\sum_i N_i a\psi \otimes q_i = \sum_i N_i \psi\nu(a) \otimes q_i$, then into $\epsilon(\nu(a))\epsilon = m(a)\epsilon$, under the canonical isomorphism $H \otimes Q \cong H^*$.

Let A be an algebra with augmentation ϵ, ${}_AM_A$ an A-bimodule and define the k-module of left integrals in M as $\int_M^\ell := \{x \in M | ax = \epsilon(a)x\}$. For a Hopf algebra and P-Frobenius algebra H we consider the natural H-bimodule ${}_HH_H \otimes Q$ in the lemma below.

Lemma 4.3. *Given Hopf algebra H and Frobenius homomorphism ψ, $\int_{H\otimes Q}^\ell$ is a sub-bimodule freely generated by the left norm $N = \sum_i N_i \otimes q_i$ and a k-direct summand of $H \otimes Q$.*

Proof. N is left integral by Equation 19. We recall the isomorphism $H \otimes Q \stackrel{\cong}{\to} \mathrm{Hom(H, P) \otimes Q} \stackrel{\cong}{\to} \mathrm{H^*}$ given by $a \otimes q \mapsto (a\psi)q$. Given $T = \sum_i T_i \otimes q_i' \in \int_{H\otimes Q}^\ell$, denote $\phi(T) := \sum_i \psi(T_i)q_i' \in k$, and note that, for all $x \in H$,

$$\sum_i \psi(xT_i)q_i' = \epsilon(x)\phi(T) = \sum_i \psi(xN_i)q_i\phi(T).$$

Whence

$$T = \phi(T)N. \tag{27}$$

Thus, N generates $\int_{H\otimes Q}^\ell$ and the mapping of $H \otimes Q \to \int_{H\otimes Q}^\ell$ given by $x \otimes q \mapsto \psi(x)qN$ is a k-linear projection.

If $\lambda \in k$ such that $\lambda N = 0$, then

$$0 = \sum_i \psi(N_i)q_i\lambda = \epsilon(1)\lambda = \lambda,$$

so N freely generates $\int_{H\otimes Q}^\ell$. □

We similarly define right integrals in a bimodule over an augmented algebra, and prove a right-handed version of the lemma. It follows from Lemma 4.3 that $T := \sum_i \overline{S}(N_i) \otimes q_i$ is a right integral that freely generates $\int^r_{H\otimes Q}$, since $\epsilon \circ \overline{S} = \epsilon$ and $\overline{S}$ is an anti-automorphism of H. By Theorem 3.8, we compute

$$T = \sum_{i,j,(N_i)} \psi(\overline{S}(N_j)N_{i2})q_i\overline{S}(N_{i1}) \otimes q_j \quad = \quad \sum \psi(\overline{S}(N_j))q_i\overline{S}(N_{i1})\epsilon(N_{i2}) \otimes q_j$$
$$= \quad T(\sum_j q_j\psi(\overline{S}(N_j))),$$

whence

$$\sum_j q_j\psi(\overline{S}(N_j))) = 1_k. \tag{28}$$

It follows that T is a right norm in the sense that $\sum_i q_i\psi\overline{S}(N_i) = \epsilon$.

Lemma 4.4. *ψ and $\psi \circ \overline{S}$ are left and right norms in the natural H-bimodule $H^* \otimes P \cong \mathrm{Hom_k(H,P)}$.*

Proof. Proposition 4.1 shows that $N \in H \otimes Q$ is a Frobenius homomorphism for the dual Hopf algebra H^*. The concepts of left and right norm relative to N make sense in the H^*-bimodule $H^* \otimes Q$. But Equation 22 implies that $\psi \in \mathrm{Hom(H,P)} \cong \mathrm{H^* \otimes P}$ is a left norm for N. Similarly, $\sum_i S(N_i) \otimes q_i$ is a Frobenius homomorphism $H^* \to Q$ by applying the anti-automorphism S as in Theorem 2.8, and $\overline{S}\psi$ is a right norm. □

One easily checks that $\sum_i S(N_i) \otimes q_i$ is a right norm in $H \otimes Q$ for $\psi \circ \overline{S}$. Since H^* is a Q-Frobenius algebra, it has a Nakayama automorphism ν^*, which we make formal use of below.

Definition 4.5. Let $b \in H$, where H is canonically identified with H^{**}, be the left modular function defined by

$$b = \eta \circ \nu^* \tag{29}$$

where η is the counit of H^* defined by $\eta(f) = f(1)$ for every $f \in H^*$.

It follows from Equation 26 and Lemma 4.4 that for every $f \in H^*$,

$$\psi f = \psi f(b), \tag{30}$$

where $\psi \in H^* \otimes P$ has the natural H^*-bimodule structure.

5. An Application to Radford's Formula

We now compute a formula for the Nakayama automorphism of $\psi : H \to P$ in terms of the square of the antipode and m. The notation $g \rightharpoonup a := \sum a_1 g(a_2)$ and $a \leftharpoonup g := \sum g(a_1)a_2$ denotes the usual left and right module actions of the convolution algebra H^* on $H \cong H^{**}$.

Theorem 5.1. *The Nakayama automorphism ν for $\psi : H \to P$ is given by*

$$\nu(a) = \overline{S}^2(m \rightharpoonup a) = m \rightharpoonup \overline{S}^2(a) \tag{31}$$

Proof. The rightmost equation follows from noting that m is a group-like element in H^*, whence $m \circ S = m^{-1}$ and $m \circ S^2 = m$: i.e., S^2 and $\overline{S}^2$ fix m.

The leftmost equation is computed below and follows [32, Satz 3.17] until 32: for every $a \in H$,

$$\begin{aligned} S^2(\nu(a)) &= S^2(\sum \psi(N_{i2}a)q_i\overline{S}(N_{i1})) \\ &= \sum S(N_{i1})\psi(N_{i2}a)q_i \\ &= \sum S(N_{i1})N_{i2}a_1\psi(N_{i3}a_2)q_i \\ &= \sum a_1\psi(N_i a_2)q_i \qquad (32) \\ &= \sum a_1 m(a_2)\psi(N_i)q_i \\ &= m \rightharpoonup a \end{aligned}$$

by Equations 6, 17, 26 and 18, respectively. □

Since H has Frobenius system $(\psi,\ N_{i2},\ q_i,\ \overline{S}(N_{i1}),\ \nu)$, it follows from Theorem 2.8 that we obtain another Frobenius system by applying the algebra (and coalgebra) anti-automorphism $\overline{S}$:

Proposition 5.2. *A Hopf algebra H with left norm N has Frobenius system*

$$(\overline{S}\psi,\ \chi N_{i1},\ q_i,\ S(N_{i2}),\ \alpha) \tag{33}$$

where $\overline{S}\psi$ satisfies a "right integral-like equation,"

$$(\overline{S}\psi)(x) \otimes 1_H = \sum (\overline{S}\psi)(x_1) \otimes x_2 \tag{34}$$

and the Nakayama automorphism,

$$\alpha(x) = S^2(x) \leftharpoonup m \tag{35}$$

for every $x \in H$.

Proof. The dual bases 33 follows directly from Theorems 2.8 and 3.8. Equation 34 follows from Equation 17 since $\overline{S}$ is a coalgebra anti-automorphism.

To compute the Nakayama automorphism we first need to find the inverse of Equation 31: for all $a \in H$,

$$\overline{\nu}(a) = S^2(m^{-1} \rightharpoonup a) = m^{-1} \rightharpoonup S^2(a). \tag{36}$$

Next we apply Equation 13 where $\overline{S}$ is the anti-automorphism:

$$\begin{aligned} \alpha(x) &= (S \circ \overline{\nu} \circ \overline{S})(x) \\ &= S(m^{-1} \rightharpoonup S(x)) \\ &= S(\sum S(x_2)m^{-1}(S(x_1))) \\ &= S^2(x) \leftharpoonup m, \end{aligned}$$

since $m \circ S = m^{-1}$ and S^2 is an algebra and coalgebra *automorphism.* □

By the comparison theorem, we know that the two Frobenius homomorphisms ψ and $\overline{S}\psi$ are related by an invertible element d called the derivative: $\overline{S}\psi = \psi d$. The next proposition shows that d is proportional to the left distinguished group-like element b of H^*.

Proposition 5.3. *If ψ is a Frobenius homomorphism for the Hopf algebra H, then*

$$\psi \circ \overline{S} = \gamma\psi b \tag{37}$$

Proof. We first show that ψb is a right integral in the H^*-bimodule $H^* \otimes P$. Recall that $H^* \otimes P$ is canonically identified with $\mathrm{Hom_k(H, P)}$ Let $f \in H^*$, then

$$(\psi b)f = [\psi(fb^{-1})]b = [\psi((fb^{-1})(b))]b = (\psi b)f(1)$$

since $\Delta(b) = b \otimes b$.

Since $\psi \circ \overline{S}$ is a right norm it follows that there is $\lambda \in k$ such that $\psi \circ \overline{S} = \lambda(\psi b)$. But comparing Equation 28 to the application below of Equation 19:

$$\sum_i q_i(\psi b)(N_i) = \chi\epsilon(b)\epsilon(1) = \chi,$$

shows that $\lambda = \gamma$ (cf. Eq. 2). □

Theorem 5.4. *If H is a Hopf algebra with left distinguished group-like elements $b \in H$ and $m \in H^*$, then for every $a \in H$,*

$$S^4(a) = b^{-1}(m \rightharpoonup a \leftharpoonup m^{-1})b. \tag{38}$$

Proof. On the one hand, the Nakayama automorphism $\alpha : H \to H$ for the Frobenius homomorphism $\overline{S}\psi$ is by Proposition 5.2 given by

$$\alpha(a) = S^2(a) \leftharpoonup m = S^2(a \leftharpoonup m)$$

for every $a \in H$. On the other hand, the Nakayama automorphism ν of H for the Frobenius homomorphism $\psi \in H^*$ is by Theorem 5.1

$$\nu(a) = \overline{S}^2(m \rightharpoonup a) = m \rightharpoonup \overline{S}^2(a),$$

for every $a \in H$. By Proposition 5.3, $\psi \circ \overline{S} = \gamma\psi b$, so by the comparison theorem

$$\alpha(a) = b^{-1}\nu(a)b$$

for every $a \in H$.

Substituting the first two equations in the third yields,

$$S^2(a) = b^{-1}\overline{S}^2(m \rightharpoonup a)b \leftharpoonup m^{-1}$$

which is equivalent to Equation 38 since S^2 fixes b and m, and for every group-like $a \in H$, we have $m \rightharpoonup (axa^{-1}) = a(m \rightharpoonup x)a^{-1}$. □

In [12] it was proven that a group-like element g in a finite projective Hopf algebra has finite order dividing the least common multiple N of the local ranks of H. Since m and b are group-like elements, it follows from the general Radford formula and Equation 31 that the antipode S and the Nakayama automorphism $\nu : H \to H$ have finite order dividing $4N$ and $2N$ respectively.

Corollary 5.5. *Let H be a finite projective Hopf algebra over a ring k. Then* $S^{4N} = \nu^{2N} = \mathrm{Id_H}$.

Waterhouse sketches a different method of how to extend the Radford formula to a finite projective Hopf algebra and show that S has finite order [39]. As noted before, Schneider has established Radford's formula by different Frobenius methods for k = field [37]. Radford's formula is generalized to double Frobenius algebras over fields by Koppinen [17].

6. When Hopf algebras are separable

In this section we give a criterion in terms of the left norm N for when a finite projective Hopf algebra H is separable. We first need a proposition closely related to some results on when Frobenius algebras/extensions/bimodules are separable [10, 7, 11]. Let k be a commutative ground ring.

Proposition 6.1. *Suppose A is a P-Frobenius algebra with system (ψ, x_i, q_i, y_i). Then A is k-separable if and only if there is $d \in P \otimes A$ such that*

$$\sum_i x_i q_i d y_i = 1_A.$$

Proof. The forward implication is proven by first letting $\sum_j a_j \otimes b_j$ be the separability element for A. Next set $d := \sum_j \psi(a_j) \otimes b_j \in P \otimes H$. Then

$$\sum_i x_i q_i d y_i = \sum_{i,j} x_i q_i \psi(a_j) b_j y_i = \sum_j \sum_i x_i q_i \psi(y_i a_j) b_j = \sum_j a_j b_j = 1_A.$$

The reverse implication is proven by noting that $e := \sum_i x_i \otimes q_i d y_i$ is a separability element for A. By hypothesis, $\mu(e) = 1$ where $\mu : A \otimes A \to A$ is the multiplication mapping. e is in the center $(A \otimes A)^A$ of the natural A-bimodule $A \otimes A$ as a consequence of Proposition 2.6. □

Next, let P be an invertible k-module with inverse Q. We shall say that $q \in Q$ is *Morita-invertible* if there is $p \in P := Q^*$ such that $qp = 1_k$. Note that a left inverse in this sense may differ from a right inverse by a unit χ in k, since $qp = \chi pq$. More generally, we say that $\sum_i q_i \otimes a_i \in Q \otimes A$ is Morita-invertible where A is a k-algebra if there is $\sum_j p_j \otimes b_j \in P \otimes A$ such that $\sum_{i,j} q_i p_j a_i b_j = 1_A$. The next theorem generalizes results in [28, 2].

Theorem 6.2. *Suppose H is a finite projective Hopf algebra with P-Frobenius homomorphism ψ satisfying Equation 17 and left norm $N = \sum_i N_i \otimes q_i$. Then H is k-separable if and only if $\sum_i \epsilon(N_i) q_i$ is Morita-invertible.*

Proof. We make use of the dual bases $\{N_{i2}\}, \{q_i\}, \{\overline{S}(N_{i1})\}$ given by Theorem 3.8. If H is k-separable, then by the proposition above there is $d := \sum_j p_j \otimes a_j \in P \otimes H$ such that

$$\sum_{i,(N_i)} N_{i2} q_i d \overline{S}(N_{i1}) = 1_H.$$

Applying ϵ we obtain

$$\sum \epsilon(\epsilon(N_{i1}) N_{i2}) q_i p_j \epsilon(a_j) = \sum_i \epsilon(N_i) q_i \sum_j p_j \epsilon(a_j) = 1_k,$$

whence $\sum_i \epsilon(N_i) q_i$ is Morita-invertible.

Conversely, if $q := \sum_i \epsilon(N_i) q_i$ is Morita-invertible with inverse $p \in P$ such that $qp = 1_k$, then we let $d := p \otimes 1_H$. Note that

$$\sum N_{i2} q_i d \overline{S}(N_{i1}) = \sum_i \epsilon(N_i) q_i p 1_H = 1_H,$$

whence H is k-separable by Proposition 6.1. □

Next we study when separable Hopf algebras are strongly separable. Recall that an algebra A is *strongly separable* [14, 9, Kanzaki, Hattori] if there is $e := \sum_j z_j \otimes w_j \in A \otimes A$ such that $\mu(e) = \sum_j z_j w_j = 1_A$ and for every $a \in A$, we have $\sum_j z_j a \otimes w_j = \sum_j z_j \otimes a w_j$. We will call such an $e \in A \otimes A$ a *Kanzaki separability element*: one may prove that its transpose $\sum_i w_j \otimes z_j$ is an ordinary separability idempotent [9, 13]. For example, if k is an algebraically closed field of characteristic p, then A is strongly separable if it is semisimple and none of its simple modules have dimension over k divisible by p. We first need a proposition which generalizes part of [13, Prop. 4.1].

Proposition 6.3. *Suppose A is a P-Frobenius algebra with system (ψ, x_i, q_i, y_i) such that*

$$u := \sum_i q_i \otimes y_i x_i \tag{39}$$

is Morita-invertible. Then A is strongly separable.

Proof. Suppose $\sum_j p_j \otimes a_j \in P \otimes A$ satisfies $\sum_{i,j} q_i p_j y_i x_i a_j = 1_A$. From this and Proposition 2.6, we easily see that $e := \sum_i y_i \otimes x_i q_i p_j a_j$ is a Kanzaki separability element. □

Setting $u^{-1} := \sum_j p_j \otimes a_j$, we can apply Proposition 2.6 to obtain a formula for the Nakayama automorphism:

$$\nu(a) = uau^{-1}, \tag{40}$$

where we make use of the usual Morita mapping $Q \otimes P \to k$.

Recall that a Hopf algebra H is *involutive* if $S^2 = \mathrm{Id_H}$. The next theorem contains a result of Larson [19] as a special case.

Theorem 6.4. *Suppose H is a finite projective, separable, involutive Hopf algebra. Then H is strongly separable.*

Proof. If $(\psi, N_{i2}, q_i, \overline{S}(N_{i1}))$ is the P-Frobenius system for H given by Theorem 3.8, we note here that $\overline{S} = S$, so that the u-element of Proposition 6.3,

$$u := \sum_i q_i \otimes \sum_{(N_i)} S(N_{i1}) N_{i2} = \sum_i q_i \epsilon(N_i) \otimes 1_H$$

is Morita-invertible by Theorem 6.2. □

It follows from a result of Etingof and Gelaki [6] that a separable and coseparable Hopf algebra is automatically involutive, so long as $2 \in k$ is not a zero-divisor [13].

7. Hopf Subalgebras

Throughout this section, k is a commutative ring and we consider a finite projective Hopf algebra H with Hopf subalgebra K which is also finite projective as a k-module. We will show that the functors of induction and co-induction from the category $\mathcal{M}_K$ of K-modules to $\mathcal{M}_H$ are naturally isomorphic up to a Morita auto-equivalence of $\mathcal{M}_K$ determined by a relative Nakayama automorphism and a relative Picard group element. This section generalizes results in [29, 28, 36, 12].

Let R be an arbitrary ring, $\beta : R \to R$ a ring automorphism, and M_R a module over R. The β-twisted module M_β is defined by $m \cdot r := m\beta(r)$, clearly another R-module. If β is an inner automorphism, is easy to check that $M_R \cong M_\beta$. We also note that $M \otimes_R R_\beta \cong M_\beta$. It is not hard to see from this that the bimodule ${}_R R_\beta$ induces a Morita auto-equivalence of $\mathcal{M}_R$ via tensoring.

Lemma 7.1. *If A is a P-Frobenius k-algebra with Frobenius homomorphism ϕ and corresponding Nakayama automorphism ν, then we have the following bimodule isomorphisms:*

$$ {}_A A_A \cong {}_A \mathrm{Hom}(A, P)_\nu \cong {}_{\nu^{-1}} \mathrm{Hom}(A, P) \tag{41} $$

Proof. Since $a\phi = \phi\nu(a)$ in A^* for every $a \in A$, it follows that the Frobenius isomorphisms $a \mapsto a\phi$ and $a \mapsto \phi a$ induce the first and second isomorphisms above (between A and $\mathrm{Hom}(A, P)$). □

As a straightforward extension of Definition 2.1, we define P-Frobenius extension A/S, where P is an invertible S-bimodule (and $- \otimes_S P$ defines a Morita auto-equivalence of $\mathcal{M}_S$ [1]).

Definition 7.2. Suppose S is a subring of ring A and P is an invertible S-bimodule. We say A is a *P-Frobenius extension of S, or A/S is a Frobenius extension of the third kind,* if

1. A_S is a finite projective module;
2. ${}_S A_A \cong {}_S \mathrm{Hom}_S(A_S, P_S)_A$

A P-Frobenius extension has a symmetric definition, a Frobenius system like in Section 2, a Nakayama automorphism defined on the centralizer subalgebra $C_S(A)$ of A [32], and a comparison theorem, which we will not need here. As a straightforward consequence of a theorem by Morita [23, 24], we state without proof (cf. [8]):

Theorem 7.3. *A is P-Frobenius extension of S if and only if there is a natural isomorphism of right A-modules,*

$$ M \otimes_S A \cong \mathrm{Hom}_S(A_S, M \otimes_S P_S) \tag{42} $$

for every module $M \in \mathcal{M}_S$.

This equivalent condition for a P-Frobenius extension states in other words that the functors of induction and co-induction from $\mathcal{M}_S$ into $\mathcal{M}_A$ form a commutative triangle with the Morita auto-equivalence of $\mathcal{M}_S$ induced by $- \otimes_S P$.

Suppose a Frobenius algebra pair forms a projective ring extension such that the Nakayama automorphism of the overalgebra preserves the subalgebra. We now obtain a theorem that states that such a pair forms a certain P-Frobenius extension.

Theorem 7.4. *Suppose A is a P-Frobenius algebra, B is a P'-Frobenius algebra, and B is subalgebra of A such that A_B is a finite projective module, and a Nakayama automorphism ν_A of A sends B into B: $\nu_A(B) = B$. Let ν_B denote a Nakayama automorphism of B. Then A is a W-Frobenius extension of B, where*

$$ W = {}_\beta B \otimes Q' \otimes P, \tag{43} $$

$Q' = {P'}^*$ and β is the relative Nakayama automorphism given by

$$ \beta = \nu_B \circ \nu_A^{-1}. \tag{44} $$

Proof. Since A_B is assumed finite projective, we need only show that ${}_BA_A \cong {}_B\mathrm{Hom_B(A_B, W_B)}$. We compute using the hom-tensor adjointness relation and two applications of Lemma 7.1:

$$\begin{aligned} {}_BA_A &\cong {}_{\nu_A^{-1}}\mathrm{Hom(A, P)_A} \\ &\cong \mathrm{Hom_k(A \otimes_B B_{\nu_A^{-1}}, k)_A \otimes P} \\ &\cong {}_{\nu_A^{-1}}\mathrm{Hom_B(A_B, B_B^*)_A \otimes P} \\ &\cong {}_{\nu_A^{-1}}\mathrm{Hom_B(A_B, {}_{\nu_B}B_B \otimes Q')_A \otimes P} \\ &\cong {}_B\mathrm{Hom_B(A_B, {}_{\nu_B \circ \nu_A^{-1}}B_B \otimes Q' \otimes P)_A} \quad \square \end{aligned}$$

Let $K \subseteq H$ be a pair of finite projective Hopf k-algebras where K is a Hopf subalgebra of H (i.e., $\Delta(K) \subseteq K \otimes K$ and $S(K) = K$) in the next corollary. Let $P(K)^*$, $P(H)^*$ be the k-module of integrals $\int_K^\ell$, $\int_H^\ell$, respectively, ν_H, ν_K be the respective Nakayama automorphisms and m_H, m_K be the respective left modular functions.

Corollary 7.5. *If $K \subseteq H$ is a finite projective Hopf subalgebra pair, then H/K is a P-Frobenius extension where*

$$P = {}_\beta K \otimes P(K)^* \otimes P(H) \tag{45}$$

and

$$\beta = \nu_K \circ \nu_H^{-1}. \tag{46}$$

Proof. The natural module H_K is finite projective as a corollary of the Nichols-Zoeller Freeness theorem [12, Prop. 5.3] (or adapt [36, Cor. 2.5(2)]). Furthermore, the Nakayama automorphism $\nu_H^{\pm 1}(a) = m_H^{\pm 1} \rightharpoonup S^{\mp 2}(a)$ for every $a \in H$ by Equation 31, whence $\nu_H(K) = K$. Thus the hypotheses of Theorem 7.4 are satisfied. $\square$

It follows from the formulas for ν_H and ν_K in Equation 31 that for every $x \in K$,

$$\begin{aligned} \beta(x) &= m_K \rightharpoonup \overline{S}^2(m_H^{-1} \rightharpoonup S^2(x)) \\ &= (m_K * m_H^{-1}) \rightharpoonup x \end{aligned} \tag{47}$$

(cf. [8]).

Kasch makes a study in [15] of the relative homological algebra of Frobenius extensions. One can extend the results of Kasch to a Frobenius extension A/S of the third kind by taking into account some Morita theory. For example, one may show by these means that under the (rather common) additional assumption that S is S-bimodule isomorphic to a direct summand in A, the flat dimension of any S-module is equal to both the flat dimension of its induced A-module and of its co-induced A-module. In extension of [15], Pareigis [31] studies a cohomology theory for FH-algebras, showing that these have a complete cohomology with cup product, a generalized Tate duality under a certain cocommutativity condition, and a generalized Hochschild-Serre spectral sequence.

8. Embedding H into an FH-algebra

In this section we show that a finite projective Hopf algebra H is a Hopf subalgebra of an FH-algebra in two ways. We first show that H is a Hopf subalgebra of $D(H)$. We let k be a commutative ring. The *quantum double* $D(H)$ of a finite dimensional Hopf algebra, due to Drinfel'd [5], is readily extended to a finite projective Hopf algebra H over k: at the level of coalgebras it is given by

$$D(H) := H^{*\,\mathrm{cop}} \otimes_k H,$$

where $H^{*\,\mathrm{cop}}$ is the co-opposite of H^*, the coproduct being Δ^{op}. The multiplication on $D(H)$ is described in two equivalent ways as follows [22, Lemma 10.3.11]. In terms of the notation gx replacing $g \otimes x$ for every $g \in H^*, x \in H$, both H and $H^{*\,\mathrm{cop}}$ are subalgebras of $D(H)$, and for each $g \in H^*$ and $x \in H$,

$$xg := \sum (x_1 g S^{-1} x_3) x_2 = \sum g_2 (S^{-1} g_1 \rightharpoonup x \leftharpoonup g_3). \tag{48}$$

The algebra $D(H)$ is a Hopf algebra with antipode $S'(gx) := SxS^{-1}g$, the proof of this proceeding as in [16].

Theorem 8.1. *If H is a finite projective Hopf algebra, then $D(H)$ is an FH-algebra.*

Proof. It is enough to show that $\int^{\ell}_{D(H)^*} \cong k$. As an algebra, $D(H)^* \cong H^{\mathrm{op}} \otimes H^*$, the tensor product algebra of H^* and the opposite algebra of H. Now H is P-Frobenius algebra if and only if H^{op} is, since they have the same Frobenius system with a change of order in the dual base. By Proposition 4.1, H^* is a P^*-Frobenius algebra. It follows from Lemma 2.9 that $D(H)^*$ is a Frobenius algebra, since $P \otimes P^* \cong k$. Now the k-space of integrals of an augmented Frobenius algebra is free of rank one, which proves our theorem. □

Next we show that H has a ring extension to an FH-algebra $H \otimes_k K$. This will follow right away from the construction of a ring extension $k \subset K$ where K has trivial Picard group. We continue with k as a commutative ring, and let M be the set of all maximal ideals in k. Choose finite subsets $M_\alpha \subset M$, $\alpha \in I$ such that $\cup_{\alpha \in I} M_\alpha = M$ and the subsets M_α are linearly ordered with respect to inclusion: in other words, for any two indices $\alpha, \beta \in I$ either $M_\alpha \subset M_\beta$ or $M_\beta \subset M_\alpha$.

Let $m_{\alpha_1}, \ldots, m_{\alpha_n}$ be all the elements of M_α, i.e. maximal ideals in k. Then the set

$$K_\alpha = k_{m_{\alpha_1}} \oplus \cdots \oplus k_{m_{\alpha_n}}$$

is a semilocal ring and has trivial Picard group: $Pic(K_\alpha) = 0$. For any pair $M_\alpha \subset M_\beta$, we have the canonical projection $\pi_{\alpha\beta} : K_\beta \to K_\alpha$ and we may consider the inverse limit ring

$$K := \lim_{\leftarrow} (K_\alpha, \pi_{\alpha\beta}). \tag{49}$$

Furthermore, for any $\alpha \in I$ we have the canonical homomorphism $f_\alpha : k \to K_\alpha$, which is the direct sum of the corresponding localization homomorphisms. The following diagram is clearly commutative:

$$\begin{array}{ccc} & k & \\ {}^{f_\beta}\swarrow & & \searrow^{f_\alpha} \\ K_\beta & \xrightarrow{\pi_{\alpha\beta}} & K_\alpha \end{array}$$

¿From universality we obtain a homomorphism $f : k \to K$.

Lemma 8.2. *f is a monomorphism.*

Proof. Let f_m be the localization homomorphism $f_m : k \to k_m$. Then it follows easily that $\ker f = \cap_{m \in M} \ker f_m = 0$. □

Now let $\pi_\alpha : K \to K_\alpha$ be the canonical epi. Since the diagram

$$\begin{array}{ccc} & K & \\ \overset{\pi_\beta}{\swarrow} & & \overset{\pi_\alpha}{\searrow} \\ K_\beta & \overset{\pi_{\alpha\beta}}{\longrightarrow} & K_\alpha \end{array}$$

is commutative, the following diagram is commutative as well:

$$\begin{array}{ccc} & Pic(K) & \\ \overset{Pic(\pi_\beta)}{\swarrow} & & \overset{Pic(\pi_\alpha)}{\searrow} \\ Pic(K_\beta) & \overset{Pic(\pi_{\alpha\beta})}{\longrightarrow} & Pic(K_\alpha) \end{array}$$

Again from universality we obtain a homomorphism

$$\Phi : \; Pic(K) \longrightarrow \lim_{\leftarrow}(Pic(K_\alpha), Pic(\pi_{\alpha\beta}))$$

Theorem 8.3. Φ *is injective.*

Proof. We need the following result proved in [3]:

Theorem 8.4. *Suppose I is some linearly ordered set and for each ordered $\alpha, \beta \in I$, A_α is a commutative ring and there is an epimorphism $\psi_{\alpha\beta}$ such that the restriction to the group of units $\psi_{\alpha\beta} : U(A_\beta) \to U(A_\alpha)$ is a surjection. If*

$$A = \lim_{\leftarrow}(A_\alpha, \psi_{\alpha\beta}),$$

then the induced map

$$Pic(A) \to \lim_{\leftarrow}(Pic(A_\alpha), Pic(\psi_{\alpha\beta}))$$

is injective.

The hypotheses of this proposition are fulfilled by the mappings $\pi_{\alpha\beta} : K_\beta \to K_\alpha$, whence Φ is injective. □

The next corollary follows from recalling that $Pic(K_\alpha) = 0$.

Corollary 8.5. *Given a commutative ring k and K defined in Equation 49, $k \subset K$ is a ring extension with $Pic(K) = 0$.*

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