Foster-Lyapunov functions via Laplace transforms of transitions measures with applications to Ito sense differentiable Markov processes

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Abstract

Let $\{X_t\}_{t\geq 0}$ be a weakly (Ito sense) differentiable Markov process on R^+, i.e., $\forall s \in [0,\infty)$ the expression

$$\frac{1}{h}\log \mathbb{E}_{X(t_0)} e^{i[s(X(t+h)-X(t))]}$$

has a limit in the sense of convergence in probability as $h \to 0^+$ (Ito differential of the process $\{X_t\}_{t\geq 0}$ at the point t_0).

Let $s(x)$ be a real function satisfying the following equation

$$s(x)a(x) + \frac{1}{2}b^2(x)s^2(x) + \beta\frac{\mathbb{E}_x e^{s(x)U} - 1}{s(x)} = 0.$$

Using the function

$$f(x) = \int_0^x e^{\int_0^y s(u)du} dy,$$

sufficient and necessary conditions for different asymptotic behavior and in the exponential ergodic case computable bounds of rate of convergence are obtained.

Keywords: Lyapunov function, drift criteria, ergodicity, exponential ergodicty

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1 Introduction

It is natural to consider Markov processes with state space $\mathbb{R}^+$, which locally exhibit the same behavior as processes with independent increments. Let us consider the characteristic function

$$\phi(h,x,s) = \int_{R^+} e^{isu} P(h,x,du),\ h>0,$$
$$\phi(0,x,s) = 1$$

of the transition semigorup $P(h,x,B)$.

Definition 1.1. A Markov process $\{X_t\}_{t\geq 0}$ is called weakly differentiable in wide sense or (differrentiable in Ito sense at a point t) if for every $t,s\in R$ the expression

$$\frac{1}{h}\log \mathbb{E}_{X_(t)} e^{i[s[X(t+h)-X(t)]]}$$

has a limit in the sense of convergence in probability as $h\to 0^+$ (Ito differential of the process $\{X_t\}_{t\geq 0}$ at the point t), or if the function $\phi(\cdot,x,s)$ is differentiable at zero uniformly on $|s|\leq S$, i.e.,

$$\lim_{h\to 0}\frac{\phi(h,x,s)-1}{h} = g(x,s)$$

exists uniformly on $|s|\leq S$ for all $x\in R^+$ and an arbitrary $S>0$.

The value of the Ito differential is a function of s and of $\{X_t\}_{t\geq 0}$ and has the form of the characteristic function of an infinitely divisible law, so it is well known that if a Markov process is weakly differentiable, then there exists a function $a(x)$, nonnegative function $b^2(x)>0$ and a measure $q_x(du)$ on $\mathbb{R}^+$, $q_x(0)=0$ such that

$$g(x,s) = e^{ia(x)s-\frac{1}{2}b^2(x)s^2+\int_{\mathbb{R}^+}[e^{isx-1-\frac{isx}{1+x^2}}]\frac{1+x^2}{x^2}q(x,dz)},$$

and for an arbitrary function $f\in C_0^2(\mathbb{R}^+)$ it follows that

$$\begin{aligned}\mathcal{A}f(x) &= \lim_{h\to 0}\frac{\mathbb{E}_x f(X_h)-f(x)}{h}\\ &= a(x)f'(x)+\frac{1}{2}b^2(x)f''(x)+\int_{\mathbb{R}^+}[f(x+u)-f(x)]q_x(du)\end{aligned}$$

We shall assume that the measures $q_x(du)$ are uniformly bounded, so we can write $q_x(du)=\beta(x)B_x(du)$, where $B_x(\cdot)$ is probability measure.

This means that the process evolves as a regular diffusion with variance $b^2(x)$ and drift $a(x)$ in between jumps, and that jumps U occur at rate $\beta(x)$, and have distribution B_x when $X_t=x$, and $\forall x\in(0,\infty)$ $\quad B_x(0,\infty)=1$ or in another words, we obtain processes whose jumps are superposed on the continuous diffusion component, assuming that $\beta(x)=\beta$.

Suppose also that the following conditions on the continuous functions $b^2(\cdot)$ and $a(\cdot)$ are satisfied: for any $x > 0$ and some $v > 0, b(\cdot) > 0$ and $\forall \gamma \in (0,1)$

$$\inf_{\gamma < x < \gamma^{-1}} (-a(x)) > 0, \qquad \sup_{0 < x < \gamma^{-1}} (-a(x)) < \infty,$$
$$\sup_{x \in [0,\infty)} b^2(x) < \infty, \quad b^2(0) = 0, \quad \inf_{x > v} b^2(x) > 0,$$

where obviously the function $a(\cdot)$ is supposed to be non-positive, $a(0) = 0$ and

$$\int_0^x \frac{dy}{-a(y)} < \infty.$$

Hence, the process $\{X_t\}_{t\geq 0}$ is irreducible (se [1],[4] [5]), strong Markov with right continuous path.

By Prop. 7.2 in [5] it follows that in the recurrent case, each one-state point set x^* is a regeneration set for $\{X_t\}_{t\geq 0}$. In the positive recurrent case, the ergodicity is provided by the existence of the spread-out regenerations cycles [1]. We will use some drift-criteria from [4] which involve the concept of the truncation $\{X_t^m\}$ of $\{X_t\}$ and concept of the extended generator A_m of the truncated process. Let $\{O_n : n \in Z^+\}$ be a fixed family of open precompact sets for which $O_n \uparrow R^+$ as $n \to \infty$. Let T^m be defined as the first-entrance time to O_m^c and defined X_t^m by

$$X_t^m = \begin{cases} X_t, & t < T^m \\ \zeta_m, & t \geq T^m \end{cases},$$

where ζ_m is any fixed state in O_m^c.

Two next sections deal with the asymptotic properties of $\{X_t\}_{t\geq 0}$, the third section describes the one-dimensional zero-reflected diffusion, and the last one shows the behavior in the storage case.

The purpose of this paper is to construct Lyapunov functions applicable to situations where

$$\lim_{x\to\infty} \frac{-a(x)/\beta - \mathbb{E}_x U}{b^2(x)/\beta + \mathbb{E}_x U^2} = 0,$$

and the construction is based upon roots of the equation

$$\beta \frac{\mathbb{E}_x e^{sU} - 1}{s} = -a(x) - \frac{1}{2} b^2(x) s.$$

2 Conditions for Harris recurrence and transience

1. Harris recurrence, limiting case: $\exists v > 0$ such that $s(x) \equiv 0$ on $[0, v]$ and $s(x) < 0$ on (v, ∞). $s(x) \to 0$ as $x \to \infty$.

Let $\underline{s}(\cdot)$ be an auxiliary function such that $\underline{s}(\cdot) < s(\cdot), \underline{s}(x) \to 0$ as $x \to \infty$. Suppose that $\exists \delta > 0, \exists n \in \mathbb{Z}^+$ such that for some $v > 0$

$$\sup_{x>v} \mathbb{E}_x U^{2+\delta} < \infty \tag{2.1}$$

$$\mathbb{E}_x[U; U > \frac{1}{n}x] \leq \frac{1}{2}[\mathbb{E}_x U^2 + b^2(x)] \cdot [s(x) - \underline{s}(x)] \tag{2.2}$$

Let us define functions $s^n(\cdot)$ and $f^n(\cdot)$ by

$$s^n(x) = \inf_{y \in [\frac{x}{1+\frac{1}{n}}, x]} \underline{s}(y), \quad f^n(x) = \int_0^x e^{\int_0^y s^n(u)du} dy dx.$$

Lemma 2.1. Suppose that (2.1), (2.2) hold true for the function $\underline{s}(\cdot)$ and some $n \in \mathbb{Z}^+$. If $f^n(\infty) = \infty$, then the process $\{X_t\}_{t \geq 0}$ is Harris recurrent.

Proof. Trying the norm-like function $f_n(x)$ in (CD1), [4], we get that $\forall m \in \mathbb{Z}^+, \forall x \in O_m \cap (v, \infty)$

$$\mathcal{A}_m f_n(x) \leq e^{\int_0^x s_n(y)dy} \left\{ \beta \frac{\mathbb{E}_x e^{\underline{s}(x)U} - 1}{\underline{s}(x)} + a(x) + \frac{1}{2} b^2(x) \underline{s}(x) \right.$$
$$\left. + \beta \mathbb{E}_x[U; U > \frac{1}{n}x] \right\} \leq e^{\int_0^x s_n(y)dy} \beta \left\{ \mathbb{E}_x[U; U > \frac{1}{n}x] + \frac{\mathbb{E}_x e^{\underline{s}(x)U} - 1}{\underline{s}(x)} - \frac{\mathbb{E}_x e^{s(x)U} - 1}{s(x)} \right\} \leq 0.$$

The recurrence of $\{X_t\}_{t \geq 0}$ follows by Th. 3.3 in [4]. □

Th. 2.1. The process $\{X_t\}_{t \geq 0}$ is Harris recurrent if and only if

$$\int_0^\infty e^{\int_0^x s(y)dy} dx = \infty.$$

Proof. ($\Rightarrow$) Let $\underline{s}(x) = s(x) - \frac{1}{x^\alpha}$, for some suitable $\alpha > 1$. Since (2.2) obviously holds true $\forall n \in \mathbb{Z}^+$,

$$\mathcal{A}_m f^n(x) \leq 0 \quad \forall m \in \mathbb{Z}^+, \forall x \in O_m \cap (v, \infty).$$

Then

$$\mathbb{E}_x \int_x^{x+U} e^{\int_x^y s^n(u)du} dy \leq -a(x) - \frac{1}{2} b^2(x) s^n(x)$$

for any $n \in \mathbb{Z}^+$.

Let $g_n(z) = \int_x^{x+z} e^{\int_x^y s^n(u)du} dy$. Obviously, $g_n(z) \upuparrows \underline{g}(z) = \int_x^{x+z} e^{\int_x^y \underline{s}(u)du} dy$, as $n \to \infty$ and by monotone convergence theorem we get that

$$\lim_{n \to \infty} \left\{ \mathbb{E}_x \int_x^{x+U} e^{\int_x^y s^n(u)du} dy + \frac{1}{2} b^2(x) s^n(x) \right\} =$$
$$= \mathbb{E}_x \int_x^{x+U} e^{\int_x^y \underline{s}(u)du} + \frac{1}{2} b^2(x) \underline{s}(x) \leq -a(x).$$

Thus,

$$\mathcal{A}_m \underline{f}(x) = \mathcal{A}_m \left(\int_0^x e^{\int_0^y \underline{s}(u)du} dy \right) \le 0 \quad \forall m \in \mathbb{Z}^+, \forall x \in O_m \cap (v, \infty)$$

and the function $\underline{f}(\cdot)$ is norm-like if and only if

$$\int_0^\infty e^{\int_0^x s(y)dy} dx = \infty.$$

($\Leftarrow$) There is an obvious duality in the consideration of the transient behavior of $\{X_t\}_{t\ge 0}$, and we omit the proof. □

Corollary 2.1. Let $\theta \in (0,1)$. Then the process $\{X_t\}_{t\ge 0}$ is

(i) Harris recurrent if

$$\int_0^\infty e^{\int_0^x \frac{2(1+\theta)[-\frac{a(y)}{\beta} - \mathbb{E}_y U]}{b^2(y)/\beta + \mathbb{E}_y U^2} dy} dx = \infty,$$

(ii) transient if

$$\int_0^\infty e^{\int_0^x \frac{2(1-\theta)[-a(y)/\beta - \mathbb{E}_y U]}{b^2(y)/\beta + \mathbb{E}_y U^2} dy} dx < \infty.$$

3 Conditions for ergodicity and exponential ergodicity

1. Ergodicity, limiting case: $\forall x > v$ and some $v > 0$, $s(x) \equiv 0$ on $[0, v]$, $s(x) > 0$ on (v, ∞), $s(x) \to 0$ as $x \to \infty$ and

$$\sup_{x \in R^+} \mathbb{E}_x \int_x^{x+U} e^{-\int_0^y \frac{2a(u)}{b^2(u)} du} dy < \infty \tag{3.1}$$

$$\inf_{x > v} \mathbb{E}_x U^2 > 0 \tag{3.2}$$

Let us suppose that for an auxiliary function $\underline{s}(\cdot)$ and some $n \in \mathbb{Z}^+$

$$\mathbb{E}_x \left[\int_x^{x+U} e^{-\int_0^y \frac{2a(u)}{b^2(u)} du} dy; U > \frac{1}{n} x \right] \le$$

$$\le \frac{1}{2} [\mathbb{E}_x U^2 + b^2(x)][s(x) - \underline{s}(x)] \tag{3.3}$$

and define the functions $f^n(\cdot)$ and $f_\epsilon(\cdot)$ by

$$f_\epsilon(x) = \int_0^x e^{\int_0^y [s(u)-\epsilon(u)]du} dy, \quad f_\epsilon^n(x) = \int_0^x e^{\int_0^y [s^n(u)-\epsilon^n(u)]du} dy,$$

where

$$\epsilon^n(x) = e^{-\int_0^x s^n(y)dy}, \quad \epsilon(x) = e^{-\int_0^x s(y)dy}.$$

Lemma 3.1. Suppose that (2.1), (3.1), (3.2) and (3.3) hold true. If

$$\int_0^\infty e^{-\int_0^x s^n(y)dy} dx < \infty,$$

then the process $\{X_t\}_{t\geq 0}$ is Haris ergodic.

The extended generator of the truncated process is shown to satisfy [4]: $\forall f \in C_0^2[0,\infty)$

$$\begin{aligned}\mathcal{A}_m f^\epsilon(x) &= \beta \int_0^m [f(x+y) - f(x)] B_x(dy) + c[f(x+m) - f(x)][1 - B_x(m)] \\ &+ \frac{1}{2} b^2(x) f''(x) + a(x) f'(x).\end{aligned}$$

Trying the function $f_\epsilon^n(x)$ in (CD2)[4], we get that $\forall x \in O_m \cap (v, \infty)$

$$\begin{aligned}\mathcal{A}_m f_\epsilon^n(x) &\leq e^{\int_0^x [s^n(u)-\epsilon(u)]du} \Big\{ a(x)] + \\ &+ \frac{1}{2}[s(x) - \epsilon(x)] b^2(x) + \beta \frac{\mathbb{E}_x e^{[s(x)-\epsilon(x)]U} - 1}{s(x) - \epsilon(x)} \Big\}\end{aligned}$$

for all $m \in Z^+$. Since (3.2) and (3.3)

$$\begin{aligned}&-\frac{1}{2}[s(x) - \epsilon(x)] b^2(x) - \beta \frac{\mathbb{E}_x e^{[s(x)-\epsilon(x)]U} - 1}{s(x) - \epsilon(x)} + \frac{1}{2} s(x) b^2(x) + \beta \frac{\mathbb{E}_x e^{s(x)U} - 1}{s(x)} \\ &\geq \frac{1}{2}[b^2(x) + \beta \mathbb{E}_x U^2] \epsilon(x),\end{aligned}$$

$$\mathcal{A}_m f_\epsilon^n(x) \leq -\gamma$$

for some $\gamma > 0$ and $\forall x \in O_m \cap (v, \infty)$, $\forall m \in \mathbb{Z}^+$. Since (3.1), the assumptions of Th. 4.4 in [4] are satisfied.

Th. 3.1. The process $\{X_t\}_{t\geq 0}$ is Harris ergodic if

$$\int_0^\infty e^{-\int_0^x s(y)dy} dx < \infty.$$

Proof. The inequality (3.3) holds true for $\underline{s}(x) = s(x) - \frac{1}{x^\alpha}$ for some suitable $\alpha > 1$ and any $n \in \mathbb{Z}^+$. Applying the assumptions of the theorem and the monotone convergence theorem, we get that $\exists \gamma_0 > 0$ such that

$$\mathcal{A}_m f_\epsilon(x) < -\gamma_0 \quad \forall x \in O_m \cap (v, \infty) \quad \forall m \in \mathbb{Z}^+.$$

Since (3.1),

$$\sup_{x \in [0,v]} \mathcal{A}_m f_\epsilon(x) < \infty.$$

□

2. Exponential ergodicity, limiting case: for some $\lambda > 0$ $s(x) > \lambda$ on (v, ∞) and $s(x) \equiv 0$ on $[0, v]$, $s(x) \to \lambda$ as $x \to \infty$.

Let us assume that (2.1), (3.1), (3.2) and (3.3) hold true.

Lemma 3.2. The process is exponentially ergodic if

$$\int_0^\infty e^{-\int_0^x [s^n(y) - \lambda] dy} dx < \infty.$$

Proof. Testing the norm-like function $f_\lambda^n(x) = \int_0^x e^{\int_0^y [s^n(u) - \lambda] du} dy$ in (CD3), [4], we get that $\forall m \in \mathbb{Z}^+, \forall x \in O_m \cap (v, \infty)$

$$\mathcal{A}_m f_\lambda^n(x) \le -\frac{\lambda[\frac{1}{2}b^2(x) + \beta(3-e)\mathbb{E}_x U^2]}{\int_0^\infty e^{-\int_0^x [s^n(y) - \lambda] dy}} f_\lambda^n(x).$$

(We have used that $\forall s > 0 \quad \varphi'(s) = \left(\frac{e^{sU} - 1}{s}\right)' > (3-e)U^2$). □

Th. 3.2. The process $\{X_t\}_{t \ge 0}$ is exponentially ergodic if

$$\int_0^\infty e^{-\int_0^x [s(y) - \lambda] dy} dx < \infty,$$

and the convergence rate ρ satisfies

$$0 < \rho \le \lambda \frac{[\beta(3-e)\inf_{x>v} \mathbb{E}_x U^2 + \frac{1}{2}\inf_{x>v} b^2(x)]}{\int_0^\infty e^{-\int_0^x [s(y) - \lambda] dy} dx} \tag{3.4}$$

Proof. Similar to the proof of Th. 3.1. □

Corollary 3.1. Let $0 < \theta < 1$. Then the process $\{X_t\}_{t \ge 0}$ is

(i) Harris ergodic if

$$\int_0^\infty e^{-\int_0^x \frac{2\theta[-a(y)/\beta - \mathbb{E}_y U]}{b^2(y) + \mathbb{E}_y U^2} dy} dx < \infty,$$

and (ii) exponentially ergodic with the convergence rate ρ satisfying (3.4) if

$$\int_0^\infty e^{-\int_0^x \frac{2\theta[-a(y)/\beta - \mathbb{E}_y U - \lambda \mathbb{E}_y U^2 e^{\lambda U}]}{b^2(y) + \mathbb{E}_y U^2} dy} dx < \infty.$$

4 Application: one dimension time-homogeneous diffusion processes

If diffusion and drift coefficients of a diffusion process are $\sigma^2(x)$ and $a(x)$, then the diffusion process will be in the form of a solution to the equation

$$d(\Phi_t) = a(\Phi(t)) + \sigma(\Phi(t))dB(t)$$

B_t is standard Brownian motion. We assume that the $\{\Phi_t\}_{t\geq 0}$ is governed on $[0,\infty)$ and the reflection at $\{0\}$ is done in such a manner that $\{\Phi_t\}_{t\geq 0}$ has continuous sample paths. It follows from [3] that the asymptotic behavior, for $\{\Phi_t\}_{t\geq 0}$ can be obtained by studying the unreflected generator $\mathcal{A}$, i.e., $\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}[\sigma(x)]^2 f''(x) \quad \forall f \in C^2[0,\infty)$.
1. Conditions for recurrence and transience.

Th. 4.1. The process $\{\Phi_t\}_{t\geq 0}$ is (i) Harris recurrent $\Leftrightarrow$

$$\int_0^\infty e^{-\int_0^y \frac{2a(x)}{\sigma^2(x)}dx} dy = \infty,$$

and (ii) transient $\Leftrightarrow$

$$\int_0^\infty e^{-\int_0^y \frac{2a(x)}{\sigma^2(x)}dx} dy < \infty$$

Proof. Obviously the function $f(x) = \int_0^x e^{-\int_0^y \frac{2a(u)}{\sigma^2(u)}du} dy$ is harmonic for $\{\Phi_t\}_{t\geq 0}$. Under the assumptions of theorem, it is norm-like in the recurrent case and bounded in the transient one. □

Note 4.1. In [2] the following result gives the criterion for recurrence and transience of $\{\Phi'_t\}_{t\geq 0}$ on R. Let the process $\{\Phi'_t\}_{t\geq 0}$ be defined by the equation

$$d\Phi'_t = a(\Phi'_t)dt + \sigma(\Phi'_t)dB(t),$$

where a and σ are continuously differentiable in $\mathbb{R}$ and $\sigma^2(x) \neq 0$.

Consider the functions

$$\begin{aligned} Q(x) &= e^{-2\int_0^x \frac{a(x)}{\sigma^2(x)}dz} \\ f(x) &= \int_0^x Q(y)dy. \end{aligned}$$

It is easily seen that

$$\mathcal{A}f = 0.$$

If, moreover,

$$\lim_{x\to\infty} f(x) = \infty$$
$$\lim_{x\to-\infty} f(x) = -\infty,$$

then the process $\{\Phi'_t\}_{t\geq 0}$ is recurrent, and transient otherwise. $\square$

2. Conditions for ergodicity and exponential ergodicty.

Th. 4.2. Suppose that $\inf_{x>v}\sigma^2(x) > 0$ the process $\{\Phi_t\}_{t\geq 0}$ is (i) Harris ergodic if and only if

$$\int_0^\infty e^{\int_0^x \frac{2a(y)}{\sigma^2(y)}dy}dx < \infty,$$

and (ii) exponentially ergodic with convergence rate ρ satisfying

$$0 < \rho \leq \frac{\frac{1}{2}\inf_{x>v}\sigma^2(x)\cdot\lambda}{\int_0^\infty e^{\int_0^x[\frac{2a(y)}{\sigma^2(y)}+\lambda]dy}dx}$$

if

$$\int_0^\infty e^{\int_0^y[\frac{2a(u)}{\sigma^2(u)}+\lambda]du}dy < \infty.$$

Proof. (i) ($\Leftarrow$) We shall test the function$f^\epsilon(x) = \int_0^x e^{\int_0^y[-\frac{2a(u)}{\sigma^2(u)}-\epsilon(u)]du}dy$ where $\epsilon(y) = e^{\int_0^y \frac{2a(u)}{\sigma^2(u)}du}$. Then

$$\mathcal{A}f^\epsilon(x) = e^{-\int_0^x[\frac{2a(y)}{\sigma^2(y)}+\epsilon(y)]dy}\left\{a(x) + \frac{1}{2}\sigma^2(x)\cdot[-\frac{2a(x)}{\sigma^2(x)} - \epsilon(x)]\right\} =$$
$$= -\frac{1}{2}e^{-\int_0^x[\frac{2a(u)}{\sigma^2(u)}+\epsilon(u)]du}\cdot\sigma^2(x)\epsilon(x) \leq -\frac{1}{2}\inf_{x>v}\sigma^2(x)\int_0^\infty e^{\int_0^x\frac{2a(y)}{\sigma^2(y)}dy}dx.$$

The process is Harris ergodic by Th. 4.4. in [4].

($\Rightarrow$) Th. 3.8 in [2].

(ii) Trying the function $f_\lambda(x) = \int_0^x e^{-\int_0^y[\frac{2a(u)}{\sigma^2(u)}+\lambda]du}dy$, we get

$$\mathcal{A}f_\lambda(x) = e^{-\int_0^x[\frac{2a(y)}{\sigma^2(y)}+\lambda]dy}\{a(x) + \frac{1}{2}\sigma^2(x)[-\frac{2a(x)}{\sigma^2(x)} - \lambda]\} =$$
$$= -\frac{1}{2}\frac{\sigma^2(x)\lambda\cdot e^{-\int_0^x[\frac{2a(y)}{\sigma^2(y)}+\lambda]dy}\cdot f_\lambda(x)}{f_\lambda(x)} =$$
$$= -\frac{1}{2}\sigma^2(x)\lambda f_\lambda(x)\Big/\int_0^\infty e^{\int_0^x[\frac{2a(y)}{\sigma^2(y)}+\lambda]dy}dx,$$

and assumptions of Th. 6.1 in [4] are satisfied.

$\square$

Example 1. Th. 6.1 in [3]. Affine drift. Assume that $a(x) \le -a(1+x)$ and $|\sigma(x)| \le \gamma$.

By Th. 4.2 (ii) we have that convergence rate ρ satisfy

$$0 < \rho \le \frac{1}{2}\gamma^2 \cdot \lambda \Big/ \int_0^\infty e^{\int_0^x [\frac{-2a(1+y)}{\gamma^2} + \lambda] dy} dx \le \frac{1}{2}\sigma^2 \cdot \frac{2a}{\sigma^2} = a$$

Example 2. Th 6.2 in [3]. Constant drift, $b^2(x) \equiv 1$. Using Th. 4.2 (ii), we get

$$0 < \rho \le \frac{1}{2}\mu^2$$

□

5 Application: storage model

Let $\{X_t\}_{t\ge 0}$ be a continuous time Markov process on the state space $[0,\infty)$, satisfying the storage equation

$$X_t = X_0 + A_t - \int_0^t r(X_s)ds$$

where $\{A_t\}_{t\ge 0}$ is an input process which we shall assume to be a compound Poisson process without drift term,

$$A_t = \sum_{i=0}^{N_t} U_i,$$

where $\{N_t\}_{t\ge 0}$ is a Poisson process with jump rate β and $\{U_i\}_0^\infty$ is a sequence of independent identically distributed random variables with

$$\mathbb{P}\{U \le x\} = G(x)$$

and independent of $\{N_t\}_{t\ge 0}$. (Here $U > 0$, i.e., $G(0) = 0$). The function $r : [0,\infty) \to [0,\infty), r(0) = 0$ is called the release rate for the system. The release rate being $r(x)$ at content x means that in between jumps, $\{X_t\}_{t\ge 0}$ should satisfy the differential equation

$$d\dot{x} = -r(x)dt,$$

where $d\dot{x}$ means left derivative. We shall assume that r is strictly positive, left continuous and has strictly positive right limits everywhere on $[0,\infty)$. We also assume that

$$0 < \inf_{\epsilon < x < \epsilon^{-1}} r(x),$$
$$\sup_{0 < x < \epsilon^{-1}} r(x) < \infty$$

for any $\epsilon \in (0,1)$ and

$$\int_0^x r(y)^{-1} dy < \infty, \quad x > 0.$$

The construction is based upon roots of the equation

$$s(x) = \beta \frac{\mathbb{E} e^{s(x)U} - 1}{r(x)}.$$

Let us suppose that $s(x) \equiv 0$ on $[0, v]$, $s(x) < 0$ on (v, ∞) and $s(x) \to 0$ as $x \to \infty$, we give next results without proofs.

Th. 5.1. If (i)

$$\int_0^\infty e^{\int_0^x s(y) dy} dx = \infty,$$

then $\{Y_n\}_0^\infty$ and $\{X_t\}_{t \geq 0}$ are Harris recurrent;
(ii) If

$$\int_0^\infty e^{\int_0^x s(y) dy} dx < \infty,$$

then $\{Y_n\}_0^\infty$ and $\{X_t\}_{t \geq 0}$ are transient;

Let us suppose that $s(x) \equiv 0$ on $[0, v]$ $s(x) > 0$ for some $v > 0$ and $s(x) \to 0$ as $x \to \infty$.

Th. 5.2. Then the process $\{X_t\}_{t \geq 0}$ is Harris ergodic if

$$\int_0^\infty e^{-\int_0^x s(y) dy} dx < \infty.$$

□

Th. 5.3. Let us assume that $\exists \lambda > 0$ such that $s(x) \equiv 0$ on $[0, v]$, $s(x) > \lambda$ on (v, ∞) and $s(x) \to \lambda$ as $x \to \infty$. Then the process $\{X_t\}_{t \geq 0}$ is exponentially ergodic if

$$\int_0^\infty e^{-\int_0^x [s(y) - \lambda] dy} dx < \infty,$$

and the convergence rate ρ satisfies

$$0 < \rho < \frac{(3 - e)\lambda \beta \mathbb{E} U^2}{\int_0^\infty e^{-\int_0^x [s(y) - \lambda] dy} dx}.$$

□

Corollary 5.1. Suppose that $\mathbb{P}\{U < x\} = 1 - e^{-\delta x}, x \geq 0$. Then the process $\{X_t\}_{t\geq 0}$ is

(i) Harris recurrent if and only if

$$\int_0^\infty e^{\int_0^x [\delta - \frac{\beta}{r(u)}] dy} dx = \infty,$$

(ii) transient if and only if

$$\int_0^\infty e^{\int_0^x [\delta - \frac{\beta}{r(u)}] dy} dx < \infty,$$

(iii) Harris ergodic if

$$\int_0^\infty e^{-\int_0^x [\delta - \frac{\beta}{r(y)}] dy} dx < \infty,$$

(iv) exponentially ergodic if for some $\lambda > 0$

$$\int_0^\infty e^{-\int_0^x [\delta - \frac{\beta}{r(y)} - \lambda] dy} dx < \infty.$$

□

6 Comments

The construction of the test functions presented in this paper will also work for discrete parameter chains, where the characteristic function of transition semigroup is of infinitely divisible law [6].

The asymptotic behavior of the storage and risk processes is described in [7]. The assumption of the existence of the exponential moments is quite restrictive, but the relaxation of this assumption is presented in [7].

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