On the Milne problem and the hydrodynamic limit for a steady Boltzmann equation model.

L. Arkeryd * & A. Nouri †. ‡

Abstract. For a stationary nonlinear Boltzmann equation in a slab with a particular truncation in the collision operator, the Milne problem for the boundary layer together with a weak type of hydrodynamic behaviour in the interior of the slab, are studied by non-perturbative methods in the small mean free path limit.

1 Introduction

Solutions to half-space problems for the Boltzmann equation play an important role as boundary layers in the study of hydrodynamic limits for solutions to the Boltzmann equation when the mean free path tends to zero. Such problems have been extensively studied in the linear context, using functional analytic and energy methods ([BCN1], [C1], [C2], [G], [GP], [Gu] and others). In the discrete velocity case for the Boltzmann equation a number of problems have been investigated, among them half-space problems for the Broadwell model in [BT] and weak shock wave solutions in [BIU]. The existence of solutions to the half-space problem with given indata at one end was proven in [CIPS], as well as their convergence to a set of Maxwellians at infinity. The question whether the limit Maxwellian can be fixed a priori was answered positively in [U] for a fixed Maxwellian at infinity which is close to the given indata.
For the BGK and Boltzmann equations, a wide range of similar questions have been addressed by the Kyoto group around Y. Sone and K. Aoki in a perspective of asymptotic analysis and numerical studies. Among their papers in this area we mention [S1], [S2], [SOA1], [SOA2], where extensive

*Department of Mathematics, Chalmers Institute of Technology, Gothenburg, Sweden.
†UMR 5585, INSA, 20 av. A. Einstein, 69621 Villeurbanne Cedex, France.
‡EU-support under contract ERB FMRX CT97 0157 is acknowledged

references can also be found.
So far there are few purely theoretical results on the half-space problem and the hydrodynamic limit for the fully nonlinear Boltzmann equation with continuous velocities. An existence theorem for the half-space problem was established [GPS] for small data and specular reflexion boundary conditions. The hydrodynamic limit of solutions of the (evolutionary) Boltzmann equation [DL] towards solutions of the incompressible Navier-Stokes equations was performed for smooth solutions in [DEL] and for weak solutions in [BGL1], [BGL2], complemented in [BU]. In [BCN2], a kinetic description of a gas between two plates at different temperatures and no mass flux was given in the case of a small mean free path for the nonlinear stationary Boltzmann equation under diffuse reflection boundary conditions.
In this paper, we address the half-space problem for the stationary nonlinear Boltzmann equation in the slab with given indata, for a collision operator truncated for large velocities and for small values of the velocity component in the slab direction. Instead of considering the half-space problem in isolation, it is here studied within a frame of hydrodynamic limits for solutions to the nonlinear stationary Boltzmann equation in the slab. This avoids explicitly dealing with what type of Maxwellians that are permitted at infinity in the half space problem (cf. e.g. [AC], [CGS]). An earlier paper [AN1] considered in the same spirit a fluid approximation inside a bounded domain together with initial and boundary layers for an evolutionary linear Boltzmann model of condensation and evaporation.
Existence of solutions to the nonlinear Boltzmann equation in a bounded slab is proved in [AN2], [AN3] (see also [AN4] and [P] for the related stationary Povzner equation). By the conservation properties of the Boltzmann collision operator, there are in general at most two Maxwellians with the same fluxes as the limit of such solutions, when the mean free path tends to zero. In that limit the existence is proven of solutions to the Milne problem with given indata at the boundary point, and either convergence to one of those two Maxwellians, or collapse at small velocities at spatial infinity. One main ingredient in the techniques of the paper is the use of a kinetic inequality, deduced from the smallness of the entropy production term, for measuring the distance to the set of Maxellians, see [A], [N].
The plan of the paper is as follows. Section 2 is devoted to preliminaries and a statement of the main results. In Section 3 the existence of solutions to the half-space problem is proven. Section 4 describes the asymptotic behaviour of such half-space solutions, in particular a possible convergence to one of the at most two Maxwellians having the same fluxes as the solution.

Finally Section 5 studies a limiting behaviour with hydrodynamic aspects in the interior of the slab, when the mean free path tends to zero.

2 Preliminaries and statement of results.

An integrable cylindrically symmetric Maxwellian

$$M(v) := \frac{\rho}{(2\pi T)^{\frac{3}{2}}} e^{-\frac{(\xi-u)^2+\eta^2+\zeta^2}{2T}}, \quad v = (\xi, \eta, \zeta) \in I\!R^3,$$

(with $\rho \geq 0$ and $T > 0$), is uniquely determined by its three moments

$$\rho = \int M(v)dv, \quad \rho u = \int \xi M(v)dv, \quad \rho(u^2+T) = \int v^2 M(v)dv.$$

However, it is well known that for nonzero bulk velocity, there can be zero, one or two Maxwellians with given fluxes

$$\int \xi M(v)dv, \quad \int \xi^2 M(v)dv, \quad \int \xi v^2 M(v)dv,$$

as stated in the following lemma.

Lemma 2.1 *Let $(c_i)_{1\leq i\leq 3}$, with $c_1 \neq 0$, be given.*
(i) If $c_2 \leq 0$ or $c_1c_3 \leq 0$ or $c_1c_3 > \frac{25}{16}c_2^2$, there is no Maxwellian with fluxes $(c_i)_{1\leq i\leq 3}$.
(ii) If $c_1c_3 = \frac{25}{16}c_2^2$, there is a unique Maxwellian with fluxes $(c_i)_{1\leq i\leq 3}$.
(iii) If $0 < c_2^2 < c_1c_3 < \frac{25}{16}c_2^2$, there are two Maxwellians with fluxes $(c_i)_{1\leq i\leq 3}$.
(iv) If $c_2^2 \geq c_1c_3 > 0$, there is a unique Maxwellian with fluxes $(c_i)_{1\leq i\leq 3}$.

For the convenience of the reader, we recall a short proof.
<u>Proof of Lemma 2.1</u> The unknown ρ, u, T defining an integrable Maxwellian M, are solutions to the system

$$\rho \geq 0,\ T > 0,\ \rho u = c_1,\ \rho(u^2+T) = c_2,\ \rho u(u^2+5T) = c_3. \tag{2.1}$$

Since $c_1 \neq 0$, there are no positive solutions ρ and T when $c_2 \leq 0$ or $c_1c_3 \leq 0$. Since $c_1 \neq 0$,

$$\rho = \frac{c_1}{u}, \quad T = \frac{c_2}{c_1}u - u^2,$$

where u is a solution to

$$4c_1u^2 - 5c_2u + c_3 = 0, \tag{2.2}$$

$$c_1u > 0, \quad u \in]0, \frac{c_2}{c_1}[, \tag{2.3}$$

and $]0, \frac{c_2}{c_1}[$ denotes the open interval with end points 0 and $\frac{c_2}{c_1}$. For $c_1c_3 > \frac{25}{16}c_2^2$, there is no real solution u to equation (2.2). For $c_1c_3 = \frac{25}{16}c_2^2$, the solution $\frac{5c_2}{8c_1}$ to equation (2.2) satisfies (2.3). For $0 < c_2^2 < c_1c_3 < \frac{25}{16}c_2^2$, both solutions to equation (2.2),

$$u_\epsilon = \frac{5c_2 + \epsilon\sqrt{25c_2^2 - 16c_1c_3}}{8c_1}, \quad \epsilon \in \{-,+\}, \tag{2.4}$$

satisfy (2.3). For $c_2^2 \geq c_1c_3 > 0$, only $u = \frac{5c_2 - \sqrt{25c_2^2 - 16c_1c_3}}{8c_1}$ satisfies (2.3).$\square$

Remark. We note for $c_1 > 0$ that $0 \leq u_- \leq u_+$, $T_+ \leq \frac{3\rho^2}{5} \leq T_-$. The Mach number is defined by $M_\epsilon^2 = \frac{3u_\epsilon^2}{5T_\epsilon}$. Then

$$u_\epsilon^2 = \frac{c_3 M_\epsilon^2}{c_1(3 + M_\epsilon^2)}, \ T_\epsilon = \frac{3c_3}{5c_1(3 + M_\epsilon^2)}.$$

With

$$\sin^2\theta = \frac{16c_1c_3}{25c_2^2}, \ 0 \leq \theta \leq \frac{\pi}{2},$$

we get

$$M_-^2(\theta) = \frac{3}{4\mathrm{ctg}^2\frac{\theta}{2} - 1}, \ M_+^2(\theta) = \frac{3}{4\mathrm{tg}^2\frac{\theta}{2} - 1},$$

where $M_-(\theta)$ is subsonic and $M_+(\theta)$ is supersonic.

Define for $0 < \mu < \lambda$

$$V_\lambda := \{v \in I\!R^3; \mid v \mid \leq \lambda\}, \quad V'_\lambda = \{v \in V_\lambda; \mu \leq \mid \xi \mid\}.$$

By a perturbative argument there are $\lambda_0 < \infty$ and $0 < \mu_0$, so that in the sense of the following lemma, for $\lambda \geq \lambda_0, 0 < \mu < \mu_0$, (iii-iv) of Lemma 2.1 hold for the Maxwellian fluxes, also when the integrals are truncated with respect to V'_λ.

Lemma 2.2 *Let $(c_i)_{1 \leq i \leq 3}$, with $0 < c_1c_3 < \frac{25}{16}c_2^2$ and $c_1c_3 \neq c_2^2$ be given. There are $\lambda_0 < \infty$ and $\mu_0 > 0$, such that for $\lambda \geq \lambda_0$, $\mu \in]0, \mu_0[$, (iii-iv) of Lemma 2.1 hold for the truncated Maxwellian fluxes*

$$(c_1, c_2, c_3) = (\int_{V'_\lambda} \xi M(v)dv, \int_{V'_\lambda} \xi^2 M(v)dv, \int_{V'_\lambda} \xi v^2 M(v)dv). \tag{2.5}$$

In the case $c_1c_3 = c_2^2$, *let* (ρ_-, u_-, T_-) *be the values of* (ρ, u, T) *for* $\lambda = \infty, \mu = 0$ *when* $\epsilon = -$ *in* (2.4), *and correspondingly* (ρ_+, u_+, T_+) *with* $T_+ = 0$ *for* $\epsilon = +$. *Given any neighbourhoods* $\mathbf{O}_-$ *and* $\mathbf{O}_+$ *of* (ρ_-, u_-, T_-) *and* (ρ_+, u_+, T_+) *respectively, then* $(\rho(\lambda,\mu), u(\lambda,\mu), T(\lambda,\mu))$ *is either in* $\mathbf{O}_-$ *or in* $\mathbf{O}_+$ *for* λ, μ^{-1} *large enough. Moreover,* $(\rho(\lambda,\mu), u(\lambda,\mu), T(\lambda,\mu))$ *is uniquely determined in the* $\mathbf{O}_-$*-case.*

Proof of Lemma 2.2. We discuss the case $c_1 > 0$. The case $c_1 < 0$ is analogous.
For $(c_i)_{1\leq i\leq 3}$ with $0 < c_1c_3 < \frac{25}{16}c_2^2$, consider

$$F(\lambda, \mu, \rho, u, T) = (F_1, F_2, F_3)(\lambda, \mu, \rho, u, T),$$

where

$$F_1(\lambda, \mu, \rho, u, T) := \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \int_{V'_\lambda} \xi e^{-\frac{|v-u|^2}{2T}},$$

$$F_2(\lambda, \mu, \rho, u, T) := \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \int_{V'_\lambda} \xi^2 e^{-\frac{|v-u|^2}{2T}},$$

$$F_3(\lambda, \mu, \rho, u, T) := \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \int_{V'_\lambda} \xi v^2 e^{-\frac{|v-u|^2}{2T}}.$$

At $(\lambda, \mu, \rho, u, T)$ with $\rho \geq 0, T > 0, \lambda^{-1} = \mu = 0$, it holds that

$$\frac{\partial F_1}{\partial \rho} = u, \quad \frac{\partial F_1}{\partial u} = \rho, \quad \frac{\partial F_1}{\partial T} = 0,$$
$$\frac{\partial F_2}{\partial \rho} = u^2 + T, \quad \frac{\partial F_2}{\partial u} = 2\rho u, \quad \frac{\partial F_2}{\partial T} = \rho,$$
$$\frac{\partial F_3}{\partial \rho} = u(u^2 + 5T), \quad \frac{\partial F_3}{\partial u} = \rho(3u^2 + 5T), \quad \frac{\partial F_3}{\partial T} = 5\rho u,$$

and so the Jacobian J with respect to (ρ, u, T) of F at $(\infty, 0, \rho, u, T)$ is equal to

$$J = \rho^2 u(3u^2 - 5T).$$

At any (ρ_0, u_0, T_0) of Lemma 2.1 such that $T_0 \neq 0$, and such that

$$F(\infty, 0, \rho_0, u_0, T_0) = (c_1, c_2, c_3)$$

with $0 < c_1c_3 < \frac{25}{16}c_2^2$, it holds that

$$J = 8c_1^2(u_0 - \frac{5c_2}{8c_1}) \neq 0.$$

Consequently, by the implicit function theorem, there are neighborhoods V_1 of $(\infty, 0)$ with $V_1 = \{\lambda > \lambda_0, 0 \leq \mu < \mu_0\}$, and V_2 of $(\infty, 0, \rho_0, u_0, T_0)$ respectively, and a C^1 function G from V_1 to R^3, so that for $(\lambda, \mu) \in V_1$, $T_0 > 0$, it holds that

$$(c_1, c_2, c_3) = F(\lambda, \mu, \rho, u, T), \quad \text{and} \qquad (\lambda, \mu, \rho, u, T) \in V_2$$
$$iif \ (\rho, u, T) = G(\lambda, \mu).$$

The neighbourhoods can be taken locally constant with respect to c. Assume that there are other solutions than the above local perturbations for arbitrarily large λ and for arbitrarily small μ. Then there is a sequence $(\lambda_n, \mu_n)_{n\in IN}$ tending to $(\infty, 0)$, when $n \to +\infty$ and a sequence $(\rho_n, u_n, T_n)_{n\in IN}$, satisfying

$$c = (c_1, c_2, c_3) = F(\lambda_n, \mu_n, \rho_n, u_n, T_n), \quad \rho_n > 0, \quad T_n > 0. \tag{2.6}$$

By the positivity of c_1 it follows that $u_n > 0$. Writing V'_{λ_n} for V'_λ with $\lambda = \lambda_n, \mu = \mu_n$, we discuss separately the cases, when (ρ_n, u_n, T_n) is bounded and unbounded.

Case 1. (ρ_n, u_n, T_n) is bounded, hence converges (up to a subsequence) to some (ρ_*, u_*, T_*).

Case 1(i). $T_* > 0$. Passing to the limit when n tends to $+\infty$ in (2.6) implies that $F(\infty, 0, \rho_*, u_*, T_*) = c$. Hence, by Lemma 2.1, (ρ_*, u_*, T_*) is one of the (at most) two solutions of the case $\lambda = \infty, \mu = 0$. Then, for n large enough, (ρ_n, u_n, T_n) is in a given neighbourhood of (ρ_*, u_*, T_*). Hence in this case only the previous local perturbative Maxwellians exist.

Case 1(ii). $T_* = 0, u_* \neq 0$. Then $\frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}}e^{-\frac{|v-u_n|^2}{2T_n}}$ converges for the weak* topology of bounded measures to $\rho_*\delta_{v=u_*}$. Hence,

$$c_2^2 = (\lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V'_{\lambda_n}} \xi^2 e^{-\frac{|v-u_n|^2}{2T_n}} dv)^2 = (\rho_* u_*^2)^2$$

$$= (\rho_* u_*)(\rho_* u_*^3) = (\lim_{n\to\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V'_{\lambda_n}} \xi e^{-\frac{|v-u_n|^2}{2T_n}} dv)$$

$$(\lim_{n\to\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V_{\lambda_n}} \xi v^2 e^{-\frac{|v-u_n|^2}{2T_n}} dv) = c_1 c_3,$$

and $(\rho_*, u_*, T_*) = (\rho_+, u_+, T_+)$ in the notation of (2.4).
Case 1(iii). $T_* = 0, u_* = 0$. Then

$$\begin{aligned} c_1 &= \lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V'_{\lambda_n}} \xi e^{-\frac{|v-u_n|^2}{2T_n}} dv \\ &\leq \lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V'_{\lambda_n}} \mid \xi \mid e^{-\frac{|v-u_n|^2}{2T_n}} dv \\ &\leq \lim_{n\to+\infty} \frac{\rho_n}{\sqrt{\pi}} \int \mid u_n + x\sqrt{2T_n} \mid e^{-x^2} dx = 0, \end{aligned}$$

which contradicts the assumption $c_1 > 0$.

Case 2(i). The sequence (ρ_n, u_n, T_n) is unbounded with - for a subsequence - $\lim_{n\to+\infty} u_n = +\infty$. Then, for any $A > 0$ there is $n_A \in I\!N$, such that for $n \geq n_A$,

$$\frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{c_1\xi > 2A, |v| < \lambda_n} \xi e^{-\frac{|v-u_n|^2}{2T_n}} dv > \frac{c_1}{2}.$$

Then,

$$\begin{aligned} c_2 &= \lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{|v|<\lambda_n, |\xi| > \mu_n} \xi^2 e^{-\frac{|v-u_n|^2}{2T_n}} dv \\ &> \lim_{n\to+\infty} \frac{2A\rho_n}{c_1 (2\pi T_n)^{\frac{3}{2}}} \int_{c_1\xi > 2A, |v| < \lambda_n} \xi e^{-\frac{|v-u_n|^2}{2T_n}} dv > \frac{c_1}{2} \frac{2A}{c_1} = A, \end{aligned}$$

which contradicts the finiteness of c_2.

Case 2(ii) $\lim_{n\to+\infty} T_n = +\infty, \lim_{n\to+\infty} u_n = u_* \geq 0$ and finite. Then

$$\begin{aligned} 0 < c_1 &= \lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V'_{\lambda_n}} \xi e^{-\frac{|v-u_n|^2}{2T_n}} dv \\ &\leq \lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V_{\lambda_n}} \mid \xi \mid e^{-\frac{|v-u_n|^2}{2T_n}} dv. \end{aligned}$$

Analogously,

$$\begin{aligned} c_2 &= \lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V'_{\lambda_n}} \xi^2 e^{-\frac{|v-u_n|^2}{2T_n}} dv \\ &\geq \frac{1}{2} \lim_{n\to+\infty} \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V_{\lambda_n}} \xi^2 e^{-\frac{|v-u_n|^2}{2T_n}} dv. \end{aligned}$$

In this case the integral representation of a bound from below of $\frac{c_2}{c_1}$ has infinite limit when $n \to \infty$, which leads to a contradiction.

Case 2(iii). The sequence (ρ_n, u_n, T_n) is unbounded and the sequences (u_n) and (T_n) are bounded. Then

$$(c_1, c_2, c_3) = \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{V'_{\lambda_n}} \xi(1, \xi, v^2) e^{-\frac{|v-u_n|^2}{2T_n}} dv$$

coincides with the limit when $n \to \infty$ of

$$(c_1^n, c_2^n, c_3^n) = \frac{\rho_n}{(2\pi T_n)^{\frac{3}{2}}} \int_{|\xi|>\mu_n} \xi(1, \xi, v^2) e^{-\frac{|v-u_n|^2}{2T_n}} dv.$$

For $T_* > 0$ this contradicts the boundedness of c_2. If $T_* = 0, u_* > 0$, then

$$0 = \lim_{n\to\infty} \frac{c_2^n}{\rho_n} = \lim_{n\to+\infty} \frac{1}{(2\pi T_n)^{\frac{3}{2}}} \int_{|\xi|>\mu_n} \xi^2 e^{-\frac{|v-u_n|^2}{2T_n}} dv = \lim_{n\to+\infty} u_n^2 = u_*^2 > 0,$$

which is impossible. It remains the case $T_* = u_* = 0$. Then

$$c_1 = \lim_{n\to+\infty} \frac{\rho_n}{\sqrt{2\pi T_n}} \int_{\mu_n}^{+\infty} \xi(e^{-\frac{(\xi-u_n)^2}{2T_n}} - e^{-\frac{(\xi+u_n)^2}{2T_n}}) d\xi.$$

Hence, by an integration by parts,

$$c_1 = \lim_{n\to+\infty} \{\rho_n \sqrt{\frac{T_n}{2\pi}} (e^{-\frac{(\mu_n-u_n)^2}{2T_n}} - e^{-\frac{(\mu_n+u_n)^2}{2T_n}}) + \frac{\rho_n u_n}{\sqrt{2\pi T_n}} \int_{\mu_n}^{+\infty} (e^{-\frac{(x-u_n)^2}{2T_n}} + e^{-\frac{(x+u_n)^2}{2T_n}}) dx\}. \tag{2.7}$$

Analogously,

$$c_2 = \lim_{n\to+\infty} [\rho_n \sqrt{\frac{T_n}{2\pi}} \{\mu_n (e^{-\frac{(\mu_n-u_n)^2}{2T_n}} + e^{-\frac{(\mu_n+u_n)^2}{2T_n}}) + \int_{\mu_n}^{+\infty} (e^{-\frac{(x-u_n)^2}{2T_n}} + e^{-\frac{(x+u_n)^2}{2T_n}}) dx\} + u_n c_1]. \tag{2.8}$$

Finally,

$$\begin{aligned} c_3 = \lim_{n\to+\infty} [(u_n^2 + 2T_n)c_1 + u_n \rho_n \mu_n \sqrt{\frac{T_n}{2\pi}} (e^{-\frac{(\mu_n-u_n)^2}{2T_n}} + e^{-\frac{(\mu_n+u_n)^2}{2T_n}}) \\ + \rho_n (\mu_n^2 + 2T_n) \sqrt{\frac{T_n}{2\pi}} (e^{-\frac{(\mu_n-u_n)^2}{2T_n}} - e^{-\frac{(\mu_n+u_n)^2}{2T_n}}) \\ + 3\rho_n u_n \sqrt{\frac{T_n}{2\pi}} \int_{\mu_n}^{+\infty} (e^{-\frac{(x-u_n)^2}{2T_n}} + e^{-\frac{(x+u_n)^2}{2T_n}}) dx\}. \end{aligned} \tag{2.9}$$

By (2.7) and (2.8),

$$u_n\rho_n\mu_n\sqrt{\frac{T_n}{2\pi}}(e^{-\frac{(\mu_n-u_n)^2}{2T_n}}+e^{-\frac{(\mu_n+u_n)^2}{2T_n}})<c_2u_n,$$

$$\rho_n(\mu_n^2+2T_n)\sqrt{\frac{T_n}{2\pi}}(e^{-\frac{(\mu_n-u_n)^2}{2T_n}}-e^{-\frac{(\mu_n+u_n)^2}{2T_n}})<c_1(\mu_n^2+2T_n),$$

$$\rho_nu_n\sqrt{\frac{T_n}{2\pi}}\int_{\mu_n}^{+\infty}(e^{-\frac{(x-u_n)^2}{2T_n}}+e^{-\frac{(x+u_n)^2}{2T_n}})<c_1T_n.$$

Consequently, in the limit when $n\to+\infty$ in (2.9), $c_3=0$ which contradicts the hypotheses.
We have thus proved that for $c_1c_3\neq c_2^2$ there are only the perturbative Maxwellian solutions. For $c_1c_3=c_2^2$ and (λ,μ) close enough to $(\infty,0)$, the above proof implies that either the Maxwellian is of perturbative type connected to T_-, or that the $\mathbf{O}_+$-situation holds. □
<u>Remark.</u> When $c_1c_3=c_2^2$, it is the discussion in 1(ii) that only leads to the $\mathbf{O}_+$-control instead of the stronger uniqueness results that follows from the implicit function theorem in the other cases. We do not exclude that a more detailed analysis also in this case might prove the same type of uniqueness as when $c_1c_3\neq c_2^2$.
For $c_1c_3<0$ or $c_1c_3>\frac{25}{16}c_2^2$, and (λ,μ) close enough to $(\infty,0)$, the above proof implies that there is no Maxwellian with such c-values and satisfying (2.5). For $c_1c_3=\frac{25}{16}c_2^2$ and any neighbourhood $\mathbf{O}$ of $(\rho_-,u_-,T_-)=\rho_+,u_+,T_+)$, the above proof implies that $(\rho(\lambda,\mu),u(\lambda,\mu),T(\lambda,\mu))$ is in $\mathbf{O}$ for λ,μ^{-1} large enough.

For the stationary Boltzmann equation in a slab with given indata on the boundary the following result was proved in [AN3].

Lemma 2.3 *Consider a slab $-a\leq x\leq a$ with ξ the component of the velocity $v\in\mathbb{R}^3$ in the $x-$direction. Let indata f_b be given on the boundary with*

$$\int_{-\epsilon\xi>0}[\xi(1+|v|^2+|\log f_b|)+(1+|v|^\beta)]f_b(\epsilon a,v)dv<\infty,\ \ \epsilon\in\{-1,1\}.$$

Given β with $0\leq\beta<2$ and $m>0$, there is a weak solution to the stationary Boltzmann equation such that $\int(1+\mid v\mid)^\beta f(x,v)dxdv=m$, and with indata

kf_b for some $k > 0$.
Given β with $-3 < \beta < 0$ and $m > 0$, there is a mild solution to the stationary Boltzmann equation such that $\int f(x,v)dxdv = m$, and with indata kf_b for some $k > 0$.

In the sequel we shall also need a result relating the distance of density functions from the set of Maxwellians, to the magnitude of the collision integrand.

Lemma 2.4 *Consider a set of non-negative functions f that is weakly compact in $L^1(\mathbb{R}^3)$. Given $\epsilon, \eta > 0$, there is $\delta > 0$, such that if*

$$\mid ff_* - f'f'_* \mid < \delta$$

in $V'_\lambda \times V'_\lambda \times S^2$ outside of some subset of measure smaller than δ, then for some Maxwellian M_f (depending on f),

$$\int_{V'_{\lambda-\eta}} \mid f - M_f \mid dv < \epsilon.$$

Lemma 2.4 was proved in the $\mathbb{R}^3$ case in [A] and [N]. From those proofs the present local version can be obtained similarly to the way the corresponding result for the functional equation $ff_* - f'f'_* = 0$ was localized in [W].

Denote by (ξ, η, ζ) the three components of $v \in \mathbb{R}^3$ and set $\sigma = \sqrt{\eta^2 + \zeta^2}$. In this paper, the hydrodynamic limit is considered for subsequences of f^ϵ, solutions to

$$\xi \frac{\partial f^\epsilon}{\partial x} = \frac{1}{\epsilon} Q(f^\epsilon, f^\epsilon), \quad x \in]-1,1[, \quad v \in \mathbb{R}^3, \tag{2.10}$$

$$f^\epsilon(-1,v) = M_l(v),\ \xi > 0, \quad f^\epsilon(1,v) = M_r(v),\ \xi < 0, \tag{2.11}$$

when the mean free path ϵ tends to zero. Here

$$M_l(v) := \frac{\rho_l}{(2\pi T_l)^{\frac{3}{2}}} e^{-\frac{v^2}{2T_l}}, \ M_r(v) := \frac{\rho_r}{(2\pi T_r)^{\frac{3}{2}}} e^{-\frac{v^2}{2T_r}},$$

and

$$Q(f,f)(x,v) := \int_{\mathbb{R}^3 \times S^2} b(\theta)\chi(v, v_*, \omega) \mid v - v_* \mid^\beta (f'f'_* - ff_*) dv_* d\omega,$$

$\theta \in (0,\pi)$ is the azimuthal angle between $v - v_*$ and ω,

$$f_* = f(x,v_*), \quad f' = f(x,v'), \quad f'_* = f(x,v'_*),$$
$$v' = v - (v - v_*,\omega)\omega, \quad v'_* = v_* + (v - v_*,\omega)\omega.$$

Moreover,

$$\chi(v,v_*,\omega) = 0 \quad \text{if} \quad | v | \geq \lambda, \text{or } | v_* | \geq \lambda, \text{or } | v' | \geq \lambda, \text{or } | v'_* | \geq \lambda,$$

$$\text{or} \quad | \xi | \leq \mu, \text{or } | \xi_* | \leq \mu, \text{or } | \xi' | \leq \mu, \text{or } | \xi'_* | \leq \mu,$$

$$\chi(v,v_*,\omega) = 1 \quad \text{else}, \quad \beta \in [0,2[, \quad b \in L^1_+(0,\pi), \quad b(\theta) \geq c > 0, \quad a.e..$$

For λ finite, the factor $|v - v_*|^\beta$ only introduces minor changes in the arguments, so we shall only discuss the case $\beta = 0$. Under the boundary conditions (2.11), there are cylindrically symmetric (with respect to the variables (ξ,σ)) functions f^ϵ solutions to (2.10-11). Only such solutions are considered in the following. In particular,

$$\int \xi\eta f^\epsilon(x,v)dv = \int \xi\zeta f^\epsilon(x,v)dv = 0$$

under the cylindrical symmetry. By Green's formula the fluxes

$$(c_i^\epsilon)_{1\leq i\leq 3} = (\int_{|\xi|\geq\mu, |v|\leq\lambda} \xi(1,\xi,v^2)f^\epsilon(x,v)dv)$$

are constant in x with ϵ-independent bounds determined by M_l and M_r. Denote by $(c_i^{\epsilon_j})_{1\leq i\leq 3}$ a converging subsequence with limit $(c_i(\lambda,\mu))_{1\leq i\leq 3}$, when $\epsilon_j \to 0$. Either $c_1(\lambda,\mu) = 0$ or $c_1(\lambda,\mu) \neq 0$. In this paper we only discuss such sequences of solutions with $c_1(\lambda,\mu) \neq 0$, and then - possibly after a change of x-direction - take $c_1(\lambda,\mu) > 0$, also requiring $c_1^{\epsilon_j} > 0$ for all j. Such systems can be considered to model an evaporation-condensation situation with evaporation at $x = -1$ and condensation at $x = 1$. We shall further assume (for a subfamily in λ,μ) the existence of $\lim_{\lambda,\mu^{-1}\to\infty} c_i(\lambda,\mu) = c_i$, for $i = 1,2,3$, with $c_1 > 0$. The quantities λ_0 and μ_0 as defined in Lemma 2.2 may be taken locally constant with respect to (c_1,c_2,c_3) satisfying the conditions of the lemma, and with λ_0, μ_0^{-1} so large that negative T's are excluded. From here on we only consider such $\lambda \geq \lambda_0, 0 < \mu \leq \mu_0$, and $0 < c_1c_3 < \frac{25}{16}c_2^2$.

The main results established in this paper are contained in the following three theorems.

Theorem 2.5 *Denote by*

$$g^{\epsilon}(\frac{x+1}{\epsilon}, v) := f^{\epsilon}(x,v), \quad a.a.\ x \in]-1,1[, \quad v \in I\!R^3.$$

Then there is a sequence (ϵ_j) *with* $\lim_{j\to\infty}\epsilon_j = 0$*, such that* (g^{ϵ_j}) *converges weakly in* $L^1([-1,1]\times I\!R^3)$ *to a function* g*, which is a weak solution to the half-space problem*

$$\xi\frac{\partial g}{\partial x} = Q(g,g), \quad x \geq 0, \quad v \in I\!R^3,$$
$$g(0,v) = M_l(v), \quad \xi > 0,$$

in the sense that for any $x_0 > 0$*, for any test function* φ *with support in* $[0, x_0[\times V'_\lambda$

$$\int_{\xi>0} \xi M_l(v)\varphi(0,v)dv + \int_0^{x_0}\int_{I\!R^3}(\xi g\frac{\partial\varphi}{\partial x} + Q(g,g)\varphi)dxdv = 0.$$

<u>Remark.</u> In this paper test functions are L^∞-functions, differentiable in the x-variable for a.e. $v \in V'_\lambda$ with $\varphi(0,v) = 0$ for $\xi < 0$.
An analogous result holds for $h^{\epsilon}(\frac{1-x}{\epsilon}, v) := f_{\epsilon}(x,v)$ and M_r.

Theorem 2.6 *Denote by* S_δ *the union of* $\{v \in V'_\lambda; \mu \leq|\ \xi\ |\leq \mu+\delta, \sigma \leq 4\mu+\delta\}$ *and* $\{v \in V'_\lambda; \mu \leq|\ \xi\ |\leq 3\mu+\delta, \sigma \leq \delta\}$*. If* $c_1c_3 = c_2^2$*, then include in* S_δ *also a* δ*-neighbourhood in* V'_λ *of* $(\frac{c_2}{c_1}, 0, 0)$*. Either for all* $\delta > 0$

$$\lim_{x\to\infty}\int_{V'_\lambda\setminus S_\delta} g(x,v)dv = 0,$$

or

$$\lim_{x\to\infty}\int |\ g(x,v) - M_-(v)\ |\ dv = 0,$$

or

$$\lim_{x\to\infty}\int |\ g(x,v) - M_+(v)\ |\ dv = 0$$

in the case $c_1c_3 \neq c_2^2$*, respectively*

$$\lim_{x\to\infty}\inf\int |g(x,v) - M_{\lambda,\mu}(v)|dv = 0$$

in the case $c_1c_3 = c_2^2$*. Here* M_-, M_+ *are those defined in Lemma 2.2, and in the notations of that lemma the infimum is taken over* $M_{\lambda,\mu}$ *corresponding to* $\mathbf{O}_+$ *and satisfying* (2.5).

Remark. The solution g of the half-space problem in Theorem 2.5 satisfies the Milne problem in the sense of Theorem 2.6. The M_+-alternative is only possible in the case (iii) of Lemma 2.2.

Theorem 2.7 *Suppose $c_1c_3 \neq c_2^2$. There is a sequence (ϵ_j) such that $\lim_{j\to\infty} \epsilon_j = 0$, and (f^{ϵ_j}) converges in weak* measure sense to a non-negative element f of $L^1((-1,1);\mathcal{M}(\mathcal{V}'_\lambda))$ that satisfies*

$$\int_{V'_\lambda} \xi(1,\xi,v^2)f(x,v)dv = (c_1(\lambda,\mu), c_2(\lambda,\mu), c_3(\lambda,\mu)).$$

Moreover, there are measurable non-negative functions $\theta_-(x), \theta_+(x)$ with $0 \leq \theta_-(x) + \theta_+(x) \leq 1, -1 \leq x \leq 1$, such that for test functions ϕ with support in $V'_\lambda \setminus S_\delta$ for some $\delta > 0$,

$$\int \phi f(x,v)dv = \int (\theta_- M_- + \theta_+ M_+)\phi dv.$$

Here we have written $f(x,v)dv$ for the measure in the $v-$variable defined by $f(x,.)$.

Remark. Also for this theorem, there is a (more involved) version in the case $c_1c_3 = c_2^2$.

3 Boundary layer analysis and the half-space problem.

This section is devoted to a proof of Theorem 2.5. The theorem is an immediate consequence of Lemma 3.1-3 below.

Lemma 3.1 *The family (g^ϵ) is weakly compact in $L^1((0,x_0)\times V'_\lambda)$.*

Proof of Lemma 3.1. Since

$$\int g^\epsilon(x,v)dxdv \leq \frac{1}{\mu^2}\int \xi^2 g^\epsilon(x,v)dxdv,$$

(g^ϵ) is uniformly bounded in $L^1((0,x_0)\times V'_\lambda)$. It remains to prove the uniform equiintegrability of (g^ϵ) in $L^1((0,x_0)\times V'_\lambda)$. If (g^ϵ) were not uniformly equiintegrable on $(0,x_0)\times V'_\lambda$, then there would be a number $\eta>0$, a sequence of subsets V_j of $(0,x_0)\times V'_\lambda$, and a subsequence of (g^ϵ), denoted by (g_j), such that

$$\epsilon_j \to 0, \mid V_j \mid < \frac{1}{j^2},\ \mathit{and}\ \int_{V_j} g_j(x,v)dxdv > \eta. \tag{3.1}$$

Denote by

$$B_j = \{x \in (0,x_0); \mid \{v \in V'_\lambda; (x,v) \in V_j\} \mid \geq \frac{1}{j}\},$$

and by

$$W_j(x) = \{v \in V'_\lambda; (x,v) \in V_j\}, \quad x \in (0,x_0).$$

Then $\mid B_j \mid \leq \frac{1}{j}$, so that

$$\int_{B_j^c}\int_{W_j(x)} g_j(x,v)dvdx > \frac{\eta}{2},$$

for j large enough. Only consider j so large that

$$\frac{c_2(\lambda,\mu)}{2} \leq \int \xi^2 g_j(x,v)dv \leq 2c_2(\lambda,\mu).$$

Set

$$B'_j = \{x \in B_j^c; \int_{W_j(x)} g_j(x,v)dv > \frac{\eta}{4x_0}\},$$

$$W'_j(x) = \{v \in W_j(x); g_j(x,v) > \frac{j\eta}{8x_0}\}.$$

It holds that

$$\int_{B'_j} dx \int_{W_j(x)} g_j(x,v)dv > \frac{\eta}{4}.$$

And so for $x \in B'_j$,

$$\int_{W'_j(x)} g_j(x,v)dv > \frac{\eta}{8x_0},$$

together with

$$\int_{B'_j} dx \int_{W'_j(x)} g_j(x,v)dv > \frac{\eta}{8}.$$

Using the exponential form of the equation for g_j, it follows for $x \leq x_0$, $\lambda \geq \xi \geq \mu$, that $g_j(x,v) \geq e^{-cx_0} M_l(v)$, where $c > 0$ is independent of j. Set $V_j'(x) = \{v \in V_\lambda'; g_j(x,v) \leq \frac{2c_2}{\delta\mu^2}\}$. Then $| V_\lambda' - V_j'(x) | < \delta$. Let $x \in B_j'$, $v \in W_j'(x)$ be given. It follows that there is a subset $W_j(x,v) \subset V_\lambda' \times S^2$ with measure uniformly in x, v, j bounded from below by a positive constant, such that for $(v_*, \omega) \in W_j(x,v)$ it holds that

$$g_j(x,v_*) \geq e^{-cx_0} \inf_{V_\lambda'} M_l(v),$$

and that

$$\frac{2c_2}{\delta\mu^2} \geq \max(g_j(x,v'), g_j(x,v_*')).$$

Hence,

$$g_j(x,v)g_j(x,v_*) - g_j(x,v')g_j(x,v_*') \geq cg_j(x,v), \qquad \frac{g_j(x,v)g_j(x,v_*)}{g_j(x,v')g_j(x,v_*')} \geq cj. \tag{3.2}$$

Integrating (3.2) over

$$K_j := \{(x,v,v_*,\omega); x \in B_j', v \in W_j'(x), (v_*,\omega) \in W_j(x,v)\},$$

leads to

$$c\eta < \frac{1}{\ln j} \int_{K_j} b(\theta)\chi(v,v_*,\omega)(g_j(x,v)g_j(x,v_*) - g_j(x,v')g_j(x,v_*')) \ln \frac{g_j(x,v)g_j(x,v_*)}{g_j(x,v')g_j(x,v_*')} dx dv dv_* d\omega < \frac{c'}{\ln j},$$

which is impossible for j large enough. □

Lemma 3.2 *The family $(Q^\pm(g^\epsilon, g^\epsilon))$ is weakly compact in any $L^1((0,x_0) \times V_\lambda')$.*

<u>Proof of Lemma 3.2.</u> It is sufficient to prove the weak compactness of $(Q^-(g^\epsilon, g^\epsilon))$. Then the compactness of $(Q^+(g^\epsilon, g^\epsilon))$ will follow from the weak compactness of $(Q^-(g^\epsilon, g^\epsilon))$ together with the boundedness of the entropy production term. And so it is enough to prove the weak L^1 compactness of $(g^\epsilon g_*^\epsilon)$. If this family were not compact, then for some $\eta > 0$, there would be a family (ϵ_j) and a family of sets $B_j \subset [0,x_0] \times V_\lambda' \times V_\lambda'$ with

$$| B_j | < \frac{1}{j^2} \text{ and } \int_{B_j} g_j g_{j*} dx dv dv_* > \eta.$$

But then for each j, there would be a subset of $[0, x_0]$ of measure not exceeding $\frac{1}{j}$, outside of which the set A_x of (v, v_*) such that $(x, v, v_*) \in B_j$, has measure bounded by $\frac{1}{j}$. Since $\int g_j(x,v)dv \leq \frac{2c_2}{\mu^2}$, the integral of $g_j g_{j*}$ over the first set is of magnitude $\leq \frac{1}{j}$. For x from the second set,

$$\int_{A_x} g_j g_{j*} dv dv_* \leq \frac{2c_2}{\mu^2} \sup_{A'_x} \int_{A'_x} g_j dv,$$

for $A'_x \subset V'_\lambda, \mid A'_x \mid < j^{-\frac{1}{2}}$. An application of Lemma 3.1 completes the proof.
□
Also using the regularizing properties of the equation, we get

Lemma 3.3 *Denote by g the weak L^1 limit in $(0, x_0) \times V'_\lambda$ of a converging sequence of (g_j) with $\lim \epsilon_j = 0$. For any test function φ defined on $(0, x_0) \times V'_\lambda$,*

$$\lim_{j \to +\infty} \int_{(0,x_0)\times V'_\lambda} \varphi Q^{\pm}(g_j, g_j)(x,v) dx dv = \int_{(0,x_0)\times V'_\lambda} \varphi Q^{\pm}(g, g)(x,v) dx dv.$$

This can be proved similarly to the corresponding (more involved) result in the time-dependent case [DL].
In Section 4 an entropy dissipation estimate for the half-space solution g will be needed.

Lemma 3.4 $\int b\chi(gg_* - g'g'_*) \log \frac{gg_*}{g'g'_*} dx dv dv_* d\omega \leq c$, *where the constant c only depends on the boundary values.*

This easily follows from the corresponding inequality for g^ϵ, the weak L^1 compactness of $(g^\epsilon g^\epsilon_*)$ in the proof of Lemma 3.2, and the convexity of the entropy-dissipation integrand.

4 The behaviour at infinity in the boundary layer.

This section is devoted to a proof of Theorem 2.6.
Proof of Theorem 2.6. By the weak L^1 convergence and by the conservation properties, in the notation of Lemma 3.3,

$$\int (\xi, \xi^2, \xi v^2) g(x,v) dv = \lim_{j \to +\infty} \int (\xi, \xi^2, \xi v^2) g_j(x,v) dv.$$

Recall that in the present setup, by Lemma 2.1-2 there are at most two Maxwellians M_- and M_+ of perturbation type such that

$$\int (\xi, \xi^2, \xi v^2) M_i(v) dv = \int (\xi, \xi^2, \xi v^2) g(x,v) dv = (c_1, c_2, c_3),$$

$$x \in I\!\!R^+, i \in \{+,-\}.$$

Recall that for $c_j = \lim_{\lambda, \mu^{-1} \to \infty} c_j(\lambda, \mu)$ with $0 < 16 c_1 c_3 < 25 c_2^2$, (cf Section 2 just before the statement of Theorem 2.6), and with $c_1 c_3 \neq c_2^2$, the constants $0 < \mu_0$ and $0 < \lambda_0$ are fixed, so that for $0 < \mu \leq \mu_0$, $\lambda_0 \leq \lambda$ the corresponding $c_j(\lambda, \mu), j = 1,2,3$, still define the same number (one or two) of Maxwellians. In the case $c_1 c_3 = c_2^2$, the set S_δ as defined in Theorem 2.6 also contains a δ-neighbourhood of $(\frac{c_2}{c_1}, 0, 0)$, and M_+ is replaced by an $\mathbf{O}_+$ based family $\mathcal{F}$ of λ, μ-truncated Maxwellians satisfying (2.5).
Either for all $\delta > 0$

$$\lim_{x \to \infty} \int_{V'_\lambda \setminus S_\delta} g(x,v) dv = 0, \tag{4.1}$$

or for some $\delta > 0$ and some sequence (x_j) tending to infinity,

$$\int_{V'_\lambda \setminus S_\delta} g(x_j, v) dv > 2\delta. \tag{4.2}$$

In the latter case $g(x,.)$ converges in $L^1(V'_\lambda)$ to either M_-, M_+, or in the the case $c_1 c_3 = c_2^2$ to the family $\mathcal{F}$, as will now be proved.
Uniform continuity of $g(x,.)$ in the $L^1(V'_\lambda)$-norm follows from the equation for g and from $\sup_x \int \mid Q(g,g)(x,v) \mid dv < \infty$. This means that given $\alpha > 0$, there is $a(\alpha) > 0$ such that

$$\int_{V'_\lambda} \mid g(x,v) - g(y,v) \mid dv < \alpha,$$

for $\mid x - y \mid < a(\alpha)$. Take $\alpha = \delta$ so that for $a_1 = a(\delta)$ by (4.2),

$$\int_{V'_\lambda \setminus S_\delta} g(x,v) dv > \delta \tag{4.3}$$

for $\mid x - x_j \mid \leq a_1, j \in I\!\!N$. Set

$$G(x) = \int b\chi (g' g'_* - g g_*) \log \frac{g' g'_*}{g g_*} (x, v, v_*, \omega) dv dv_* d\omega,$$

and (if necessary) take a subsequence (x_j) so that $x_{j+1} - x_j > a_1, j \in I\!N$. It follows from $\int_0^{+\infty} G(x)dx < +\infty$ that

$$\Sigma_j \int_{x_j - a_1}^{x_j + a_1} G(x)dx = \int_{-a_1}^{a_1} (\Sigma_j G(y + x_j))dy < +\infty,$$

hence $\lim_{j\to+\infty} G(y + x_j) = 0$ a.e. in $[-a_1, a_1]$. For such an y, the subsequence $(g(x_j + y, .))$ is weakly $L^1(V'_\lambda)$ compact. Only the uniform equi-integrability has to be discussed, and that follows similarly to the proof of Lemma 3.1, but using the estimate (4.2) instead of estimating $g(x, v_*)$ from below using boundary values. For this we start from such an y with

$$\int_{V'_\lambda \setminus S_\delta} g(x_j + y, v_*)dv_* > \delta, \quad \lim_{j\to+\infty} G(x_j + y) = 0.$$

Write $g_j(v) := g(x_j + y, v)$. A Dirac measure limit for a subsequence of g_j at $v = v_0$ implies $\xi_0 \geq \mu + \delta, \sigma_0 = 0$, and is excluded when $c_1c_3 \neq c_2^2$, and by the condition on S_δ also when $c_1c_3 = c_2^2$. Instead the following holds for some $d \in]0, \delta[$. For all $v_0 \in V'_\lambda$ and all $j \in I\!N$,

$$\int_{|v-v_0|\geq d} g_j(v)dv \geq d. \tag{4.4}$$

If (g_j) is not uniformly equiintegrable, then there are a constant $\eta > 0$ and a sequence of subsets (V_j) of V'_λ with $| V_j |\leq \frac{1}{j^2}$, such that

$$\int_{V_j} g_j(v)dv > \eta.$$

Similarly to Lemma 3.1 this is contradicted using an entropy dissipation argument. Consider the following three cases.
Either

$$\int_{W_{j1}} g_j(v)dv \geq \frac{\eta}{3}, \tag{4.5}$$

where $W_{j1} := \{v \in V_j \cap S_{\frac{\delta}{2}}; \sigma \leq 10^{-3}d\}$. Or

$$\int_{W_{j2}} g_j(v)dv \geq \frac{\eta}{3}, \tag{4.6}$$

where $W_{j2} := V_j \setminus S_{\frac{\delta}{2}}$. Or

$$\int_{W_{j3}} g_j(v)dv \geq \frac{\eta}{3}, \tag{4.7}$$

where $W_{j3} := \{v \in V_j \cap S_{\frac{\delta}{2}}; \sigma \geq 10^{-3}d\}$. For any v in $W_{jk}, k = 1, 2, 3$, the contradiction follows from delineating for each j a set of v_*'s with volume uniformly bounded from below by a positive constant, where $g_j(v_*)$ is uniformly in v_* and j bounded away from zero, together with for each j and v_* a set of ω's in S^2 of measure uniformly bounded away from zero, that generate (from above) uniformly bounded $g_j(v'), g_j(v'_*)$, so that the entropy dissipation argument applies.

Case 1(i). The bound (4.5) holds. Also using (4.3), assume in this case that $\int_{W_*} g_j(v_*)dv_* \geq \frac{\delta}{4}$, where

$$W_* := \{v_* \in V'_\lambda \setminus S_\delta; \sigma \geq 10^{-3}d, g_j(v_*) > \frac{\delta}{4 \mid V'_\lambda \mid}\}.$$

W_* is invariant under rotation around the ξ-axis. Taking $v \in W_{j1}$, there is sufficient volume of $v_* \in W_*$ and $\omega \in S^2$ for the entropy dissipation argument to apply and to exclude this case.

Case 1(ii). The bound (4.5) holds, but the second condition of 1(i) is violated. And so using (4.3), $\int_{W_*} g_j(v_*)dv_* \geq \frac{\delta}{2}$, where $W_* = \{v_* \in V'_\lambda \setminus S_\delta; \sigma_* < 10^{-3}d\}$. For $v \in W_{j1}$ and $v_* \in W_*$, notice that $\mid v - v_* \mid \geq \frac{\delta}{2}$. Write $W_* = A_1 \cup A_2 \cup A_3$, with three disjoint subsets A_1, A_2, A_3, such that

$$\inf_{A_1} g_j(v_*) \geq \sup_{A_2} g_j(v_*) \geq \inf_{A_2} g_j(v_*) \geq \sup_{A_3} g_j(v_*) \quad \text{and}$$

$$\int_{A_1} g_j(v_*)dv_* = \int_{A_2} g_j(v_*)dv_* = \int_{A_3} g_j(v_*)dv_* \geq \frac{\delta}{6}.$$

Analogously split W_{j1} into three disjoint sets B_1, B_2, B_3 with the same properties. Denote by $S(v, v_*)$ the subset of $\omega \in S^2$ such that $v', v'_* \in V'_\lambda, \sigma' \geq \frac{d}{8}$ and $\sigma'_* \geq \frac{d}{8}$. We shall discuss the case when $\sup_{A_2} g_j(v_*) \leq \inf_{B_1} g_j(v)$ for an infinite sequence of j's. The converse case is analogous after changing the roles of v and v_*. There is a positive uniform bound from below C_ω for the measure of $S(v, v_*)$. Suppose $2 \int g_j(v_*)dv_* \geq \int_{A_2} g_j(v_*)dv_*$ where the first integral is taken over those $v_* \in A_2$ for which $g_j g_{j*} \geq 2 g'_j g'_{j*}$ for at least half of $S(v, v_*)$. This cannot hold for infinitely many j's since

$$\frac{\eta}{3} \leq \int_{W_{j1}} g_j(v)dv \leq CG(x_j + y) \to 0, j \to \infty.$$

So the converse holds for infinitely many j's. Then $g_j(v_*) \leq \sqrt{2} g_j(v')$ on at least $\frac{1}{4}$th of $S(v, v_*)$. And so $g_j(v_*) \leq \frac{4\sqrt{2}}{C_\omega} \int g_j(v')d\omega$ where the integral is

over $\frac{1}{4}$th of $S(v,v_*)$. The Jacobian of the Carleman transformation of the gain term is uniformly bounded with respect to the relevant v, v_*, and ω in respectively B_1, A_2, and $S(v,v_*)$.
Then switch from v_* to such v' at a distance $\geq \frac{d}{8}$ from the ξ-axis. Cylindrical symmetry can be applied for v' to generate enough volumes in $V'(v,v',v'_*)$ and $V'_*(v,v',v'_*)$ for the entropy dissipation argument to apply, excluding this case when j is large enough.
Case 2(i). The bound (4.6) holds, and $\int_W g_j(v)dv > \frac{\eta}{6}$, where $W = \{v \in W_{j2}; \sigma \geq 10^{-3}d\}$. Then denote by $\bar{W_{j2}}$ the image set of W by rotation around the ξ -axis. Moreover, given v, by (4.4)

$$\int_{W_{*,v}} g_j(v_*)dv_* > \frac{d}{2}, \quad \text{where } W_{*,v} := \{v_*; \mid v_* - v \mid \geq d, g_j(v_*) > \frac{d}{2 \mid V'_\lambda \mid}\}.$$

Taking $v \in \bar{W_{j2}}$, $v_* \in W_{*,v}$ and using rotation around the ξ-axis in $\bar{W_{j2}}$, generates volumes bounded from below for which $g_j(v'), g_j(v'_*)$ are uniformly bounded from above and for which the entropy dissipation argument applies, excluding this case when j is large enough.
Case 2(ii). The bound (4.6) holds, and $\int_{W_j} g_j(v)dv > \frac{\eta}{6}$, where $W_j = \{v \in W_{j2}; \sigma < 10^{-3}d\}$. Then, either given v, by (4.4)

$$\int_{|v_*-v|\geq d, \sigma_* \geq 10^{-3}d} g_j(v_*)dv_* > \frac{d}{2}.$$

For such $v \in W_j$, taking v_* in the image set by rotation around the ξ-axis of $\{v_* \in V'_\lambda; \mid v_* - v \mid \geq d, \sigma_* \geq 10^{-3}d, g_j(v_*) > \frac{d}{4|V'_*|}\}$, and taking suitable $\omega \in S^2$, gives a setting where the entropy dissipation argument applies. Or otherwise

$$\int_{|v_*-v|\geq d, \sigma_* < 10^{-3}d} g_j(v_*)dv_* > \frac{d}{2},$$

and the argument of Case 1(ii) can be used. And so this type of concentration is excluded.
Case 3. The bound (4.7) holds. Denote by $\bar{W_{j3}}$ the image set of W_{j3} by rotation around the ξ-axis. Then $\int_{\bar{W_{j3}}} g_j(v)dv > \frac{\eta}{3}$. Taking $v \in \bar{W_{j3}}$, suitable $v_* \in V'_\lambda \setminus S_\delta$ and $\omega \in S^2$, generates using (4.3), a setting where the entropy dissipation argument applies as in the earlier cases, thus excluding also this final possibility.
We conclude that (g_j) is uniformly equiintegrable. It then follows for y with

$\lim_{j\to\infty} G(y+x_j)=0$ from the weak L^1-compactness just proved and using Lemma 2.4, that there is a sequence $(M_j(v))$ of Maxwellians such that

$$\lim_{j\to\infty}\int \mid g(x_j+y,v)-M_j(v)\mid dv=0.$$

Moreover,

$$\lim_{j\to\infty}\int(\xi,\xi^2,\xi v^2)M_j(v)dv=(c_1(\lambda,\mu),c_2(\lambda,\mu),c_3(\lambda,\mu)).$$

Except in the case $c_1c_3=c_2^2$ and $T_*=0$, this implies (for a subsequence) that $\lim_{j\to\infty}\int \mid M_j(v)-M_-(v)\mid dv=0$ or $\lim_{j\to\infty}\int \mid M_j(v)-M_+(v)\mid dv=0$. It then remains to prove that

$$\lim_{x\to\infty}\int \mid g(x,v)-M_-(v)\mid dv=0$$

in the first case, and that

$$\lim_{x\to\infty}\int \mid g(x,v)-M_+(v)\mid dv=0$$

in the second case. We carry out the proof in the first case. Let

$$0<\epsilon<\min[\delta,10^{-1}\int \mid M_-(v)-M_+(v)\mid dv,\quad 10^{-1}\int_{V'_\lambda\setminus S_\delta}M_-(v)dv]\tag{4.8}$$

be given. Let us prove that for x large enough, $\int \mid g(x,v)-M_-(v)\mid dv<2\epsilon$, when we already know that there is j_0, such that

$$\int \mid g(x_j+y,v)-M_-(v)\mid dv<\epsilon,\quad j\geq j_0.$$

Let a_2 be such that

$$\int \mid g(r,v)-g(s,v)\mid dv<\epsilon,\quad \mid r-s\mid<a_2.$$

We have

$$\int_{V'_\lambda\setminus S_\delta} g(x_j+y,v)dv\geq\int_{V'_\lambda\setminus S_\delta}M_-(v)dv$$

$$-\int_{V'_\lambda\setminus S_\delta}\mid g_j(x_j+y,v)-M_-(v)\mid dv\geq 10\epsilon-\epsilon=9\epsilon,\quad j\geq j_0.$$

And so,

$$\int_{V'_\lambda \setminus S_\delta} g(z,v)dv > 8\epsilon, \quad z \in [x_j + y - a_2, x_j + y + a_2], j \geq j_0.$$

Now $\lim_{X\to+\infty} \int_X^{+\infty} G(x)dx = 0$, and so given $\eta > 0$, there is $X_1 > 0$ such that $\mathrm{meas}\{z > X_1; G(z) > \eta\} < \frac{a_2}{6}$. So for $j \geq j_0$ there is $z_j \in [x_j + y + \frac{2a_2}{3}, x_j + y + a_2]$ such that $G(z_j) < \eta$. Here $\eta > 0$ has by the previous discussion been chosen small enough, so that

$$\min(\int \mid g(z_j,v) - M_-(v) \mid dv, \int \mid g(z_j,v) - M_+(v) \mid dv) < \epsilon.$$

But

$$\int \mid g(z_j,v) - M_+(v) \mid dv > \int \mid M_+(v) - M_-(v) \mid dv$$

$$-(\int \mid M_-(v) - g(x_j + y),v) \mid dv + \int \mid g(x_j + y, v) - g(z_j,v) \mid dv) > 8\epsilon.$$

Hence

$$\min(\int \mid g(z_j,v) - M_-(v) \mid dv, \int \mid g(z_j,v) - M_+(v) \mid dv) = \int \mid g(z_j,v) - M_-(v) \mid dv < \epsilon,$$

$$\int \mid g(z,v) - M_-(v) \mid dv < \int \mid g(z,v) - g(z_j,v) \mid dv$$

$$+ \int \mid g(z_j,v) - M_-(v) \mid dv < 2\epsilon, \quad z \in [x_j + y, x_j + y + a_2].$$

We can now repeat the above discussion from $x_j + y + a_2$ instead of $x_j + y$, and in a finite number of iterations reach $x_{j+1} + y$.
In the remaining case when $c_1c_3 = c_2^2$ and $T_+ = 0$, we replace $\int |M_-(v) - M_+(v)|dv$ with $\inf \int |M_-(v) - M_{\lambda,\mu}|dv$. Here the infimum is taken over the family $\mathcal{F}$ of (λ, μ)-truncated Maxwellians $M_{\lambda,\mu}$ according to Lemma 2.2. Using (2.5) we choose (λ_0, μ_0) so that the infimum in (4.8) is positive in this case. Then the above proof can be repeated in the case $T_+ = 0$, if M_+ is replaced by relevant members $M_{\lambda,\mu}$ from the family $\mathcal{F}$. $\square$

5 A hydrodynamic limit in the slab.

This section is devoted to a proof of Theorem 2.7.
Proof of Theorem 2.7. Let $0 < \epsilon_j, j \in I\!N$, be a decreasing sequence with $\Sigma_j \epsilon_j < \infty$, and with f_{ϵ_j} converging in weak* measure sense to a measure f. Write $f_{\epsilon_j} = f_j$. Set

$$G_\epsilon(x) = \int b\chi(f'_\epsilon f'_{\epsilon *} - f_\epsilon f_{\epsilon *}) \log \frac{f'_\epsilon f'_{\epsilon *}}{f_\epsilon f_{\epsilon *}}(x, v, v_*, \omega) dv dv_* d\omega$$

The above hypotheses imply that

$$\int_{-1}^{1} \Sigma_j G_{\epsilon_j}(x) dx \leq C \Sigma_j \epsilon_j < \infty.$$

And so for a.e. x,

$$\Sigma_j G_{\epsilon_j}(x) < \infty,$$

and in particular $\lim_{j\to\infty} G_{\epsilon_j}(x) = 0$. Moreover, given $m \in I\!N$, in the complement I_m of some set of measure less than m^{-1} in $[-1,1]$, G_{ϵ_j} converges uniformly to 0 when j tends to infinity. Let $\chi_m(x)$ be the characteristic function of I_m. Let $\chi_{jn}(x,v)$ be the characteristic function (in x) of those $x \in [-1,1]$, for which $\int_{V'_\lambda \setminus S_{n^{-1}}} f_j(x,v) dv > n^{-1}$, multiplied with the characteristic function (in v) of $V'_\lambda \setminus S_{n^{-1}}$. We may take the sequence (f_j) so that for each $m, n \in I\!N$, the sequence $(f_j \chi_{jn} \chi_m)_{j \in I\!N}$ converges in weak* measure sense. By a variant of the reasoning in Section 4, the sequence is also weakly compact in $L^1((-1,1) \times V'_\lambda)$, and with a weak* measure limit $\theta_-^{nm} M_- + \theta_+^{nm} M_+$. Here $\theta_-^{nm}, \theta_+^{nm}$ are increasing as functions of m, n with limits θ_-, θ_+, and as functions of x they satisfy $0 \leq \theta_-^{nm}(x), \theta_+^{nm}(x), \theta_-^{nm}(x) + \theta_+^{nm}(x) \leq 1$ with $\theta_+ \equiv 0$ in the case $c_2^2 > c_1 c_3$. For test functions ϕ with support in $V'_\lambda \setminus S_{n^{-1}}$,

$$|\int f_j (1 - \chi_{jn}) \phi dx dv| \leq 2 \|\phi\|_\infty n^{-1}.$$

Also

$$\int f_j (1 - \chi_m) dx dv \leq m^{-1} \frac{c_2}{\mu^2}.$$

And so

$$\int \phi f dx dv = \lim_{j\to\infty} \int f_j \phi dx dv = \lim_{j\to\infty} \int f_j (1 - \chi_m) \phi dx dv$$

$$+\lim_{j\to\infty}\int f_j(1-\chi_{jn})\chi_m\phi dxdv + \lim_{j\to\infty}\int f_j\chi_{jn}\chi_m\phi dxdv$$

$$=\int(\theta_-M_- + \theta_+M_+)\phi dxdv + O_{nm},$$

where O_{nm} tends to zero when $n, m \to \infty$. And so

$$\int \phi f\, dxdv = \int(\theta_-M_- + \theta_+M_+)\phi dxdv.$$

By the same argument for any $\delta > 0$, for any test function ϕ and with χ the characteristic function of $V'_\chi \setminus S_\delta$,

$$\int \phi\chi f\, dxdv = \int(\theta_-M_- + \theta_+M_+)\chi\phi dxdv.$$

It follows that f is composed of a singular measure on $[-1,1]\times S_0$, together with a Lebesgue absolutely continuous measure with density $\theta_-M_- + \theta_+M_+$. Finally $\int \phi f dv$ is Lebesgue measurable in x and

$$\int \xi(1,\xi,v^2)f dv = (c_1, c_2, c_3)(\lambda,\mu).\Box$$

Acknowledgement. We thank A. Bobylev and Y. Sone for useful discussions.

References

A Arkeryd, L., A kinetic inequality, Non Standard Analysis, Kluwer,1997.

AN1 Arkeryd, L., Nouri, A., 'A condensation-evaporation problem', SIAM J. Math. Anal. 29, 1998, 30-47.

AN2 Arkeryd, L., Nouri, A., 'L^1 solutions to the stationary Boltzmann equation in a slab, preprint, 1998.

AN3 Arkeryd, L., Nouri, A., 'The stationary Boltzmann equation in the slab with given weighted mass for hard and soft forces', Ann. Scuola Norm. Sup. Pisa 27, 1998, 533-556.

AN4 Arkeryd, L., Nouri, A., 'On the stationary Povzner equation in $I\!R^n$', J. Math. Kyoto Univ. 39, 1999, 115-153.

AC Arthur, M.D., Cercignani, C., 'Nonexistence of a steady rarefied supersonic flow in a half-space', Z. Angew. Math. Phys. 31, 1980, 634-645.

BCN1 Bardos, C., Caflish, E., Nicolaenko, B., 'The Milne and Kramers problems for the Boltzmann equation of a hard spere gas', Comm. Pure Appl. Math. 39, 1986, 323-352.

BCN2 Bardos, C., Caflish, E., Nicolaenko, B., 'Thermal layer solutions of the Boltzmann equation', Progr. Phys. 10, 235-251, Birkhäuser, Mass., 1985.

BGL1 Bardos, C., Golse, F., Levermore, C.D., 'Fluid dynamic limits of kinetic equations I', J. Stat. Phys. 63, 1991, 323-344.

BGL2 Bardos, C., Golse, F., Levermore, C.D., 'Fluid dynamic limits of kinetic equations II', Comm. Pure Appl. Math. 46, 1993, 667-753.

BU Bardos, C., Ukai, S., 'The classical incompressible Navier-Stokes limit of the Boltzmann equation', Math. Mod. Meth. Appl. Sci. 1, 1991, 235-257.

BT Bobylev, A. V., Toscani, G., 'Two-dimensional half space problems for the Broadwell discrete velocity model', Cont. Mech. Thermodyn. 8, 1996, 257-274.

BIU Bose, C., Illner, R., Ukai, S., 'On shock wave solutions for discrete velocity models of the Boltzmann equation', Transp. Th. Stat. Phys. 27, 1998, 35-66.

C1 Cercignani, C., Mathematical methods in kinetic theory, Plenum Press, New-York, 1969.

C2 Cercignani, C., 'Half-space problems in the kinetic theory of gases', Trends in applications of pure mathematics to mechanics, 35-50, Springer Verlag, Berlin, 1986.

CIPS Cercignani, C., Illner, R., Pulvirenti, M., Shinbrot, M., 'On nonlinear stationary half-space problems in discrete kinetic theory', J. Stat. Phys., 52, 1988, 885-896.

CGS Coron, F., Golse, F., Sulem, C., 'Classification of kinetic layer problems', Comm. Pure Appl. Math. 41, 1988, 409-435.

DEL De Masi, A., Esposito, R., Lebowitz, J.L., 'Incompressible Navier-Stokes and Euler limits of the Boltzmann equation', Comm. Pure Appl. Math. 42, 1989, 1189-1214.

DL DiPerna, R. J., Lions, P. L., 'On the Cauchy problem for Boltzmann equations: Global existence and weak stability', Ann. Math. 130, 1989, 321-366.

GP Golse, F., Poupaud, F., 'Stationary solutions of the linearized Boltzmann equation in a half space', Math. Meth. Appl. Sci. 11, 1989, 483-502.

GPS Golse, F., Perthame, B., Sulem, C., 'On a boundary layer problem for the nonlinear Boltzmann equation', Arch. Rat. Mech. Anal., 103, 1988, 81-96.

G Grad, H., 'Asymptotic theory of the Boltzmann equation', Phys. Fluids, 6, 1963, 147-181.

Gu Guiraud, J.P., 'Equation de Boltzmann linéaire dans un demi-éspace', C.R. Acad. Sci. Paris, 274, 1974, 417-419.

N Nouri, A., 'On a Cauchy inequality for the Boltzmann equation', Preprint 1999.

P Panferov, V., 'On the stationary Povzner equation', in preparation.

S1 Sone, Y., 'Kinetic theory of evaporation and condensation- Linear and nonlinear problems' J. Phys. Soc. Jap. 45, 1978, 315-320.

S2 Sone, Y., 'Kinetic theoretical studies of the half-space problem of evaporation and condensation', Transp. Th. Stat. Phys., 1999, to appear.

SOA1 Sone, Y., Ohwada, T., Aoki, K., 'Evaporation and condensation on a plane condensed phase: Numerical analysis of the linearized Boltzmann equation for hard-sphere molecules', Phys. Fluids A1, 1989, 1395-1405.

SOA2 Sone, Y., Ohwada, T., Aoki, K., 'Evaporation and condensation of a rarefied gas between its two parallel condensed phases with different temperature',Math. aspects of fluid plasma dynamics - Proceedings, Lecture Notes in Math. 1460, 186-202, Springer Verlag, Berlin, 1991.

U Ukai, S., 'On the half-space problem for the discrete velocity model of the Boltzmann equation', Advances in Nonlinear PDEs and Stochastics, 160-174, World Scientific, Singapore, 1998.

W Wennberg, B., 'On an entropy dissipation inequality for the Boltzmann equation' C.R. Acad. Sci. Paris 315, 1992, 1441-1446.