Hermitian Lattices over Rings with Involutions

ANETTE WIBERG

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Department of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
AND GÖTEBORG UNIVERSITY
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Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31-7721000

Matematiskt centrum
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Abstract

Let R be an integral domain with quotient field K and let $\Lambda \supseteq R$ be an R–algebra finitely generated as an R–module. Let $x \mapsto \bar{x}$ be an R–involution on Λ, that is, an R–linear mapping of Λ such that $\overline{xy} = \bar{y}\bar{x}$ and $\bar{\bar{x}} = x$. A left R–module M has a hermitian structure if there is an R–bilinear mapping $h : M \times M \to \Lambda$ such that $h(\lambda x, y) = \lambda h(x,y)$ and $h(y,x) = \overline{h(x,y)}$ for all $\lambda \in \Lambda$ and $x, y \in M$.

This paper presents a survey of some of the known results concerning classification of hermitian lattices when Λ is the ring of integers S in a quadratic field extension over a local field or a maximal order in a quaternion algebra over such a field. We also investigate relations between classification of hermitian S–lattices (M, h) and classification of certain classes of orders over R contained in the endomorphism ring $\mathrm{End}_S(M)$ and invariant with respect to an involution corresponding to h. For example, we prove that under suitable assumptions on the extensions $R \subset S$, two lattices (M, h) and (M', h') are isomorphic if and only if the corresponding orders are isomorphic.

Keywords: hermitian lattice, hermitian form, involution, endomorphism ring, quaternion order.

AMS 1991 Subject classification: 11E39, 16H05, 16S50, 16W10.

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Contents

1 Introduction

Consider an integral domain R with quotient field K and an R–algebra Λ finitely generated as an R–module. We assume that $R \subseteq \Lambda$ and do not exclude the case when $R = K$. Let $x \mapsto \bar{x}$ be an R–involution on Λ, that is, an R–linear mapping of Λ such that $\overline{xy} = \bar{y}\bar{x}$ and $\bar{\bar{x}} = x$. A left R–module M has a hermitian structure if there is an R–bilinear mapping $h : M \times M \to \Lambda$ such that $h(\lambda x, y) = \lambda h(x, y)$ and $h(y, x) = \overline{h(x, y)}$ for all $\lambda \in \Lambda$ and $x, y \in M$.

Hermitian modules (M, h) over algebras with involutions play a very important role in many situations. They are studied in connection with different types of applications as well as for their own sake. Let us only mention extensive presentations of this subject in [8], [9] and [15]. In some cases, there is a close relation between hermitian and quadratic modules. If the set of fixed points for the involution $x \mapsto \bar{x}$ on Λ is a ring S contained in the center of Λ, then every hermitian module (M, h) over Λ defines a quadratic module (M, q) over S, where $q(x) = h(x, x)$ for $x \in M$. A well-known theorem of N. Jacobson (see [7] or [5]) says that if Λ is a field or a quaternion division algebra, then two hermitian modules over Λ are isomorphic (as modules with hermitian structure) if and only if the corresponding quadratic modules are isomorphic over S.

Even if the theory of hermitian modules is much better developed over fields (that is, in the case $R = K$) than over more general rings, there are several papers concerned with arithmetical questions related to hermitian modules over R–algebras, where R is a Dedekind domain. In particular, R. Jacobowitz gives in [6] a complete classification of hermitian Λ–lattices M (that is, M is finitely generated and projective as an R–module) when R is a complete discrete valuation ring and Λ is a quadratic or quaternion R–algebra with the standard R–involution (that is, $x + \bar{x}, x\bar{x} \in R$). Hermitian lattices over maximal quaternion R–orders, where R is the ring of all integers in a local or a global field, were investigated by G. Shimura in [16] in order to prove approximation theorems for their similitude groups (an R–order is a finitely generated projective R–algebra).

The present paper is concerned with hermitian lattices over R–algebras with involution, where R is a Dedekind domain. The purpose of this paper is to give a survey of some of the known results in this area and also to investigate a class of R–algebras closely related to the endomorphism rings of hermitian lattices in order to get information about relations between the hermitian lattices and the corresponding R–algebras with involution.

For the remainder of this introduction, let $K \subset L$ be a separable quadratic field extension and let S be the integral closure of R in L. If A is a quaternion L–algebra (that is, a central simple algebra of dimension 4 over L), then according to a theorem of A. Albert, A has an involution σ of the second kind (that is, $\sigma_{|L}$ is the non-trivial automorphism of L over K) if and only if there is a quaternion K–algebra A_σ such that $A = KA_\sigma$. We get such a situation in the particular case, when (V, h) is a hermitian space of dimension 2 over L and $A = \mathrm{End}_L(V)$. According to the general theory (see Proposition 3.7), h defines an involution σ_h on A with the desired property. A_{σ_h} can always be defined as a subring of A on which the involution σ_h coincides with the standard S–involution on A.

Our interest in this situation is motivated by a theorem of H. Franke in [4], saying that two skew-hermitian lattices (M, h) and (M', h') of rank 2 over the ring of integers S in a quadratic extensions over the rational numbers whose discriminant is a prime p congruent to 1 modulo 4, are equivalent if and only if the determinants of these lattices are equal assuming that the determinants are not divisible by p^2. If the last condition is not satisfied, then there exist exactly two equivalence classes of lattices with given determinants. This result shows that the determinant is a very strong invariant. It can be expected that the determinant of the lattice (M, h) on a hermitian space (V, h) over L is closely related to the discriminant of its endomorphism ring or some rings associated to it. This is the reason why we study S–orders in A and R–orders in A_{σ_h} which correspond to (M, h). In fact, we define in Section 3

$$\Lambda(M, h) = \mathrm{End}_S(M) \cap \sigma_h(\mathrm{End}_S(M)),$$

which is an S–order in $\mathrm{End}_L(V)$ invariant under the involution σ_h. We compute the discriminant of this S–order and the R–order $\Lambda(M, h)_\sigma = \Lambda(M, h) \cap A_{\sigma_h}$, when R is a local ring and (V, h) is a hermitian space of dimension 2 over L. Equivalence of hermitian lattices can be translated to isomorphism of the corresponding orders at least for some types of extensions $K \subset L$ (see Corollaries 6.12 and 6.14).

The paper is organized as follows. Assume that R is a Dedekind domain with quotient field K, such that $\mathrm{char}(K) \neq 2$. In Section 3, we give a generalization of Theorem 4.2 in [9], which says that, given a regular ϵ–hermitian form h (that is, $\overline{h(x,y)} = \epsilon h(y,x)$, where $\epsilon = \pm 1$) on a Λ–lattice M, there is a bijection between the set of certain classes of ϵ–hermitian forms on M and the set of involutions on the endomorphism ring of M.

In Section 4, we mention some well-known properties of the quaternion algebras and we present the theorem of A. Albert described above. We also study the corresponding arithmetic situation, where we show that given an S–order Λ in A, invariant under an involution σ of the second kind, we get an R–order $\Lambda_\sigma = A_\sigma \cap \Lambda$ in A_σ.

In Section 5, we give a summary of R. Jacobowitz's paper [6]. We present the theorem of N. Jacobson mentioned above. We give a proof of a corresponding result in the case of hermitian lattices over a complete DVR using the results due to R. Jacobowitz [6]. In the case of hermitian spaces over a quaternion algebra, with respect to an involution of the *second kind*, we show that a similar attempt to reduce the classification problem of hermitian spaces to quadratic spaces, fails. We exhibit non-isomorphic hermitian spaces whose corresponding quadratic spaces are isomorphic.

In Section 6, we study how the S–order $\Lambda(M,h)$ and the R–order $\Lambda(M,h)_\sigma$ mentioned above depend on the hermitian lattice. In Section 6.3, we assume that $K \subset L$ is an unramified or a ramified non-dyadic separable quadratic extension of local fields. We prove a statement similar to the theorem of H. Franke in [4] described above, where we classify certain hermitian lattices using their determinants (see Proposition 6.8).
Furthermore, let (M,h) and (M',h) be integral (that is, for all $x,y \in M$ $h(x,y) \in S$) hermitian S–lattices on a regular hermitian space (V,h) over L. We show that (M,h) and (M',h) are similar (that is, there is an $r \in R, r \neq 0$, such that (M,rh) and (M',h) are isomorphic as hermitian S–lattices) if and only if $\Lambda(M,h)$ and $\Lambda(M',h)$ are isomorphic S–orders, assuming that $\Lambda(M,h)$ is not a maximal order in $M_2(L)$ in the case $K \subset L$ is a ramified non-dyadic extension. If the last condition is not satisfied, then there exist exactly two classes of similar hermitian lattices with corresponding maximal S-orders. We show that (M,h) and (M',h) are similar if and only if $\Lambda(M,h)_\sigma$ and $\Lambda(M',h)_\sigma$ are isomorphic R–orders. We examine to what extent the determinant of the hermitian lattice determines the S–order $\Lambda(M,h)$. We also study the relationship between the orders $\Lambda(M,h)$ and $\Lambda(M,h)_\sigma$.

Section 2 contains definitions and some well-known properties of the objects considered in this paper.

2 Basic definitions

We shall start by giving definitions and some well-known properties of the basic objects considered in this paper.

Let R be a Dedekind domain with quotient field K.

2.1. Definition. Let Λ be an R–algebra with $R \subseteq \Lambda$. An R–**involution** on Λ is an R–linear map $\sigma : \Lambda \to \Lambda$ such that for all $x, y \in \Lambda$,

$$\begin{aligned} \sigma(x+y) &= \sigma(x)+\sigma(y), \\ \sigma(xy) &= \sigma(y)\sigma(x), \\ \sigma^2(x) &= x. \end{aligned}$$

2.2. Definition. Let Λ be an R–algebra with an R–involution σ and M a left Λ–module. Let $\epsilon = \pm 1$. An **ϵ–hermitian form** on M with respect to σ is a map $h : M \times M \to \Lambda$ such that for all $x, y, z \in M$ and a, $b \in \Lambda$,

$$\begin{aligned} h(x+z, y) &= h(x,y)+h(z,y), \\ h(ax, by) &= ah(x,y)\sigma(b), \\ h(x,y) &= \epsilon\sigma(h(y,x)). \end{aligned}$$

If $\sigma = \mathrm{id}_\Lambda$, then Λ is commutative and h is bilinear. In such a case, h is called symmetric if $\epsilon = 1$ and skew-symmetric if $\epsilon = -1$. If $\sigma \neq \mathrm{id}_\Lambda$, h is called hermitian if $\epsilon = 1$ and skew-hermitian if $\epsilon = -1$.

2.3. Definition. Let M be a left R–module. A map $q : M \to R$ is called a **quadratic form** if $q(rm) = r^2 m$ for all $r \in R$, $m \in M$, and

$$b_q(x,y) = q(x+y) - q(x) - q(y), \;\; x, y \in M$$

is a bilinear form. We call b_q **the bilinear form associated with** q.

2.4. Definition. Let V be a finite dimensional vector space over K. An R–**lattice on** V is a finitely generated R–module $M \subseteq V$ such that $KM = V$.

Note that if M is an R–lattice on a finite dimensional vector space V over K, then M is R–torsion free and hence R–projective, since R is a Dedekind domain (see [3], Introduction, §4D).

Moreover, any finitely generated R–torsion free R–module M can be considered as an R–lattice on a finite dimensional vector space V over K. Namely, choose $V = K \otimes_R M$. Since M is R–torsion free the map $m \mapsto 1 \otimes m$ is injective. Let us denote the image of M in $K \otimes_R M$ also by M. Then $V = KM$.

2.5. Definition. An R–**order** in a K–algebra A is a subring Λ of A, containing the unity element of A, and such that Λ is an R–lattice on A.

2.6. Definition. Let Λ be an R–order in a K–algebra A. We say that an R–lattice M on V is a Λ–**lattice on** V if $\Lambda M = M$.

Let Λ be a finitely generated R–torsion free R–algebra with an R–involution σ. Then Λ can be considered as an R–order in $A = K \otimes_R \Lambda$. We can extend σ uniquely to a K–involution on A by $\sigma'(a) = \frac{1}{r}\sigma(ra)$ for some $r \in R$, $r \neq 0$, such that $ra \in \Lambda$. It is easily checked that σ' is well defined. We also denote the extension σ' by σ.

Let M be a finitely generated left Λ–module, which is R–torsion free, and let $h : M \times M \to \Lambda$ be an ϵ–hermitian form. Thus M is a Λ–lattice on $V = K \otimes_R M$ and V is clearly an A–module. We can extend, in the same manner as above, the ϵ–hermitian form h to V with respect to the extension of σ to A.

2.7. Definition. Let A be a K–algebra with a K–involution σ and let Λ be an R–order invariant under the involution. Let V be a finitely generated left A–module with an ϵ–hermitian form $h : V \times V \to A$. By an ϵ–**hermitian** Λ–**lattice** (M, h) **on** (V, h) we mean a Λ–lattice M on V with h restricted to M. Assume that $R \subset K$. We call (M, h) **integral** if $h(x, y) \in \Lambda$ for all $x, y \in M$.

We denote the ϵ–hermitian Λ–lattice just by M when it is clear from the context which ϵ–hermitian form is considered. We also omit the Λ, when it is clear over which ring we consider the lattice. If $R = K$ in the above definition, then $\Lambda = A$ and $M = V$, and we say that (V, h) is an ϵ–hermitian **space** over A.

In a similar way, given a finite dimensional vector space V with a quadratic form $q : V \to K$, a quadratic R–lattice (M, q) on (V, q) is an R–lattice M on V with q restricted to M.

Let Λ be an R–algebra with an R–involution σ and let M be a left Λ–module. Then $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$ is a right Λ–module by $(f\lambda)(x) = f(x)\lambda$. We can consider M^* as a left Λ–module by defining $\lambda f = f\sigma(\lambda)$. We will always consider M^* as a left Λ–module in this sense. Given an ϵ–hermitian form $h : M \times M \to \Lambda$, we get a Λ-homomorphism of left Λ-modules by

$$\hat{h} : M \to M^*, \text{ where } \hat{h}(m)(x) = h(x, m),$$

for all $m, x \in M$.

2.8. Definition. We say that an ϵ-hermitian form h is **regular** on M if $\hat{h}$ is an isomorphism. We say that a quadratic form q is **regular** if b_q is regular.

2.9. Definition. Let (M, h) and (M', h') be ϵ–hermitian Λ–lattices. We say that (M, h) and (M', h') are **isometric**, which we denote by $(M, h) \cong (M', h')$, if there exists a Λ–isomorphism $\varphi : M \to M'$ such that for all $x, y \in M$ $h'(\varphi(x), \varphi(y)) = h(x, y)$.

In a similar way, we say that two quadratic R–lattice (M, q) and (M', q') are isometric, which we denote by $(M, q) \cong (M', q')$, if there is an R–isomorphism $\varphi : M \to M'$ such that for all $x \in M$, $q'(\varphi(x)) = q(x)$.

For any matrix $T = [t_{ij}]$ in $M_{m\times n}(\Lambda)$, $m, n \in \mathbb{N}$, let ${}^tT$ denote the transpose and T^σ the matrix $[\sigma(t_{ij})]$.

Let (M, h) be a free left ϵ–hermitian Λ–lattice and let $m_1, \dots, m_n$ be a basis for M over Λ. If $x \in M$ and $x = \sum_{i=1}^n x_i m_i$, let $\mathbf{x}$ denote the row vector $[x_1, \dots, x_n]$. Let H be the matrix $[h(m_i, m_j)]$, $1 \le i, j \le n$. Then $h(x, y) = \mathbf{x}H{}^t\mathbf{y}^\sigma$.
If the matrix H is diagonal, we say that M has an orthogonal basis with respect to h and we write

$$M = (a_1) \oplus \cdots \oplus (a_n),$$

where $h(m_i, m_i) = a_i$ for $i = 1, \dots, n$. In this case, we sometimes denote H by $< a_1, \dots, a_n >$.
Let (M, h) and (M', h') be isometric free ϵ–hermitian lattices of rank n over Λ. Let H and H' be the matrices of h, respectively h', with respect to some bases of M and M'. The definition of isometry implies that there is a $T \in GL_n(\Lambda)$ such that $H' = TH{}^tT^\sigma$.

Let us now look at a commutative situation. For the remainder of this section, let L be a field with an involution σ such that $K = \{l \in L : \sigma(l) = l\}$. Let S be the integral closure of R in L. Note that S is invariant under σ.

2.10. Definition. Let (M, h) and (M', h') be ϵ–hermitian S–lattices. We say that (M, h) and (M', h') are **similar**, which we denote by $(M, h) \sim (M', h')$, if there exists an $r \in R, r \neq 0$, such that $(M, rh) \cong (M', h')$.

Let (M, h) be a free ϵ–hermitian S–lattice. Given two matrices H and H' of h with respect to two bases for M over S, $\det H$ and $\det H'$ differ by a factor from $\mathrm{Nr}_{L/K}(S^\times) = \{x\sigma(x) : x \in S^\times\}$.

2.11. Definition. With the notations above, we denote the class of $\det H$ in $K^\times/\mathrm{Nr}_{L/K}(S^\times)$ by $\mathbf{d}(h)$ and call it the **determinant** of h.

We get a well-defined map, $(M, h) \mapsto \mathbf{d}(h)$, which sends free ϵ–hermitian S–lattices into $K^\times/\mathrm{Nr}_{L/K}(S^\times)$. In a similar way we get a well-defined map from ϵ–hermitian spaces over L into $K^\times/\mathrm{Nr}_{L/K}(L^\times)$.
For a quadratic S–lattice (M, q), we define $\mathbf{d}(q)$ to be the determinant of the associated symmetric bilinear form b_q.

2.12. Definition. Let (M, h) be an ϵ–hermitian S–lattice on a regular ϵ–hermitian space (V, h) over L. We define the **dual** $M^\#$ of M as

$$M^\# = \{v \in V \ : \ \forall m \in M \ \ h(v, m) \in S\}.$$

Then $M^\#$ is an S–lattice which is isomorphic to $M^* = \mathrm{Hom}_S(M, S)$, since $M^\#$ is the inverse image of M^* under the isomorphism $\hat{h} : V \to V^*$, where $\hat{h}(v)(x) = h(x, v)$ for all $v, x \in V$. Since S is a Dedekind domain, $M^{\#\#} = M$ ([15], Chapter 6, §1).

2.13. Definition. Let M and M' be S–lattices on a vector space V over L. By the **index** $[M' : M]$, we mean the S–ideal in L generated by determinants $\det \psi$ of all L–linear transformations $\psi : V \to V$ such that $\psi(M') \subseteq M$.

If $M \subseteq M'$, then $[M' : M] \subseteq S$ and it is easily checked that $[M' : M] = S$ if and only if $M' = M$.

2.14. Definition. Let (V, h) be a regular ϵ–hermitian space over L. Let (M, h) be an ϵ–hermitian S–lattice on (V, h). The index $[M^\# : M]$ is denoted by $\mathrm{D}(M)$ and called the **discriminant** of M.

If M is free over S with basis $m_1 \ldots, m_n$, then one checks easily that $\mathrm{D}(M) = (\det[h(m_i, m_j)])$.

2.15. Definition. Assume that R is a UFD and (V, h) is an ϵ–hermitian space over L. Let (M, h) be a free integral ϵ–hermitian S–lattice on (V, h). We say that h is R–**primitive** if there is no $r \in R$, $r \neq 0$, such that $(M, \frac{1}{r}h)$ is an integral ϵ–hermitian S–lattices on $(V, \frac{1}{r}h)$.

Let (M, h) be a free ϵ–hermitian S–lattice of rank 2, where h is R–primitive. This means that, given a basis for M over S, the matrix of h is of the form,

$$H = \begin{bmatrix} a_0 + a_1\omega & b_0 + b_1\omega \\ c_0 + c_1\omega & d_0 + d_1\omega \end{bmatrix},$$

where $S = R[\omega]$, $a_i, b_i, c_i, d_i \in R$, $i = 0, 1$, and the non-zero elements among the a_i, b_i, c_i, d_i, $i = 0, 1$, are relatively prime in R.

3 Involutions and forms

In this section, we are going to see that certain classes of ϵ–hermitian forms on a left Λ–lattice M correspond bijectively to involutions on the endomorphism ring of M.
Moreover, given a hermitian lattice M over a commutative ring and an involution σ on the endomorphism ring of $V = K \otimes_R M$, we define an order invariant under the involution σ, which is contained in the endomorphism ring of M. In Section 6, we investigate, in some special cases, to what extent this order determines the hermitian lattice. We start with some well-known results concerning involutions on algebras.

Throughout this section, let R be a Dedekind domain with quotient field K such that $\mathrm{char}(K) \neq 2$.

3.1 Involutions on algebras

Let Λ be an R–algebra with $R \subseteq \Lambda$. It is easily checked that the center of Λ, $\mathcal{Z}(\Lambda)$, is mapped to itself by any R–involution. The R–involutions are divided into two classes according to the following definition.

3.1. Definition. We say that an R–involution on an R–algebra Λ is of the **first kind** if its restriction to the center of Λ is the identity. If the restriction to the center is not the identity, we say that the involution is of the **second kind**.

In order to study the two kinds of involutions at the same time, we use the following notation.

3.2. Definition. Let Λ be an R–algebra . By an S/R–**involution** σ on Λ, we mean an R–involution on Λ, where $S = \mathcal{Z}(\Lambda)$ and $R = \{s \in S : \sigma(s) = s\}$.

The following theorem from [15], Chapter 8, §7, describes all involutions on a central simple algebra over a field given one fixed involution.

3.3. Theorem. *Let A be a finite dimensional central simple L–algebra with an L/K–involution σ.*

(1) *If* $\eta \in L$ *satisfies* $\eta\sigma(\eta) = 1$ *and* $a \in A^\times$ *is such that* $a = \eta\sigma(a)$, *then* $\sigma_a : A \to A$, *defined by* $\sigma_a(b) = a^{-1}\sigma(b)a$, *is an* L/K*–involution on* A.

(2) *If conversely* τ *is an arbitrary* L/K*–involution on* A, *then there is a unit* $a = \pm\sigma(a)$ *in* A *such that* $\tau = \sigma_a$.

(3) *If the involution* σ *is of the first kind,* a *is uniquely determined up to a scalar factor* $l \in L^\times$. *If the involution* σ *is of the second kind, one can find an* a *such that* $a = \sigma(a)$, *and this* a *is uniquely determined up to a scalar factor* $k \in K^\times$.

Proof. (1) is easily checked.

(2), (3) If τ and σ are L/K–involutions then $\sigma \circ \tau$ is an inner automorphism on A. According to the Skolem-Noether Theorem ([15], Chapter 8, §4) there is $c \in A^\times$ such that $(\sigma \circ \tau)(b) = c^{-1}bc$ for all $b \in A$. Then $\tau(b) = a^{-1}\sigma(b)a$ for all $b \in A$ where $a = \sigma(c)^{-1}$. We get that

$$b = \tau(\tau(b)) = a^{-1}\sigma(\tau(b))a = a^{-1}\sigma(a)b\sigma(a)^{-1}a.$$

This implies that $\sigma(a)^{-1}a = \eta \in L^\times$. ¿From $a = \sigma(a)\eta$, we get that $\sigma(a) = a\sigma(\eta) = \sigma(a)\eta\sigma(\eta)$ and hence $\sigma(\eta)\eta = 1$ since $\eta \in L^\times$. We have $\tau = \sigma_a$.

If σ is of the first kind, $\eta\sigma(\eta) = \eta^2 \in K^\times$ and hence $\eta = \pm 1$. It is clear that b is uniquely determined up to a scalar factor in $L^\times$.
If σ is of the second kind, then we can choose according to Hilbert's Theorem 90 ([11], Chapter VI, §6), $\alpha \in L^\times$ such that $\alpha\sigma(\alpha)^{-1} = \eta$. We replace a by $c = \alpha^{-1}a$. Then $\tau = \sigma_a = \sigma_c$ and $\sigma(c) = c$. This c is uniquely determined up to a factor in $K^\times$. □

We now turn to the arithmetic situation. Let A and σ be as in Theorem 3.3. Let S be the integral closure of R in L and let Λ be an S–order in A.

3.4. Lemma. *With the notations as above,* $\mathcal{Z}(\Lambda) = S$.

Proof. Observe first that $\mathcal{Z}(\Lambda) \subseteq \mathcal{Z}(A) = L$. Every $\lambda \in \Lambda$ is integral over S and since S is integral over R, $\mathcal{Z}(\Lambda)$ is integral over R. But S is the integral closure of R in L and hence $\mathcal{Z}(\Lambda) = S$. □

In general, an L/K–involution σ on A does not have to preserve Λ (see Example 3.15), but we can always find an S–order in A contained in Λ on which σ is an R–involution.

3.5. Proposition. *Let A be a finite dimensional central simple L–algebra with an L/K–involution σ. Let Λ be an S–order in A. Then σ is an S/R–involution on the S–order $\Lambda \cap \sigma(\Lambda)$ in A.*

Proof. It is clear that $\Lambda \cap \sigma(\Lambda)$ is a subring of A containing the unity element of A and that $\Lambda \cap \sigma(\Lambda)$ is finitely generated over S. Now $L(\Lambda \cap \sigma(\Lambda)) = A$, since for every $a \in A$ there are $s, s' \in S$ such that $sa \in \Lambda$ and $s'a \in \sigma(\Lambda)$ and hence $ss'a \in \Lambda \cap \sigma(\Lambda)$.

Let $x \in \Lambda \cap \sigma(\Lambda)$, then $\sigma(x) \in \Lambda \cap \sigma(\Lambda)$. Thus σ is an R–involution on $\Lambda \cap \sigma(\Lambda)$. ¿From Lemma 3.4, we get that $\mathcal{Z}(\Lambda \cap \sigma(\Lambda)) = S$. We need to prove that $R = \{s \in S : \sigma(s) = s\}$. It is clear that every element in R is fixed by σ. Let $s \in S$ such that $\sigma(s) = s$, so $s \in K$ and since R is integrally closed, $s \in R$. We have showed that σ is an S/R–involution on $\Lambda \cap \sigma(\Lambda)$. □

3.6. Definition. Let Λ and Λ' be R–algebras with involution σ and σ' respectively. A **homomorphism of algebras with involution** $f : (\Lambda, \sigma) \to (\Lambda', \sigma')$ is an R–homomorphism $f : \Lambda \to \Lambda'$ such that $f \circ \sigma = \sigma' \circ f$.

3.2 The adjoint involution

In this section, let Λ be a finitely generated R–torsion free R–algebra containing R, with an S/R–involution σ, where $S = \mathcal{Z}(\Lambda)$ is a Dedekind domain. Let M be a finitely generated left Λ–module such that M is R–torsion free. We are now going to study a connection between ϵ–hermitian forms on M and involutions on the endomorphism ring of M.

Note that we have a map from $\mathcal{Z}(\Lambda)$ to $\mathrm{End}_\Lambda(M)$ given by $\lambda \mapsto \varphi_\lambda$, where $\varphi_\lambda(m) = \lambda m$ for $m \in M$. The map is injective since M is R–torison free. We will always consider $\mathcal{Z}(\Lambda)$ embedded in $\mathrm{End}_\Lambda(M)$ in this sense.

The following proposition is a generalization of Proposition 4.1 in [9], Chapter 1.

3.7. Proposition. *Let the notations be as above. Then for each regular ϵ–hermitian form h on M, there exists a unique R–involution σ_h on $\mathrm{End}_\Lambda(M)$ such that $\sigma_h(a) = \sigma(a)$ for all $a \in \mathcal{Z}(\Lambda)$ and*

$$h(x, \sigma_h(f)(y)) = h(f(x), y) \text{ for all } x,\ y \in M \text{ and } f \in \mathrm{End}_\Lambda(M).$$

The involution σ_h is called the adjoint involution of $\mathrm{End}_\Lambda(M)$ *with respect to* h.

Proof. Let h be a regular ϵ–hermitian form on M, that is, the map $\hat{h} : M \to M^*$ is an isomorphism. Given $f \in \mathrm{End}_\Lambda(M)$, we define its transpose $f^t \in \mathrm{End}_\Lambda(M^*)$ by $f^t(\varphi) = \varphi \circ f$ for $\varphi \in M^*$. The desired involution is given by

$$\sigma_h(f) = \hat{h}^{-1} \circ f^t \circ \hat{h},$$

since $h(x, (\hat{h}^{-1} \circ f^t \circ \hat{h})(y)) = ((f^t \circ \hat{h})(y))(x) = (\hat{h}(y) \circ f)(x) = h(f(x), y)$ for all $x, y \in M$ and it is clear that σ_h is an R–involution. From the fact that h is regular, it follows that σ_h is unique and that $\sigma_h(a) = \sigma(a)$ for all $a \in \mathcal{Z}(\Lambda)$. □

Observe that all the rh, with $r \in R$, $r \neq 0$, define the same involution on $\mathrm{End}_\Lambda(M)$. Therefore we consider classes of ϵ–hermitian forms. We say that two hermitian forms h and h' on M belong to the same class if and only if there are $r, r' \in R, r, r' \neq 0$, such that $rh = r'h'$. Denote the class of h by $[h]$. Note that if $h' \in [h]$ and h is ϵ–hermitian, then h' is also ϵ–hermitian.

3.8. Lemma. *With notations as above, let h' be an ϵ'–hermitian form on M with respect to σ and h a regular ϵ–hermitian form on M. Let $g = \hat{h}^{-1} \circ \hat{h}' \in$* $\mathrm{End}_\Lambda(M)$ *then*

$$h'(x, y) = h(x, g(y)) \text{ for all } x, y \in M.$$

Proof. We have $\hat{h}' = \hat{h} \circ g$ and

$$h'(x, y) = \hat{h}'(y)(x) = ((\hat{h} \circ g)(y))(x) = h(x, g(y)).$$

□

Assume that $A = K \otimes_R \Lambda$ is a simple K–algebra. Thus $\mathcal{Z}(A) = L$ is the quotient field of S. It is easily checked that the extension of σ from Λ to A is an L/K–involution on A of the same kind as σ.
Let $V = K \otimes_R M$. From Lemma 3.4, we get that $\mathcal{Z}(\mathrm{End}_\Lambda(M)) = \mathcal{Z}(\Lambda)$, since $\mathrm{End}_\Lambda(M)$ is an S–order in the central simple L–algebra $\mathrm{End}_A(V)$.

Consider ϵ–hermitian forms on M whose extensions to V are regular. The classes of such forms on V define K–involutions on $\mathrm{End}_A(V)$ according to

Proposition 3.7, and we get the following generalization of Theorem 4.2 in [9], Chapter 1.

3.9. Theorem. *With the notations above, assume that there exists a regular ϵ–hermitian form $h_0 : M \times M \to \Lambda$.*

(1) If σ is of the first kind, the map $[h] \mapsto \sigma_h$ defines a one-to-one correspondence between the set of classes of ϵ–hermitian and classes of $-\epsilon$–hermitian forms on M regular on V such that σ_h preserves $\mathrm{End}_\Lambda(M)$, and the set of involutions of the first kind on $\mathrm{End}_\Lambda(M)$.

(2) If σ is of the second kind, the map $[h] \mapsto \sigma_h$ defines a one-to-one correspondence between the set of classes of ϵ–hermitian forms on M regular on V such that σ_h preserves $\mathrm{End}_\Lambda(M)$, and the set of involutions τ of the second kind on $\mathrm{End}_\Lambda(M)$ such that $\tau(x) = \sigma(x)$ for all $x \in \mathcal{Z}(\Lambda)$.

Proof. ¿From the remark following Proposition 3.7, the map $[h] \mapsto \sigma_h$, which sends a class of ϵ–hermitian forms to the adjoint involution on $\mathrm{End}_\Lambda(M)$, is well-defined.
¿From the fact that $\mathcal{Z}(\Lambda) = \mathcal{Z}(\mathrm{End}_\Lambda(M)) \subseteq L$ and $\sigma(a) = \sigma_h(a)$ for all $a \in L$, it follows that σ_h on $\mathrm{End}_\Lambda(M)$ is of the same kind as of σ on Λ.

Let us now prove the injectivity of the maps in (1) and (2). Let $\eta, \eta' = \pm 1$. Let h be an η–hermitian form and let h' be an η'–hermitian form on M, regular over V. Suppose that $\sigma_h(f) = \sigma_{h'}(f)$ for all $f \in \mathrm{End}_\Lambda(M)$. Then the equality holds for all $f \in \mathrm{End}_A(V)$ since $K\mathrm{End}_\Lambda(M) = \mathrm{End}_A(V)$. Lemma 3.8 gives us

$$(*) \qquad h'(x, y) = h(x, g(y)),$$

with $g = \hat{h}^{-1} \circ \hat{h}' \in \mathrm{End}_A(V)^\times$. This implies that $g \circ \sigma_{h'}(f) = \sigma_h(f) \circ g$ for all $f \in \mathrm{End}_A(V)$. So $g \circ \sigma_{h'}(f) = \sigma_{h'}(f) \circ g$ for all $f \in \mathrm{End}_A(V)$ which means that $g \in \mathcal{Z}(\mathrm{End}_A(V)) = \mathcal{Z}(A) = L$, since A is a simple algebra. By $(*)$, $h' = h\sigma(g)$.
If the involution σ on Λ is of the first kind, then the extension of σ to A is also of the first kind and hence $L = K$. Then h and h' belong to the same class of forms, since $h' = h\sigma(g)$ and $\sigma(g) \in K$.
If the involution σ on Λ is of the second kind, the equality $h' = h\sigma(g)$ with $g \in L$ implies that g is fixed by the involution and hence $g \in K$. Therefore h and h' belong to the same class of forms. So the injectivity is proved in (1) and (2).

We now want to prove the surjectivity. Let us start with (1) and assume that σ is an involution on Λ of the first kind, so $R = \mathcal{Z}(\Lambda)$. Let τ be an involution on $\mathrm{End}_\Lambda(M)$ of the first kind. We want to show that there is a class of ϵ–hermitian forms or a class of $-\epsilon$–hermitian forms with τ as adjoint involution. Let σ_0 be the adjoint involution on $\mathrm{End}_\Lambda(M)$ with respect to h_0. Now extend τ and σ_0 to involutions on $\mathrm{End}_A(V)$. From Theorem 3.3, it follows that there is $v \in \mathrm{End}_A(V)^\times$ such that $\sigma_0(v) = \pm v$ and $\tau(f) = v^{-1}\sigma_0(f)v$. Define for $x, y \in V$,

$$(**) \qquad h(x,y) = h_0(x, v(y)).$$

Then h is an ϵ–hermitian form if $\sigma(v) = v$ and a $-\epsilon$–hermitian form if $\sigma(v) = -v$. It is clear that h is regular on V. We get $h(f(x), y) = h(x, \tau(f)(y))$ since

$$\begin{aligned} h(f(x),y) &= h_0(f(x), v(y)) = h_0(x, \sigma_0(f)(v(y))) = \\ &= h_0(x, v(\tau(f)(y))) = h(x, \tau(f)(y)), \end{aligned}$$

for $x, y \in V$ and $f \in \mathrm{End}_A(V)$. Take $r \in R, r \neq 0$, such that $rv \in \mathrm{End}_\Lambda(M)$. Then rh is an ϵ–hermitian form or a $-\epsilon$–hermitian such that $rh(x,y) \in \Lambda$ when $x, y \in M$. The class of rh has τ as its image.

Finally, the surjectivity of (2). Assume that σ is an S/R–involution of the second kind on Λ. Let τ be an involution on $\mathrm{End}_\Lambda(M)$ of the second kind such that $\tau(x) = \sigma(x)$ for all $x \in \mathcal{Z}(\Lambda) = S$. ¿From this, and the fact that $\mathcal{Z}(\Lambda) = \mathcal{Z}(\mathrm{End}_\Lambda(M))$, it follows that $R = \{f \in \mathcal{Z}(\mathrm{End}_\Lambda(M)) : \tau(f) = f\}$. In other words, τ is an S/R–involution of the second kind on $\mathrm{End}_\Lambda(M)$. Let σ_0 be the adjoint involution on $\mathrm{End}_\Lambda(M)$ with respect to h_0. We can extend σ_0 and τ to L/K–involutions on the central simple L–algebra $\mathrm{End}_A(V)$. We can now proceed as in the proof of (1) with the only difference that σ_0 is of the second kind and therefore according to Theorem 3.3 there is a $v \in \mathrm{End}_A(V)^\times$ such that $\sigma_0(v) = v$. ¿From this it follows that h, defined as in $(**)$, is an ϵ–hermitian form. Exactly as in (1), we get that the class of rh, $r \in R$, $r \neq 0$, has τ as its image. □

In the previous theorem, we proved that given an involution of the first kind on Λ and a regular ϵ–hermitian form on M, certain classes of ϵ–hermitian and classes of $-\epsilon$–hermitian forms on M correspond to involutions of the first kind on $\operatorname{End}_\Lambda(M)$. We will now describe the involutions on $\operatorname{End}_\Lambda(M)$ corresponding to classes of ϵ–hermitian forms respectively classes of $-\epsilon$– hermitian forms.

Suppose $\sigma = \mathrm{id}_\Lambda$ in Theorem 3.9 so $\Lambda = \mathcal{Z}(\Lambda) = R$. Given a regular bilinear symmetric or skew-symmetric form on M, we get a correspondence between classes of symmetric and classes of skew-symmetric bilinear forms $b : M \times M \to R$, regular on V, such that σ_b preserves $\operatorname{End}_R(M)$, and involutions of the first kind on $\operatorname{End}_R(M)$.
Moreover, if we put $R = K$, we see that classes of regular symmetric and classes of regular skew-symmetric bilinear forms $b : V \times V \to K$ correspond to involutions of the first kind on $\operatorname{End}_K(V)$. With this in mind, we have the following definition:

3.10. Definition. Let A be a central simple L–algebra. An involution σ on A of the first kind is said to be of **orthogonal type** if for any splitting field L' of A and any isomorphism of algebras with involution $(A \otimes_L L', \sigma \otimes id_{L'}) \cong (\operatorname{End}_{L'}(V), \sigma_b)$, the bilinear form b is symmetric. If b is skew-symmetric then σ is said to be of **symplectic type**.
Let Λ be an S–algebra, where $S = \mathcal{Z}(\Lambda)$, and such that $A = \Lambda \otimes_S L$ is a central simple L–algebra. We say that an involution σ on Λ of the first kind, is of orthogonal (symplectic) type if the extension of σ to A is of orthogonal (symplectic) type.

It is possible to show ([9], Chapter 1) that the property of b being symmetric or skew-symmetric only depends on the involution and not on the choice of the splitting field L' nor on the choice of b.

Take $R = K$ in Theorem 3.9. Thus $\Lambda = A$ and $M = V$ and we get a part of Theorem 4.2 in [9]. If σ is of the first kind, we get that classes of ϵ–hermitian and classes of $-\epsilon$–hermitian forms $h : V \times V \to A$ correspond to involutions of the first kind on $\operatorname{End}_A(V)$. The following is also proved in [9], Theorem 4.2:

3.11. Proposition. *With notations as above, let σ be an involution on A of the first kind.*

(1) *If $\sigma = id_A$, then σ on A and σ_h on* $\operatorname{End}_A(V)$ *are of the same type if and only if h is a symmetric bilinear form.*

(2) *If $\sigma \neq id_A$, then σ on A and σ_h on* $\operatorname{End}_A(V)$ *are of the same type if and only if h is a hermitian form.*

Using this result and Definition 3.10, we get the following corollary to Theorem 3.9.

3.12. Corollary. *With notations as in Theorem 3.9, let σ be an involution on Λ of the first kind.*

(1) *If $\sigma = id_\Lambda$, then σ on Λ and σ_h on $\mathrm{End}_\Lambda(M)$ are of the same type if and only if h is a symmetric bilinear form.*

(2) *If $\sigma \neq id_\Lambda$, then σ on Λ and σ_h on $\mathrm{End}_\Lambda(M)$ are of the same type if and only if h is a hermitian form.*

3.13. Endomorphism and matrix algebras. We now introduce some matrix notations and translate involutions on endomorphism algebras into involutions on matrix algebras.

Let A be a simple K–algebra with a K–involution σ. Let V be a free left A–module and let $v_1, \ldots, v_n$ be a basis for V over A. If $f \in \mathrm{End}_A(V)$ and $f(v_i) = \sum_{j=1}^n t_{ij} v_j$ then $f(x) = \mathbf{x}T$ where $T = [t_{ij}] \in M_n(A)$. This gives us an isomorphism between $\mathrm{End}_A(V)$ and $M_n(A)$.

For any matrix T in $M_n(A)$ recall that ${}^tT$ denotes the transpose and T^σ the matrix $[\sigma(t_{ij})]$. We can construct an involution on $M_n(A)$ by $T \mapsto {}^tT^\sigma$.
Suppose σ is an L/K–involution on A, where $L = \mathcal{Z}(A)$. Given a regular hermitian form h on V, we get an involution σ_h on $\mathrm{End}_A(V)$ by Theorem 3.9, such that $h(f(x), y) = h(x, \sigma_h(f)(y))$. This corresponds to the involution on $M_n(A)$, which maps T to $H^tT^\sigma H^{-1}$. We also denote this involution by σ_h.

3.14. Definition of an invariant order. Let σ be an involution on a field L and let S be the ring of integers in L. Let (M, h) be an integral hermitian S–lattice on a regular hermitian space (V, h) over L. Consider $A = \mathrm{End}_L(V)$ and $\Lambda = \mathrm{End}_S(M)$. Then Λ is an S–order in the central simple L–algebra A. The involution σ_h does not have to preserve $\mathrm{End}_S(M)$ as the following example shows.

3.15. Example. Let $V = \mathbb{Q}v_1 + \mathbb{Q}v_2$, where $v_1, v_2 \in V$ is a basis for V over $\mathbb{Q}$ and let $M = \mathbb{Z}v_1 + \mathbb{Z}v_2$. Thus $A = \mathrm{End}_\mathbb{Q}(V) \cong M_2(\mathbb{Q})$ and $\Lambda = \mathrm{End}_\mathbb{Z}(M) \cong M_2(\mathbb{Z})$. Let $\sigma = \mathrm{id}_\mathbb{Q}$ and let h be a hermitian form on V with respect to σ with a matrix $H =< 1, 2 >$. The adjoint involution of h on $M_2(\mathbb{Q})$ is given by $\sigma_h(T) = H\,{}^tTH^{-1}$, for $T \in M_2(\mathbb{Q})$. But σ_h does not preserve $M_2(\mathbb{Z})$ since

$$\sigma\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} \notin M_2(\mathbb{Z}).$$

Let us denote

$$\Lambda(M,h) = \mathrm{End}_S(M) \cap \sigma_h(\mathrm{End}_S(M)). \tag{3.16}$$

This S–order is invariant under the involution σ_h according to Proposition 3.5. We study $\Lambda(M,h)$ in Section 6 when $\Lambda(M,h)$ is an order in a quaternion algebra in order to get information about the hermitian lattice. This order is also considered by W. Plesken in [13], where he studies sets of certain forms on a lattice M over this order and classifies these sets of forms by means of the endomorphism ring of $M \oplus M^*$. The following proposition gives us a different way to look at $\Lambda(M,h)$.

3.17. Proposition. *With the notations above,*

$$\Lambda(M,h) = \mathrm{End}_S(M) \cap \mathrm{End}_S(M^{\#}).$$

Proof. Given $f \in \mathrm{End}_S(M) \cap \sigma_h(\mathrm{End}_S(M))$, then for all $v \in M^{\#}$ and $m \in M$,

$$h(f(v),m) = h(v, \sigma_h(f)(m)) \in S,$$

since $\Lambda(M,h)$ is invariant under the involution. So $f \in \mathrm{End}_S(M^{\#})$ and hence $\Lambda(M,h) \subseteq \mathrm{End}_S(M) \cap \mathrm{End}_S(M^{\#})$. Let $f \in \mathrm{End}_S(M) \cap \mathrm{End}_S(M^{\#})$. We need to show that $\sigma_h(f) \in \mathrm{End}_S(M)$. For all $m \in M$ and $v \in M^{\#}$,

$$h(\sigma_h(f)(m), v) = h(m, f(v)) \in S.$$

So $\sigma_h(f)(m) \in M^{\#\#} = M$, since S is Dedekind, and hence $\sigma_h(f) \in \mathrm{End}_S(M)$. $\square$

3.18. Remark. If $M = M^{\#}$, or in other words if h is regular on M, then σ_h preserves $\mathrm{End}_S(M)$ according to Proposition 3.7 or Proposition 3.17. In Remark 6.6, we show that the converse is not true in general, that is, there exists involutions σ_h which preserve $\mathrm{End}_S(M)$ with h not regular on M.

4 Quaternion algebras

We now present a theorem due to Albert [1], Chapter X, §10, which describes an interesting property of the involutions on quaternion algebras: Given an involution σ of the second kind on a quaternion algebra A, let K be the field consisting of all elements in the center of A fixed by σ. Then the set of all elements in A, for which σ coincides with the canonical involution on A, forms a quaternion algebra A_σ over K. In this section, we also study the corresponding arithmetic situation, where we take an order Λ in A, invariant under σ and define an order Λ_σ in A_σ. In Section 6, we will study the orders Λ and Λ_σ when the quaternion algebra A splits. In that situation, Λ is the order already considered in (3.16) in a particular case.
We start by mentioning some well-known properties of quaternion algebras, which we will need in Section 5, where we study hermitian spaces over quaternion algebras and hermitian lattices over the maximal order in a quaternion division algebra.

Let S be a Dedekind domain with quotient field L.

4.1. Definition. A **quaternion algebra** A over L is a central simple algebra of dimension 4 over L.

Let us assume that $\mathrm{char}(L) \neq 2$ for the remainder of this section. It can be showed that there is a base of A, denoted by $1, i, j, k$, such that

$$i^2 = a,\ j^2 = b,\ ij = -ji,$$

where $a, b \in L^\times$. The quaternion algebra A is denoted by $(a, b)_L$. We can embed $(a, b)_L$ in the matrix algebra $M_2(L(\sqrt{a}))$ by

$$i \mapsto \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix},\ j \mapsto \begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix}. \tag{4.2}$$

If two quaternion algebras A and A' are isomorphic, we denote this by $A \cong A'$. We mention some properties of quaternion algebras (see [15], Chapter 2, §11).

4.3. Proposition. *For all $a, b, c, \alpha, \beta \in L^\times$:*

(1) $(a, b)_L \cong (a\alpha^2, b\beta^2)_L$,

(2) $(a,b)_L \cong (b,a)_L$,

(3) $(1,1)_L \cong (1,a)_L \cong (b,-b)_L \cong (c,1-c)_L \cong M_2(L)$,

(4) $(a,a)_L \cong (a,-1)_L$.

There is an involution of the first kind $\bar{\ } : A \to A$ defined by

$$\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mapsto \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k,$$

such that $\overline{a}a \in L$ for all $a \in A$. If $A = M_2(L)$, then $\bar{\ }$ is given by

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mapsto \begin{bmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{bmatrix},$$

where $\alpha_{rs}, \alpha_t \in L$, for $r, s = 1, 2$, $t = 1, \ldots 4$. It is well-known that this is the only involution of the first kind on A with the property that $\overline{a}a \in L$ for all $a \in A$. We call $\bar{\ }$ the **canonical involution** on A. We have $\mathrm{Tr}_{A/L}(a) = \overline{a} + a$ and $\mathrm{Nr}_{A/L}(a) = \overline{a}a$, where $\mathrm{Tr}_{A/L}$ is the reduced trace and $\mathrm{Nr}_{B/L}$ is the reduced norm in A ([9], Chapter 1, §2C).
It is possible to regard $(a,b)_L$ as a cyclic algebra $(L(\sqrt{a})/L, \sigma, b)$. So $(a,b)_L \cong M_2(L)$ if and only if $b \in \mathrm{Nr}_{L(\sqrt{a})/L}(L(\sqrt{a})^\times)$, see [15], Chapter 8, §12.

Here is the theorem of Albert [1] mentioned above. The proof presented here is from [15], Chapter 8, §11.

4.4. Theorem. *Let A be a quaternion algebra over L. Let $K \subset L$ be a separable quadratic extension. Assume that σ is an L/K–involution on A of the second kind. Then there exists exactly one quaternion algebra $A_\sigma \subset A$ over K such that the restriction of σ to A_σ is the canonical involution on A_σ. Conversely, if A_1 is a quaternion algebra over K contained in A, then there is exactly one L/K–involution σ on A such that the restriction of σ to A_1 is the canonical involution on A_1.*

Proof. Let $\bar{\ }$ be the canonical involution on A and σ be an involution of the second kind on A. Let us first note that these two involutions commute. Put $\tau(x) = \sigma(\overline{\sigma(x)})$ for $x \in A$. This is an involution of the first kind such that $\tau(x)x \in L$ for all $x \in A$. So τ is the canonical involution and hence $\overline{\sigma(x)} = \sigma(\overline{x})$ for all $x \in A$.
Let

$$A_\sigma = \{x \in A \,:\, \sigma(x) = \overline{x}\}.$$

It is clear that A_σ is a K–subalgebra of A and $\sigma(A_\sigma) = A_\sigma$. The idea is to show that $A \cong A_\sigma \otimes_K L$, since this implies that A_σ is a central simple algebra over K. The universal property of tensor products of algebras gives us a unique L–algebra homomorphism $f : A_\sigma \otimes_K L \to A$. Write $L = K(\alpha)$ with $\sigma(\alpha) = -\alpha$. We show that $A = A_\sigma \oplus \alpha A_\sigma$. It will then follow that f is an isomorphism.
Let $x = \alpha z \in A_\sigma \cap \alpha A_\sigma$ where $x,\ z \in A_\sigma$. Then $\sigma(x) = \sigma(\alpha)\sigma(z) = -\alpha\overline{z} = -\overline{x}$ which implies that $x = 0$. Therefore $A_\sigma \cap \alpha A_\sigma = \{0\}$.
Since σ commutes with the canonical involution on A, we get that for all $x \in A,\ x + \overline{\sigma(x)} \in A_\sigma$ and

$$x = \frac{1}{2}(x + \overline{\sigma(x)}) + \frac{1}{2}\alpha(\alpha^{-1}(x - \overline{\sigma(x)})$$

shows that $A = A_\sigma \oplus \alpha A_\sigma$. It is clear that $\sigma|_{A_\sigma} =^{-} |_{A_\sigma}$ is the canonical involution and that A_σ is uniquely determined.
Conversely, let $A_1 \subset A$ be a quaternion algebra over K. Let σ be the canonical involution on A_1 and ρ the non-trivial automorphism on L. Then there is a unique extension of σ to $A_1 \otimes_K L \cong LA_1 = A$ by $\sigma(b \otimes \alpha) = \sigma(b) \otimes \rho(\alpha)$. It is clear that σ is an L/K–involution of the second kind on A and $A_1 = A_\sigma$.

□

4.5. Remark. This theorem also holds in the case $\mathrm{char}(L) = 2$, see [15], Chapter 8, §11.

For every unit a in A such that $\sigma(a) = a$, we get an L/K–involution of the second kind on A according to Theorem 3.3, by letting

$$\sigma_a(x) = a\sigma(x)a^{-1},\ x \in A.$$

Then $A_{\sigma_a} = \{x \in A : \sigma_a(x) = \overline{x}\}$ is a quaternion algebra over K by Theorem 4.4.

4.6. Proposition. *With the notations above, A_{σ_a} and A_{σ_b} are isomorphic if and only if there exist $k \in K^\times$ and $c \in A^\times$ such that $a = kcb\sigma(c)$.*

Proof. Suppose that there exists a K–algebra isomorphism $\varphi : A_{\sigma_a} \to A_{\sigma_b}$. Extend φ to an L–algebra automorphism of A. From the Skolem-Noether Theorem ([15], Chapter 8, §4), it follows that there is a $c \in A^\times$ such that $\varphi(x) = c^{-1}xc$. For all $x \in A_{\sigma_a}$, $\varphi(x)$ is an element of A_{σ_b}, that is, $b\sigma(\varphi(x))b^{-1} = \overline{\varphi(x)}$. Thus $b\sigma(c)\sigma(x)\sigma(c)^{-1}b^{-1} = c^{-1}\overline{x}c$, using that $c^{-1} = \mathrm{Nr}_{L/K}(c)^{-1}\overline{c}$. Now $\overline{x} = a\sigma(x)a^{-1}$ and this give us that

$$b\sigma(c)\sigma(x)\sigma(c)^{-1}b^{-1} = c^{-1}a\sigma(x)a^{-1}c$$

for all $x \in A_{\sigma_a}$, hence $a^{-1}cb\sigma(c) \in K^{\times}$. We have showed that there is $k \in K^{\times}$ and $c \in A^{\times}$ such that $a = kcb\sigma(c)$.
Conversely, assume that $a = kcb\sigma(c)$, where $k \in K^{\times}$, $c \in A^{\times}$, and define $\varphi : A_{\sigma_a} \to A_{\sigma_b}$ by $\varphi(x) = c^{-1}xc$. Similar arguments as above show that φ is a K–algebra isomorphism. □

Let us now turn our attention to orders in quaternion algebras. One can check that the discriminant of an order in a quaternion algebra, with respect to the reduced trace, is always a square (see for example [2]).

4.7. Definition. Let A be a quaternion L–algebra and let Λ be an S–order in A. The **reduced discriminant** $d(\Lambda)$ of Λ is defined to be the square root of $\mathrm{D}(\Lambda)$, where $\mathrm{D}(\Lambda)$ is the discriminant of Λ with respect to the reduced trace $\mathrm{Tr}_{A/L}$ according to Definition 2.14.

For the rest of this section, let σ be an L/K–involution of the second kind on a quaternion algebra A, and let A_σ be the quaternion algebra over K given in Theorem 4.4. Let R be the ring of integers in K. Note that, for any S–order Λ in A, $\overline{\Lambda} = \Lambda$. This follows from the fact that $\mathrm{Tr}_{A/L}(x) \in S$ for all $x \in \Lambda$ ([14], Chapter 2, Theorem 8.6) and $\overline{x} = \mathrm{Tr}_{A/L}(x) - x \in \Lambda$. As seen in Example 3.15, σ does not have to preserve an S–order Λ in A, but we can always consider the invariant S–order $\Lambda \cap \sigma(\Lambda)$ from Proposition 3.5. We now introduce the following R–order.

4.8. Proposition. *Let Λ be an S–order in A invariant under σ. Then*

$$\Lambda_\sigma = \{x \in \Lambda \,:\, \sigma(x) = \overline{x}\}$$

is an R–order in the quaternion algebra A_σ over K.

Proof. It is clear that $\Lambda_\sigma \subseteq A_\sigma$ is an R–algebra. Since Λ is finitely generated over R and $\Lambda_\sigma \subseteq \Lambda$, it follows that Λ_σ is finitely generated over R.
It remains to show that Λ_σ is an R–lattice on A_σ. Take $x \in A_\sigma$. There is an

$s \in S$, $s \neq 0$, such that $sx \in \Lambda$. Since S is invariant under σ, $\sigma(s)sx \in \Lambda$. Now $\sigma(s)s \in R$, since it is fixed by σ and $\sigma(\sigma(s)sx) = \sigma(x)\sigma(s)s = \overline{x}\,\overline{\sigma(s)s} = \overline{(\sigma(s)sx)}$ implies that $\sigma(s)sx \in \Lambda_\sigma$. Thus we have shown that $K\Lambda_\sigma = A_\sigma$. $\square$

Given an L/K-involution σ of the second kind on A, there is an quaternion algebra A_σ contained in A by Theorem 4.4, an invariant S-order Λ in A by Proposition 3.5 and an R-order Λ_σ in A_σ by Proposition 4.8. The following figure shows the various inclusions.

(4.9)

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & A \arrow[rr, dash] \arrow[dd, dash] & & A_\sigma \arrow[dd, dash] \\
L \arrow[ur, dash] \arrow[rr, dash, crossing over] \arrow[dd, dash] & & K \arrow[ur, dash] & \\
 & \Lambda \arrow[rr, dash] & & \Lambda_\sigma \\
S \arrow[ur, dash] \arrow[rr, dash] & & R \arrow[ur, dash] \arrow[uu, dash, crossing over] &
\end{tikzcd}
$$

We have a surjective map from the set of S–orders in A invariant under the involution σ to the set of R–orders in A_σ, given by $\Lambda \mapsto \Lambda_\sigma$: If Λ' is an R–order in A_σ, then $S\Lambda'$ is an S–order invariant under σ on A and $(S\Lambda')_\sigma = \Lambda'$.

4.10. Proposition. *Let Λ and Λ' be S–orders in A invariant under σ and let $\Psi : \Lambda \to \Lambda'$ be an S–algebra isomorphism. Then the restriction of Ψ to Λ_σ is an isomorphism between Λ_σ and Λ'_σ if and only if $\Psi \circ \sigma = \sigma \circ \Psi$ on Λ_σ.*

Proof. Note that $\overline{\Psi(x)} = \Psi(\overline{x})$ since after we extend Ψ to A, the Skolem-Noether Theorem gives us a $c \in A^\times$ such that $\Psi(x) = c^{-1}xc$ for all $x \in A$, and $\overline{\Psi(x)} = c^{-1}\overline{x}c = \Psi(\overline{x})$, using that $c^{-1} = \mathrm{Nr}_{L/K}(c)^{-1}\overline{c}$. ¿From this we get that $\Psi(x) \in \Lambda'_\sigma$ for all $x \in \Lambda_\sigma$ if and only if $\sigma(\Psi(x)) = \overline{\Psi(x)} = \Psi(\overline{x}) = \Psi(\sigma(x))$ for all $x \in \Lambda_\sigma$. If $\Psi(x) \in \Lambda'_\sigma$ for all $x \in \Lambda_\sigma$, it is clear that Ψ is an R–algebra isomorphism. $\square$

In Section 6, we study the situation in figure (4.9) in a special case when the quaternion algebra A splits. In Section 6.3, we give an example where $S\Lambda_\sigma = \Lambda$. We also give an example where the σ–invariant S–orders Λ and Λ' are isomorphic even though the corresponding R–orders Λ_σ and Λ'_σ are not isomorphic.

5 Hermitian lattices

In this section, we present some of the cases when the classification problem of hermitian lattices over rings is solved. In Section 5.1, we give a short summary of the results of Jacobowitz in [6], where hermitian lattices over a complete DVR and over the maximal order in a quaternion division algebra over a local field are classified.
One of the methods that was used in the classification of hermitian spaces over fields and over quaternion division algebras with respect to the canonical involution ([15], Chapter 10) was to relate hermitian spaces to quadratic spaces. This is based on a theorem of Jacobson [7], which we present in Section 5.2. We also show a corresponding arithmetic result in the case of hermitian lattices over a complete DVR using the results due to Jacobowitz [6]. In the case of hermitian spaces over a quaternion algebra with respect to an involution of the second kind, we exhibit a counter example to show that a similar statement would not be true.
Franke showed in [4], Chapter 1, §2, the following simple criterion to determine if two integral skew-hermitian lattices of rank 2 over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{p})]$, where $p \in \mathbb{N}$ is a prime congruent to 1 modulo 4, are isometric:

5.1. Theorem. *Let $p \in \mathbb{N}$ be a prime congruent to 1 modulo 4. Let (M,h) and (M',h') be two integral $\mathbb{Z}$–primitive skew-hermitian lattices of rank 2 over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{p})]$. If $\mathbf{d}(h) \not\equiv 0 \mod p^2$ and $\mathbf{d}(h') \not\equiv 0 \mod p^2$ then*

$$(M,h) \cong (M',h') \text{ if and only if } \mathbf{d}(h) = \mathbf{d}(h').$$

There are exactly two classes of isometric $\mathbb{Z}$–primitive skew-hermitian lattices (M,h) of rank 2 over $\mathbb{Z}[\frac{1}{2}(1+\sqrt{p})]$ with $\mathbf{d}(h) \equiv 0 \mod p^2$.

We show a similar result in Section 6.3 in a slightly different setting.

In Section 5.1, we are going to see that every hermitian lattice over a complete DVR is an orthogonal sum of submodules of rank 1 or 2. For hermitian spaces over division algebras, we have the following result.

5.2. Proposition. *Let A be a division algebra with an involution σ. Assume that $\mathrm{char}(A) \neq 2$. Let (V,h) be a hermitian space over A. Then (V,h) has an orthogonal basis.*

Proof. The statement holds if $h = 0$ or $\dim_A V = 1$. Assume that $h \neq 0$ and that $h(x,x) = 0$ for all $x \in V$. Since $h \neq 0$ there are $x, y \in V$ such that $h(x,y) \neq 0$. Put $z = h(x,y)^{-1}x$. We get

$$h(y+z, y+z) = h(y,z) + h(z,y) = 2 \neq 0$$

This is a contradiction. This means that there is $x \in V$ such that $h(x,x) \neq 0$. Put $U = \{v \in V : h(x,v) = 0\}$. Since $Ax \cap U = \{0\}$ and $v - h(v,x)h(x,x)^{-1}x \in U$ for all $v \in V$, we get that $V = Ax \oplus U$. Now we can proceed by induction. $\square$

5.1 Hermitian lattices over complete rings

We now give a summary of the paper of Jacobowitz [6]. We will see an application of the theory presented here in Section 5.2 and in the study of certain orders in Sections 6.2 and 6.3.

We shall begin with some well-known properties of complete discrete valuation rings and of quaternion algebras over local fields. For more information see [14], Chapter 3.

Let R be a complete DVR, that is, R is a principal ideal domain with a unique maximal ideal $\mathfrak{p} = \pi R$, and R is complete with respect to the standard $\mathfrak{p}$–adic norm which will be denoted by $|\cdot|$. Assume that the residue class field $R/\mathfrak{p}$ is finite. Let K be the quotient field of R, so K is a local field. We are going to assume that either,

1) A is a quaternion division algebra over K and $^-$ is the canonical involution, or

2) $A \supset K$ is a quadratic field extension and $^-$ is the non-trivial automorphism of A.

In both cases, it is well-known that we can extend the norm to A in a unique way by $|x| = |\mathrm{Nr}_{A/K}(x)|^{\frac{1}{2}}$ making A complete ([14], Chapter 2, Corollary 13.6). Let $\Lambda = \{x \in A : |x| \leq 1\}$ be the unique maximal order in A. Every Λ–ideal in A is two-sided and principal. Let $\mathfrak{P} = \{x \in A : |x| < 1\}$ be the unique two-sided maximal prime ideal in Λ with a generator Π. Then $\Lambda/\mathfrak{P}$ is a finite field ([14], Chapter 3, Theorem 14.3).
If $|2| < 1$, then A will be called dyadic, if $|2| = 1$, then A will be called non-dyadic.

Throughout this section let (M,h) be a hermitian Λ–lattice on a regular hermitian space (V,h) over A.

Observe that M is free over Λ : Since Λ is maximal, M is isomorphic to a direct sum of left ideals in Λ ([14], Chapter 1, Theorem 2.44). Every ideal in Λ is principal and hence M is free over Λ.

5.3. Definition. Let $\alpha \in K^{\times}$. By a hermitian lattices (M,h) **scaled** by α, we mean the lattice M with a hermitian form given by αh. A scalar $\alpha \in A$ is said to be **represented** by M if there is an element x in M such that $h(x,x) = \alpha$. A non-zero element $x \in M$ is called **isotropic** if $h(x,x) = 0$. Let the **scale** of M, sM, be the left Λ–ideal in A generated by all $h(x,y)$ with $x, y \in M$. By the **norm** nM of M we mean the left Λ–ideal in A generated by all $h(x,x)$ with $x \in M$. Note that $nM \subseteq sM$. If $nM = sM$ we say that M is **normal**, otherwise we call M **subnormal**.

5.4. Definition. For any non-zero scalar a, let $aM = \{ax : x \in M\}$. We say that an element x in M is **maximal** if $x \notin \Pi M$.

Every element in a basis for M is maximal in M and a maximal element in M can always be extended to a basis of M.

5.5. Definition. Let $i \in \mathbb{Z}$. A hermitian lattice M is called Π^i**-modular** if $h(x,M) = \Pi^i\Lambda$ for every maximal element $x \in M$.

If M is a Π^i–modular lattice then $sM = \Pi^i\Lambda$. We call hermitian lattices of rank 1 and 2 lines and planes. Examples of Π^i–modular lattices are lines Λm with $|h(m,m)| = |\Pi|^i$ and planes $\Lambda m_1 + \Lambda m_2$ with $|h(m_1,m_2)| = |\Pi^i|$ and $|h(m_k,m_k)| < |\Pi^i|$, $k = 1,2$.
For each $i \in \mathbb{Z}$, let $\mathrm{H}(i)$ denote the Π^i–modular plane with hermitian form

$$\begin{bmatrix} 0 & \Pi^i \\ \overline{\Pi^i} & 0 \end{bmatrix}.$$

5.6. Definition. A direct sum $M = M_1 \oplus \cdots \oplus M_k$ is called a **Jordan splitting** if all the M_i are modular and $sM_k \subset \cdots \subset sM_1$.
Let $M = M_1 \oplus \cdots \oplus M_k$ and $M' = M'_1 \oplus \cdots \oplus M'_{k'}$ be Jordan splittings. They are of the same **type** if $k = k'$ and for each $i \in \{1 \ldots k\}$, $sM_i = sM'_i$, $\mathrm{rank}_\Lambda M_i = \mathrm{rank}_\Lambda M'_i$ and M_i and M'_i are both normal or both subnormal.

Jordan splittings of isometric hermitian lattices are of the same type, in particular, two Jordan splittings of the same hermitian lattice are of the same type. If two hermitian lattices M and M' have Jordan splittings of the same type, we say that M and M' are of the same type.

5.7. Definition. Let $M = \oplus M_j$ be a Jordan splitting. Let $s(j)$ denote the integer such that $sM_j = \Pi^{s(j)}\Lambda$.

We have the following result from [6], §4:

5.8. Proposition. *Let M be a hermitian Λ–lattice with a basis $m_1, \dots, m_k$. If $a_i = -\frac{h(m_i,m_1)}{h(m_1,m_1)}$ is an integer for each $i \geq 2$, then $M = m_1\Lambda \oplus (m_1\Lambda)^\perp$ with $(m_1\Lambda)^\perp = \sum_{i=2}^k (m_i + a_i m_1)\Lambda$.*

If

$$
\begin{aligned}
b_i &= \frac{h(m_i,m_1)h(m_2,m_2) - h(m_i,m_2)h(m_2,m_1)}{h(m_2,m_1)h(m_1,m_2) - h(m_1,m_1)h(m_2,m_2)}, \\
c_i &= \frac{h(m_i,m_2)h(m_1,m_1) - h(m_i,m_1)h(m_1,m_2)}{h(m_2,m_1)h(m_1,m_2) - h(m_1,m_1)h(m_2,m_2)}
\end{aligned}
$$

are integers for each $i \geq 3$, then $M = (m_1\Lambda + m_2\Lambda) \oplus (m_1\Lambda + m_2\Lambda)^\perp$ and $(m_1\Lambda + m_2\Lambda)^\perp = \sum_{i=3}^k (m_i + b_i m_1 + c_i m_2)\Lambda$.

5.9. Corollary. *M can always be written as an orthogonal sum of lines and modular planes.*

Proof. Let $M = m_1\Lambda + \cdots + m_k\Lambda$ and consider the maximum of the set $X = \{|h(m_i,m_j)| : 1 \leq i,j \leq k\}$. If there is an $i \in \{1, \dots, k\}$ such that $|h(m_i,m_i)|$ is the maximum of X, then $M = m_i\Lambda \oplus (m_i\Lambda)^\perp$. Otherwise, there are $i,j \in \{1,\dots,k\}$, $i \neq j$, such that $|h(m_i,m_j)|$ is the maximum of X and thus $M = (m_i\Lambda + m_j\Lambda) \oplus (m_i\Lambda + m_j\Lambda)^\perp$. It is clear that $m_i\Lambda + m_j\Lambda$ is a modular plane. Now we can proceed by induction. □

We are going to classify hermitian lattices over Λ by their Jordan splittings and, in the commutative case, their determinants. In order to do this we study the possibility of improving the result of Corollary 5.9, to find in some sense, the "nicest" possible basis for a modular plane. We first look at the non-commutative case as it turns out to be the easiest.

5.1.1 Non-commutative case

Assume that A is a quaternion division algebra and $^-$ the canonical involution. Up to isomorphism there is just one quaternion division algebra over a local field. We can therefore assume that $A = (\pi, \epsilon)_K$, where ϵ is a non-square unit of the form $1 + 4\eta$, $\eta \in R^\times$ ([12], Chapter VI, §63B). In other words, with notations as in Section 4, A has a basis $1, i, j, k$, over K where $i = \Pi$ and $i^2 = \Pi^2 = \pi$, $j^2 = \epsilon$, $k = ij$.

5.10. Proposition. *Let M be a Π^i–modular lattice of rank n. Then*

(1) $M \cong (\pi^{i/2}) \oplus \cdots \oplus (\pi^{i/2})$ *if i is even,*

(2) $M \cong \mathrm{H}(i) \oplus \cdots \oplus \mathrm{H}(i)$ *if i is odd.*

Proof. By scaling M with an appropriate power of π we may assume that $i = 0$ or $i = 1$.
(1) Case $i = 0$. By Corollary 5.9, we can assume that $n = 2$. Suppose $M = \Lambda m_1 + \Lambda m_2$. If $|h(m_1, m_2)| \leq |h(m_k, m_k)|$ for $k = 1$ or $k = 2$ we are done according to Proposition 5.8. Let us assume that $|h(m_1, m_2)| > |h(m_k, m_k)|$, $k = 1, 2$. Let $a = 1$ in the non-dyadic case and let $a = \frac{1}{2}(1 + j)$ in the dyadic case. It is easily checked that a has the property that $|a| = |a + \overline{a}| = 1$. Changing the base to $m_1' = am_1 + m_2$ and $m_2' = m_2$ gives us that $|h(m_1', m_1')| = 1$ and we can apply Proposition 5.8.
Left to show is that given $m \in M$ such that $|h(m, m)| = 1$, Λm represents 1. But this is true for Am since, considering the quadratic space (AM, q) over K with $q(x) = h(x, x)$, $x \in M$, we have $\dim_K(AM, q) = 4 \dim_A(AM, h)$ and every quadratic space of dimension at least 4 over a local field represents every non-zero scalar (see [15], Chapter 6, §4). So there is $a \in A$ such that $h(am, am) = 1$. Thus $|h(am, am)| = |a|^2 = 1$, so $a \in \Lambda$ and hence Λm represents 1.

(2) Case $i = 1$. Let us first note that M is Π–modular if and only if its orthogonal components are Π–modular. Therefore M must be an orthogonal sum of Π–modular planes, since lines never are Π–modular. It thus suffices to assume that M is a Π–modular plane. Considering (AM, q) and the fact that every quadratic space of dimension at least 5 is isotropic, it follows that AM is isotropic. Let $m_1 \in M$ be a maximal isotropic element. Consider a base $m_1, m_2 \in M$. Let $b = -\frac{1}{2\Pi}h(m_2, m_2)$ in the non-dyadic case and let $b = -\frac{1}{2\Pi}(1 + j)h(m_2, m_2)$ in the dyadic case. Then $b \in \Lambda$ and changing the

base to $m_1' = m_1$ and $m_2' = bm_1 + m_2$ gives us $M = \mathrm{H}(1)$. $\square$

5.11. Theorem. *Suppose A is a quaternion division algebra over K. Then two hermitian Λ–lattices are isometric if and only if they are of the same type.*

5.1.2 Commutative case

In this section, we assume A to be commutative, that is, $A \supset K$ is a quadratic extension of local fields. Let us denote A by L, Λ by S and $^{-}$ by σ. If M is a modular plane, we have the following possibilities of finding a nice basis for M depending whether the extension $K \subset L$ is unramified or ramified and dyadic or non-dyadic.

5.12. Proposition. *Let M be a Π^i–modular plane with $i = 0$ or $i = 1$. Then we have the following.*

(I) *If $K \subset L$ is an unramified extension then M has an orthogonal basis.*

(II) *Suppose $K \subset L$ is a ramified extension.*

1) *If $K \subset L$ is non-dyadic, then*

a) *there is an orthogonal basis for M if $i = 0$,*

b) *$M \cong \mathrm{H}(i)$ if $i = 1$.*

2) *If $K \subset L$ is dyadic, then either*

a) *M is normal and M has orthogonal basis or,*

b) *M is subnormal and h is of the form*

$$\begin{bmatrix} \pi^m & \Pi^i \\ \sigma(\Pi)^i & \epsilon\pi^{m+l} \end{bmatrix},$$

where $nM = \pi^m S$, $m, l \in \mathbb{N}$, $m \neq 0$, and $\epsilon \in R^\times \cup \{0\}$.

Proof. See Proposition 4.4 and §7 in [6] for a proof of (I).
If $K \subset L$ is a non-dyadic ramified extension, see Proposition 8.1 in [6], and if $K \subset L$ is a dyadic ramified extension, see Proposition 10.2 in [6] for a proof of (II). $\square$

5.13. Theorem. *Let $K \subset L$ be an unramified quadratic field extension. Then two hermitian S–lattices are isometric if and only if they are of the same type.*

Proof. See §7 in [6]. □

5.14. Theorem. *Let $K \subset L$ be a ramified quadratic extension of non-dyadic fields. Then two hermitian S–lattices (M, h) and (M', h') are isometric if and only if the following conditions holds.*

(1) *M and M' are of the same type.*

(2) *Given Jordan splittings, $M = \oplus_j M_j$ and $M' = \oplus_j M'_j$, then for each index j such that $s(j)$ is even, we have $\mathbf{d}(h_{|M_j}) = \mathbf{d}(h_{|M'_j})$.*

Proof. See Theorem 8.3 in [6]. □

5.15. Remark. A similar classification theorem is proved in [6], §11, in the case of a ramified quadratic dyadic field extension.

5.2 Quadratic and hermitian lattices

Let R be a Dedekind domain with quotient field K such that $\mathrm{char}(K) \neq 2$. As in Section 5.1, we assume that either A is, a quaternion division algebra over K and $^-$ is the canonical involution, or $A \supset K$ is a quadratic field extension and $^-$ is the non-trivial automorphism of A.

Let (V, h) be a regular hermitian space over A with respect to $^-$. The hermitian form h over A gives rise to a quadratic form over K by

$$q(x) = h(x, x),\ x \in V. \tag{5.16}$$

In this section, we present a theorem of Jacobson [7] which shows that classification of hermitian spaces over A can be reduced to classification of quadratic spaces over K defined by (5.16).

We study this theorem in two different aspects. First we look at hermitian lattices over a complete DVR with finite residue class field and their corresponding quadratic lattices, using the results in Section 5.1. Then we consider hermitian spaces over a quaternion algebra with respect to an involution of the *second kind*. We define a similar correspondence between hermitian and quadratic spaces as defined in (5.16) and show that there exists non-isometric hermitian forms for which the corresponding quadratic spaces are isometric.

5.2.1 Jacobson's Theorem

In order to prove Jacobson's Theorem, we need the following cancellation theorem which is due to Witt (for a proof see [15], Chapter 1, §6).

5.17. Theorem. *Let k be a local ring in which 2 is invertible and let M be a finitely generated projective k–module. If (M,q) (M,q_1) (M,q_2) (M,q') are regular quadratic k–lattices such that*

$$(M,q) \oplus (M,q_1) \cong (M,q') \oplus (M,q_2)$$

and

$$(M,q) \cong (M,q'),$$

then $(M,q_1) \cong (M,q_2)$.

Using the notations introduced in the beginning of Section 5.2, Jacobson's Theorem says:

5.18. Theorem. *Two regular hermitian spaces (V,h) and (V',h') over A are isometric if and only if the corresponding quadratic spaces (V,q) and (V',q') over K are isometric.*

Proof. Evidently, if (V,h) and (V',h') are isometric then (V,q) and (V',q') are isometric.
Assume that (V,q) and (V',q') are isometric over K, that is, there is a K–isomorphism $\varphi : V \to V'$ such that $q'(\varphi(x)) = q(x)$ for all $x \in V$. ¿From Proposition 5.2, it follows that there is $x \in V$ such that $h(x,x) = q(x) \neq 0$. Consider $V = Ax \oplus (Ax)^{\perp}$, where $(Ax)^{\perp}$ is the orthogonal complement to Ax with respect to h. When A is a field, we can also regard $(Ax)^{\perp}$ as all vectors in V which are orthogonal, with respect to b_q, to x and $\sqrt{d}x$. When A is quaternion algebra, with basis $1, i, j, k$ over L, we can regard $(Ax)^{\perp}$ as all vectors in V which are orthogonal, with respect to b_q, to x, ix, jx, kx. In a similar way, consider $V' = Ax' \oplus (Ax')^{\perp}$, where $x' = \varphi(x)$. Note that defining

$\psi : Ax \to Ax'$ by $\psi(ax) = ax'$ for all $a \in A$, we get that (Ax, h) and (Ax', h') are isometric.
Since $(Ax, q) \oplus ((Ax)^{\perp}, q) \cong (Ax', q') \oplus ((Ax')^{\perp}, q')$ and $(Ax, q) \cong (Ax', q')$, $((Ax)^{\perp}, q) \cong ((Ax')^{\perp}, q')$ according to Theorem 5.17. By induction, (V, h) and (V', h') are isometric. □

Let $K \subset L$ be an unramified separable quadratic extension of local fields. Let R, $\mathfrak{p} = \pi R$ and $|\cdot|$ be as explained in the beginning of Section 5.1. Let S be the integral closure of R in L and let $\mathfrak{P}$ be the unique maximal ideal in S with a generator Π.

5.19. Proposition. *Two regular integral hermitian S–lattices (M, h) and (M', h') are isometric if and only if the corresponding quadratic R–lattices (M, q) and (M', q') are isometric.*

Proof. Obviously, if (M, h) and (M', h') are isometric then (M, q) and (M', q') are isometric.
Assume that (M, q) and (M', q') are isometric over R, that is, there is an R–isomorphism $\varphi : M \to M'$ such that $q'(\varphi(x)) = q(x)$ for all $x \in M$. ¿From Proposition 5.8 and Proposition 5.12, we know that it is possible to diagonalize (M, h) over S. Then there is a basis $x_1, \ldots, x_n$ of M over S such that $|q(x_1)| = \max\{|h(x_i, x_j)| : 1 \le i, j \le n\}$. It is easy to check that for all $x, y \in M$, $|h(x, y)| \le |q(x_1)|$. Then $M = Sx_1 \oplus (Sx_1)^{\perp}$, where $(Sx_1)^{\perp}$ is the orthogonal complement to Sx_1 with respect to h. We can also regard $(Sx_1)^{\perp}$ as all vectors in M which are orthogonal, with respect to b_q, to x_1 and ωx_1, where $S = R[\omega]$.

Consider $x_1' = \varphi(x_1)$. First note that x_1' is a maximal vector in M' since if $x_1' = \pi m'$, for some $m' \in M'$, then $|q(x_1)| = |h'(x_1', x_1')| = |\pi|^2 |h'(m', m')| < |q(\varphi^{-1}(m'))|$ which is a contradiction. Extend x_1' to a basis $x_1', \ldots, x_n'$ for M'.

We want to show that $M' = Sx_1' \oplus (Sx_1')^{\perp}$. According to the proof of Proposition 5.8, we need to show that $|q(x_1')| = \max\{|h(x_i', x_j')| : 1 \le i, j \le n\}$. Note first that for all $x \in M'$, $|h'(x, x)| = |h(\varphi^{-1}(x), \varphi^{-1}(x))| \le |h(x_1, x_1)| = |h'(x_1', x_1')|$. Let $x, y \in M'$. We can assume that $h'(x, y) = \pi^k$, $k \in \mathbb{N}$. Then $h'(x + y, x + y) = h'(x, x) + h'(y, y) + 2\pi^k$. ¿From this it follows that $|h'(x, y)| \le |h'(x_1', x_1')|$ and hence $M' = Sx_1' \oplus (Sx_1')^{\perp}$. We can now proceed as in the proof of Theorem 5.18. □

5.2.2 A counter example

Let R be a Dedekind domain with quotient field K such that $\mathrm{char}(K) \neq 2$. Assume that K is a global field. Let A be a quaternion algebra over L with an L/K–involution σ. As in the beginning of Section 5.2, let $^-$ denote the canonical involution on A. Let (V, h) be a regular hermitian space over A with respect to σ. The hermitian form h over A gives rise to the following quadratic form over K

$$q(x) = h(x, x) + \overline{h(x, x)},\ x \in V. \tag{5.20}$$

Note that if $\sigma = {}^-$, this quadratic form is twice the quadratic form defined in (5.16). Assume that σ is an L/K–involution of the second kind on A. We will show that the quadratic form defined in (5.20) is almost independent of the hermitian form h. We give an example of two non-isometric hermitian spaces over $(-1, -1)_{\mathbb{Q}(\sqrt{3})}$ with isometric corresponding quadratic spaces over $\mathbb{Q}$ defined by (5.20).

We start by mentioning, without proof, a classification theorem for quadratic spaces over global fields. First we need to define invariants which determine whether two quadratic spaces over a global field are isometric. We also state some of the properties of the invariants.

Let (V, q) be a quadratic space over $\mathbb{R}$. It is well-known that $(V, q) = (V^+, q) \oplus (V^-, q)$, where $q(x) > 0$ for all $x \in V^+, x \neq 0$, and $q(x) < 0$ for all $x \in V^-, x \neq 0$. The dimensions of V^+ and V^- are independent of the choice of orthogonal decomposition.

5.21. Definition. Let (V, q) be a quadratic space over $\mathbb{R}$. The **signature** of (V, q) is defined by

$$\mathrm{sign}(V, q) = \dim_{\mathbb{R}}(V^+) - \dim_{\mathbb{R}}(V^-).$$

We sometimes use the shorter notation $\mathrm{sign}(q)$ for the signature when it is clear on which vector space we consider the quadratic form q.

Let $\mathrm{Br}(K)$ denote the Brauer group of the field K.

5.22. Definition. Let $(V, q) = (a_1) \oplus \cdots \oplus (a_n)$ be a quadratic space over K. The algebra $S(q) = \otimes_{1 \leq i < j \leq n}(a_i, a_j)_K$ is called the Hasse algebra. Let $\mathbf{s}(q)$ denote the class of $S(q)$ in $\mathrm{Br}(K)$. The class $\mathbf{s}(q)$ is called the **Hasse invariant** of (V, q).

The Hasse algebra is independent of the choice of diagonalization of q ([15], Chapter 2, §13). Note that quaternion algebras over K have order 2 in $\mathrm{Br}(K)$. We are going to use the following properties of the Hasse invariant (for a proof see [10], Chapter 5, §3).

5.23. Proposition. *Let (V, q) and (V', q') be quadratic spaces over K.*

(1) $\mathbf{s}(q \oplus q')_K = \mathbf{s}(q)\mathbf{s}(q')(\det q, \det q')_K$.

(2) *If* $\dim_K V = n$ *is even, then for* $\alpha \in K$ $\mathbf{s}(\alpha q) = \mathbf{s}(q)(\alpha, (-1)^{\frac{n(n-1)}{2}} \det q)_K$.

Let $\mathfrak{p}$ denote an archimedean or a non-archimedean prime spot of K. Let $\hat{K}_{\mathfrak{p}}$ denote the corresponding completion of K. Thus $\hat{K}_{\mathfrak{p}}$ is a local field or is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, in which case we call $\mathfrak{p}$ a real, respectively a complex, prime spot. Let $V_{\mathfrak{p}} = V \otimes_K \hat{K}_{\mathfrak{p}}$.

We are now able to state the theorem which tells us when two quadratic spaces over K are isometric (for a proof see [15], Chapter 6 , §6).

5.24. Proposition. *Let K be a global field. Then two quadratic spaces (V, q) and (V', q') over K are isometric if and only if* $\dim_K V = \dim_K V'$, $\mathbf{d}(q) = \mathbf{d}(q')$, $\mathbf{s}(q) = \mathbf{s}(q')$ *and* $\mathrm{sign}(V_{\mathfrak{p}}, q) = \mathrm{sign}(V'_{\mathfrak{p}}, q')$ *for all real prime spots $\mathfrak{p}$ in K.*

Let $L = K(\sqrt{d})$ where $d \in K^\times$ is square free. Suppose $A = (\eta, \gamma)_L$ where $\eta, \gamma \in K^\times$ and let $\sigma : A \to A$ be an involution given by

$$\sigma(l_0 + l_1 i + l_3 j + l_3 k) = \sigma(l_0) - \sigma(l_1)i - \sigma(l_3)j - \sigma(l_3)k, \tag{5.25}$$

where $l_n \in L$ and $\sigma_{|L}$ is the non-trivial automorphism on L. Thus σ is an L/K–involution of the second kind and $A = A_\sigma \oplus \sqrt{d}A_\sigma$, where $A_\sigma = (\eta, \gamma)_K$ according to Theorem 4.4. Put $A_+ = \{a \in A : \sigma(a) = a\}$. It is easy to check that A_+ is a vector space of dimension 4 over K with basis $1, \sqrt{d}i, \sqrt{d}j, \sqrt{d}k$. Let (V, h) be a regular hermitian space over A of rank 1. The hermitian form h is given by an element in $A_+^\times$, that is, $h(be, ce) = bu\sigma(c)$ for all $b, c \in A$, where $V = Ae$ and $u = h(e, e) \in A_+^\times$. With a slight abuse of notations, denote u also by h. Let us write $h = \alpha + a\sqrt{d} \in A_+^\times$, where $\alpha \in K$ and a is a pure quaternion in A_σ, that is, $a = \alpha_1 i + \alpha_2 j + \alpha_3 k$, $\alpha_n \in K$. The corresponding quadratic space (V, q) is of dimension 8 over K.

Assume $\mathrm{Tr}_{A/L}(h) \neq 0$. Straightforward calculations, with respect to the basis $e, ie, je, ke, \sqrt{d}e, \sqrt{d}ie, \sqrt{d}je, \sqrt{d}ke$ of V over K, give us

$$(V, q) \cong (V, \alpha^2 q) = (W, 2\alpha f) \oplus (W, -2\alpha\, d\, \mathrm{Nr}_{A/L}(h) f),$$

where (W, f) is the quadratic space $(1) \oplus (-\eta) \oplus (-\gamma) \oplus (\eta\gamma)$ over K. ¿From this it follows that $\mathbf{d}(q) = 1$, independently of h. Using Proposition 5.23, Proposition 4.3 and $\mathbf{d}(f) = 1$, we get

$$\begin{aligned} \mathbf{s}(q) &= \mathbf{s}(2\alpha f)\mathbf{s}(-2\alpha\, d\, \mathrm{Nr}_{A/L}(h) f) \\ &= \mathbf{s}(f)(2\alpha, \det f)_K\, \mathbf{s}(f)(-2\alpha d\, \mathrm{Nr}_{A/L}(h), \det f)_K \\ &= \mathbf{s}(f)^2 (2\alpha, 1)_K\, (-2\alpha d\, \mathrm{Nr}_{A/L}(h), 1)_K \\ &= 1. \end{aligned}$$

5.26. Proposition. *Let $A = (\eta, \gamma)_L$, where $L = \mathbb{Q}(\sqrt{d})$ with $\eta, \gamma, d \in \mathbb{Q}^\times$, d is square free. Let σ be the involution on A defined in (5.25). Let (V, h) and (V', h') be two regular hermitian spaces of rank 1 over A. Assume that $\mathrm{Tr}_{A/L}(h)\mathrm{Tr}_{A/L}(h') \neq 0$. Then $(V, q) \cong (V', q')$ if and only if at least one of η, γ, d and $\mathrm{Tr}_{A/L}(h)\mathrm{Tr}_{A/L}(h')$ is greater than zero.*

Proof. According to Proposition 5.24 and the previous discussion, we only need to consider the signature of $(V \otimes_{\mathbb{Q}} \mathbb{R}, q)$. It is easy to check, working through all possibilities, that if $\eta, \gamma, d < 0$, then $\mathrm{sign}(q) = 8\frac{\alpha}{|\alpha|}$. In all other cases $\mathrm{sign}(q) = 0$, independently of h.
Let $h' = \alpha' + a'\sqrt{d} \in A_+$, where $\alpha' \in \mathbb{Q}^\times$ and a' is a pure quaternion in A_σ. If $\eta, \gamma, d < 0$ and $\mathrm{Tr}_{A/L}(h)\mathrm{Tr}_{A/L}(h') = 4\alpha\alpha' < 0$, then $\mathrm{sign}(q) \neq \mathrm{sign}(q')$ and hence $(V, q) \not\cong (V', q')$. In all other cases, we get that $(V, q) \cong (V', q')$. ☐

5.27. Example. Let $A = (-1, -1)_{\mathbb{Q}(\sqrt{3})}$ and let σ be defined as in (5.25). So $A_\sigma = (-1, -1)_{\mathbb{Q}}$. We claim that $h = 1 + (i + j + k)\sqrt{3}$ and $h' = 1$ are not isometric over A. Yet corresponding quadratic spaces (V, q) and (V, q') are isometric according to Proposition 5.26

Assume that $(V, h) \cong (V', h')$, that is, there exists $c \in A^\times$ such that $h = ch'\sigma(c) = c\sigma(c)$. Let $c = c_0 + c_1\sqrt{3}$, where $c_0, c_1 \in A_\sigma$. Then $h = c\sigma(c)$ implies

$$1 = \mathrm{Nr}_{A_\sigma/\mathbb{Q}}(c_0) - 3\,\mathrm{Nr}_{A_\sigma/\mathbb{Q}}(c_1), \tag{5.28}$$

$$i + j + k = c_1\overline{c}_0 - c_0\overline{c}_1. \tag{5.29}$$

Set $t = c_1\overline{c}_0$. By (5.29), $t = s + \frac{2}{i+j+k}$ for some $s \in \mathbb{Q}$. Note that $c_0 \neq 0$ and hence $c_0 \in A_\sigma^\times$, since A_σ is a division algebra. Put $c_1 = t/\overline{c}_0$ in (5.28). Thus

$$\mathrm{Nr}_{A_\sigma/\mathbb{Q}}(c_0) \quad = \quad \mathrm{Nr}_{A_\sigma/\mathbb{Q}}(c_0)^2 - 3\mathrm{Nr}_{A_\sigma/\mathbb{Q}}(t).$$

Using that $\mathrm{Nr}_{A_\sigma/\mathbb{Q}}(t) = s^2 + 3/4$, it follows that the quadratic equation $X^2 - X - 3(s^2 + 3/4) = 0$ has a rational root $X = \mathrm{Nr}_{A_\sigma/\mathbb{Q}}(c_0)$. Hence $10 + 12s^2 = r^2$ for some $r \in \mathbb{Q}^\times$. Thus the equation

$$10X^2 + 12Y^2 = Z^2$$

has an integer solution. It is easy to check that this equation has no integer solutions. We have a contradiction and hence $(V, h) \ncong (V', h')$.

6 Hermitian lattices and orders in quaternion algebras

Let R be a Dedekind domain with quotient field K such that $\mathrm{char}(K) \neq 2$. Let $L = K(\sqrt{d})$ where $d \in K^\times$ is square free and let σ be the non-trivial automorphism of L over K. Let S be the integral closure of R in L. Let (M, h) be an integral hermitian S–lattice on a regular hermitian space (V, h) of dimension 2 over L.

¿From Theorem 3.9, we get an L/K–involution σ_h of the second kind on $A = \mathrm{End}_L(V)$. We can consider the S–order $\Lambda(M, h) = \mathrm{End}_S(M) \cap \sigma_h(\mathrm{End}_S(M))$ from (3.16) as an S–order in the quaternion algebra $A = \mathrm{End}_L(V) \cong M_2(L)$ over L. According to Theorem 4.4, the set A_{σ_h} of all elements for which σ_h coincides with the canonical involution on A, forms a quaternion algebra over K. Let $\Lambda(M, h)_\sigma$ denote the R–order in A_{σ_h} given in Proposition 4.8. In this section, we study how the orders $\Lambda(M, h)$ and $\Lambda(M, h)_\sigma$ depend on the hermitian lattice.

In Section 6.3, we assume that S is a complete DVR with finite residue class field. We prove a similar statement to Theorem 5.1, classifying certain hermitian S–lattices with respect to the determinants. We determine to what extent we can relate classification of certain hermitian lattices of rank 2 over S to classification of orders in quaternion algebras over local fields. We also study the relations between the orders $\Lambda(M, h)$ and $\Lambda(M, h)_\sigma$.

6.1 The endomorphism ring of a hermitian lattice

By the correspondence between hermitian forms and involutions from Section 3, the following proposition in [16], Chapter 1, §4, can be deduced from Theorem 4.4 and Proposition 4.6.

6.1. Proposition. *Let* $A = \mathrm{End}_L(V)$ *and*

$$A_{\sigma_h} = \{f \in \mathrm{End}_L(V) \;:\; \forall x, y \in V \;\; h(f(x), y) = h(x, \overline{f}(y))\},$$

where $^-$ *is the canonical involution on* $\mathrm{End}_L(V)$*. Then the following holds:*

(1) A_{σ_h} *is a quaternion algebra over* K*.*

(2) $A_{\sigma_h} \otimes_K L \cong \mathrm{End}_L(V)$*.*

(3) *The restriction of* $^-$ *to* A_{σ_h} *is the canonical involution on* A_{σ_h}.

(4) *The isomorphism class of* A_{σ_h} *is completely determined by* $\mathbf{d}(h)$ *and vice versa. In particular,* A_{σ_h} *is isomorphic to* $M_2(K)$ *if* $\mathbf{d}(h)$ *is represented by* -1.

Proof. We have an L/K–involution σ_h on $\mathrm{End}_L(V)$, according to Proposition 3.7. Given a basis for V over L, we can consider $\mathrm{End}_L(V) \cong M_2(L)$ as a quaternion algebra. The involution σ_h on $M_2(L)$ is given by $\sigma_h(T) = H\,{}^tT^\sigma H^{-1}$ for $T \in M_2(L)$. Then A_{σ_h} consists of all $T \in M_2(L)$ such that $\sigma_h(T) = \overline{T}$, where $^-$ is the canonical involution on $M_2(L)$. Theorem 4.4 gives us that A_{σ_h} is a quaternion algebra over K with $A_{\sigma_h} \otimes_K L \cong M_2(L)$ and σ_h restricted to A_{σ_h} is the canonical involution on A_{σ_h}. So (1), (2) and (3) are proved.

Left to prove is (4). According to Proposition 5.2, there is a basis of V over L such that the matrix of h is $H =< a, ab >$ for some $a, b \in K^\times$. Take another form h' and $H' =< a', a'b' >$ with $a', b' \in K^\times$ and define $A_{\sigma_{h'}}$ similarly. ¿From Proposition 4.6 we have that A_{σ_h} and $A_{\sigma_{h'}}$ are isomorphic if and only if $H = kCH'\,{}^tC^\sigma$, for some $k \in K^\times$ and $C \in GL_2(L)$, which implies that $\mathbf{d}(h) = \mathbf{d}(h')$.
Conversely, if $\mathbf{d}(h) = \mathbf{d}(h')$, then $b = c\sigma(c)b'$ where $c \in L^\times$. Then $H = a(a')^{-1}CH'\,{}^tC^\sigma$ where $C =< 1, c >$ and hence A_{σ_h} and $A_{\sigma_{h'}}$ are isomorphic.

Straightforward calculations give us

$$A_{\sigma_h} = \left\{ \begin{bmatrix} s & t \\ -b\sigma(t) & \sigma(s) \end{bmatrix} \; : \; s,\, t \in L \right\}.$$

This is the image of $(d, -b)_K$ under the embedding in $M_2(L)$ described in (4.2). It is clear that A_{σ_h} is isomorphic to $M_2(K)$ if and only if $\mathbf{d}(h) = -1$. $\square$

For the remainder of this paper, let us fix a regular hermitian space (V, h) of dimension 2 over L. Given an integral hermitian S–lattice (M, h) on (V, h), let $\Lambda(M, h)_\sigma = \{f \in \Lambda(M, h) \; : \; \sigma_h(f) = \overline{f}\}$ be the R–order in A_{σ_h} from Proposition 4.8, where $^-$ denotes the canonical involution on A. A natural question to ask is how $\Lambda(M, h)$ and $\Lambda(M, h)_\sigma$ vary with the lattice. In the following proposition, we show that isometric lattices yield isomorphic orders.

6.2. Proposition. *Let (M,h) and (M',h) be integral hermitian S–lattices on (V,h). If $(M,h) \cong (M',h)$, then $\Lambda(M,h)$ is isomorphic to $\Lambda(M',h)$ and $\Lambda(M,h)_\sigma$ isomorphic to $\Lambda(M',h)_\sigma$.*

Proof. Suppose that there is an S–isomorphism $\psi : M \to M'$ such that $h(\psi(x),\psi(y)) = h(x,y)$. We claim that the map $\Psi : \Lambda(M,h) \to \Lambda(M',h)$, where

$$\Psi(f) = \psi \circ f \circ \psi^{-1},$$

is an S–algebra isomorphism. First, it is clear that $\psi \circ f \circ \psi^{-1} \in \mathrm{End}_S(M')$ since ψ is S–linear. ¿From the fact that σ_h is the unique adjoint involution it follows easily that $\psi \circ \sigma_h(f) \circ \psi^{-1} = \sigma_h(\psi \circ f \circ \psi^{-1})$ and hence $\Psi(f) \in \Lambda(M',h)$. It is clear that Ψ is an S–homomorphism of algebras and we can construct an inverse map by $\Phi(g) = \psi^{-1} \circ g \circ \psi$ for $g \in \Lambda(M',h)$. Hence $\Lambda(M,h)$ and $\Lambda(M',h)$ are isomorphic.
Moreover, since $\Psi \circ \sigma_h = \sigma_h \circ \Psi$ and Ψ is an S–algebra isomorphism, it follows from Proposition 4.10 that $\Lambda(M,h)_\sigma$ and $\Lambda(M',h)_\sigma$ are isomorphic as R–orders. □

We will now study the converse of the previous proposition when S is a complete DVR with finite residue field using the results in Section 5.1. We also study the relation between the orders $\Lambda(M,h)$ and $\Lambda(M,h)_\sigma$. The first task will be to compute the discriminant of these orders.

6.2 The discriminants in the complete case

For the remainder of this, we assume that $K \subset L$ is a separable quadratic extension of local fields. Assume R, $\mathfrak{p} = \pi R$ and $|\cdot|$ to be as in the beginning of Section 5.1. Let S be the integral closure of R in L and let $\mathfrak{P}$ be the unique maximal ideal in S with a generator Π.

In order to calculate the discriminants of the orders $\Lambda(M,h)$ and $\Lambda(M,h)_\sigma$ when $K \subset L$ is a ramified dyadic extension, we need some more technical facts.

6.2.1 Auxiliary results in the ramified dyadic case

Assume that $K \subset L$ is a dyadic ramified extension. Following [6], we divide this case into two different subcases.

"Ramified-prime" case: $L = K(\sqrt{\pi})$, where $\mathfrak{p} = \pi R$ and $\mathfrak{P} = \Pi S$ with $\Pi = \sqrt{\pi}$.
"Ramified-unit" case: $L = K(\sqrt{1+\delta\pi^{2k+1}})$ with $k \in \mathbb{N}$, $\delta \in R^{\times}$, $|4| < |\pi^{2k+1}| < 1$, $\mathfrak{p} = \pi R$ and $\mathfrak{P} = \Pi S$ where $\Pi = \frac{1+\sqrt{1+\delta\pi^{2k+1}}}{\pi^k}$.

Note that in both cases, $S = R[\Pi]$.

Assume that M is an Π^i–modular plane with $i = 0$ or $i = 1$. Write $nM = \pi^m S$, $m \in \mathbb{N}$. Then $nM \subseteq sM \subseteq S$ implies that $m \geq 0$ if $i = 0$ and $m > 1$ if $i = 1$. According to Proposition 9.1 in [6], $n\mathrm{H}(i) \subseteq nM \subseteq S$. We have in the ramified-prime case $n\mathrm{H}(i) = 2\pi^i S$ and in the ramified-unit case $n\mathrm{H}(i) = 2\pi^{-k}S$ for $i = 0, 1$. The inclusions $n\mathrm{H}(i) \subseteq nM \subseteq S$ imply that in the ramified-prime case $\frac{2}{\pi^{m-i}} \in S$ for $i = 0, 1$, and in the ramified-unit case $\frac{2}{\pi^{m+k}} \in S$. This will be used in the proof of the following lemma.

6.3. Lemma. *Let $K \subset L$ be a ramified quadratic dyadic extension and let M be a Π^i–modular plane, $i = 0$ or $i = 1$, and $nM = \pi^m S$, $m \in \mathbb{N}$. With π and Π defined as above,*

$$\frac{1}{\pi^m}(\Pi + \sigma(\Pi)) \ , \frac{2\Pi^i\sigma(\Pi^i)}{\pi^m} \ , \frac{\sigma(\Pi^i)}{\pi^m}(\Pi - \sigma(\Pi))$$

are elements in S.

Proof. First, note that $\mathrm{Tr}_{L/K}(\Pi) = \Pi + \sigma(\Pi) = 0$ in the ramified-prime case and $\mathrm{Tr}_{L/K}(\Pi) = \frac{2}{\pi^k} \in R$ in the ramified-unit case. Then

$$\frac{1}{\pi^m}(\Pi + \sigma(\Pi)) = \frac{1}{\pi^m}\mathrm{Tr}_{L/K}(\Pi) = \frac{2}{\pi^{m+k}} \in S.$$

We have

$$\frac{2(\Pi\sigma(\Pi))^i}{\pi^m} = \begin{cases} \frac{2(-\Pi)^{2i}}{\pi^m} = \pm\frac{2}{\pi^{m-i}} \in S & \text{in the ramified-prime case} \\ \\ \frac{2(-\delta\pi)^i}{\pi^m} \in S & \text{in the ramified-unit case} \end{cases}$$

and finally,

$$\frac{\sigma(\Pi^i)}{\pi^m}(\Pi - \sigma(\Pi)) = \begin{cases} \frac{\sigma(\Pi^i)2\Pi^i}{\pi^m} = \pm\frac{2\Pi}{\pi^{m-i}} \in S & \text{in the ramified-prime case} \\ \frac{\sigma(\Pi^i)(2\Pi - \mathrm{Tr}_{L/K}(\Pi))}{\pi^m} \in S & \text{in the ramified-unit case} \end{cases}$$

□

6.2.2 Computing the discriminants

Assume that $K \subset L$ is a separable quadratic extension of local fields. Let (M, h) be an integral hermitian S–lattice on the fixed regular hermitian space (V, h) of dimension 2 over L. Let

$$[h] = \{h' : M \times M \to S \,:\, \exists\, r, r' \in R,\ r, r' \neq 0,\ \ r'h' = rh\}$$

be the class of h on M. Note that $\Lambda(M, h) = \Lambda(M, h')$ for $h' \in [h]$, since $\sigma_h = \sigma_{h'}$, by the remark following Proposition 3.7.

6.4. Definition. We call $h_0 \in [h]$ an R–**primitive representative** of the class of h on M if it is R–primitive and such that, if M has an orthogonal basis with respect to h_0, 1 is represented by M.

It is always possible to find an R–primitive representative of $[h]$. It is clear that $h = rh'$, where h' is an R–primitive form and r is a suitable element in R. Since M is free over S, $h'(x, y) \in S$ for all $x, y \in M$. If $h' =< \epsilon, \eta r >$ with $\epsilon, \eta \in R^\times, r \in R$, with respect to some basis of M over S, then $h_0 = \epsilon^{-1}h'$ is R–primitive, $h_0 \in [h]$ and 1 is represented by (M, h_0), so h_0 is an R–primitive representative of $[h]$.

The norm of M and the discriminant of M are, by definition, ideals in S. But since (M, h) is integral, there are generators in R, so we can also view them as ideals in R.

Note that given a basis for M over S there is an isomorphism between $\mathrm{End}_S(M)$ and $M_2(S)$. Let H be the matrix of h_0 with respect to this basis. Under this isomorphism, we have

$$\begin{aligned} \Lambda(M, h) &= \{T \in M_2(S) \,:\, H^t T^\sigma H^{-1} \in M_2(S)\}, \\ \Lambda(M, h)_\sigma &= \{T \in \Lambda(M, h) \,:\, \overline{T} H = H^t T^\sigma\}. \end{aligned}$$

Recall that the discriminant $\mathrm{D}(M)$ of the hermitian lattice (M, h_0) is generated by the determinant of h_0.

6.5. Proposition. *Let $K \subset L$ be a separable quadratic extension of local fields. Consider the integral hermitian S–lattice (M, h_0), where h_0 is an R–primitive representative of the class of h on M. Then*

$$\begin{aligned} d(\Lambda(M,h)) &= (sM)^{-2}\mathrm{D}(M), \\ d(\Lambda(M,h)_\sigma) &= (nM)^{-2}\mathrm{D}(M)\mathrm{D}(S). \end{aligned}$$

Proof. ¿From Corollary 5.9, we have that either M is an orthogonal sum of lines or M is a modular plane.

Let us first suppose that M is an orthogonal sum of lines, that is, $M = (1) \oplus (\eta r)$, $r \in R\backslash\{0\}$, $\eta \in R^\times$, since h_0 is primitive. Straightforward calculations from the definition of $\Lambda(M,h)$ and $\Lambda(M,h)_\sigma$ give us that

$$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis of $\Lambda(M,h)$ over S and

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} \omega & 0 \\ 0 & \sigma(\omega) \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 1 \\ -r & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & \omega \\ -r\sigma(\omega) & 0 \end{bmatrix}$$

is basis of $\Lambda(M,h)_\sigma$ over R, where $S = R[\omega]$. Let $A = M_2(L)$. We get $\mathrm{D}(\Lambda(M,h)) = (\det[\mathrm{Tr}_{A/L}(F_iF_j)]) = r^2S$, so $d(\Lambda(M,h)) = rS = \mathrm{D}(M)S = (sM)^{-2}\mathrm{D}(M)$, since $sM = S$. In a similar way, we get $d(\Lambda(M,h)_\sigma) = r\mathrm{D}(S) = \mathrm{D}(M)\mathrm{D}(S) = (nM)^{-2}\mathrm{D}(M)\mathrm{D}(S)$, since $nM = R$.

Let us now assume that M is a Π^i–modular plane, $i \in \mathbb{N}$. Then $i = 0$ or $i = 1$, since h_0 is primitive. The possibility of finding a nice basis for h_0 depends on the extension. If $K \subset L$ is an unramified extension, then according to Proposition 5.12, M has an orthogonal basis.

Suppose now that $K \subset L$ is a ramified extension. Let us first look at the case when $\mathfrak{p}$ is a non-dyadic prime ideal. If M is a Π^i–modular plane with $i = 0$

then, according to Proposition 5.12, M has an orthogonal basis. If M is a Π–modular plane then, by Proposition 5.12, $M = \mathrm{H}(1)$ is a hyperbolic plane. Straightforward calculations from the definition of $\Lambda(M,h)$ and $\Lambda(M,h)_\sigma$ give us that $\Lambda(M,h) = M_2(S)$ and $\Lambda(M,h)_\sigma = M_2(R)$. Since we have a ramified non-dyadic extension, $S = R[\sqrt{\pi}]$ and from this it follows that $\mathrm{D}(M) = \Pi^2$, $\mathrm{D}(S) = \Pi^2 R$ and $nM = \Pi^2 R$. Then we get $d(\Lambda(M,h)) = S = (sM)^{-2}\mathrm{D}(M)$, since $sM = \Pi S$ and $d(\Lambda(M,h)_\sigma) = R = (nM)^{-2}\mathrm{D}(M)\mathrm{D}(S)$.

Now let $\mathfrak{p}$ be a dyadic prime ideal. If M is a normal plane, then according to Proposition 5.12 (II), M has an orthogonal basis. Assume that M is subnormal. By Proposition 5.12 (II 2(b)), the matrix of h_0 is of the form

$$H_0 = \begin{bmatrix} \pi^m & \Pi^i \\ \sigma(\Pi)^i & \epsilon\pi^{m+l} \end{bmatrix},$$

with $nM = \pi^m S$, $m, l \in \mathbb{N}$, $m \neq 0$, $\epsilon \in R^\times \cup \{0\}$ and $i = 0$ or $i = 1$.

Suppose that M is a Π^i–modular plane with $i = 0$. Then $\mathbf{d}(h_0) \in R^\times$ and hence $[M^\# : M] = S$. Then $M = M^\#$ and by Remark 3.18, $M_2(S)$ is invariant under the involution. Therefore $d(\Lambda(M,h)) = S = (sM)^{-2}\mathrm{D}(M)$, since $sM = S$.
Let us consider $\Lambda(M,h)_\sigma$. Let

$$(*) \qquad T = \begin{bmatrix} s & t \\ u & v \end{bmatrix},$$

with $s, t, u, v \in S$ and $t = t_0 + t_1\Pi$, $s = s_0 + s_1\Pi$, where $s_0, s_1, t_0, t_1 \in R$. So $T \in \Lambda(M,h)_\sigma$ if and only if the following four equations hold

$$\begin{aligned} \pi^m(v - \sigma(s)) &= t + \sigma(t) = 2t_0 + (\Pi + \sigma(\Pi))t_1, \\ \pi^m(\sigma(u) + t\pi^n\epsilon) &= v - \sigma(v), \\ \pi^m(u + \sigma(t)\pi^n\epsilon) &= s - \sigma(s) = (\Pi - \sigma(\Pi))s_1, \\ \epsilon\pi^{m+n}(s - \sigma(v)) &= u + \sigma(u). \end{aligned}$$

¿From this it follows that $T \in \Lambda(M,h)_\sigma$ if and only if

$$\begin{aligned} v &= \frac{2}{\pi^m}t_0 + \frac{1}{\pi^m}(\Pi + \sigma(\Pi))t_1 + \sigma(s), \\ u &= \frac{1}{\pi^m}(\Pi - \sigma(\Pi))s_1 - \sigma(t)\pi^n\epsilon, \end{aligned}$$

since from Lemma 6.3 we have that $\frac{2}{\pi^m}$, $\frac{1}{\pi^m}(\Pi+\sigma(\Pi))$, $\frac{1}{\pi^m}(\Pi-\sigma(\Pi)) \in S$. We then get the following basis for $\Lambda(M,h)_\sigma$ over R

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} \Pi & 0 \\ \frac{1}{\pi^m}(\Pi-\sigma(\Pi)) & \sigma(\Pi) \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 1 \\ -\epsilon\pi^n & \frac{2}{\pi^m} \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & \Pi \\ -\epsilon\pi^n\sigma(\Pi) & \frac{1}{\pi^m}(\Pi+\sigma(\Pi)) \end{bmatrix}.$$

Calculating $\det[\mathrm{Tr}_{A/L}(E_iE_j)]$ and taking the square root gives us $d(\Lambda(M,h)_\sigma)$ $= (nM)^{-2}\mathrm{D}(M)\mathrm{D}(S)$.

Suppose that M is a Π–modular plane. Then $\mathbf{d}(h_0) = \Pi^2\eta$, $\eta \in R^\times$, since $|\Pi\sigma(\Pi)| = |\pi|$. ¿From straightforward calculations, we get ${H_0}^t T^\sigma {H_0}^{-1} \in M_2(S)$ for all $T \in M_2(S)$ using that $|\Pi^2| = |\sigma(\Pi^2)| = |\pi|$. Thus $d(\Lambda(M,h)) = S = (sM)^{-2}\mathrm{D}(M)$, since $sM = \Pi S$.
Let us consider $\Lambda(M,h)_\sigma$. Let $T \in M_2(S)$ be as in $(*)$. Then $T \in \Lambda(M,h)_\sigma$ if and only if the following four equations hold

$$\begin{aligned} \pi^m(v-\sigma(s)) &= \Pi\sigma(t)+\sigma(\Pi)t = (\Pi+\sigma(\Pi))t_0 + 2\Pi\sigma(\Pi)t_1, \\ \pi^m(\sigma(u)+t\pi^n\epsilon) &= \Pi(v-\sigma(v)), \\ \pi^m(u+\sigma(t)\pi^n\epsilon) &= \sigma(\Pi)(s-\sigma(s)) = \sigma(\Pi)(\Pi-\sigma(\Pi))s_1, \\ \epsilon\pi^{m+n}(s-\sigma(v)) &= \sigma(\Pi)\sigma(u)+\Pi u. \end{aligned}$$

We get that $T \in \Lambda(M,h)_\sigma$ if and only if

$$\begin{aligned} v &= \frac{1}{\pi^m}(\Pi+\sigma(\Pi))t_0 + \frac{2}{\pi^m}\Pi\sigma(\Pi)t_1 + \sigma(s), \\ u &= \frac{1}{\pi^m}\sigma(\Pi)(\Pi-\sigma(\Pi))s_1 - \sigma(t)\pi^n\epsilon, \end{aligned}$$

since by Lemma 6.3, $\frac{1}{\pi^m}(\Pi+\sigma(\Pi))$, $\frac{2\Pi\sigma(\Pi)}{\pi^m}$, $\frac{\sigma(\Pi)}{\pi^m}(\Pi-\sigma(\Pi)) \in S$. We get the following basis for $\Lambda(M,h)_\sigma$ over R

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} \Pi & 0 \\ \frac{\sigma(\Pi)}{\pi^m}(\Pi-\sigma(\Pi)) & \sigma(\Pi) \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 1 \\ -\epsilon\pi^n & \frac{1}{\pi^m}(\Pi+\sigma(\Pi)) \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & \Pi \\ -\epsilon\pi^n\sigma(\Pi) & \frac{2}{\pi^m}\Pi\sigma(\Pi) \end{bmatrix}.$$

Calculating $\det[\mathrm{Tr}_{A/L}(E_iE_j)]$ and taking the square root gives us $d(\Lambda(M,h)_\sigma) = (nM)^{-2}\mathrm{D}(M)\mathrm{D}(S)$. $\square$

6.6. Remark. By Proposition 3.17, we know that $M^\# = M$ implies that $\mathrm{End}_S(M)$ is invariant under the involution. The converse is not true in general: consider $M = \mathrm{H}(1)$ when $K \subset L$ is a ramified non-dyadic extension or a Π–modular plane M when $K \subset L$ is a ramified dyadic extension. In these cases, we showed in the proof of the previous proposition, that $\mathrm{End}_S(M)$ is invariant under the involution, but $[M^\# : M] \neq S$ and hence $M^\# \neq M$.

6.3 Correspondence between hermitian lattices and orders in quaternion algebras in the complete case

Assume that $K \subset L$ is an unramified or a ramified non-dyadic quadratic extension of local fields. Throughout this section, let (M, h_0) and (M', h_0') be integral hermitian S–lattices on the fixed regular hermitian space (V, h), where h_0 and h_0' are R–primitive representatives of the class of h on M, respectively of the class of h on M', according to Definition 6.4. In this section, we prove a statement similar to Theorem 5.1 of H. Franke, which describes to what extent the determinant of h_0 determines the isometry class of (M, h_0). Furthermore, we show that (M, h) and (M', h) are similar if and only if $\Lambda(M, h)$ and $\Lambda(M', h)$ are isomorphic S–orders, assuming that $\Lambda(M, h)$ is not a maximal order in $M_2(L)$ if $K \subset L$ is a ramified non-dyadic extension. If the last condition is not satisfied, then there exist exactly two classes of similar hermitian lattices with corresponding maximal S-orders. We show that (M, h) and (M', h) are similar if and only if $\Lambda(M, h)_\sigma$ and $\Lambda(M', h)_\sigma$ are isomorphic R–orders. We investigate to what extent the determinant of h_0 determines $\Lambda(M, h)$. We also study the relationship between the orders $\Lambda(M, h)$ and $\Lambda(M, h)_\sigma$.

Under the above assumptions, Propositions 5.8 and 5.12 give us that

$$(M, h_0) \cong (1) \oplus (\eta\pi^j) \text{ or } (M, h_0) \cong \mathrm{H}(1), \tag{6.7}$$

where $\eta \in R^\times$, $j \in \mathbb{N}$. In order to prove the results in this section, we will use the following table, where we have computed the determinant of the lattice and the discriminants of the orders in the different cases using Proposition 6.5:

Table 1 Determinants and discriminants.

$K \subset L$	unramified	ramified non-dyadic	
(M, h_0)	$(1) \oplus (\eta\pi^j)$	$(1) \oplus (\eta\pi^j)$	$\mathrm{H}(1)$
$\mathbf{d}(h_0)$	$\eta\pi^j$	$\eta\pi^j$	π
$d(\Lambda(M, h))$	$\pi^j S$	$\pi^j S$	S
$d(\Lambda(M, h)_\sigma)$	$\pi^j R$	$\pi^{j+1} R$	R

In a similar way, $(M', h_0') \cong (1) \oplus (\eta'\pi^k)$, where $\eta' \in R^\times$, $k \in \mathbb{N}$, or $(M', h_0') \cong \mathrm{H}(1)$.

6.8. Proposition. *Let $K \subset L$ be an unramified extension or a ramified non-dyadic extension in which case we assume that $\mathbf{d}(h_0) \neq \pi$. Then*

$$(M, h_0) \cong (M', h_0') \textit{ if and only if } \mathbf{d}(h_0) = \mathbf{d}(h_0').$$

Assume that $K \subset L$ is a ramified non-dyadic extension. Then $\mathbf{d}(h_0) = \pi$ if and only if (M, h_0) is isometric to one of the non-isometric lattices $(1) \oplus (\pi)$ or $\mathrm{H}(1)$.

Proof. It is clear that isometric hermitian lattices have equal determinants. Conversely, suppose $\mathbf{d}(h_0) = \mathbf{d}(h_0')$ and assume that $\mathbf{d}(h_0) \neq \pi$ if $K \subset L$ is a ramified non-dyadic extension. By Table 1, M and M' have orthogonal bases. The orthogonal decomposition of M in (6.7) is a Jordan splitting if and only if $j \neq 0$, otherwise a Jordan splitting of M is a modular plane. Similar conditions hold for M'.
If $\mathbf{d}(h_0) = \mathbf{d}(h_0') \in R^\times$, then $j = k = 0$ and M and M' are both modular planes. If $\mathbf{d}(h_0) = \mathbf{d}(h_0') \notin R^\times$, then M and M' are orthogonal sums of lines. In both cases, the type of the Jordan splitting of M, respectively M', is the same. Hence $\mathbf{d}(h_0) = \mathbf{d}(h_0')$ implies $(M, h_0) \cong (M', h_0')$ by Theorem 5.13 and 5.14.

Suppose that $K \subset L$ is a ramified non-dyadic extension and $\mathbf{d}(h_0) = \pi$. By Table 1, (M, h_0) is isometric to $(1) \oplus (\eta\pi)$ or $\mathrm{H}(1)$, where $\eta \in \mathrm{Nr}_{L/K}(S^\times)$. These integral hermitian lattices have equal determinants but they are not isometric, since their Jordan splittings are of different types. Since $(1) \oplus (\eta\pi) \cong (1) \oplus (\pi)$ by Proposition 5.14, the proposition is proved. □

6.9. Proposition. *Let $K \subset L$ be an unramified or a ramified non-dyadic extension in which case we assume that $\Lambda(M, h)$ is not a maximal S–order* *in $M_2(L)$. Then*

$$(M,h_0)\cong(M',h_0') \textit{ iff } \Lambda(M,h) \textit{ and } \Lambda(M',h) \textit{ are isomorphic } S\text{–}\textit{orders.}$$

Assume that $K\subset L$ is a ramified non-dyadic extension. Then $\Lambda(M,h)$ is a maximal S–order in $M_2(L)$ if and only if (M,h_0) is isometric to one of the non-isometric lattices $\mathrm{H}(1)$ *or* $(1)\oplus(\eta)$, $\eta\in R^\times$.

Proof. In Proposition 6.2 we showed that if $(M,h_0)\cong(M',h_0')$, then $\Lambda(M,h)$ and $\Lambda(M',h)$ are isomorphic S–orders.

First note that since h_0 and h_0' belong to the same class of hermitian forms on V, we have that $\mathbf{d}(h_0)=\mathbf{d}(h_0')$ in $K^\times/\mathrm{Nr}_{L/K}(L^\times)$. In other words, $\eta\pi^j = l\sigma(l)\eta'\pi^k$ for some $l\in L^\times$. ¿From this it follows that $\mathbf{d}(h_0)=\mathbf{d}(h_0')$ in $R/\mathrm{Nr}_{L/K}(S^\times)$ if and only if $j=k$.

Let $K\subset L$ be an unramified or a ramified non-dyadic extension in which case we assume that $\Lambda(M,h)$ is not a maximal S–order in $\mathrm{M}_2(L)$. Suppose that $\Lambda(M,h)$ and $\Lambda(M',h)$ are isomorphic S–orders, so $d(\Lambda(M,h))=d(\Lambda(M',h))$. Since $d(\Lambda(M,h))=d(\Lambda(M',h))\neq S$ in the ramified non-dyadic case, it follows from Table 1 that $\mathbf{d}(h_0)=\epsilon\mathbf{d}(h_0')$ for some $\epsilon\in R^\times$. So $j=k$ and hence $\mathbf{d}(h_0)=\mathbf{d}(h_0')$. By Proposition 6.8, $(M,h_0)\cong(M',h_0')$ if $\mathbf{d}(h_0)\neq\pi$, and $(M,h_0)\cong(M',h_0')\cong(1)\oplus(\pi)$ if $\mathbf{d}(h_0)=\pi$.

Assume that $K\subset L$ is a ramified non-dyadic extension and suppose that $\Lambda(M,h)$ is a maximal S–order in $M_2(L)$. Thus $d(\Lambda(M,h))=S$ and according to Table 1, M is isometric to $(1)\oplus(\eta)$ or $\mathrm{H}(1)$. These lattices are not isometric, since their Jordan splittings are of different types. ¿From Table 1, it follows that their corresponding S–orders are maximal. ☐

6.10. Corollary. *Let $K\subset L$ be an unramified extension or a ramified non-dyadic extension in which case we assume that $\mathbf{d}(h_0)\neq\pi$. If $\mathbf{d}(h_0)=\mathbf{d}(h_0')$, then $\Lambda(M,h)$ and $\Lambda(M',h)$ are isomorphic S–orders.*
Moreover, if $K\subset L$ is a ramified non-dyadic extension and $\mathbf{d}(h_0)=\pi$, then $\Lambda(M,h)$ is isomorphic to $M_2(S)$ or to the S–order consisting of matrices

$$\begin{bmatrix} s & \pi t \\ u & v \end{bmatrix},$$

where $s,t,u,v\in S$.

Proof. Let $K \subset L$ be an unramified extension or a ramified non-dyadic extension in which case we assume that $\mathbf{d}(h_0) \neq \pi$. By Proposition 6.8, $\mathbf{d}(h_0) = \mathbf{d}(h'_0)$ implies that $(M, h_0) \cong (M', h'_0)$ and hence $\Lambda(M, h)$ and $\Lambda(M', h)$ are isomorphic S–orders.

Assume that $K \subset L$ is a ramified non-dyadic extension. If $\mathbf{d}(h_0) = \pi$ then (M, h_0) is isometric to $H(1)$ or $(1) \oplus (\pi)$. The result follows from the proof of Proposition 6.5. $\square$

6.11. Remark. Note that the converse of the first statement in Corollary 6.10 is not true, that is, there exists isomorphic S-orders where the determinants of their corresponding hermitian lattices are not equal. For example, assume that $K \subset L$ is a ramified non-dyadic extension. Consider $(M, h_0) = (1) \oplus (-1)$ and $(M', h'_0) = H(1)$. Thus $\Lambda(M, h)$ and $\Lambda(M', h)$ are maximal S–orders, and therefore isomorphic, but $\mathbf{d}(h_0) \neq \pi$ and $\mathbf{d}(h_0) \neq \mathbf{d}(h'_0)$. This shows also that the converse of the second statement in Corollary 6.10 is not true, since $\Lambda(M, h)$ is isomorphic to $M_2(S)$ but $\mathbf{d}(h_0) \neq \pi$.

6.12. Corollary. *Let $K \subset L$ be an unramified or a ramified non-dyadic extension in which case we assume that $\Lambda(M, h)$ is not a maximal S–order in $M_2(L)$. Then*

$$(M, h) \sim (M', h) \text{ iff } \Lambda(M, h) \text{ and } \Lambda(M', h) \text{ are isomorphic } S\text{–orders.}$$

Assume that $K \subset L$ is a ramified non-dyadic extension. Then $\Lambda(M, h)$ is a maximal S–order in $M_2(L)$ if and only if (M, h) is similar to one of the non-isometric lattices H(1) *or* $(1) \oplus (\eta)$, $\eta \in R^\times$.

Proof. By the discussion following Definition 6.4, $h_{|M} = rh_0$ and $h_{|M'} = r'h'_0$, where $r, r' \in R, r, r' \neq 0$, and h_0 and h'_0 are R–primitive representatives of the class of h on M, respectively of the class of h on M'. If $\frac{r}{r'} \in R$ then according to Proposition 6.9, $(M, h) \cong (M', \frac{r}{r'}h)$ if and only if $\Lambda(M, h)$ and $\Lambda(M', h)$ are isomorphic S–orders. We get a similar result if $\frac{r'}{r} \in R$. The last statement follows immediately from Proposition 6.9 $\square$

6.13. Proposition. *Let $K \subset L$ be an unramified or a ramified non-dyadic extension. Then*

$$(M, h_0) \cong (M', h'_0) \text{ iff } \Lambda(M, h)_\sigma \text{ and } \Lambda(M', h)_\sigma \text{ are isomorphic } R\text{–orders.}$$

Proof. If $K \subset L$ is an unramified extension or a ramified non-dyadic extension, in which case we assume that $\Lambda(M,h)$ is not a maximal S–order in $\mathrm{M}_2(L)$, the proof is exactly as in Proposition 6.9.

Let us assume that $K \subset L$ is a ramified non-dyadic extension. By Table 1, $d(\Lambda(M,h)_\sigma) = \mathrm{D}(M)\mathrm{D}(S) = \pi^{j+1}R$ when $M \cong (1) \oplus (\eta\pi^j)$, $\eta \in R^\times$, $j \in \mathbb{N}$. ¿From this it follows that $d(\Lambda(M,h)_\sigma) = R$ if and only if $M \cong \mathrm{H}(1)$. Therefore the discriminant determines the hermitian lattice and hence if $\Lambda(M,h)_\sigma$ and $\Lambda(M',h)_\sigma$ are isomorphic R–orders then $(M,h_0) \cong (M',h_0')$. □

With the same argument as in the proof of Corollary 6.12, we get the following corollary of the above proposition.

6.14. Corollary. *Let $K \subset L$ be an unramified or a ramified non-dyadic extension. Then*

$$(M,h) \sim (M',h) \text{ iff } \Lambda(M,h)_\sigma \text{ and } \Lambda(M',h)_\sigma \text{ are isomorphic } R\text{–orders.}$$

Let us now look at the relationship between the two orders $\Lambda(M,h)$ and $\Lambda(M,h)_\sigma$. Consider the case when $K \subset L$ is a ramified non-dyadic extension. Let $(M,h_0) = (1) \oplus (-1)$ and $(M',h_0') = \mathrm{H}(1)$. These are non-isometric lattices over S with $\Lambda(M,h)$ and $\Lambda(M',h)$ conjugated, since they are maximal. But $\Lambda(M,h)_\sigma$ and $\Lambda(M',h)_\sigma$ are non-isomorphic R–orders in $A_{\sigma_h} = M_2(K)$ according to Proposition 6.13. Note that $\Lambda(M,h) = \alpha\Lambda(M',h)\alpha^{-1}$, $\alpha \in A^\times$ where $\alpha \notin A_{\sigma_h}$ since otherwise $\Lambda(M,h)_\sigma$ and $\Lambda(M',h)_\sigma$ would be isomorphic by Proposition 4.10.

In the case of an unramified extension, we are going to compare $S\Lambda(M,h)_\sigma$ and $\Lambda(M,h)$. We need the following:

6.15. Lemma. $\mathrm{D}(S \otimes_R \Lambda(M,h)_\sigma) = S \otimes_R \mathrm{D}(\Lambda(M,h)_\sigma) = S\mathrm{D}(\Lambda(M,h)_\sigma)$.

Proof. To simplify the notations in this proof, let $\otimes_R$ be denoted by $\otimes$, $\Lambda(M,h)$ by Λ and $\Lambda(M,h)_\sigma$ by Λ_σ. We claim that $\mathrm{D}(S \otimes \Lambda_\sigma)$ and $S \otimes \mathrm{D}(\Lambda_\sigma)$ are generated over S (where S is identified with $S \otimes \mathrm{id}_\Lambda$) by the same elements. Let $e_1, \dots, e_4$ be a basis for Λ_σ over R. Then $\mathrm{D}(\Lambda_\sigma) = (\det[\mathrm{Tr}_{A_{\sigma_h}/K}(e_ie_j)])$ and thus $S \otimes \mathrm{D}(\Lambda_\sigma)$ is generated by $1 \otimes \det[\mathrm{Tr}_{A_{\sigma_h}/K}(e_ie_j)]$ over S.
We have that $1 \otimes e_1, \dots, 1 \otimes e_4$ is a basis for $S \otimes \Lambda_\sigma$ over S and since $\mathrm{Tr}_{A/L}((1 \otimes e_i)(1 \otimes e_j)) = \mathrm{Tr}_{A_{\sigma_h}/K}(e_ie_j)$ ([14], Chapter 2, Theorem 9.27),

$\mathrm{D}(S \otimes \Lambda_\sigma) = S \otimes \mathrm{D}(\Lambda_\sigma)$.
Since $\mathrm{D}(\Lambda_\sigma)$ is R–torison free as an R–module, we can identify $\mathrm{D}(\Lambda_\sigma)$ with $1 \otimes \mathrm{D}(\Lambda_\sigma)$, and hence $S \otimes \mathrm{D}(\Lambda_\sigma) = S(1 \otimes_R \mathrm{D}(\Lambda_\sigma)) = S\mathrm{D}(\Lambda_\sigma)$. □

6.16. Proposition. *Let $K \subset L$ be an unramified quadratic extension of local fields. Then*

$$\Lambda(M,h) = S \otimes_R \Lambda(M,h)_\sigma = S\Lambda(M,h)_\sigma.$$

Proof. It is clear that $S\Lambda(M,h)_\sigma \subseteq \Lambda(M,h)$ and according to Lemma 6.15 and Table 1,

$$d(S\Lambda(M,h)_\sigma) = Sd(\Lambda(M,h)_\sigma) = d(\Lambda(M,h))$$

and hence, $S\Lambda(M,h)_\sigma = \Lambda(M,h)$. □

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