Percolation on Transitive Graphs as a Coalescent Process: Relentless Merging Followed by Simultaneous Uniqueness

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Dedicated to Harry Kesten

Abstract

Consider i.i.d. percolation with retention parameter p on an infinite graph G. There is a well known critical parameter $p_c \in [0,1]$ for the existence of infinite open clusters. Recently, it has been shown that when G is quasi-transitive, there is another critical value $p_u \in [p_c, 1]$ such that the number of infinite clusters is a.s. ∞ for $p \in (p_c, p_u)$, and there is a unique infinite cluster for $p > p_u$. We prove a simultaneous version of this result in the canonical coupling of the percolation processes for all $p \in [0,1]$; in particular, a.s. *simultaneous uniqueness* holds for all $p > p_u$. Simultaneously for all $p \in (p_c, p_u)$, we also prove that each infinite cluster has uncountably many ends. For $p > p_c$ we prove that all infinite clusters are *indistinguishable by robust properties.* Under the additional assumption that G is unimodular, we prove that a.s. for all $p_1 < p_2$ in (p_c, p_u), every infinite cluster at level p_2 contains *infinitely many* infinite clusters at level p_1. We also show that any product G of d infinite connected graphs of bounded degree satisfies $p_u(G) \le p_c(\mathbf{Z}^d)$.

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1 Introduction

We consider i.i.d. bond percolation with retention parameter $p \in [0,1]$ on an infinite locally finite connected graph $G = (V, E)$. This means that each edge is independently assigned the value 1 (open) with probability p, and the value 0 (closed) with probability $1-p$. We write $\mathbf{P}_p^G$ for the resulting probability measure on $\{0,1\}^E$. All our results and proofs may be adapted to site percolation as well.

Percolation theory deals with the structure of the connected components of open edges, especially infinite connected components (clusters). By Kolmogorov's zero-one law, the existence of *at least one* infinite cluster has probability 0 or 1, and one defines

$$p_c(G) = \inf\{p \in [0,1] : \mathbf{P}_p^G(\exists \text{ an infinite cluster}) = 1\}.$$

Following Benjamini and Schramm [9], we also define

$$p_u(G) = \inf\{p \in [0,1] : \mathbf{P}_p^G(\exists \text{ a unique infinite cluster}) = 1\}.$$

For $G = \mathbf{Z}^d$, Aizenman, Kesten and Newman [1] showed that whenever an infinite cluster exists, it is a.s. unique, so that $p_c = p_u$; subsequently, shorter proofs were given in [14] and [11]. (With the usual abuse of notation, we write $\mathbf{Z}^d$ for the graph whose vertex set is $\mathbf{Z}^d$ and whose edge set consists of the pairs of Euclidean nearest neighbors.) For general graphs uniqueness no longer holds, but for a large class of graphs, including the *quasi-transitive* ones, (see Definition 1.1 below), the number of infinite clusters is an a.s. constant (depending on p) which may be either 0, 1 or ∞. As noted independently by several authors, this follows from the arguments of Newman and Schulman [27].

The pioneering paper of Grimmett and Newman [16] revealed that surprising new phenomena appear when one goes beyond lattices in Euclidean space, and Benjamini and Schramm [9] indicated the right level of generality to study these phenomena. Nevertheless, as we shall see in Section 8, certain deep results for percolation in $\mathbf{Z}^d$ (uniqueness in orthants, and estimates of p_c) have significant implications beyond the Euclidean setting.

Write $\mathrm{Aut}(G)$ for the group of graph automorphisms of the graph G.

Definition 1.1 *A graph $G = (V, E)$ is called* **transitive** *if for any $x, y \in V$ there exists a $\gamma \in Aut(G)$ which maps x to y. The graph G is called* **quasi-transitive** *if V can be partitioned into finitely many sets (orbits) $V_1, \dots, V_k$, so that for $x \in V_i$ and $y \in V_j$, there exists $\gamma \in Aut(G)$ mapping x to y iff $i = j$.*

Clearly, a transitive graph is quasi-transitive.

Benjamini and Schramm [9] conjectured that for quasi-transitive graphs, a.s. uniqueness of the infinite cluster holds for all $p > p_u$. This was proved for Cayley graphs (and, more generally, for quasi-transitive unimodular graphs; see Definition 6.1) by Häggström and Peres [18], and in full generality by Schonmann [29].

Theorem 1.2 ([18], [29]) *Consider bond percolation on a connected, infinite, locally finite, quasi-transitive graph G. Then $\mathbf{P}_p^G$-a.s., the number N of infinite clusters satisfies*

$$N = \begin{cases} 0 & \text{if} \quad p \in [0, p_c) \\ \infty & \text{if} \quad p \in (p_c, p_u) \\ 1 & \text{if} \quad p \in (p_u, 1] \ . \end{cases}$$

The parameter space $[0, 1]$ is thus split into three qualitatively different intervals, separated by the two critical values p_c and p_u. Some of the intervals may be degenerate or empty (e.g., for $\mathbf{Z}^d$ we have $p_c = p_u$, and for trees we have $p_u = 1$). Grimmett and Newman [16] presented the first example of a transitive graph where all three regimes are nondegenerate: the product of a regular tree and $\mathbf{Z}$. Other examples were given by Benjamini and Schramm [9] and Lalley [22].

There is a natural way to couple the percolation processes for all p simultaneously. Equip the edges of G with i.i.d. random variables $\{U(e)\}_{e \in E}$, uniform in $[0, 1]$, and write Ψ^G for the resulting product measure on $[0, 1]^E$. For each p, the edge set $\{e \in E : U(e) \leq p\}$ has the same distribution as the set of open edges under $\mathbf{P}_p^G$. This yields a *coalescent process* which has turned out to be a fruitful object of study in Erdős–Rényi random graph theory (see e.g. [19, 2]) and which has recently attracted more attention also in percolation theory (e.g. [10]): When $p = 0$ every vertex is its own connected component. As the parameter p is increased, more and more edges become open, causing connected components to coalesce, until finally, when $p = 1$ all edges are open.

By Theorem 1.2 and Fubini's Theorem, we have Ψ^G-a.s. that the number of infinite clusters is ∞ for (Lebesgue-)a.e. $p \in (p_c, p_u)$, and 1 for a.e. $p \in (p_u, 1]$. However, it is not obvious that the quantifier "a.e." can be strengthened to "every" in these statements. Alexander [3] demonstrated that this strengthening holds for $G = \mathbf{Z}^d$ and other Euclidean lattices, and Häggström and Peres [18] handled the case where G is quasi-transitive and unimodular. Here we prove the simultaneous version of Theorem 1.2 for all quasi-transitive graphs.

Theorem 1.3 *Let G be an infinite, locally finite, connected, quasi-transitive graph, and let p_c and p_u be as in Theorem 1.2. Consider the coupling Ψ^G of the percolation processes on G for all $p \in [0,1]$ simultaneously, and let $N(p)$ be the number of infinite clusters determined by the edge set $\{e \in E : U(e) \le p\}$. With Ψ^G-probability 1, we then have*

$$N(p) = \begin{cases} 0 & \textit{for all} \quad p \in [0, p_c) \\ \infty & \textit{for all} \quad p \in (p_c, p_u) \\ 1 & \textit{for all} \quad p \in (p_u, 1]\,. \end{cases}$$

This is an immediate consequence of the following result.

Theorem 1.4 *Let G be an infinite, locally finite, connected, quasi-transitive graph, With Ψ^G-probability 1, for all $p_1 < p_2$ in $(p_c, 1]$, every infinite p_2-cluster contains an infinite p_1-cluster.*

This sharpens a result of Schonmann [29], which gives the same assertion except that the order of the quantifiers "with Ψ^G-probability 1" and "for all $p_1 < p_2$" is interchanged.

Theorems 1.3 and 1.4 imply that as the parameter p increases, infinite clusters are "born" only at, or immediately after, level p_c. For larger p, infinite clusters grow and merge, but no new ones are formed from finite clusters. Our next result shows that infinite clusters "merge relentlessly" in the intermediate regime (p_c, p_u). We can only prove this result for quasi-transitive graphs under the additional assumption of unimodularity (see Definition 6.1), but we believe it holds for all quasi-transitive graphs.

Theorem 1.5 *Let G be an infinite, locally finite, connected, quasi-transitive unimodular graph, and let p_c and p_u be as in Theorem 1.2. Then, with Ψ^G-probability 1, for any $p_1 < p_2$ in (p_c, p_u), any infinite cluster at level p_2 contains infinitely many infinite clusters at level p_1.*

The next result also concerns the intermediate regime (p_c, p_u). Say that two infinite self-avoiding paths ξ_1 and ξ_2 in the same infinite cluster $\mathcal{C}$ are *equivalent* if for any finite set $\{e_1, \ldots, e_n\}$ of edges in $\mathcal{C}$, both paths are eventually in the same connected component of $\mathcal{C} \setminus \{e_1, \ldots, e_n\}$. Equivalence classes of self-avoiding paths in $\mathcal{C}$ are called *ends* of $\mathcal{C}$. The following theorem is proved in Section 4.

Theorem 1.6 *Let G be an infinite, locally finite, connected, quasi-transitive graph. Then, Ψ^G-a.s., for all $p \in (p_c, p_u)$ every infinite p-cluster has precisely $2^{\aleph_0}$ many ends.*

The proof extends to the case $p = p_u$ if there are multiple infinite clusters at that level; thus, this theorem confirms Conjecture 5 of Benjamini and Schramm [9] (except for the case $p = p_c$, if infinitely many infinite clusters can exist there). The fixed-p unimodular case was proved in the original version of [18].

When there are infinitely many infinite clusters, can they be qualitatively different? To make this question precise, we need some definitions.

Definition 1.7 *Let $G = (V, E)$ be a quasi-transitive graph. By a* subgraph *of G, we mean a collection of edges. A set Q of subgraphs of G is called a* **property** *if for every $p \in (0, 1)$ and every vertex x, the event that the open cluster of x at level p belongs to Q is $\mathbf{P}_p$-measurable.*

- *Q is an* **invariant** *property if for every $\gamma \in \mathrm{Aut}(G)$ and $E_0 \in Q$, necessarily $\gamma(E_0) \in Q$.*
- *Q is* **monotone** *if whenever $E_1 \in Q$ and $E_1 \subset E_2$, then also $E_2 \in Q$.*
- *Q is* **robust** *if for for every infinite connected subgraph $\mathcal{C}$ of G and every edge $e \in \mathcal{C}$, we have the equivalence: $\mathcal{C} \in Q$ iff there is an infinite connected component of $\mathcal{C} \setminus \{e\}$ that satisfies Q.*

Suppose that Q is a robust property and $\mathcal{C}$ is an infinite cluster satisfying Q. If an edge adjacent to $\mathcal{C}$ is opened, then the resulting cluster will satisfy Q, and if an edge in $\mathcal{C}$ is closed, then at least one of the resulting infinite clusters will satisfy Q.

Transience (for simple random walk) is a robust, monotone, invariant property of subgraphs that has been studied extensively. An invariant property of interest, that is robust but not monotone, is $\{\mathcal{C} : \exists \text{ infinitely many encounter points in } \mathcal{C}\}$ (see [11], [8], or the end of the present paper for the definition and significance of encounter points). A monotone invariant property for which robustness is not known is $\{\mathcal{C} : p_c(\mathcal{C} \times \mathbf{Z}) < p_0\}$, where $p_0 < 1$ is fixed.

Following a question of O. Schramm (personal communication), Häggström and Peres [18] showed that for G quasi-transitive and unimodular, if Q is any monotone invariant property, then $\mathbf{P}_p$-a.s, infinite clusters with and without Q cannot coexist, except possibly at one value of p. This result was substantially improved by Lyons and Schramm [24], who showed there is no exceptional p, and the monotonicity assumption on Q can be dropped. Thus on quasi-transitive unimodular graphs, [24] shows that infinite clusters are *indistinguishable by invariant properties.* as noted there, this strong result fails without unimodularity; see Section 6.

Nevertheless, on any quasi-transitive graph, infinite clusters cannot be distinguished by *robust* invariant properties. The following theorem is proved in Section 5.

Theorem 1.8 *Let G be an infinite, locally finite, connected, quasi-transitive graph, and let $p \in (p_c, p_u]$. If Q is a robust invariant property of subgraphs of G such that* $\mathbf{P}_p(\exists \text{ an infinite cluster satisfying } Q) > 0$*, then $\mathbf{P}_p$-a.s., all infinite clusters in G satisfy Q.*

Next, we present an upper bound on p_u for products of infinite graphs. For d graphs $\{G_i = (V_i, E_i)\}_{i=1}^d$, define the **product graph** $G = G_1 \times \cdots \times G_d$ as the graph with vertex set $V = V_1 \times \cdots \times V_d$, and edge set E consisting of pairs $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ such that x_i and y_i are neighbors in G_i for exactly one coordinate $i \in \{1, \ldots, d\}$, and $x_j = y_j$ for all other coordinates j. Clearly, a product of two or more quasi-transitive graphs is quasi-transitive. Some of the most natural examples (such as the Grimmett–Newman example) arise this way.

Theorem 1.9 *Let $G_1, \dots, G_d$ be infinite connected graphs with bounded degree, and let G be their product $G_1 \times \cdots \times G_d$. Then, for bond percolation on G with parameter $p > p_c(\mathbf{Z}^d)$, we have $\mathbf{P}_p^G$-a.s. that the number of infinite clusters is exactly 1. Moreover, in the coupling Ψ^G, uniqueness of the infinite cluster holds a.s. simultaneously for all $p > p_c(\mathbf{Z}^d)$.*

In particular, if $G_1, \dots, G_d$ are infinite connected graphs with bounded degree, then

$$p_u(G_1 \times \cdots \times G_d) \le p_c(\mathbf{Z}^d).$$

The rest of the paper is organized as follows. In the next section we state an extension of Theorem 1.4. We prove this extension in Section 3, by combining the approach of Schonmann [29] with invasion percolation ideas. Theorem 1.6 on ends is established by similar means in Section 4. We prove Theorem 1.8 (indistinguishability by robust properties) in Section 5. In Section 6, we define unimodularity and recall a technique known as the mass-transport method, which we then use in Section 7 to prove Theorem 1.5. In Section 8 we prove Theorem 1.9, building on classical results for percolation in $\mathbf{Z}^d$, and a result in [29]. Lower bounds on p_u are also discussed there. Section 9 contains examples, remarks, and unsolved problems.

2 Uniform percolation and semi-transitive graphs

In this section we will extend Theorem 1.4. To state this extension, we will need the notion of *uniform percolation* from [29]. The ball $B(x, R)$ of radius R centered at $x \in V$, is defined as the set of edges in G which have both endpoints within (graph-theoretic) distance R from x.

Definition 2.1 *A graph $G = (V, E)$ exhibits* **uniform percolation at level** *p if*

$$\lim_{R\to\infty} \inf_{x \in V} \mathbf{P}_p(\textit{some infinite } p\textit{-cluster intersects } B(x,R)) = 1\,. \tag{1}$$

It is easy to see that any quasi-transitive graph exhibits uniform percolation at all levels $p > p_c$. In fact, this holds in the larger class of *semi-transitive* graphs.

Definition 2.2 *A graph $G = (V, E)$ is called* **semi-transitive** *if there is a finite set $V_F \subset V$ such that for any vertex $x \in V$, there is a vertex $y \in V_F$ and an injective graph homomorphism of G that maps y to x.*

The simplest examples of semi-transitive graph that are not quasi-transitive are the nearest-neighbor graph on the positive integers $\mathbf{Z}_+$, and d-ary trees where the root has degree d and all other vertices have degree $d+1$. More generally, the "super-periodic" trees that discussed in Lyons and Peres [23] are semi-transitive; an example is the subtree of the binary tree consisting of all vertices such that the path from the root to v has at least as many left turns as right turns. These trees are closely related to the "super self-similar" sets studied by Falconer [13]. A class of graphs, mentioned in [29], which are semi-transitive but not quasi-transitive, are products $G \times \mathbf{Z}_+$, where G is quasi-transitive.

The next result extends Theorem 1.4.

Theorem 2.3 *Let G be an infinite connected graph with bounded degree, that exhibits uniform percolation at level p_*. With Ψ^G-probability 1, for all $p_1 < p_2$ in $(p_*, 1]$, every infinite p_2-cluster contains some infinite p_1-cluster. In particular, there is Ψ^G-a.s. a unique infinite cluster at level p for all $p > \max(p_u, p_*)$.*

This will be proved in the next section using invasion percolation. Here, we show how it implies a generalization of Theorem 1.4.

Proof of Theorem 1.4, generalized to semi-transitive G: Let $p > p_c(G)$. Since the existence of an infinite cluster has $\mathbf{P}_p$-probability 1, we have for each fixed $x \in V$ that

$$\lim_{R\to\infty} \mathbf{P}_p(\text{some infinite } p_1\text{-cluster intersects } B(x,R)) = 1\,.$$

The infimum in (1) is attained for some y in the finite set V_F specified in Definition 2.2, and it follows that (1) holds for any $p > p_c(G)$. Invoking Theorem 2.3 completes the proof. □

Theorem 1.3 may fail in the semi-transitive setting, because there exist semi-transitive graphs where with positive probability, the number of infinite clusters is finite but greater than one. (An example of this, due to O. Schramm, is described in the final section.) Nevertheless, Theorem 1.3 does extend to semi-transitive graphs G where $\mathrm{Aut}(G)$ has an infinite orbit (e.g. $G = G_1 \times G_2$ where G_1 is quasi transitive and G_2 is semi-transitive), since a standard argument shows that in such graphs G, for each parameter p the number of infinite clusters is $0, 1$ or ∞ a.s.

3 Invasion clusters hit infinite percolation clusters

A key idea in proving Theorem 2.3 is to use *invasion percolation*, which is a sequential construction based on the same uniform random variables $\{U(e)\}_{e \in E}$ as the canonical coupling of the ordinary percolation processes. Here we give only a brief description of invasion percolation; we refer to Chayes, Chayes and Newman [12] for a general introduction to the model, and to [25, 28] for some interesting recent applications in statistical mechanics.

The invasion cluster of a vertex $x \in V$ is built up sequentially by constructing an increasing sequence of edge sets $I_1^x \subset I_2^x \subset \cdots$ as follows. Let I_1^x consist of the single edge e which minimizes $U(e)$ among all edges incident to x. When I_i^x is constructed, I_{i+1}^x is taken to be $I_i^x \cup \{e\}$, where e is the edge which minimizes $U(e)$ among all edges e that are not in I_i^x but are adjacent to some edge in I_i^x. The invasion cluster of x is the edge set

$$I_\infty^x = \bigcup_{i=1}^{\infty} I_i^x .$$

Proposition 3.1 *Let $G = (V, E)$ be an infinite, connected graph with bounded degrees. If G exhibits uniform percolation at level p_*, then Ψ^G-a.s. for any $p > p_*$ and any $x \in V$, the invasion cluster I_∞^x intersects some infinite p-cluster.*

This proposition was proved by Chayes, Chayes and Newman [12] for $\mathbf{Z}^d$, by Alexander [4] for other Euclidean graphs, and by O. Schramm (personal communication) for transitive unimodular graphs. Before proving Proposition 3.1, we explain how it implies Theorem 2.3.

Proof of Theorem 2.3: For $p \in [0, 1]$ and a vertex x, let $\mathcal{C}(x, p)$ denote the cluster at level p containing x. Also let $\Omega_{x,p}$ denote the event that (i) all the edges in G are assigned distinct labels $U(e)$, and (ii) the invasion cluster I_∞^x hits some infinite p-cluster. Fix $p > p_*$ and an edge labeling $\{U(e)\}_{e\in E}$ in $\Omega_{x,p}$. For any parameter $p_2 > p$ such that the cluster $\mathcal{C}(x,p_2)$ is infinite, it must contain the invasion cluster I_∞^x, and hence $\mathcal{C}(x,p_2)$ must intersect some infinite p-cluster $\mathcal{C}(y,p)$. Obviously, $\mathcal{C}(x,p_2)$ then intersects some infinite p_1-cluster for any $p_1 \in [p,p_2)$. Proposition 3.1 ensures that $\Psi^G(\cap_{x,p}\Omega_{x,p}) = 1$, where the intersection ranges over all $x \in V$ and all rational $p > p_*$, and this proves the theorem. □

The proof of Proposition 3.1 is based on an adaptation to invasion percolation of the proof of the main result in [29]. The following lemma is needed.

Lemma 3.2 *Let $G = (V,E)$ be an infinite, connected graph with bounded degrees, and let $R > 0$ be an integer. With Ψ^G-probability 1, the invasion cluster I_∞^x contains a ball of radius R.*

Proof: Denote by D the maximum degree of vertices in G. The standard inequality

$$p_c(G) \geq \frac{1}{D-1} > 0 \tag{2}$$

(see, e.g., [15]), will be used at the end of the proof. Let $(v_1, v_2, \ldots)$ be an arbitrary enumeration of the vertex set V. For $n = 1, 2, \ldots$, set $L_n = n(2R+1)$, and define

$$\tau_n := \min\{k : I_k^x \text{ comes within distance } R \text{ from some } y \in V \setminus B(x, L_n)\}.$$

Since the invasion cluster I_∞^x is infinite, τ_n is a.s. finite for every n. For each n, define y_n to be the vertex in $V \setminus B(x, L_n)$ at minimal distance from $I_{\tau_n}^x$ (in case of a tie, y_n is the one with minimal index in the above enumeration). Finally, consider the events

$$A_n = \{U(e) < p_c \text{ for all } e \in B(y_n, R)\}.$$

Since there are a.s. no infinite p-clusters for $p < p_c$, on A_n the invasion cluster I_∞^x must contain the ball $B(y_n, R)$. Thus it suffices to prove that $\Psi^G\Big(\cap_{i=1}^\infty A_i^c\Big) = 0$. The conditional probability of A_n given $A_1^c, \ldots, A_{n-1}^c$ and the invasion process up to time τ_n, is at least $p_c^{D^{R+1}}$. Therefore

$$\Psi^G\Big(\cap_{i=1}^n A_i^c\Big) \leq \Big(1 - p_c^{D^{R+1}}\Big)^n,$$

and the right-hand side tends to 0 as $n \to \infty$ by (2). □

Proof of Proposition 3.1: Fix p_*, p and x as in the proposition. Define the random variable $\xi^x_{p_*}$ as the number of edges that have one endpoint in I^x_∞ and the other in some infinite p_*-cluster. Our proof consists of first showing that

$$\Psi^G(\xi^x_{p_*} = \infty) = 1 \tag{3}$$

and then showing that for $p > p_*$,

$$\Psi^G(I^x_\infty \text{ intersects some infinite } p\text{-cluster} \mid \xi^x_{p_*} = \infty) = 1\,. \tag{4}$$

Letting $p \downarrow p_*$ through a countable sequence then proves the proposition.

By the uniform percolation assumption, we can, for any $\varepsilon > 0$, pick an R so large that

$$\inf_{y \in V} \Psi^G(\text{some infinite } p_*\text{-cluster intersects } B(y,R)) \geq 1 - \varepsilon\,. \tag{5}$$

Let τ denote the smallest k for which I^x_k contains a ball of radius R; by Lemma 3.2, $\tau < \infty$ a.s.

For an edge set $E_0 \subset E$, set

$$V(E_0) = \{y \in V \,:\, y \text{ is an endpoint of some } e \in E_0\}\,.$$

If E_0 is finite and contains some ball of radius R, then by (5) we have with probability at least $1-\varepsilon$ that some vertex in $V(E_0)^c$ at distance 1 from $V(E_0)$ has an open path to infinity at level p_* via vertices in $V(E_0)^c$ only. Since the invasion cluster up to time τ gives no information about the set of edges not adjacent to I^x_τ, we may apply the above reasoning with $E_0 = I^x_\tau$ to deduce that the conditional probability that there is some infinite p_*-cluster within distance 1 from I^x_τ is at least $1 - \varepsilon$. This shows that $\Psi^G(\xi^x_{p_*} = 0) \leq \varepsilon$, and since ε was arbitrary we have

$$\Psi^G(\xi^x_{p_*} = 0) = 0\,. \tag{6}$$

The next step is to rule out the possibility of having $\xi^x_{p_*} = n$ for any finite n. Note that on the event $(\xi^x_{p_*} = n)$ we can move into the event $(\xi^x_{p_*} = 0)$ by changing the status of finitely many edges. It is easy to see that this implies that if $\Psi^G(\xi^x_{p_*} = n) > 0$, then $\Psi^G(\xi^x_{p_*} = 0) > 0$ holds as well. But this would contradict (6), so we have

$$\Psi^G(\xi^x_{p_*} = n) = 0 \tag{7}$$

for any $n < \infty$, and (3) is established.

To prove (4), consider the following "coloring followed by invasion percolation" procedure. First mark every edge blue which is in some infinite p_*-cluster. Then mark every edge red which is not blue but is adjacent to some blue edge. Given the coloring information, start to build the invasion cluster at x in the usual way. For the event $(\xi^x_{p_*} = \infty)$ to happen, the invasion cluster has to meet (become adjacent to) infinitely many colored edges. If the invasion cluster ever meets any of the blue edges, then we are done (i.e. the invasion cluster intersects some infinite p-cluster). Otherwise (still on the event $(\xi^x_{p_*} = \infty)$) the invasion cluster has to meet infinitely many red edges. Suppose now that a given red edge e is met for the first time. Then the conditional distribution of $U(e)$ (given the coloring information and the invasion cluster so far) is uniform on $(p_*, 1]$. Thus the event that $U(e) < p$ (which obviously implies that I^x_∞ intersects some infinite p-cluster) has conditional probability $\frac{p-p_*}{1-p_*} > 0$. Since this conditional probability is the same every time a red edge is encountered by the invasion cluster for the first time, we have (4), and the proof is complete. □

4 Uncountably many ends

Proof of Theorem 1.6: We shall prove that for any $p_1 < p_2$ in (p_c, p_u) we have

$$\Psi^G(\forall p \in [p_1, p_2], \text{ all infinite } p\text{-clusters have } 2^{\aleph_0} \text{ ends}) = 1\,. \tag{8}$$

Sending $p_1 \downarrow p_c$ and $p_2 \uparrow p_u$ through countable sequences then proves the theorem.

Fix p_1 and p_2 as above, and set $p_0 = \frac{p_c+p_1}{2}$. To prove (8), it is (due to Theorem 1.4) enough to show, for any $x_0 \in V$, that

$$\Psi^G(H^{x_0}_{p_0,p_1,p_2}) = 0 \tag{9}$$

where $H^{x_0}_{p_0,p_1,p_2}$ is the event that x_0 is in an infinite p_0-cluster and for some $p \in [p_1, p_2]$ it is in an infinite p-cluster with less than $2^{\aleph_0}$ ends. Also define $\tilde{H}^{x_0}_{p_0,p_1,p_2}$ as the event that x_0 is in an infinite p_0-cluster and for some $p \in [p_1, p_2]$ it is in an infinite p-cluster with just one end. If a given realization $\eta \in [0,1]^E$ of the variables $\{U(e)\}_{e \in E}$ is in $H^{x_0}_{p_0,p_1,p_2}$, then (arguing as in

Benjamini and Schramm [9], p. 76) one can change finitely many of the variables to obtain a realization η' which is in $\tilde{H}^{x_0}_{p_0,p_1,p_2}$. Thus, (9) follows easily once we show

$$\Psi^G(\tilde{H}^{x_0}_{p_0,p_1,p_2}) = 0\,. \tag{10}$$

Fix $k \geq 1$. Since $p_2 \in (p_c, p_u)$, and each infinite p_2-cluster contains some infinite p_1-cluster Ψ_G-a.s., for every $x \in V$ we have

$$\lim_{R\to\infty} \Psi^G\Big(\#\Big\{p_2\text{-clusters containing infinite } p_1\text{-clusters that intersect } B(x,R)\Big\} \geq k\Big) = 1\,,$$

Thus given $\varepsilon > 0$, for every $x \in V$ there is an R such that

$$\Psi^G\Big(\#\Big\{p_2\text{-clusters containing infinite } p_1\text{-clusters that intersect } B(x,R)\Big\} \geq k\Big) \geq 1-\varepsilon. \tag{11}$$

Since G is quasi-transitive, there exists an R that satisfies (11) for all $x \in V$. (R may depend on p_1, p_2, k and ε, but not on x.) Fix such an R, and grow the invasion cluster of x_0 until the first time τ for which $I^{x_0}_\tau$ contains some ball of radius R; Lemma 3.2 ensures that this happens a.s. for some finite τ. Let $\partial I^{x_0}_\tau$ be the set of edges in $E \setminus I^{x_0}_\tau$ that are adjacent to $I^{x_0}_\tau$. Using (11) and arguing as in the proof of Proposition 3.1, we have that

$$\Psi^G(A^{x_0}_k \,|\, \text{the invasion process up to time } \tau) \geq 1 - \varepsilon\,,$$

where $A^{x_0}_k$ is the event that the percolation process restricted to the edge set $E \setminus (I^{x_0}_\tau \cup \partial I^{x_0}_\tau)$ has at least k infinite p_1-clusters that

(i) are contained in separate p_2-clusters, and

(ii) contain some vertex at distance 1 from $I^{x_0}_\tau$.

If x_0 is in an infinite p_0-cluster, then

$$U(e) \leq p_0 \text{ for every } e \in I^{x_0}_\tau\,. \tag{12}$$

Given (12) and the invasion process up to time τ, each $e \in \partial I^{x_0}_\tau$ is open at level p_1 independently with conditional probability at least $\frac{p_1-p_0}{1-p_0}$. If $A^{x_0}_k$ happens, we may pick $e_1, \ldots e_k \in \partial I^{x_0}_\tau$

adjacent to k different p_1-clusters with the properties (i), (ii) above. These properties guarantee that if x_0 is in an infinite p_0-cluster and at least two of the edges $e_1, \ldots, e_k$ are open at level p_1, then x_0 is in an infinite p-cluster with at least two ends for all $p \in [p_1, p_2]$. Hence

$$\Psi^G(\tilde{H}^{x_0}_{p_0,p_1,p_2}) \leq \varepsilon + \left(\frac{1-p_1}{1-p_0}\right)^k + k\left(\frac{p_1-p_0}{1-p_0}\right)\left(\frac{1-p_1}{1-p_0}\right)^{k-1} .$$

Sending $\varepsilon \to 0$ and $k \to \infty$ proves (10), and thus also (9) and (8), so the proof is complete. □

We end this section by noting the following very simple corollary to Theorem 1.4. It is the natural analogue for the uniqueness regime $(p_u, 1)$ of Theorem 1.6.

Corollary 4.1 *Let G be an infinite, locally finite, connected, semi-transitive graph. Then, Ψ^G-a.s., for all $p \in (p_u, 1)$ the (unique) infinite p-cluster has a single end.*

Proof: Suppose for contradiction that with positive Ψ^G-probability, there exists some $p \in (p_u, 1)$ for which the infinite cluster has more than one end. Any realization $\eta \in [0,1]^E$ of the $\{U(e)\}_{e \in E}$ variables for which this happens at level p can be modified into a configuration η' in which uniqueness of the infinite cluster fails at level p, by changing the status of just finitely many edges. It follows that with positive Ψ^G-probability, there is some $p \in (p_u, 1)$ for which uniqueness of the infinite cluster fails, contradicting Theorem 1.4. □

5 Indistinguishability by Robust Properties

Proof of Theorem 1.8: Fix $p_0 \in (p_c, p)$. Since, by Theorem 1.4, Ψ^G-a.s. any infinite p-cluster contains an infinite p_0-cluster, it suffices to show that for all $x \in V$,

$$\Psi^G[\mathcal{C}(x, p_0) \text{ is infinite and } \mathcal{C}(x,p) \notin Q] = 0 . \tag{13}$$

Define the random variable ξ^x as the number of edges that are adjacent to, or contained in, I^x_∞ and are also adjacent to, or contained in, some infinite p-cluster which satisfies Q. We will establish (13) by proving the following two statements:

$$\Psi^G[\xi^x < \infty] = 0 , \tag{14}$$

and

$$\Psi^G[\mathcal{C}(x,p_0) \text{ is infinite, } \xi^x = \infty \text{ and } \mathcal{C}(x,p) \notin Q] = 0\,. \tag{15}$$

We first prove (14). By the 0-1 law for automorphism-invariant events,

$$\Psi^G[\exists \text{ an infinite } p\text{-cluster satisfying } Q] = 1\,.$$

Therefore, for any $\varepsilon > 0$, we can pick an R so that

$$\inf_{y\in V} \Psi^G\Big(\text{no infinite } p\text{-cluster with property } Q \text{ intersects } B(y,R)\Big) < \varepsilon\,. \tag{16}$$

Let τ be the smallest m for which I_m^x contains a ball of radius R; by Lemma 3.2, $\tau < \infty$ a.s.

For an edge set $E_0 \subset E$, let ∂E_0 be the set of edges outside E_0 that are adjacent to E_0, and denote by $\mathcal{S}(E_0)$ the set of edges in ∂E_0 that are adjacent to an infinite connected component of $\{e \notin \partial E_0 \,:\, U(e) \le p\}$ which has property Q.

If a finite edge set E_0 intersects an infinite p-cluster that has property Q, then robustness of Q implies that $\mathcal{S}(E_0) \neq \emptyset$. Therefore, any finite edge set E_0 that contains a ball of radius R, satisfies $\Psi^G\Big(\mathcal{S}(E_0) = \emptyset\Big) < \varepsilon$ by (16).

Since the invasion cluster I_τ^x gives no information about the labels on edges not in $I_\tau^x \cup \partial I_\tau^x$, we may apply the above reasoning with $E_0 = I_\tau^x$ to deduce that $\Psi^G\Big(\mathcal{S}(I_\tau^x) = \emptyset\Big) < \varepsilon$. In particular, $\Psi^G(\xi^x = 0) < \varepsilon$, and since ε was arbitrary, $\Psi^G(\xi^x = 0) = 0$. On the event $(\xi^x = n)$, we can move into the event $(\xi^x = 0)$ by changing the labels $U(e)$ on finitely many edges to be greater than p; it follows that $\Psi^G(\xi^x = n) = 0$ for any $n < \infty$, and (14) is established.

To prove (15), observe that if $\mathcal{C}(x,p_0)$ is infinite, then $I_\infty^x \subset \mathcal{C}(x,p_0)$. Therefore, if also $\xi^x = \infty$ and $\mathcal{C}(x,p) \notin Q$, then the following event, which we call F, must happen: $\mathcal{C}(x,p) \notin Q$ but there are infinitely many edges in $\partial\mathcal{C}(x,p_0)$ which are adjacent to p-clusters $\mathcal{C} \in Q$ such that $\mathcal{C} \cap \mathcal{C}(x,p_0) = \emptyset$. (Note that these p-clusters $\mathcal{C}$ are connected components in G.) To establish (15), it suffices to show that $\Psi^G(F) = 0$. To prove this, write $\mathcal{C}_\ell(x,p_0)$ for the collection of edges that can be reached from x by a path which is open at level p_0 and is contained in the ball $B(x,\ell)$. Consider the event $F_{\ell,k}$ that $\mathcal{C}(x,p) \notin Q$ and $\partial\mathcal{C}_\ell(x,p_0)$ contains at least k edges that are adjacent to infinite p-clusters $\mathcal{C} \in Q$ such that $\mathcal{C} \cap \mathcal{C}(x, p_0) = \emptyset$. Clearly, for every k,

$$F \subset \cup_{\ell=1}^{\infty} F_{\ell,k}\,, \text{ whence } \Psi^G(F) \le \lim_{\ell \to \infty} \Psi^G(F_{\ell,k})\,.$$

The proof will therefore be complete once we establish that for any ℓ, k,

$$\Psi^G(F_{\ell,k}) \le \left(\frac{1-p}{1-p_0}\right)^k. \tag{17}$$

Clearly,

$$F_{\ell,k} \subset \Big\{\mathcal{C}(x,p) \notin Q \text{ and } |\mathcal{S}(\mathcal{C}_\ell(x,p_0))| \ge k\Big\}\,.$$

(In fact these events coincide, but we do not need this.) Therefore, by robustness of Q,

$$F_{\ell,k} \subset \Big\{|\mathcal{S}(\mathcal{C}_\ell(x,p_0))| \ge k \text{ and } \forall e \in \mathcal{S}(\mathcal{C}_\ell(x,p_0)) \quad U(e) \ge p\Big\}. \tag{18}$$

Denote by $\mathcal{F}_\ell$ the σ-field generated by $\mathcal{C}_\ell(x,p_0)$ and the labels $\{U(e) \,:\, e \notin \partial\mathcal{C}_\ell(x,p_0)\}$. Then $\mathcal{S}(\mathcal{C}_\ell(x,p_0))$ is $\mathcal{F}_\ell$-measurable, and the remaining labels $\{U(e) \,:\, e \in \partial\mathcal{C}_\ell(x,p_0)\}$ are conditionally independent and uniform on $[p_0, 1]$ given $\mathcal{F}_\ell$. Thus (17) follows from (18). □

6 Unimodularity and the Mass-Transport Principle

For $x \in V$, define the **stabilizer** $S(x) = \{\gamma \in \mathrm{Aut}(G) : \gamma(x) = x\}$, and for $y \in V$, define $S(x)y = \{\gamma(y) \,:\, \gamma \in S(x)\}$. Let $|A|$ denote the cardinality of a set A.

Definition 6.1 *A quasi-transitive graph G is called* **unimodular** *if for any two vertices x, y in the same orbit, we have $|S(x)y| = |S(y)x|$.*

For equivalent definitions of unimodularity, see Trofimov [31] and Benjamini, Lyons, Peres and Schramm [7]. Most quasi-transitive graphs that come up naturally are unimodular. In particular, the Cayley graph of any finitely generated group is transitive and unimodular. A transitive graph $\widehat{T}_d$ which is *not* unimodular can be constructed by considering a regular tree T_d with degree $d \ge 3$, fixing an end ξ of T_d, and for each vertex x adding an edge between x and its ξ-grandparent; see [31] or [7]. In this example, $p_u = 1$, and for $p \in (p_c, 1)$ every infinite cluster $\mathcal{C}$ has a unique vertex $v(\mathcal{C},\xi)$ that is “closest” to ξ. Thus, as noted in [24], the invariant property $\{\mathcal{C} : v(\mathcal{C},\xi)$ has degree 1 in $\mathcal{C}\}$ distinguishes some infinite clusters in $\widehat{T}_d$ from others. By utilizing more of the local structure of a cluster $\mathcal{C}$ as seen from $v(\mathcal{C},\xi)$, *any* two infinite clusters in $\widehat{T}_d$ may be invariantly distinguished.

The significance of unimodularity for us is that it allows a certain mass-transport technique, which was introduced in percolation theory in [17] and systematically developed in [7]. Central to the mass-transport method is Theorem 6.2 below, which was proved (in a more general setting) in [7]. For any graph G, every automorphism in $\mathrm{Aut}(G)$ acts as a measure-preserving transformation on the probability space $(\{0,1\}^E, \mathbf{P}_p^G)$. Let $m(x,y,\omega)$ be a nonnegative function of three variables: two vertices x, y in the same orbit of $\mathrm{Aut}(G)$, and $\omega \in \{0,1\}^E$. Intuitively, $m(x,y,\omega)$ represents the mass transported from x to y given the configuration ω. We suppose that $m(\cdot,\cdot,\cdot)$ is invariant under the diagonal action of $\mathrm{Aut}(G)$, i.e., $m(x,y,\omega) = m(\gamma x, \gamma y, \gamma\omega)$ for all x, y, ω and $\gamma \in \mathrm{Aut}(G)$.

Theorem 6.2 (The Mass-Transport Principle) *Suppose that $G = (V,E)$ is unimodular and quasi-transitive. Given $m(\cdot,\cdot,\cdot)$ as above, let*

$$M(x,y) = \int_{\{0,1\}^E} m(x,y,\omega)\, d\mathbf{P}_p^G(\omega)\,.$$

Then the expected total mass transported out of any vertex x equals the expected total mass transported into x, i.e.,

$$\forall x \in V \quad \sum_{y\in V} M(x,y) = \sum_{y\in V} M(y,x)\,. \tag{19}$$

We remark that (19) fails in the nonunimodular case; see [7]. The key element in a successful application of the mass-transport method is to make a suitable choice of the transport function $m(\cdot,\cdot,\cdot)$; examples can be found e.g. in [17, 7, 8, 18], and also in Section 7 below.

7 Relentless merging

The main step in proving Theorem 1.5 is showing that for $p \in (p_c, p_u)$, any infinite p-cluster will a.s. come within distance 1 from other infinite p-clusters in infinitely many places, as stated in the following proposition. The result assumes quasi-transitivity and unimodularity; we conjecture the latter condition to be removable.

Proposition 7.1 *Consider bond percolation on an infinite, locally finite, connected, quasi-transitive unimodular graph G with retention parameter $p \in (p_c, p_u)$. Then, $\mathbf{P}_p^G$-a.s., for any infinite cluster $\mathcal{C}$ there exist infinitely many closed edges e with one endpoint in $\mathcal{C}$ and the other endpoint in some other infinite cluster (which may depend on e).*

Proof: Assume for contradiction that with positive $\mathbf{P}_p^G$-probability there is some infinite cluster $\mathcal{C}$ which comes within distance exactly 1 from the set of other infinite clusters in at most finitely many locations. As in the argument for (7) in the proof of Proposition 3.1, it follows that with positive probability there is some infinite cluster $\mathcal{C}$ in which exactly one vertex x is at distance 1 from the set of other infinite clusters. Call such an infinite cluster $\mathcal{C}$ a *kingdom*, call x its *king*, and consider the following mass transport. If a vertex y is in a kingdom, and its king is in the same orbit as y (recall Definition 1.1), then y sends unit mass to its king. If y is in a kingdom but not in the same orbit as the king, then y sends unit mass which it distributes equally among the vertices in its orbit which are closest in G to the king. Otherwise, no mass is sent from y. The expected mass sent from each vertex is then at most 1, whereas the expected mass received has to be ∞ for some vertices. By the Mass-Transport Principle (Theorem 6.2) we have the desired contradiction. □

Proof of Theorem 1.5: We first prove the assertion of the theorem with the quantifiers interchanged, i.e. that

$$\text{for all } p_1 < p_2 \text{ in } (p_c, p_u), \text{ we have } \Psi^G\text{-a.s. that any infinite } p_2\text{-cluster contains infinitely many infinite } p_1\text{-clusters.} \tag{20}$$

We know from Theorem 1.2 that any infinite p_2-cluster contains some infinite p_1-cluster. Hence, it suffices to show that any infinite p_1-cluster $\mathcal{C}$ gets connected to infinitely many infinite p_1-clusters disjoint from $\mathcal{C}$ as we raise the percolation level to p_2. Fix a vertex x, let $\mathcal{C}(x, p_1)$ be the p_1-cluster containing x, and assume that $\mathcal{C}(x, p_1)$ is infinite. Call an edge e *pivotal* if it is closed at level p_1, has one endpoint in $\mathcal{C}$ and the other endpoint in some other infinite p_1-cluster. By

Proposition 7.1, there are Ψ^G-a.s. infinitely many pivotal edges. As we raise the level to p_2, each pivotal edge gets turned on independently with probability $\frac{p_2-p_1}{1-p_1}$, whence at least one of them gets turned on a.s., so that $\mathcal{C}(x,p_1)$ gets connected to at least one other infinite cluster a.s.

Now pick $k \geq 2$ and $q_1,\ldots,q_k$ such that $p_1 = q_1 < q_2 < \cdots < q_k = p_2$. The above reasoning shows that the infinite cluster $\mathcal{C}(x,p_1)$ gets connected to at least one additional infinite p_1-cluster for each interval (q_i, q_{i+1}), so that $\mathcal{C}(x,p_2)$ contains at least $k-1$ infinite p_1-clusters. Since k was arbitrary, we have established (20).

It remains to change the order of the quantifiers "for all $p_1 < p_2$" and "Ψ^G-a.s." in (20). To do this, note first that we can apply (20) to all rational $p_1 < p_2$ in (p_c, p_u) simultaneously. The assertion of the theorem now follows easily using Theorem 1.3 and the observation that for any $p_1 < p_2$, we can find two distinct rational numbers between p_1 and p_2. $\square$

8 Product graphs and estimates on p_u

Let us collect the results from the literature that are needed to prove Theorem 1.9. The first one concerns percolation on the orthant $\mathbf{Z}_+^d$. For $d=2$, it follows from the work of Kesten [20], while for general d it was first obtained by Barsky, Grimmett and Newman [6].

Theorem 8.1 ([20], [6]) *For bond percolation on $\mathbf{Z}_+^d$, $d \geq 2$, we have*

(a) $p_c(\mathbf{Z}_+^d) = p_c(\mathbf{Z}^d)$, and

(b) for $p > p_c(\mathbf{Z}_+^d)$, there is $P_p^{\mathbf{Z}_+^d}$-a.s. a unique infinite cluster.

The following result is due to Schonmann [29]. There it was formulated in the setting of quasi-transitive graphs, but that proof goes through unchanged in the generality stated here.

Theorem 8.2 ([29]) *Let G be any bounded degree graph, and pick $p \in [0,1]$. If*

$$\lim_{R\to\infty} \inf_{x,y\in V} \mathbf{P}_p^G(B(x,R) \leftrightarrow B(y,R)) = 1\,, \tag{21}$$

then for all $p' > p$, there is $\mathbf{P}_{p'}^G$-a.s. exactly one infinite cluster.

Remark: The conclusion of this theorem can now be strengthened to " Ψ^G-a.s. there is exactly one infinite cluster at each level $p' > p$ ". Indeed, (21) implies uniform percolation at level p, so Theorem 2.3 applies.

In the following proof, we shall work with more than one graph, and therefore write $B_G(x,R)$ for $B(x,R)$ to indicate in which graph the ball sits.

Proof of Theorem 1.9: We first prove uniqueness of the infinite cluster on G for fixed $p > p_c(\mathbf{Z}^d)$. By Theorem 8.2, it is sufficient to show that (21) holds for all $p > p_c(\mathbf{Z}^d)$. Fix such a p. Theorem 8.1 implies that

$$\lim_{R\to\infty} \mathbf{P}_p^{\mathbf{Z}_+^d}(B_{\mathbf{Z}_+^d}(0,R) \leftrightarrow \infty) = 1\,, \tag{22}$$

where $(B_{\mathbf{Z}_+^d}(0,R) \leftrightarrow \infty)$ is the event that there is an infinite open cluster intersecting $B_{\mathbf{Z}_+^d}(0,R)$. Pick $\varepsilon > 0$, and R large enough so that

$$\mathbf{P}_p^{\mathbf{Z}_+^d}(B_{\mathbf{Z}_+^d}(0,R) \leftrightarrow \infty) \geq 1-\varepsilon\,.$$

Now let $x = (x_1,\ldots,x_d)$ and $y = (y_1,\ldots,y_d)$ be arbitrary vertices of G. For $i = 1,\ldots,d$, let T_x^i be some infinite self-avoiding path in G_i starting in x_i. Also let T_y^i be some infinite self-avoiding path in G_i from y_i which eventually coincides with T_x^i. Such a path is easily seen to exist: just take a path from y_i to x_i, concatenate it with T_x^i, and erase any circuits. Finally, let z_i be the first vertex on T_x^i with the property that T_x^i and T_y^i coincide from z_i to infinity, and define T_z^i to be the self-avoiding path starting at z_i that is contained in T_x^i. Define the product graph $G_x^* = T_x^1 \times \cdots T_x^d$, and define G_y^* and G_z^* analogously. Note that G_x^*, G_y^* and G_z^* are all isomorphic to $\mathbf{Z}_+^d$, and furthermore that they are all subgraphs of G and that G_z^* is a subgraph both of G_x^* and of G_y^*.

Let $D_{R,x}$ be the event that some vertex in $B_{G_x^*}(x,R)$ has an open path to infinity in G_x^*. Define $D_{R,y}$ analogously, and set $D_{R,x,y} = D_{R,x} \cap D_{R,y}$. Using (22), we get

$$\mathbf{P}_p^G(D_{R,x,y}) \geq 1-2\varepsilon\,.$$

By Theorem 8.1, we have $\mathbf{P}_p^G$-a.s. that G_x^* and G_y^* each have a unique infinite cluster, and that both these infinite clusters contain the unique infinite cluster of G_z^*. Hence, we have $\mathbf{P}_p^G$-a.s. on the event $D_{R,x,y}$ that there is an open path in G connecting $B_G(x,R)$ and $B_G(y,R)$, so that

$$\mathbf{P}_p^G(B_G(x,R) \leftrightarrow B_G(y,R)) \geq 1 - 2\varepsilon \, .$$

Note that this is a uniform bound for all vertices x and y in G. Since ε was arbitrary we have (21), so the proof for fixed p is complete.

The asserted simultaneous uniqueness is implied by the remark following Theorem 8.2. $\square$

Next, we discuss lower bounds for p_u. Benjamini and Schramm [9, Theorem 4] proved that any quasi-transitive graph G satisfies $p_u(G) \geq (D\rho_G)^{-1}$, where D is the maximal degree in G, and ρ_G is the spectral radius for simple random walk on G. Their proof was based on coupling the percolation process with a branching random walk. In fact, a simple counting argument yields a slightly better bound.

Given a locally finite connected graph G, let $A_{x,y}^n$ denote the number of paths of length n which connect x to y. It is easy to see that

$$\lambda_G := \limsup_{n\to\infty} (A_{x,y}^n)^{1/n}$$

does not depend on x and y.

Proposition 8.3 *Let G be an infinite, locally finite, connected graph. Then for $p < \lambda_G^{-1}$, for any $x, y \in V$,*

$$\mathbf{P}_p^G[x \leftrightarrow y] \leq \frac{(p\lambda_G)^{d(x,y)}}{1 - p\lambda_G}.$$

If G is also quasi-transitive, then $p_u(G) \geq \lambda_G^{-1}$.

This bound on p_u coincides with the bound in [9] for quasi-transitive graphs with constant degree, and improves upon it if the degree is nonconstant.

Proof: Clearly,

$$A_{x,x}^{m+n} \geq A_{x,y}^m A_{y,x}^n \tag{23}$$

for any two sites x, y, and any non-negative integers m, n. Since $A^{2k}_{x,x} > 0$, it follows (see, e.g., Section 8 of [21]) that

$$\lim_{k\to\infty} (A^{2k}_{x,x})^{\frac{1}{2k}} = \tilde{\lambda}_G = \sup_{k\geq 1} (A^{2k}_{x,x})^{\frac{1}{2k}} .$$

Using symmetry and (23), we obtain

$$\forall x, y \in V \quad \forall k \geq 1 \quad A^k_{x,y} \leq \left(A^{2k}_{x,x}\right)^{1/2} \leq \tilde{\lambda}^k_G . \tag{24}$$

Note that (24) implies that $\lambda_G = \tilde{\lambda}_G$. Denote by $[x \leftrightarrow y]$ the event that there is an open path connecting the sites x and y, and let $[x \leftrightarrow \infty]$ denote the event that x belongs to an infinite p-cluster. Let $\mathcal{N}^{(k)}_{x,y}$ be the number of self-avoiding paths of length k which connect x to y. Then

$$\mathbf{P}_p[x \leftrightarrow y] \leq \sum_{k=d(x,y)}^{\infty} \mathcal{N}^{(k)}_{x,y} p^k \leq \sum_{k=d(x,y)}^{\infty} p^k A^k_{x,y} \leq \sum_{k=d(x,y)}^{\infty} (p\lambda_G)^k = \frac{(p\lambda_G)^{d(x,y)}}{1 - p\lambda_G} , \tag{25}$$

provided $p < \lambda_G^{-1}$.

Suppose now that G is quasi-transitive. In this case $\theta(p) = \inf_{z\in V} \mathbf{P}_p[z \leftrightarrow \infty] > 0$ for any $p > p_c$. If $p > p_u$, then for all $x, y \in V$,

$$\mathbf{P}_p[x \leftrightarrow y] \geq \mathbf{P}_p[x \leftrightarrow \infty, y \leftrightarrow \infty] \geq \mathbf{P}_p[x \leftrightarrow \infty]\mathbf{P}_p[y \leftrightarrow \infty] = \theta(p)^2 > 0 , \tag{26}$$

where the second inequality is an instance of the Harris inequality. Comparing (25) to (26) gives $p_u \geq \lambda_G^{-1}$. □

Remark. O. Schramm (personal communication) has obtained a sharper lower bound for p_u. He showed that $p_u \geq \gamma_G^{-1}$, where $\gamma_G := \sup_{x\in V} \limsup_{k\to\infty} \left(\mathcal{N}^k_x\right)^{\frac{1}{k}}$ and $\mathcal{N}^k_x$ is the number of self-avoiding cycles that start and end at x.

The final topic of this section is the relation between the number of ends of a quasi-transitive graph and the critical parameters p_c and p_u. It is well known that a quasi-transitive graph G can only have 1, 2 or uncountably many ends (see, e.g., Section 6 in [26]). In case the number of ends is more than 1, one can use the converse of Theorem 8.2 to show that $p_u = 1$. This converse states that for quasi-transitive graphs,

$$\forall p > p_u \qquad \lim_{R\to\infty} \inf_{x,y\in V} \mathbf{P}^G_p\Big(B(x,R) \leftrightarrow B(y,R)\Big) = 1 . \tag{27}$$

(This is a consequence of the Harris inequality, see Theorem 3.1 of [29]). The removal of an appropriate finite set of sites and edges from a graph G which has more than 1 end breaks the graph into more than one infinite connected component. Therefore the limit in (27) cannot be 1 when $p < 1$, and we must have $p_u = 1$. If the number of ends of G is uncountable, then $p_c(G) < 1$, since then G has a positive Cheeger constant (see Proposition 6.2 in [26] and Theorem 2 in [9]). If G has 2 ends, then G is just a "finite extension" of $\mathbf{Z}$, so $p_c(G) = 1$. More precisely, by the proof of Proposition 6.1 in [26], there is a doubly infinite sequence $(..., A_{-2}, A_{-1}, A_0, A_1, A_2, ...)$ of pairwise disjoint and isomorphic finite subgraphs of G, such that any infinite self-avoiding path in G must intersect either all the graphs A_j with large enough j, or all the graphs A_{-j} with large enough j.

The case of a single end is more delicate. Babson and Benjamini [5] proved that $p_u(G) < 1$ if G is the Cayley graph of a finitely presented group with one end. The question stated in [9], whether $p_u(G) < 1$ whenever G is a quasi transitive graph with one end, is still unsolved.

9 Examples and questions

A variant of the following example was shown to us by O. Schramm.

Example: A semi-transitive graph where exactly 2 infinite clusters can coexist. Let T be a binary tree with root ρ, i.e. T is the tree in which ρ is incident to exactly two edges, and all other vertices are incident to exactly three edges. Let H be the product graph $T \times \mathbf{Z}$ with an additional distinguished vertex v^* joined by a single edge to the vertex $(\rho, 0)$ of $T \times \mathbf{Z}$. Theorem 1.9 implies that $p_u(H) = p_u(T \times \mathbf{Z}) < 1$. Finally, let G consist of two copies H_1, H_2 of H, glued together at their distinguished vertices (so these vertices v_1^* and v_2^* are identified, and the resulting vertex of G is denoted w^*). It is easy to see that G is semi-transitive, with V_F consisting of the distinguished vertex w^* only. For all $p > p_u(H)$, it follows that bond percolation on G can have one or two infinite clusters with positive probability: If at least one of the two edges incident to w^* is closed, then a.s. there are exactly two infinite clusters, one contained in H_1 and the other in H_2. On the other hand, clearly the infinite clusters of H_1 and H_2 can connect to each other with positive probability. $\square$

Next, we discuss briefly yet another phase transition. Let G be a nonunimodular quasi-transitive graph, and denote by μ the left Haar measure on $\mathrm{Aut}(G)$. Recall the definition of stabilizer $S(x)$ from Section 6. Say that a cluster $\mathcal{C}$ with vertex set $V(\mathcal{C})$ is **heavy** if $\sum_{x \in V(\mathcal{C})} \mu[S(x)] = \infty$; otherwise, we say that $\mathcal{C}$ is **light**. Theorem 1.8 implies that heavy and light infinite clusters cannot coexist at any fixed level p, so it is natural to define

$$p_h := \inf \Big\{ p : \mathbf{P}_p[\text{ there is a heavy infinite cluster}] > 0 \Big\} . \tag{28}$$

The mass transport method can be used effectively in heavy clusters. For instance, Theorem 1.5 can be easily extended to nonunimodular graphs, provided that the parameters p_1, p_2 considered there are greater than p_h. For the nonunimodular example $\widehat{T}_d$ mentioned in Section 6 (a tree with additional edges leading to ξ-grandparents) it is easy to see that $p_c < p_h = p_u = 1$. On the other hand, let T_k denote the k-regular tree. For any graph G_0 with bounded degrees, if k is large enough, then $G = G_0 \times T_k$ satisfies

$$p_h(G) < p_u(G) . \tag{29}$$

The proof is similar to an argument in [9, Sect. 4]. Let D_0 be the maximal degree in G_0, and let ρ_G be the spectral radius for simple random walk on G. It is easy to see that if $p > p_c(T_k)$, then any infinite p-cluster in $G = G_0 \times T_k$ is heavy. Therefore,

$$\lambda_G \cdot p_h(G) \le (D_0 + k) \cdot \rho_G \cdot p_h(G) \le (D_0 + k) \cdot \rho_G \cdot p_c(T_k) = \frac{D_0 + k}{k - 1} \rho_G . \tag{30}$$

Since $\rho_G = \rho_{G_0 \times T_k} \to 0$ as $k \to \infty$, it follows from [9, Theorem 4], or from Proposition 8.3 above, that for large enough k (when the right hand side of (30) is less than 1) we have (29). We expect that there exist transitive graphs where $p_c < p_h < p_u < 1$, but we do not have an explicit example.

We end the paper with some **questions:**

1. Is $p_c < p_h$ for every nonunimodular quasi-transitive graph? (p_h is defined in (28).) What geometric properties of G guarantee that $p_h < p_u$?
2. Can one drop the unimodularity assumption made in Theorem 1.5 and Proposition 7.1?

3. Let u, v be vertices of a transitive graph G, and denote by $[u \leftrightarrow v]$ the event that there is some open path connecting u and v. Is the function $p \mapsto \mathbf{P}_p^G[u \leftrightarrow v]$ continuous on (p_c, p_u)? (By monotonicity, the set of points of discontinuity is at most countable.) Left-continuity of this function on $(0, 1]$ and its continuity on $(0, p_c)$ and on $(p_u, 1]$ are easily established using well-known arguments (see Grimmett [15], p. 118). On the other hand, results in [30], where the Grimmett–Newman example is studied, imply that this function can be discontinuous at p_u.

4. A graph G is said to exhibit **cluster repulsion** at level p if, for i.i.d. percolation with retention parameter p on G, any two infinite clusters can come within unit distance from each other in at most finitely many places. Does a quasi-transitive graph necessarily exhibit cluster repulsion for any $p \in [0, 1]$? It is not hard to show that cluster repulsion at level p follows from continuity at p of the pair connectivity function discussed in Question 3; hence cluster repulsion can fail at most for countably many values of p. If cluster repulsion holds, then it follows that for every R, any two infinite clusters come within distance R from each other at most finitely often a.s. We have an example of a (non-semi-transitive) graph for which cluster repulsion fails for certain p.

5. Let G be a quasi-transitive graph. A site x is an **encounter point** of the cluster $\mathcal{C}$ if removing the edges incident to x from $\mathcal{C}$ yields at least three infinite connected components. For $p \in (p_c, p_u)$, is $\mathbf{P}_p[\exists$ an infinite cluster with infinitely many encounter points$] > 0$? By Theorem 1.8, this would imply that *every* infinite cluster has infinitely many encounter points $\mathbf{P}_p$-a.s. This question has a positive answer in the unimodular case (see [8] or [24]), which readily extends to heavy clusters in the nonunimodular case. A general answer would be a significant step toward determining whether

$$\mathbf{P}_{p_c}[\exists \text{ infinite open clusters }] = 0 \tag{31}$$

for quasi-transitive graphs with a nonamenable automorphism group. (Under the additional assumption of unimodularity, (31) is proved in [8].)

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