Absence of mutual unbounded growth for almost all parameter values in the two-type Richardson model

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Abstract

We study the two-type Richardson model on $\mathbf{Z}^d$, $d \geq 2$, in the asymmetric case where the two particle types have different infection rates. Starting with a single particle of each type, and fixing the infection rate for one of the types, we show that mutual unbounded growth has probability 0 for all but at most countably many values of the other type's infection rate.

1 Introduction

The study of interacting particle systems has become one of the most fruitful branches of probability theory in the last couple of decades, see e.g. [11], [4] and [12] for overviews. One of the simplest such systems is the Richardson model [14], which can be described as the $\{0,1\}^{\mathbf{Z}^d}$-valued Markov process $\{\Xi_t\}_{t\geq 0}$ where no two sites $x, y \in \mathbf{Z}^d$ flip (change their value) simultaneously, and $x \in \mathbf{Z}^d$ flips at rate

$$\begin{cases} \lambda n_x(\Xi_t) & \text{if } \Xi_t(x) = 0 \\ 0 & \text{if } \Xi_t(x) = 1; \end{cases}$$

here $\lambda > 0$ is a fixed parameter and $n_x(\Xi_t)$ is the number of nearest neighbors y of x with value $\Xi_t(y) = 1$ (two sites in $\mathbf{Z}^d$ are considered nearest neighbors if their Euclidean distance is 1). Although this model makes sense also for $d = 1$, we will always assume that $d \geq 2$. If we think of 1's as occupied sites and 0's as empty sites, then this defines a pure growth model, where a particle at x gives birth at rate λ at each of its empty neighboring sites. The main questions that have been studied for this model concern the asymptotic growth of the set of occupied vertices starting from a single occupied vertex at time 0. The fundamental result says roughly that the set of occupied vertices has a non-random asymptotic shape as $t \to \infty$; this will be given a precise formulation in Theorem 2.1 below and then heavily used throughout the rest of the paper.

Recently, in [7], we introduced the **two-type Richardson model**, which is a generalization of the above model, where two kinds of particles compete for space in $\mathbf{Z}^d$. This new model has state space $\{0,1,2\}^{\mathbf{Z}^d}$, two parameters $\lambda_1, \lambda_2 > 0$, and the following flip rates: 1's and 2's never flip, while a 0 flips to a 1 (resp. a 2) at rate λ_1 (λ_2) times the number of nearest neighbors with value 1 (2).

Suppose that the two-type Richardson model is started at time 0 with the site $\mathbf{0} = (0, 0, \ldots, 0)$ occupied by a 1, the site $\mathbf{1} = (1, 0, \ldots, 0)$ occupied by a 2, and all other sites being vacant (i.e. having value 0). It is easily seen that with probability 1, all sites will eventually be occupied by a particle. Thus, one of the following three scenarios must take place:

(i) The set of 1's at some point surrounds (strangles) the set of 2's, so that only finitely many sites are eventually turned into 2's, and the rest of $\mathbf{Z}^d$ is filled with 1's.

(ii) The set of 2's similarly strangles the set of 1's.

(iii) Both the set of 1's and the set of 2's keep growing indefinitely.

It is easy to see that scenario (i) happens with positive probability for any choice of λ_1 and λ_2, and similarly for scenario (ii). The main question studied here and in [7] is whether scenario (iii) also has positive probability. Writing G^1 (G^2) for the event that the number of 1's (2's) grows unboundedly, and $\mathbf{P}_{\mathbf{0},\mathbf{1}}^{\lambda_1,\lambda_2}$ for the probability law of this process, we thus ask whether or not

$$\mathbf{P}_{\mathbf{0},\mathbf{1}}^{\lambda_1,\lambda_2}(G^1 \cap G^2) > 0. \tag{1}$$

We remark that the particular choice of starting configuration is irrelevant for whether or not the event in (1) has positive probability (unless one of the sets is already strangled by the other): For two disjoint subsets ξ^1 and ξ^2 of $\mathbf{Z}^d$, write $\mathbf{P}_{\xi^1,\xi^2}^{\lambda_1,\lambda_2}$ for the law of the process starting with the sites in ξ^1 having value 1, those in ξ^2 having value 2, and the rest value 0. A straightforward generalization of the proof of Proposition 1.2 of [7] shows that (1) is equivalent to having

$$\mathbf{P}_{\xi^1,\xi^2}^{\lambda_1,\lambda_2}(G^1 \cap G^2) > 0$$

for any choice of finite ξ^1 and ξ^2 except those where one of the sets is already strangled by the other.

In [7], we showed that (1) holds for $d = 2$ and $\lambda_1 = \lambda_2$, i.e. that mutual unbounded growth has positive probability if the two particle types have equal infection rates. We also stated the conjecture that mutual unbounded growth does not happen in the two-dimensional asymmetric case where $\lambda_1 \neq \lambda_2$. We now extend this conjecture to arbitrary $d \geq 2$:

Conjecture 1.1 *For the two-type Richardson model on $\mathbf{Z}^d$, $d \geq 2$, we have*

$$\mathbf{P}_{\mathbf{0},\mathbf{1}}^{\lambda_1,\lambda_2}(G^1 \cap G^2) = 0$$

whenever $\lambda_1 \neq \lambda_2$.

The problem of deciding whether or not (1) holds for $\lambda_1 = \lambda_2$ and $d \geq 3$ is also open. It seems reasonable to expect that (1) should hold in this case, although we are slightly less confident about this than about Conjecture 1.1.

The following result, which is the main result of the present paper, is a weak form of Conjecture 1.1. Note that by time-scaling, it is enough to consider the case $\lambda_1 = 1$.

Theorem 1.2 *For the two-type Richardson model on $\mathbf{Z}^d$, $d \geq 2$, we have*

$$\mathbf{P}_{\mathbf{0},\mathbf{1}}^{1,\lambda_2}(G^1 \cap G^2) = 0$$

for all but at most countably many choices of λ_2.

This strongly suggests that Conjecture 1.1 should be true. To see why, consider the following heuristic argument. The event $G^1 \cap G^2$ of mutual unbounded growth reflects some kind of "power balance" between the two types, and therefore it is reasonable to expect that $\mathbf{P}^{1,\lambda_2}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2)$ should decrease as λ_2 moves away from 1. For any fixed $\lambda_2 > 1$ there exists, by Theorem 1.2, a $\lambda_2' \in (1, \lambda_2)$ such that $\mathbf{P}^{1,\lambda_2'}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2) = 0$, so that (if the above intuition is correct) $\mathbf{P}^{1,\lambda_2}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2) = 0$ as well. Mutual unbounded growth for $\lambda_2 < 1$ is ruled out by the same argument, or by noting that symmetry plus time-scaling implies that

$$\mathbf{P}^{1,\lambda_2}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2) = \mathbf{P}^{1,1/\lambda_2}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2). \tag{2}$$

Unfortunately, we have not been able to turn the above heuristics into a proof.

Theorem 1.2 puts current knowledge about the two-type Richardson model in a situation analogous to those of certain models of statistical mechanics in $d \geq 3$ dimensions, such as the random-cluster model and the parity-dependent hard-core model (see [5] and [6], respectively). For both these models, phase transition (nonuniqueness of Gibbs measures) is known to occur for at most countably many points along certain one-dimensional curves in the parameter space. Both models are, however, very different in spirit from the two-type Richardson model, and our proof of Theorem 1.2 has little in common with the proofs of the analogous results in the other models.

Neuhauser [13] has considered a model which generalizes the standard contact process in exactly the same way that the two-type Richardson model generalizes the ordinary Richardson model. The two-type Richardson model would arise by taking death rate 0 in Neuhauser's model. We doubt, however, that this observation can be used to draw any conclusions about the problem studied here.

The rest of this paper is organized as follows. In Section 2, we recall the asymptotic shape theorem and formulate a proposition which plays a key role in the proof of Theorem 1.2. After some preliminaries in Section 3, we prove this key proposition in Section 4, and finally in Section 5 we use it to prove Theorem 1.2.

2 The shape theorem and a key proposition

Let us recall the classical shape theorem. For $\lambda > 0$ and $\xi \subset \mathbf{Z}^d$, we write $\mathbf{P}^{\lambda}_{\xi}$ for the law of the Richardson model with parameter λ, started at time 0 with 1's in ξ and 0's in $\mathbf{Z}^d \setminus \xi$. We furthermore write $\eta(t)$ for the set of 1's at time t, i.e.

$$\eta(t) := \{x \in \mathbf{Z}^d : \Xi_t(x) = 1\}.$$

Also define the "smoothed out" version $\bar{\eta}(t) \subset \mathbf{R}^d$ of $\eta(t)$ as

$$\bar{\eta}(t) := \{x + y : x \in \eta(t), y \in (-\tfrac{1}{2}, \tfrac{1}{2}]^d\}.$$

Theorem 2.1 (the Shape Theorem) *There exists a nonrandom compact convex set $B_0 \subset \mathbf{R}^d$ which is invariant under permutation of and reflection in the coordinate hyperplanes, and has a nonempty interior, such that for any $\lambda > 0$ and any $\varepsilon > 0$ we have*

$$\mathbf{P}^{\lambda}_{\mathbf{0}}\left[\exists T < \infty,\ \forall t > T,\ (1-\varepsilon)B_0 \subseteq \frac{\bar{\eta}(t)}{\lambda t} \subseteq (1+\varepsilon)B_0\right] = 1.$$

An "in probability" version of this a.s. limit theorem appears already in Richardson [14]. As pointed out by Cox and Durrett [3], the a.s. version follows by combining Richardson's results with that of Kesten [8]. Subsequently more general shape theorems have been obtained by Kesten [9] and Boivin [2]. The exact shape of B_0 remains unknown.

B_0 defines a norm $|\cdot|$ as $|x| := \inf\{t : x \in tB_0\}$. Define the modulus of a set A by $|A| := \inf\{t : A \subseteq tB_0\}$ and the dual modulus by $|A|_* := \sup\{t : tB_0 \cap A^c = \emptyset\}$. If A is a subset of the integer lattice, the complementation refers to complementation in $\mathbf{Z}^d$, not in $\mathbf{R}^d$. The following key result, which will be proved in Section 4, says roughly that in the long run even a relatively small advantage (in terms of the set of occupied sites) for the stronger type is enough to doom the weaker type.

Proposition 2.2 *Fix $\lambda_2 > 1$ and let $1 < a < b$. Let $S(u,v)$ denote the set of pairs of configurations (ξ^1, ξ^2) such that $|\xi^1| \leq u$ and $|\xi^2| \geq v$. Then*

$$\lim_{t\to\infty} \sup_{(\xi^1,\xi^2)\in S(ta,tb)} \mathbf{P}^{1,\lambda_2}_{\xi^1,\xi^2}(G^1) = 0\,.$$

3 Comparison results

We first record a few easy facts about stochastic domination and the relation of the one- and two-type models. Analogously to the notation for the one-type model in the previous section, we write $\eta^1(t)$ and $\eta^2(t)$ for the set of 1's resp. 2's in the two-type model at time t.

Lemma 3.1 *Assume that $\lambda_1 \leq \tilde{\lambda_1}$, $\lambda_2 \geq \tilde{\lambda_2}$, $\xi^1 \subseteq \tilde{\xi}^1$ and $\xi^2 \supseteq \tilde{\xi}^2$, and consider the two processes $\{\eta^1(t), \eta^2(t)\}$ and $\{\tilde{\eta}^1(t), \tilde{\eta}^2(t)\}$ with respective distributions $\mathbf{P}^{\lambda_1,\lambda_2}_{\xi^1,\xi^2}$ and $\mathbf{P}^{\tilde{\lambda}_1,\tilde{\lambda}_2}_{\tilde{\xi}^1,\tilde{\xi}^2}$. These can be coupled in such a way that for all t we get*

$$\eta^1(t) \subseteq \tilde{\eta}^1(t) \tag{3}$$

and

$$\eta^2(t) \supseteq \tilde{\eta}^2(t)\,. \tag{4}$$

Proof: For $t = 0$, (3) and (4) are trivial. The flip rates of the two systems can be paired according to the so called "basic coupling" (see e.g. Section III.1 of Liggett [11]), and this is easily seen to preserve (3) and (4). Alternatively, the "omni-λ" coupling $\mathbf{Q}$ in Section 5 may be used. □

An instance of this lemma is that the set of 1's at time t in the two-type process $\mathbf{P}^{1,\lambda_2}_{\xi^1,\xi^2}$ is stochastically dominated by the set of 1's at time t in the single-type process $\mathbf{P}^1_{\xi^1}$. Similarly, we have for any t that the set of 2's in $\mathbf{P}^{1,\lambda_2}_{\xi^1,\xi^2}$ is dominated by the set of 1's in $\mathbf{P}^{\lambda_2}_{\xi^2}$.

Lemma 3.2 *Fix $\lambda \in (0,1)$, and consider the two processes $\{\eta(t)\}$ and $\{\tilde{\eta}^1(t), \tilde{\eta}^2(t)\}$ with respective distributions $\mathbf{P}^\lambda_{\mathbf{0}}$ and $\mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}$. These can be coupled in such a way that*

$$\eta(t) \subseteq \tilde{\eta}^1(t) \cup \tilde{\eta}^2(t) \tag{5}$$

for all t.

Proof: Again the case $t = 0$ is trivial. The flips can be coupled as follows. Each ordered pair (x, y) of nearest neighbors is equipped with an independent unit rate Poisson process. At the times of the Poisson process assigned to (x, y), we toss an independent biased coin with head-probability $1 - \lambda$, and also check in both processes whether x is infected and y is not. Whenever this is the case, y is infected by x, except if the infection at x is of type 1 and the coin came up heads. This preserves (5). □

For a subset R of $\mathbf{Z}^d$, define its **boundary** ∂R to be the set of sites in R^c that have some nearest neighbor in R. To describe the next result, we define a modification of the two-type Richardson model as follows, with state space $\{0, 1, 2, 2^*\}^{\mathbf{Z}^d}$. A 0 flips to a 1 (resp 2, 2^*) at rate λ_1 (resp. λ_2, λ_2) times the number of neighboring 1's (resp. 2's, 2^*'s). A 1 and a 2^* stays put forever, while a 2 flips to a 2^* at rate λ_2 times the number of neighboring 2^*'s. The first thing to note is that if we observe this process without distinguishing 2's and 2^*'s, then we see the usual two-type Richardson model. Now let R be any fixed subset of $\mathbf{Z}^d$, and start the modified process from some configuration $(\xi^1, \xi^2, \xi^{2^*})$ such that $\xi^1 \cap R = \emptyset$ and $\xi^{2^*} \subset R$. Let τ be the first time that a site in ∂R flips to a 1 or a site in $\partial(R^c)$ flips to a 2^*. Then the growth of the set of 2^*'s up to time τ is a version of the rate λ_2 single-type model. More precisely, the modified two-type model starting from $(\xi^1, \xi^2, \xi^{2^*})$ can be coupled with the single-type process $\mathbf{P}^{\lambda_2}_{\xi^{2^*}}$ in such a way that up to time τ, the set of 2^*'s in the modified process equals the set of occupied sites in the single-type process. This fact, which we call the **separator lemma**, is trivial, but easy to state incorrectly.

The point of having two subtypes of type 2 is that it is useful to have the stopping time τ as large as possible. When we apply the separator lemma, we let ξ^{2^*} be the set of type-2 particles we know about, thus not allowing others to trigger τ.

4 Proof of key proposition

This section is devoted to the proof of Proposition 2.2. The geometrical picture to have in mind, in the setup of the proposition, is that with high probability, the particles of type 2 outside the bt-ball grow into ever-widening spherical shells, until eventually they completely fill a sphere surrounding all the weaker particles. The following definitions make precise the shape of this growth, which is roughly a d-dimensional Archimedian spiral. Recall that $|\cdot|$ is the norm defined by the shape B_0 in Theorem 2.1.

Definition 4.1 *Given $\varepsilon > 0$ and $x \in \mathbf{R}^d$ with $|x| = 1$, define sets $C_n = C_n(x, \varepsilon)$ recursively by $C_0 = \{x\}$ and*

$$C_{n+1} = \{y : |y| = 1 \text{ and } |y - z| \le \varepsilon \text{ for some } z \in C_n\}.$$

Given additionally $b > 1$, $\delta, t > 0$ and $y \in \mathbf{Z}^d$ with $y/|y| = x$ and $|y| = bt$, define sets $A_n = A_n(b, \varepsilon, \delta, t, y)$ by letting $A_0 = \{y\}$ and for $n \ge 1$,

$$A_n = \{z \in \mathbf{Z}^d : b(1+\delta)^{n-1} t \le |z| \le b(1+\delta)^n t \text{ and } z/|z| \in C_n\}.$$

Record for later use the following geometrical fact:

Lemma 4.2 *For fixed b, δ, y and any $\delta' > \delta$, an $\varepsilon > 0$ may be chosen sufficiently small so that each point of $A_n(b, \varepsilon, \delta, t, y)$ is within*

$$(b\delta')t(1+\delta)^{n-1}$$

of some point in $A_{n-1}(b, \varepsilon, \delta, t, y)$, for all n and all sufficiently large t.

Proof: Let $\tilde{A}_n$ be the corresponding set in $\mathbf{R}^d$, i.e.,

$$\tilde{A}_n = \{z \in \mathbf{R}^d : bt(1+\delta)^{n-1} \leq |z| \leq bt(1+\delta)^n \text{ and } z/|z| \in C_n\}.$$

As $\varepsilon \to 0$, $d(C_n, C_{n-1}) \to 0$ in the Hausdorff metric, uniformly in y and n. Rescaling by $(1+\delta)^{n-1}$, we then see that the lemma is true with $\tilde{A}_n$ in place of A_n, using

$$d(\tilde{A}_n, \tilde{A}_{n-1}) \leq bt(1+\delta)^{n-1} d(C_n, C_{n-1}) + bt[(1+\delta)^n - (1+\delta)^{n-1}].$$

Since $d(A_n, \tilde{A}_n)/(bt(1+\delta)^{n-1}) \to 0$ as $t \to \infty$, the lemma follows. □

Without loss of generality, it suffices to prove Proposition 2.2 in the case where $b < \lambda_2$ and ξ^2 contains some site x with $|x| = bt$, because if not, replace b by some $b' \in (1, \lambda_2)$, replace a by some $a' \in (\max\{1, ab'/b\}, b')$, and replace t by $t' = |x|/b'$, where $x \in \xi^2$ with $|x| \geq bt$; this turns $(\xi^1, \xi^2) \in S(at, bt)$ into $(\xi^1, \xi^2) \in S(a't', b't')$ with $|x| = b't'$ for some $x \in \xi^2$, reducing to the desired case. Clearly there is an $n(\varepsilon)$ such that C_n is the entire unit sphere, and hence $A_n(b, \varepsilon, \delta, t, y)$ disconnects the set $\{z : |z| \leq at(1+\delta)\}$ from infinity (for sufficiently large t). Thus, **Proposition 2.2 follows once the following result is established:**

Theorem 4.3 *For fixed* $1 < a < b < \lambda$ *there are* $\varepsilon, \delta > 0$ *such that*

$$\sup \mathbf{P}^{1,\lambda}_{\xi^1,\xi^2}(\exists k \leq n(\varepsilon) : A_k(b, \varepsilon, \delta, t, y) \not\subseteq \eta^2(t[(1+\delta)^k - 1]) \text{ or } |\eta^1(t[(1+\delta)^k - 1])| \geq at(1+\delta)^k)$$

converges to 0 as $t \to \infty$, *where the sup is over* $(\xi^1, \xi^2) \in S(ta, tb)$ *and* $x \in \xi^2$ *with* $|x| = bt$.

The one-sentence summary of the proof of Theorem 4.3 is as follows. The separator lemma keeps the strong infection starting in A_n growing unfettered up to time $t(1+\delta)^n$ unless either it or the weak cluster grows uncharacteristically fast; this forces A_{n+1} to be in the strong cluster at time $t(1+\delta)^n$ unless the strong cluster grows uncharacteristically slowly (here we start at time t in configuration (ξ^1, ξ^2)). Making this rigorous is simply a matter of stating and using a few lemmas bounding the probability of growth in the one-type model that is either uncharacteristically slow or uncharacteristically fast. These lemmas are elementary consequences of the shape theorem (Theorem 2.1); superior bounds may be possible to derive e.g. from the work of Alexander [1], Kesten [10] and Talagrand [15]. Our bounds are more powerful in one direction (ruling out slow growth) than the other because of the superadditivity of the process in time.

Lemma 4.4 *Given* $a > 1$ *and* $r > 0$, *we have*

$$\lim_{t\to\infty} \sup_{\xi : |\xi| \leq at} \mathbf{P}^1_\xi(\exists s \geq rt : |\eta(s)| \geq a(s+t)) = 0\,.$$

To prove this, we need to recall the well known *edge representation* (i.e. the first-passage percolation formulation) of the one-type process. We equip the edge set E of the $\mathbf{Z}^d$ lattice with i.i.d. random variables $\{T_e\}_{e\in E}$ that are exponentially distributed with mean one. For $x, y \in \mathbf{Z}^d$, we set $T(x, y)$ to be the infimum over all paths from x to y of the sum of the T_e's of the edges along the path. We use the convention that $T(x, x) = 0$ for all x. For fixed $\xi \in \mathbf{Z}^d$ we can define a $\{0,1\}^{\mathbf{Z}^d}$-valued process $\{\Xi_t\}_{t\geq 0}$ by for each $x \in \mathbf{Z}^d$ taking

$$\Xi_t(x) = \begin{cases} 0 & \text{for } t < \inf_{y\in\xi} T(y, x) \\ 1 & \text{for } t \geq \inf_{y\in\xi} T(y, x)\,, \end{cases}$$

so that in particular $\Xi_t(x) = 1$ for all t whenever $x \in \xi$. It is easy to see that the process $\{\Xi_t\}_{t\geq 0}$ defined in this way has precisely distribution $\mathbf{P}^1_\xi$. We now think of the one-type process as being generated by this edge representation, and take the liberty to write $\mathbf{P}^1_\xi(A)$ also for events defined in terms of the edge representation.

Fix $\varepsilon > 0$. For $s \geq rt$ and a starting configuration ξ with $|\xi| \leq at$, the event $\{|\eta(s)| \geq a(s+t)\}$ implies (for sufficiently large t), the event $A(a,r,t,\varepsilon)$ defined to be the existence of x, y and $s \geq rt$ with the following properties:

(i) $at \leq |x| \leq (1+\varepsilon)at$;

(ii) $(1-\varepsilon)a(s+t) \leq |y| \leq a(s+t)$;

(iii) $T(x,y) \leq s$.

Clearly,

$$\mathbf{P}^1_\xi(\exists s \geq rt : |\eta(s)| \geq a(s+t)) \leq \mathbf{P}^1_\xi(A(a,r,t,\varepsilon)),$$

so Lemma 4.4 follows from

Proposition 4.5 *Given $a > 1$, $r > 0$ and small enough $\varepsilon > 0$,*

$$\lim_{t\to\infty} \mathbf{P}(A(a,r,t,\varepsilon)) = 0\,.$$

Proof: Assume for contradiction that this probability does not go to zero. Then there is a sequence of times $t_n \to \infty$ for which the probabilities are at least δ, and hence the probability that $A(a,r,t_n,\varepsilon)$ happens for infinitely many t_n is at least δ. Choose ε small enough so that

$$1 < \frac{(1+2\varepsilon)a - 1}{(1-2\varepsilon)a - 1} < 1 + r.$$

The event $A(a,r,t,\varepsilon)$ implies the disjunction

$$\{|\eta((1+2\varepsilon)at)|_* \leq (1+\varepsilon)at\} \cup \{|\eta((1+2\varepsilon)at + s)| \geq (1-\varepsilon)a(s+t)\},$$

which under the assumption on ε implies

$$\{|\eta((1+2\varepsilon)at)|_* \leq (1+\varepsilon)at\} \cup \{|\eta((1-2\varepsilon)a(s+t))| \geq (1-\varepsilon)a(s+t)\}.$$

If these occur infinitely often, then the shape theorem (Theorem 2.1) is violated, finishing the proof by contradiction. □

Along the same lines we can prove a lemma for how fast the shape can creep in from beyond a sphere:

Lemma 4.6 *Let $\Delta, \varepsilon > 0$ and let $\{a_n\}$ and $\{b_n\}$ be increasing sequences with $b_n/a_n \geq 1 + \Delta$ and $b_n/b_{n-1}, a_n/a_{n-1} \geq 1 + \varepsilon$. Let $U_n = \{x : |x| \geq b_n\}$. Then for any $\Delta' < \Delta$,*

$$\sum_{n=1}^{\infty} \mathbf{P}^1_{U_n}(\exists x \in \eta(\Delta' a_n) : |x| \leq a_n) < \infty. \tag{6}$$

In fact the sum goes to zero as $a_1 \to \infty$.

Proof: For $a \in \mathbf{R}$, let $\lceil a \rceil$ denote $\min\{z \in \mathbf{Z} : z \geq a\}$. Replacing b_n by $\lceil (1+\Delta)a_n \rceil$ only increases the sum in (6), so we assume without loss of generality that $b_n = \lceil (1+\Delta)a_n \rceil$. With $M = \lceil \log(1+\Delta)/\log(1+\varepsilon) \rceil + 1$ we then have $a_{n+M} > b_n + 1$ for sufficiently large n. For such n, use the edge representation again, and let $\mathcal{F}_n$ be the σ-field of times on those edges with an endpoint x with $a_n \leq |x| \leq b_n$. For x, y in this set of vertices or on its boundary, write $T_n(x, y)$ for the infimum, over paths from x to y whose times are $\mathcal{F}_n$-measurable, of the sum of the times; thus $T_n(x, y) \in \mathcal{F}_n$. Note that $\{\mathcal{F}_{n_k}\}$ are independent if $\{n_k\}$ is a set with $n_{k+1} - n_k \geq M$. Write G_n for the event of existence of x, y with $|x| \leq a_n$, $|y| \geq b_n$ and $T_n(x, y) \leq \Delta' a_n$.

Observe that the event in the summand of (6) implies the existence of x_n, y_n for which $|x_n| \leq a_n, |y_n| \geq b_n \geq a_n + \Delta a_n$ and $T(x_n, y_n) \leq \Delta' a_n$. The path from x_n to y_n minimizing the sum of the times will have a last hit on $\partial\{x : |x| \geq a_n\}$ and a first entrance into $\{x : |x| \geq b_n\}$; the path between (and including) these steps witnesses G_n.

Assume now for contradiction that the sum in (6) is at least δ for arbitrarily large values of a_1. Then the sum restricted to n in some residue class $\{k + jM\}$ modulo M is at least δ/M for arbitrarily large a_1. Hence

$$\mathbf{E} \sum_{j=0}^{\infty} \mathbf{1}_{G_{k+jM}} \geq \frac{\delta}{M}$$

for some k and arbitrarily large a_1. Since the events G_{k+jM} are independent, this means that

$$\mathbf{P}\left(\bigcup_{j=0}^{\infty} G_{k+jM} \right) \geq 1 - e^{-\delta/M}.$$

If this holds for arbitrarily large a_1, then the probability is at least $1 - e^{-\delta/M}$ that infinitely many pairs (x_n, y_n) with $|x_n| \leq a_n$ and $|y_n| \geq (1+\Delta)a_n$ will satisfy $T(x_n, y_n) \leq \Delta' a_n$. This contradicts the shape theorem, since for each such pair, either the time to x_n is too long or the time to y_n is too short. $\square$

The next lemma is a simple large deviations result, bounding the probability of uncharacteristically slow growth.

Lemma 4.7 *For any $\beta < 1$ there are positive constants C_β and γ_β such that*

$$\mathbf{P}_{\mathbf{0}}^1(|\eta(t)|_* < \beta t) \leq C_\beta \exp(-\gamma_\beta t)$$

for all t.

Proof: It is known (see e.g. Theorem 2.1 of [1]) that $\mathbf{E}T(0, x)/|x| \to 1$ as $|x| \to \infty$, where $\mathbf{E}$ is expectation with respect to $\mathbf{P}_{\mathbf{0}}^1$. Choose M large enough so that $|x| > M$ implies

$$\mathbf{E}T(0, x) < \beta^{-1/3}|x|.$$

We may choose a finite collection A of vectors of norm at least M such that any vector $y \in \mathbf{Z}^d$ with $|y|$ sufficiently large may be written as $x_1 + \cdots + x_n$ with each $x_j \in A$ and $\sum_j |x_j| < \beta^{-1/3}|y|$. Observe that $T(0, y)$ is bounded above by the sum of independent copies of $T(0, x_j)$. We now make the following

Claim: For any $\beta < 1$, there exists a constant $s \in (0,1)$ such that for any n and any multiset $\{x_1, \dots, x_n\}$ of elements of A,

$$\mathbf{P}(S \geq \beta^{-1/3} \sum_{j=1}^{n} \mathbf{E}T(0, x_j)) \leq s^n$$

where S is the sum of independent random variables distributed as $T(0, x_j)$ for $j = 1, \dots, n$.

We prove the claim by establishing that there are $\lambda > 0$ and $s \in (0,1)$ for which

$$\sup_{x \in A} \exp\{-\lambda \beta^{-1/3} \mathbf{E}T(0,x)\} \mathbf{E}e^{\lambda T(0,x)} \leq s. \tag{7}$$

The first step in this is the observation that for each $x \in A$, the variable $T(0,x)$ has exponential tails, so that $\mathbf{E}e^{\lambda T(0,x)}$ is finite for λ in a neighborhood of 0 and we can differentiate under the integral at $\lambda = 0$ to get

$$\begin{aligned} \frac{d}{d\lambda}\Big|_{\lambda=0} \log[\mathbf{E}e^{\lambda T(0,x)}] - \lambda \beta^{-1/3} \mathbf{E}T(0,x) &= \left(\frac{\mathbf{E}[T(0,x) e^{\lambda T(0,x)}]}{\mathbf{E}e^{\lambda T(0,x)}} - \beta^{-1/3} \mathbf{E}T(0,x) \right)_{\lambda=0} \\ &= (1 - \beta^{-1/3}) \mathbf{E}T(0,x) \\ &< 0. \end{aligned}$$

When $\lambda = 0$ of course $\log[\mathbf{E}e^{\lambda T(0,x)}] - \lambda \beta^{-1/3} \mathbf{E}T(0,x) = 0$, so we see that this expression is strictly negative in some interval $(0, a(x))$. Choosing any positive $\lambda < \min_{x \in A} a(x)$ proves (7).

Now the claim follows from Markov's inequality:

$$\begin{aligned} \mathbf{P}(S \geq \beta^{-1/3} \sum_{j=1}^{n} \mathbf{E}T(0, x_j)) &\leq \exp\left\{ -\lambda \beta^{-1/3} \sum_{j=1}^{n} \mathbf{E}T(0, x_j) \right\} \mathbf{E}e^{\lambda S} \\ &= \prod_{j=1}^{n} \exp\{-\lambda \beta^{-1/3} \mathbf{E}T(0, x_j)\} \mathbf{E}e^{\lambda T(0, x_j)} \\ &\leq s^n. \end{aligned}$$

Furthermore, for any y with $|y|$ sufficiently large there is a representation as $y = \sum_{j=1}^{n} y_j$ with $y_j \in A$ and

$$\beta^{-1}|y| \geq \beta^{-2/3} \sum_{j=1}^{n} |y_j| \geq \beta^{-1/3} \sum_{j=1}^{n} \mathbf{E}T(0, y_j).$$

Thus

$$\mathbf{P}_{\mathbf{0}}^{1}(T(0,y) \geq \beta^{-1}|y|) \leq \mathbf{P}_{\mathbf{0}}^{1}(S \geq \beta^{-1/3} \sum_{j=1}^{n} \mathbf{E}T(0, y_j)).$$

By applying the claim, we see that this is at most s^n with $n \geq |y| / \max_{x \in A} |x|$. The event $\{|\eta(t)|_* \leq \beta t\}$ is contained in the event $\bigcup_{y \in W} \{T(0,y) \geq t\}$, where W is the set of points within unit distance from the boundary of the βt-ball. Hence

$$\mathbf{P}_{\mathbf{0}}^{1}(|\eta(t)|_* < \beta t) \leq \sum_{y \in W} C \exp(-\gamma |y| / \max_{z \in A} |z|)$$

for t sufficiently large. Picking any $\gamma_\beta < \beta\gamma/\max_{z\in A}|z|$, using the subexponential growth of W, and adjusting C_β to account for small values of t finishes the proof. $\square$

Equipped with these lemmas, we are now ready for the proof of Theorem 4.3.

Proof of Theorem 4.3: Begin with δ and $\varepsilon > 0$ fixed but arbitrary. The first step is to take care of the possibility that the weak cluster grows fast enough to impede the strong cluster. Let E be the event that $|\eta(s+t)| \geq a(s+t)$ for some $s \geq \delta t$. E is thus defined for the one-type process, but in (8) below we let E denote the corresponding event for the type-1 infection in the two-type process. Lemma 4.4 with $r = \delta$ (together with the strong Markov property) implies that $\mathbf{P}^1_{\mathbf{0}}(E \,|\, \eta(t) = \xi^1) \to 0$ as $t \to \infty$, when $|\xi^1| \leq at$. Using the fact that from any starting configuration η^1 is dominated by the single-type model, and shifting time t to 0, we then have

$$\begin{aligned}&\sup \mathbf{P}^{1,\lambda}_{\xi^1,\xi^2}\left(\exists k : |\eta^1(t[(1+\delta)^k - 1])| \geq at(1+\delta)^k\right)\\ \leq\ &\sup_{\xi^1 : |\xi^1| \leq at} \mathbf{P}^1_{\mathbf{0}}\left(\exists s \geq \delta t : |\eta(s+t)| \geq a(s+t) \,|\, \eta(t) = \xi^1\right)\end{aligned}$$

which tends to 0 as $t \to \infty$; the supremum on the left hand side is over $(\xi^1, \xi^2) \in S(at, bt)$ with some $x \in \xi^2$ such that $|x| = bt$. Thus it suffices to prove that

$$\sup \mathbf{P}^{1,\lambda}_{\xi^1,\xi^2}\left(E^c \cap \left\{\exists k \leq n(\varepsilon) : A_k \not\subseteq \eta^2(t[(1+\delta)^k - 1])\right\}\right) \to 0 \tag{8}$$

as $t \to \infty$.

The second step is to take care of the possibility that the growth of the strong cluster is due to some growth fast enough to violate the hypotheses of the separator lemma. We first rewrite (8) as the sum from $k = 1$ to $n(\varepsilon)$ of the probability that k is the first such k, and bound this by the sum

$$\sum_{k=1}^{n(\varepsilon)} \sup \mathbf{P}^{1,\lambda}_{\xi^1,\xi^2}\left(E^c \cap \left\{A_k \not\subseteq \eta^2(t\delta(1+\delta)^{k-1})\right\}\right) \tag{9}$$

where the supremum is restricted to ξ^2 containing A_{k-1} (we have used the Markov property to shift time by $t[(1+\delta)^{k-1} - 1]$).

Now take $k \geq 3$; the cases $k = 1, 2$ can be handled similarly. We wish to apply the separator lemma with $\xi^{2^*} = A_{k-1}$ and R_k being the complement of the ball of radius $at(1+\delta)^{k-2}$. Thus we let τ_k be the first time that a type-1 particle reaches ∂R_k or a type-2* particle reaches $\partial(R_k^c)$. By the separator lemma,

$$\begin{aligned}&\mathbf{P}^{1,\lambda}_{\xi^1,\xi^2}\left(E^c \cap \left\{A_k \not\subseteq \eta^2(t\delta(1+\delta)^{k-1})\right\}\right)\\ \leq\ &\mathbf{P}^{1,\lambda}_{\xi^1,\xi^2}\left(E^c \cap \left\{\tau_k \leq t\delta(1+\delta)^{k-1}\right\}\right) + \mathbf{P}^{\lambda}_{A_{k-1}}\left(A_k \not\subseteq \eta(t\delta(1+\delta)^{k-1})\right).\end{aligned} \tag{10}$$

On E^c, the only way for the event $\{\tau_k \leq t\delta(1+\delta)^{k-1}\}$ to occur is to have a type-2* particle reach $\partial(R_k^c)$. The first of the two terms in (10) is bounded as follows. Recalling that δ was arbitrary, we now choose δ small enough so that

$$\lambda\delta(1+\delta) < b - (1+\delta)^2 a.$$

Let $b_n = bt(1+\delta)^{n-1}$ and $a_n = at(1+\delta)^{n+1}$, and define $\Delta = b_n/a_n - 1 = (b - a(1+\delta)^2)/(a(1+\delta)^2)$ and $\Delta' = \lambda\delta/(a(1+\delta))$. Set $U_n = \{z : |z| \geq b_n\}$ so that $U_n \supseteq A_n$. By the choice of δ, we have $\Delta' < \Delta$. Lemma 4.6 says that the sum

$$\sum_{k=1}^{n(\varepsilon)} \mathbf{P}^1_{U_{k-1}}(\exists x \in \eta(\Delta' a_{k-1}) : |x| \leq a_{k-1})$$

goes to zero as $t \to \infty$. Unraveling definitions,

$$\sum_{k=1}^{n(\varepsilon)} \mathbf{P}^1_{A_{k-1}}(\exists x \in \eta(\lambda\delta t(1+\delta)^{k-1}) : |x| \leq at(1+\delta)^k) \to 0$$

as $t \to \infty$, and hence

$$\sum_{k=1}^{n(\varepsilon)} \mathbf{P}^{1,\lambda}_{\xi^1,\xi^2}\left(E^c \cap \{\tau_k \leq \delta t(1+\delta)^{k-1}\}\right) \to 0$$

as $t \to \infty$.

Having successfully bounded the first term in (10), it remains to deal with the second. We may change the infection rate to 1 and allow greater time by a factor λ, so it suffices to show that

$$\sum_{k=1}^{n(\varepsilon)} \mathbf{P}^1_{A_{k-1}}(A_k \not\subseteq \eta(\lambda t\delta(1+\delta)^{k-1})) \to 0 \qquad (11)$$

as $t \to \infty$. So far ε has been arbitrary, but we now choose $\delta' \in (\delta, \lambda\delta/b)$ (recalling that $\lambda > b$) and use Lemma 4.2 to pick ε such that for sufficiently large t, every point of A_k is within $b\delta' t(1+\delta)^{k-1}$ of some point in A_{k-1}. The event $\{A_k \not\subseteq \eta(\lambda t\delta(1+\delta)^{k-1})\}$ is therefore contained in the event

$$\left\{\bigcup_{z\in A_{k-1}} B(z, b\delta' t(1+\delta)^{k-1}) \not\subseteq \eta(\lambda t\delta(1+\delta)^{k-1})\right\},$$

where $B(z,r)$ denotes the ball of radius r centered at z. Using the edge representation of the one-type process, it is easy to see (and a standard fact) that we can view the cluster of infected sites under $\mathbf{P}^1_{A_{k-1}}$ as the union of clusters with law $\mathbf{P}^1_z$ for $z \in A_{k-1}$. The sum (11) is therefore bounded by

$$\sum_{k=1}^{n(\varepsilon)} \sum_{z\in A_{k-1}} \mathbf{P}^1_z(B(z, b\delta' t(1+\delta)^{k-1}) \not\subseteq \eta(\lambda t\delta(1+\delta)^{k-1})).$$

Translate z to the origin in each summand, we rewrite this as

$$\sum_{k=1}^{n(\varepsilon)} \sum_{z\in A_{k-1}} \mathbf{P}^1_{\mathbf{0}}\left(|\eta(\lambda t\delta(1+\delta)^{k-1})|_* \leq bt\delta'(1+\delta)^{k-1}\right).$$

Now set $\beta = b\delta'/(\lambda\delta)$ and apply Lemma 4.7 to achieve an upper bound of

$$\sum_{k=1}^{\infty} |A_{k-1}| C_\beta \exp(-\gamma_\beta t\lambda\delta(1+\delta)^{k-1})$$

for the sum in (11). The bound $|A_k| \leq C((1+\delta)^k t)^d$ shows that the supremum over $t \geq 1$ of $|A_{k-1}| C_\beta \exp(-\gamma_\beta t\lambda\delta(1+\delta)^{k-1})$ is summable in k, and also that for fixed k the summand goes to zero as $t \to \infty$. Then by dominated convergence, the sum goes to zero as $t \to \infty$, completing the proof. ☐

5 Proof of main result

In this section we finally prove Theorem 1.2. Besides the key proposition (Proposition 2.2), the other main ingredient is the following coupling of the two-type processes generated by $\mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}$ for all $\lambda \in [0,1]$ simultaneously.

To each ordered pair (x, y) of nearest neighbors in $\mathbf{Z}^d$, we assign an independent unit rate Poisson process. Also let

$$\{U_{x,y,i}\}_{x,y\in\mathbf{Z}^d, i\in\{1,2,\ldots\}}$$

be an array of i.i.d. random variables (independent also of the Poisson processes), uniformly distributed on $[0,1]$. For each $\lambda \in [0,1]$, a two-type process $\{\eta^1_\lambda(t), \eta^2_\lambda(t)\}_{t\geq 0}$ is defined by taking $(\eta^1_\lambda(0), \eta^2_\lambda(0)) = (\mathbf{0}, \mathbf{1})$, and infections as follows. For each ordered nearest neighbor pair (x, y) and each i, we check whether at the ith occurence of the Poisson process assigned to (x, y) it is the case that x is infected (i.e. has value 1 or 2) while y is not (i.e. has value 0). If that is the case, then y flips to a 1 if x is a 1 and $U_{x,y,i} \leq \lambda$, and y flips to a 2 if x is a 2. It is easy to check that for each $\lambda \in [0,1]$ the process $\{\eta^1_\lambda(t), \eta^2_\lambda(t)\}_{t\geq 0}$ has distribution $\mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}$. We write $\mathbf{Q}$ for the probability measure underlying this coupling. For $j = 1, 2$, we write G^j_λ for the event of unbounded growth for type j in the $\{\eta^1_\lambda(t), \eta^2_\lambda(t)\}_{t\geq 0}$ process. This means e.g. that

$$\mathbf{Q}(G^i_\lambda) = \mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}(G^i)\,.$$

Lemma 5.1 *For $\lambda_1 \leq \lambda_2 \in [0,1]$, we have $\mathbf{Q}$-a.s. for all t,*

$$\eta^1_{\lambda_1}(t) \subseteq \eta^1_{\lambda_2}(t) \tag{12}$$

and

$$\eta^2_{\lambda_1}(t) \supseteq \eta^2_{\lambda_2}(t)\,. \tag{13}$$

Proof: The case $t = 0$ is immediate, and it is also clear from the construction that (12) and (13) are preserved in time. $\square$

The key proposition (Proposition 2.2) comes into play in the following lemma, saying that the asymptotic shape result holds on the event of mutual unbounded growth.

Lemma 5.2 *For $\lambda \in [0,1]$, we have $\mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}$-a.s. on the event $(G^1 \cap G^2)$ that*

$$\frac{|\eta^1(t) \cup \eta^2(t)|}{t} \to \lambda$$

and

$$\frac{|\eta^1(t) \cup \eta^2(t)|_*}{t} \to \lambda$$

as $t \to \infty$.

Proof: It suffices to show that

$$\limsup_{t\to\infty} \frac{|\eta^1(t) \cup \eta^2(t)|}{t} \leq \lambda \tag{14}$$

and

$$\liminf_{t\to\infty} \frac{|\eta^1(t) \cup \eta^2(t)|_*}{t} \geq \lambda \tag{15}$$

$\mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}$-a.s. on the event $(G^1 \cap G^2)$. To see that (14) holds, note first that

$$\limsup_{t\to\infty} \frac{|\eta^1(t)|}{t} \leq \lambda \tag{16}$$

by Lemma 3.1 and Theorem 2.1. Secondly, $\limsup_{t\to\infty} \frac{|\eta^2(t)|}{t} \leq \lambda$, because if not, then there would be an $\varepsilon > 0$ such that

$$\frac{|\eta^2(t)|}{t} \geq (1+\varepsilon)\lambda \quad \text{infinitely often}$$

which in combination with (16) prevents the event G^1 due to Proposition 2.2 (where, by time scaling, $\mathbf{P}^{1,\lambda_2}_{\xi^1,\xi^2}$ can be replaced by $\mathbf{P}^{1/\lambda_2,1}_{\xi^1,\xi^2}$) and the strong Markov property. Hence, (14) is established.

On the other hand, (15) follows by invoking Lemma 3.2 and Theorem 2.1. □

Lemma 5.3 *For any $\lambda_1 < \lambda_2 \in [0,1]$, we have*

$$\mathbf{Q}(G^1_{\lambda_1} \cap G^2_{\lambda_2}) = 0\,.$$

Proof: Suppose that $G^1_{\lambda_1}$ happens. We need to show that $G^2_{\lambda_2}$ does not happen. If $G^2_{\lambda_1}$ does not happen, then we are done by Lemma 5.1, so we can assume that $G^1_{\lambda_1} \cap G^2_{\lambda_1}$ happens. Then

$$\lim_{t\to\infty} \frac{|\eta^1_{\lambda_1}(t) \cup \eta^2_{\lambda_1}(t)|}{t} = \lambda_1$$

by Lemma 5.2, so that in particular

$$\limsup_{t\to\infty} \frac{|\eta^2_{\lambda_1}(t)|}{t} \leq \lambda_1\,.$$

By another application of Lemma 5.1, we get

$$\limsup_{t\to\infty} \frac{|\eta^2_{\lambda_2}(t)|}{t} \leq \lambda_1\,. \tag{17}$$

On the other hand, Lemma 3.2 and Theorem 2.1 imply that

$$\liminf_{t\to\infty} \frac{|\eta^1_{\lambda_2}(t) \cup \eta^2_{\lambda_2}(t)|_*}{t} \geq \lambda_2\,. \tag{18}$$

But the events in (17) and (18) together imply that the $\eta^2_{\lambda_2}$ infection eventually becomes surrounded by the $\eta^1_{\lambda_2}$ infection, preventing the event $G^2_{\lambda_2}$. □

Lemma 5.4 *With $\mathbf{Q}$-probability 1, the event $(G^1_\lambda \cap G^2_\lambda)$ happens for at most one $\lambda \in [0,1]$.*

Proof: By Lemma 5.1, the set of λ for which G^1_λ happens is increasing (i.e. if $\lambda_1 < \lambda_2$ and $G^1_{\lambda_1}$ occurs, then also $G^1_{\lambda_2}$ occurs), and the set of λ for which G^2_λ happens is decreasing, $\mathbf{Q}$-a.s. Hence the set $L := \{\lambda \in [0,1] : G^1_\lambda \cap G^2_\lambda\}$ is $\mathbf{Q}$-a.s. an interval. We need to show that L consists $\mathbf{Q}$-a.s. of at most one point. If, with positive $\mathbf{Q}$-probability, L were a nondegenerate interval, then there would exist $\lambda_1 < \lambda_2$ in $[0,1]$

such that the event $(G^1_{\lambda_1} \cap G^2_{\lambda_1} \cap G^1_{\lambda_2} \cap G^2_{\lambda_2})$ has positive $\mathbf{Q}$-probability. This, however, would contradict Lemma 5.3. □

Proof of Theorem 1.2: By (2), it is enough to prove that $\mathbf{P}^{1,\lambda_2}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2) > 0$ for at most countably many $\lambda_2 \geq 1$. By time scaling, this is the same as saying that $\mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2) > 0$ for at most countably many $\lambda \in [0, 1]$. Lemma 5.4 tells us that $\mathbf{Q}(G^1_\lambda \cap G^2_\lambda) > 0$ for at most countably many $\lambda \in [0, 1]$, and since

$$\mathbf{P}^{\lambda,1}_{\mathbf{0},\mathbf{1}}(G^1 \cap G^2) = \mathbf{Q}(G^1_\lambda \cap G^2_\lambda)$$

we are done. □

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