Solutions with increasing energy for the spatially homogeneous Boltzmann equation.

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Abstract

Solutions to the spatially homogeneous Boltzmann equation can never loose energy, but it was recently discovered that in the general case there are solutions for which the energy increases. This paper shows that the same holds for all hard potentials (i.e. even without angular cutoff assumptions), and that the energy may increase continuously in time at quite arbitrary rates.

1. Introduction

The spatially homogeneous Boltzmann equation,

$$\frac{\partial}{\partial t} f(v,t) \quad = \quad Q(f,f)(v,t) \qquad\qquad (v,t) \in \mathbf{R}^3 \times (0,\infty)\,, \qquad (1)$$

describes the time evolution of the velocity distribution of a spatially homogeneous dilute gas of particles. We look for positive solutions $f \in C([0,\infty), L^1(\mathbf{R}^3))$, of equation (1) such that for all $t \in [0,\infty)$ is a density. In (1), Q is the collision operator acting on functions of velocity v :

$$Q(f,f)(v) \quad = \quad \iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)[f(v')f(v'_*) - f(v)f(v_*)]d\omega dv_*\,, \quad (2)$$

which describes the rate of change of f due to a binary collision ($Q(f,f)(v,t)$ means $Q(f(\cdot,t),f(\cdot,t))(v)$; the time enters only as a parameter in (2)). Here v, v_* are the velocities of two particles before they collide, and v', v'_* are their velocities after the collision. Leaving the details for later, we only note here that strictly hard potential interaction is the only case considered in this paper. This is not only for technical reasons: the results depend strongly on the fact that the $B(v-v_*,\omega)$ grows unboundedly with $|v-v_*|$. In the general case considered here, $B(v - v_*, \omega)$ also may be unbounded as a function of ω, but that does not have any impact on the result.

In the following section all details about the collision operator, as well as the concept of weak solutions for eq. (1) will be discussed. The remaining part of this section contains a short discussion and a background.

In a gas, each binary collision conserves the total mass, momentum and energy of the participating particles, and it is natural to expect from the solutions of the Boltzmann equation that the total energy should be conserved throughout the evolution (this is of course a very non-mathematical statement; one essential property of the Boltzmann equation is that, even though the microscopical dynamics is reversible, the Boltzmann equation itself is not). Under suitable conditions, the energy actually is conserved. It is known that in the case of hard cutoff potentials, there is a unique solution to eq. (1) that conserves energy. It is also known that there are no solutions for which the energy is decreasing ([MW],[Lu1]).

However, still under the cutoff condition, it is shown in [W3],that under very general conditions, there are *always* solutions for which the energy jumps, at time $t = 0$ or at any other time.

Clearly it is then possible to construct a solution with an arbitrary number of jumps, and one can quite arbitrarily also choose the size of the jump (always with the restriction that it be positive).

The purpose of this paper is two-fold: first to show that the energy behaves in the same way for weak solutions to the non-cutoff equation, i.e. that it is non-decreasing but may increase; secondly to construct solutions for which the energy increases continuously with time. The main results are given as Theorem 2, Theorem 3 and Theorem 5.

It is in its place to say that the solutions constructed here, and in [W3] are not the first examples of solutions to the Boltzmann equation where energy is increasing. In [TM], so-called *homo-energetic* solutions are constructed. These are special solutions to the full (spatially dependent) Boltzmann equation, for which the temperature is space independent. There are examples of homo-energetic shear flows, for which the temperature is increasing.

The paper is organised as follows. First, in Section 2, the Boltzmann equation is presented in the generality needed for our purpose. This includes a precise definition of the collision operator, the definition of weak solutions to (1) and we include some known results concerning such solutions, including a weak stability result.

In Section 3 we prove that the energy of the weak solutions may not decrease, just as in the case of mild solutions to the Boltzmann equation with cutoff, and this implies the existence of solutions for which the energy is conserved. Contrary to the case of cutoff potentials, however, this does not entail uniqueness of solutions, as we only deal with solutions which are weak limits of solutions to truncated problems. For this particular class of solutions, we also prove that moments are generated, as in the cutoff case.

Section 4, finally, contains the construction of solutions for which the energy is strictly (and continuously in time) increasing at quite arbitrary rates. This shows in particular, that without additional assumptions, the solutions are not unique.

2. The Spatially Homogeneous Boltzmann Equation

The spatially homogeneous Boltzmann equation, (1) has been studied thoroughly for a long time, and we give here only the results that are relevant for the subsequent sections.

In the collision operator in (2), the velocities before and after a collision are related by

$$\begin{aligned} v' &= v - \langle v - v_*, \omega\rangle\omega\,, \\ v'_* &= v_* + \langle v - v_*, \omega\rangle\omega\,, \qquad \omega \in \mathbf{S}^2\,, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbf{R}^3$, $|v|^2 = \langle v, v\rangle$ and $\mathbf{S}^2 = \{\omega \in \mathbf{R}^3\,;\ |\omega| = 1\}$. This implies that $v' + v'_* = v + v_*$, $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$, *i.e.* mass and energy are conserved in each collision. The rate at which each possible collision occurs is given by the collision kernel $B(z,\omega)$, which is a nonnegative function of $|z|$ and $|\langle z,\omega\rangle|$ only. For the interaction potentials of inverse power laws, $B(z,\omega)$ takes the form (see, e.g., [CIP] or [TM]):

$$B(z,\omega) = b(\theta)|z|^{\beta}, \qquad \theta = \arccos(|z|^{-1}|\langle z,\omega\rangle|), \tag{3}$$

where the exponent β is related to the potentials of interacting particles. For the so-called soft potentials ($-3 < \beta < 0$), for pseudo-Maxwellian molecules ($\beta = 0$), and for the hard potentials ($0 < \beta < 1$) and the hard sphere model ($\beta = 1$, $b(\theta) = \text{const.}\cos(\theta)$).

Except for the case of hard spheres, the angular function $b(\theta)$ has a non-integrable singularity at $\theta = \pi/2$:

$$b(\theta) \;=\; \mathcal{O}\big((\pi/2 - \theta)^{-(3-\beta)/2}\big) \tag{4}$$

as $\theta \to \pi/2$. In general, however,

$$0 \;<\; \int_0^{\pi/2} b(\theta)\cos^2(\theta)\sin(\theta)\,d\theta \;<\; \infty\,. \tag{5}$$

The so-called cutoff-assumption means that $b(\theta)$ is truncated so as to make

$$\int_0^{\pi/2} b(\theta)\sin(\theta)\,d\theta \;<\; \infty\,.$$

For hard potentials with the cutoff assumption, the collision operator can be split into two parts, $Q^+(f,f)(v)$, and $Q^-(f,f)(v)$, corresponding to the positive and negative part of (2), and solutions to (1) satisfy the mild form of the equation:

$$f(v,t) \;=\; f_0 + \int_0^t Q(f,f)(v,\tau)\,d\tau, \qquad t \geq 0\,;$$

this holds in $f \in C([0,\infty), L^1(\mathbf{R}^3))$. For these solutions it is known that

- $\int_{\mathbf{R}^3} f(v,t)\,dv = \int_{\mathbf{R}^3} f_0(v)\,dv$ (mass is conserved, so that $f(t,\cdot)$ for all t is a density)
- $\int_{\mathbf{R}^3} f(v,t)|v|^2\,dv \geq \int_{\mathbf{R}^3} f_0(v)|v|^2\,dv$ (energy is non-decreasing).
- There is one and only one solution for which energy is conserved
- $\int_{\mathbf{R}^3} f(v,t)\log(f(v,t))\,dv \leq \int_{\mathbf{R}^3} f_0 \log(f_0(v))(v)\,dv$ (the kinetic entropy is decreasing)

A rather complete discussion of this, including numerous references, can be found in [MW],[Lu1].

In the non-cutoff case, the two parts of the collision operator are not defined separately, and one is restricted to the study of weak solutions to eq. (1). These can be defined in different ways depending on the singularity of the collision kernel B. There is always a singularity in ω, as described in (4), but for soft potentials there is also a singularity where $|v-v_*| \to 0$. For *very soft potentials* ($\beta \leq -1$) the latter singularity is the more severe. A concept of weak solutions for this situation has been considered by Goudon ([Go]), and by Villani ([V]). As our main object here is the study of hard potentials, we give the definition of weak solutions as given by Arkeryd ([Ar2]), which is suitable when $-1 < \beta < 1$:

Definition 1: A non-negative function $f : [0,\infty) \rightarrow L^1(\mathbf{R}^3)$ is a weak solution to (1) if it satisfies

$$\begin{aligned}\int_{\mathbf{R}^3} f(v,t)\varphi(v,t)\,dv &= \int_{\mathbf{R}^3} f(v,0)\varphi(v,0)\,dv + \int_0^t \int_{\mathbf{R}^3} f(v,\tau)\frac{\partial}{\partial\tau}\varphi(v,\tau)\,dv\,d\tau \\ + \int_0^t \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} & B(v-v_*,\omega) f(v,t) f(v_*,t)\big(\varphi(v',t)-\varphi(v,t)\big)\,d\omega\,dv_*\,dv\,d\tau\,,\end{aligned} \tag{6}$$

for all $\varphi \in C^{1,\infty}$.

$$\begin{aligned} C^{1,\infty} &= \{\varphi \in C^\infty([0,\infty[\times\mathbf{R}^3)\;; \\ &\qquad |\varphi|_1 \equiv \sup\big(|\varphi(v,t)| + |\partial_t\varphi(v,t)| + |\nabla_v\varphi(v,t)|\big) < \infty\}\end{aligned}$$

Note that any mild solution to the Boltzmann equation with cutoff satisfies eq. (6), which guarantees the existence of weak solutions in that case. Arkeryd used a weak stability result to prove that weak solution exist also in the general case. Here, and always in the sequel of the paper, we assume that the initial data satisfy

$$\int_{\mathbf{R}^3} f_0(v)\Big(1+|v|^2+\log f_0(v)\Big)\,dv \quad < \quad \infty\,. \tag{7}$$

We also define the weighted L^1-spaces L^1_s with norm

$$\|f\|_{L^1_s} \quad = \quad \int_{\mathbf{R}^3} |f(v)|\big(1+|v|^2\big)^{s/2}\,dv\,.$$

Theorem 1 (weak stability and existence of solutions): (Arkeryd [Ar2]) *Assume that the collision kernel $B(v-v_*,\omega)$ satisfy (3) and (5), and let $B_n(v-v_*,\omega)$ be a sequence of cutoff kernels converging pointwise to $B(v-v_*,\omega)$ and satisfying $B_n(v-v_*,\omega) \le B(v-v_*,\omega)$. Assume that $f_0^n(v)$ is a sequence of initial data for the Boltzmann equation with kernel $B_n(v-v_*,\omega)$, and let $f^n(v,t)$ be the corresponding (weak) solutions that conserve mass and momentum, and that satisfy*

$$\sup_{n\ge 1, t\ge 0} \int_{\mathbf{R}^3} f^n(v,t)\big(1+|v|^2+|\log(f^n(v,t))|\big)\,dv \quad < \quad \infty\,,$$

$$f_0^n \rightharpoonup f_0 \qquad \textit{weakly in} \qquad L^1(\mathbf{R}^3)\,.$$

Then there is a subsequence f^{n_j} such that for all t, $f^{n_j}(\cdot,t) \rightharpoonup f(\cdot,t)$ weakly in $L^1(\mathbf{R}^3)$, where $f(v,t)$ is a weak solution to the Boltzmann equation with the kernel B. This weak solution conserves mass and momentum, and

$$\int_{\mathbf{R}^3} f(v,t)\log\big(f(v,t)\big)\,dv \quad\le\quad \liminf_{n\to\infty}\int_{\mathbf{R}^3} f^{n_j}(v,t)\log\big(f^{n_j}(v,t)\big)\,dv\,, \qquad (8)$$

$$\int_{\mathbf{R}^3} f(v,t)|v|^2\,dv \quad\le\quad \liminf_{n\to\infty}\int_{\mathbf{R}^3} f^{n_j}(v,t)|v|^2\,dv\,. \qquad (9)$$

Remark: The statement about the weak lower semi-continuity of entropy, i.e. about

$$H(f(\cdot,t)) \quad\equiv\quad \int_{\mathbf{R}^3} f(v,t)\log\big(f(v,t)\big)\,dv$$

is proven in [E2]. This together with the bound on the energy,

$$E(f(\cdot,t)) \quad\equiv\quad \int_{\mathbf{R}^3} f(v,t)|v|^2\,dv \;<\; C\,,$$

implies that the entropy is bounded from below.

For the remaining part, we restrict ourselves to the case $\beta > 0$, i.e. to hard potentials.

3. Energy and moment estimates for weak solutions.

Moment estimates for the Boltzmann equation are usually obtained by the use of a type of estimates that were first obtained by Povzner; later more elaborate versions have been obtained in several contexts. See e.g. [MW] or [Lu1] for references. The first statement in Lemma 1 below is quoted directly from [W1], and the second statement is a small modification of the same result:

Lemma 1: *Let $b(\theta)$ be positive, symmetric and such that $\int_{S^2} b(\theta)\cos^2(\theta)d\omega$ is convergent, and let $s > 2$ be an even integer. Then, for some positive constants, C_s and K_s*

$$\int_{S^2} \left(|v'|^s + v'_*|^s - |v|^s - |v_*|^s\right) b(\theta)\, d\omega \qquad (10)$$
$$\le C_s\left(|v|^{s-1}|v_*| + |v_*|^{s-1}|v|\right) - K_s\left(|v|^s + |v_*|^s\right).$$

And denoting $(y)^+ = \max(0,y)$,

$$\int_{S^2} \left(|v'|^s + v'_*|^s - |v|^s - |v_*|^s\right)^+ b(\theta)\, d\omega \qquad (11)$$
$$\le C_s\left(|v|^{s-1}|v_*| + |v_*|^{s-1}|v|\right).$$

The constants depend on n and on the kernel $b(\theta)$, but the estimate holds for cutoff potentials as well as for non-cutoff potentials.

The proof of this lemma is easy: It relies on writing

$$\begin{aligned} {r'}^2 &= r^2\cos^2\theta + r_*^2\sin^2\theta + 2\tau r r_*\sin\theta\cos\theta\cos\phi\,, \\ {r'_*}^2 &= r^2\sin^2\theta + r_*^2\cos^2\theta - 2\tau r r_*\sin\theta\cos\theta\cos\phi\,, \end{aligned}$$

(where $r = |v|$ etc.) for a suitable parametrisation of the sphere S^2, and using the binomial theorem. For example, expanding

$$r'^s \;=\; \left(\left[r^2\cos^2\theta + r_*^2\sin^2\theta\right] + \left[\tau r r_*\sin\theta\cos\theta\cos\phi\right]\right)^{s/2}.$$

as indicated by the square brackets yields the negative term in eq. (10) by keeping the terms not involving ϕ; the sum of these is negative by a convexity argument. The remaining terms have varying signs, but also integrating the modulus gives the bound $C_s\left(|v|^{s-1}|v_*| + |v_*|^{s-1}|v|\right)$, hence the second estimate follows trivially.

In eq. (10) replacing $|v|^s$ by $\left(1+|v|^2\right)^{s/2}$ essentially only changes the *negative* term on the right-hand side; this becomes

$$-K_n\left(\left(1+|v|^2\right)^n + \left(1+|v_*|^2\right)^n\right).$$

In [MW] and [Lu1] it was proven that the energy of the mild solutions to the cutoff hard potential Boltzmann equation is non-decreasing. Here we use the approach of Lu to extend this result to weak solutions, but still only for hard potentials.

Theorem 2. *Let the initial data to the Boltzmann equation satisfy $0 \le f_0 \in L^1_2(\mathbf{R}^3)$, and let the collision kernel $B(v - v_*, \omega)$ satisfy eq. (3) and (5) with $0 \le$*

$\beta \leq 1$. *Then for any weak solution* $f \in L^\infty([0,\infty), L^1_2(\mathbf{R}^3))$ *of the Boltzmann equation (1), the energy is always non-decreasing on* $[0,\infty)$*:*

$$E(f(\cdot,t)) = E(f_0) + \lim_{\varepsilon\to 0+} \int_0^t d\tau \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(v,\tau)f(v_*,\tau)K_\varepsilon(v,v_*)dv_*dv, \quad t\geq 0 \tag{12}$$

where

$$K_\varepsilon(v,v_*) = \frac{1}{2\varepsilon}\int_{S^2} B(v-v_*,\omega)\log\left(1+\frac{\varepsilon^2(|v'|^2|v'_*|^2-|v|^2|v_*|^2)^+}{(1+\varepsilon|v|^2)(1+\varepsilon|v_*|^2)}\right)d\omega$$

and $(y)^+ = \max\{y,0\}$. *In particular it follows from Theorem 1 that there is a solution to eq. (1) that conserves energy.*

Proof: For $\varepsilon > 0,\ \delta > 0$, let

$$\varphi_{\varepsilon,\delta}(v) = \frac{1}{\varepsilon}\log\left(1+\frac{\varepsilon|v|^2}{1+\delta|v|^2}\right), \quad \varphi_\varepsilon(v) = \frac{1}{\varepsilon}\log(1+\varepsilon|v|^2).$$

It is easily seen that $\varphi_{\varepsilon,\delta} \in C^{1,\infty} : \varphi_{\varepsilon,\delta}(v) \leq 1/\delta$ and $|\mathrm{grad}_v \varphi_{\varepsilon,\delta}(v)| \leq 1/\sqrt{\varepsilon}$ so that for $t > 0$,

$$\int_{\mathbf{R}^3} f(v,t)\varphi_{\varepsilon,\delta}(v)dv = \int_{\mathbf{R}^3} f_0(v)\varphi_{\varepsilon,\delta}(v)dv + \frac{1}{2}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} F_{\varepsilon,\delta}(v,v_*,\omega,\tau)d\omega dv_* dv \tag{13}$$

where

$$F_{\varepsilon,\delta}(v,v_*,\omega,\tau) = B(v-v_*,\omega)f(v,\tau)f(v_*,\tau)[\varphi_{\varepsilon,\delta}(v')+\varphi_{\varepsilon,\delta}(v'_*)-\varphi_{\varepsilon,\delta}(v)-\varphi_{\varepsilon,\delta}(v_*)].$$

The expression within square brackets is smaller than

$$\frac{2}{\sqrt{\varepsilon}}|\langle v-v_*,\omega\rangle|$$

and

$$\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} B(v-v_*,\omega)f(v,\tau)f(v_*,\tau)\frac{2}{\sqrt{\varepsilon}}|\langle v-v_*,\omega\rangle| d\omega dv_* dv \leq \frac{2}{\sqrt{\varepsilon}}A_0\int_0^t d\tau\left(\int_{\mathbf{R}^3} f(v,\tau)(1+|v|^2)^{(1+\beta)/2}dv\right)^2 < \infty,$$

Moreover $\varphi_{\varepsilon,\delta}(v) \leq \varphi_\varepsilon(v)$, $\varphi_{\varepsilon,\delta}(v) \to \varphi_\varepsilon(v)$ $(\delta \to 0+)$, and so the dominated convergence theorem implies that

$$\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} |F_\varepsilon(v,v_*,\omega,\tau)| d\omega dv_* dv \quad < \quad \infty \,,$$

and that

$$\int_{\mathbf{R}^3} f(v,t)\varphi_\varepsilon(v)dv = \int_{\mathbf{R}^3} f_0(v)\varphi_\varepsilon(v)dv \tag{14}$$

$$+\frac{1}{2}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} F_\varepsilon(v,v_*,\omega,\tau)d\omega dv_* dv \,, \tag{15}$$

where $F_\varepsilon(v,v_*,\omega,\tau) = F_{\varepsilon,0}(v,v_*,\omega,\tau)$. Next using the conservation of energy, $|v'|^2+|v_*'|^2 = |v|^2+|v_*|^2$ and the fact that ϕ_ε is a logarithm, we can write

$$\begin{aligned}\varphi_\varepsilon(v') + \varphi_\varepsilon(v_*') - &\varphi_\varepsilon(v) - \varphi_\varepsilon(v_*) \\ &= \frac{1}{\varepsilon}\log\left(1+\frac{\varepsilon^2(|v'|^2|v_*'|^2-|v|^2|v_*|^2)^+}{(1+\varepsilon|v|^2)(1+\varepsilon|v_*|^2)}\right) \\ &- \frac{1}{\varepsilon}\log\left(1+\frac{\varepsilon^2(|v|^2|v_*|^2-|v'|^2|v_*'|^2)^+}{(1+\varepsilon|v|^2)(1+\varepsilon|v_*|^2)}\right),\end{aligned} \tag{16}$$

thus splitting the bracket in a positive and a negative part.

The first of the terms in the right-hand side of eq. (16) is positive, and the second one tends pointwise to zero as $\varepsilon \to 0$. It remains to find a uniform bound, so as to allow using the dominated convergence theorem again.

Using the identity $|v|^2|v_*|^2 - |v'|^2|v_*'|^2 = \frac{1}{2}(|v'|^4+|v_*'|^4-|v|^4-|v_*|^4)$ and the inequality, $\frac{1}{\varepsilon}\log(1+\varepsilon^2 Z) \leq \varepsilon Z$ we find that

$$J_\varepsilon(v,v_*) \quad \equiv \quad \frac{1}{2\varepsilon}\int_{S^2} B(v-v_*,\omega)\log\left(1+\frac{\varepsilon^2(|v|^2|v_*|^2-|v'|^2|v_*'|^2)^+}{(1+\varepsilon|v|^2)(1+\varepsilon|v_*|^2)}\right)d\omega$$

is bounded by

$$\begin{aligned}\frac{1}{2}\int_{S^2} B(v-v_*,\omega)&\frac{\varepsilon(|v|^2|v_*|^2-|v'|^2|v_*'|^2)^+}{(1+\varepsilon|v|^2)(1+\varepsilon|v_*|^2)}\,d\omega \\ &\leq \frac{|v-v_*|^\beta}{2\big(1+\varepsilon(|v|^2+|v_*|^2)\big)}\int_{S^2} b(\theta)\big(|v'|^4+|v_*'|^4-|v|^4-|v_*|^4\big)^+ d\omega,\end{aligned} \tag{17}$$

and using (11) with $s=4$ gives the bound

$$C|v||v_*||v-v_*|^\beta \quad \leq \quad C(1+|v|^2)(1+|v_*|^2) \tag{18}$$

for some new constant C, that depends only on the integral in (17) Therefore

$$\int_0^t d\tau \int_{\mathbf{R}^3\times\mathbf{R}^3} f(v,\tau)f(v_*,\tau)J_\varepsilon(v,v_*)dv_* dv \quad < \quad \infty \,,$$

and so by (13)

$$\begin{aligned}\int_{\mathbf{R}^3} f(v,t)\varphi_\varepsilon(v)dv &= \int_{\mathbf{R}^3} f_0(v)\varphi_\varepsilon(v)dv \\ &+ \int_0^t d\tau \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(v,\tau)f(v_*,\tau)K_\varepsilon(v,v_*)dv_*dv \\ &- \int_0^t d\tau \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(v,\tau)f(v_*,\tau)J_\varepsilon(v,v_*)dv_*dv.\end{aligned} \tag{19}$$

Since $\lim_{\varepsilon\to 0+} J_\varepsilon(v,v_*) = 0$, for all $(v,v_*) \in \mathbf{R}^3 \times \mathbf{R}^3$, it follows by the dominated convergence theorem that

$$\lim_{\varepsilon\to 0+} \int_0^t d\tau \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(v,\tau)f(v_*,\tau)J_\varepsilon(v,v_*)dv_*dv = 0,$$

and

$$\lim_{\varepsilon\to 0+} \int_{\mathbf{R}^3} f(v,t)\varphi_\varepsilon(v)dv = E(f(\cdot,t))$$

for $t>0$. Therefore (12) follows from eq. (19) by letting $\varepsilon \to 0+$.

An early result on moments of solutions to the Boltzmann equation is that of Elmroth [E1], who showed that all moments that are initially bounded remain bounded. It was then shown, first by Desvillettes [D], that solutions of the Boltzmann equation for hard potentials gain moments. His result was then improved upon in e.g. [W2]. Here we use the estimates from the latter paper together with the weak convergence result from Section 2 to prove that also in the non-cutoff case, there are solutions that behave similarly.

We begin by choosing a sequence of truncated collision kernels, and a sequence of truncated initial data as follows:

$$\begin{aligned} B_n(|v-v_*|,\omega) &= |v-v_*|^\beta \min(b(\theta),n)\,, \\ f_0^n(v) &= e^{-|v|^2/n} f_0(v). \end{aligned}$$

Assuming that the full initial data f_0 satisfies the estimate (7), the same holds for all f_0^n, uniformly in n. In addition, all the functions f_0^n satisfy bounds on higher moments, $\|f_0^n\|_{L^1_s} \le C_{f_0,n,s}$.

Let $f^n(v,t)$ be the mild solutions of the Boltzmann equation with these initial data and collision kernels. From the results in [MW], those are unique in the class of solutions with given energy. The results from Section 2 imply that there is a subsequence (which we still denote by $f^n(v,t)$) that converges weakly to a weak solution $f(v,t)$ of (1), and the estimate in Section 3 shows that the energy of $f(v,t)$ is conserved, i.e.

$$E(f(\cdot,t)) \equiv \int_{\mathbf{R}^3} f(v,t)|v|^2\,dv = E(f_0).$$

We also know that the entropy of the solutions $f^n(v,t)$ is bounded uniformly in n, and because of the weak semi-continuity of the entropy (see e.g. [E2]),

$$H(f(\cdot,t)) \equiv \int_{\mathbf{R}^3} f(v,t)\log f(v,t)\,dv \le \liminf_{n\to\infty} H(f^n(\cdot,t))$$

From this we may draw the conclusion that all constants depending only on mass, energy and entropy of the solutions can be chosen uniformly in n.

Thus given a sequence of functions f^n that converge weakly to a weak solution of the Boltzmann equation, (1), we now wish to prove that the limiting function f at any positive time possesses moments of all orders. The method is the same as in [W2], we only need to verify that all estimates can be obtained in a uniform way with respect to n. And this in turn relies on the fact that Lemma 1 is independent of the truncation b_n.

We begin by multiplying equation (1) (with kernel $B_n(v-v_*,\omega)$) by $\left(1+|v|^2\right)^{s/2}$, where $s>2$ is an even number, and then integrate. That gives

$$\begin{aligned}
&\frac{d}{dt}\|f^n(\cdot,t)\|_{L^1_s} = \\
&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} f^n(v,t)f^n(v_*,t)|v-v_*|^\beta b_n(\theta)\Big(\left(1+|v'|^2\right)^{s/2}+\left(1+|v'_*|^2\right)^{s/2} \\
&\qquad - \left(1+|v|^2\right)^{s/2} - \left(1+|v_*|^2\right)^{s/2}\Big)\,d\omega dv\,dv_*
\end{aligned}$$

which by Lemma 1 (and the remark just after) together with an obvious symmetry argument implies

$$\begin{aligned}
&\frac{d}{dt}\|f^n(\cdot,t)\|_{L^1_s} \le \\
&- K_s \iint_{\mathbf{R}^3\times\mathbf{R}^3} f^n(v,t)f^n(v_*,t)|v-v_*|^\beta\left(1+|v|^2\right)^{s/2}dv\,dv_* \\
&+ C_s \iint_{\mathbf{R}^3\times\mathbf{R}^3} f^n(v,t)f^n(v_*,t)|v-v_*|^\beta|v|^{s-1}|v_*|\,dv\,dv_*
\end{aligned} \tag{20}$$

The uniform bound on the energy and on the entropy implies that

$$\int_{\mathbf{R}^3} f^n(v_*,t)|v-v_*|^\beta\,dv \ge C_{H,E,\beta}(1+|v|^2)^{\beta/2}\,.$$

(see e.g. Arkeryd [Ar1]). Hence the negative term in (20) is bounded from above by

$$\begin{aligned}
&-K_sC_{H,E,\beta}\|f^n(\cdot,t)\|_{L^1_{s+\beta}} \\
&\qquad \le -K_sC_{H,E,\beta}\|f^n(\cdot,t)\|_{L^1_2}^{\beta/(2-s)}\|f^n(\cdot,t)\|_{L^1_s}^{1+\beta/(s-2)}\,;
\end{aligned}$$

the last estimate follows by the Hölder inequality. The positive term in (20) can be estimated by

$$C_s\|f^n(\cdot,t)\|_{L^1_2}\|f^n(\cdot,t)\|_{L^1_s}$$

and hence, using the uniform bounds on the energy (N.B., uniform in n and in t), we obtain the differential inequality

$$\frac{d}{dt}\|f^n(\cdot,t)\|_{L^1_s} \le a_s\|f^n(\cdot,t)\|_{L^1_s} - \bar{a}_s[\|f^n(\cdot,t)\|_{L^1_s}]^{1+\beta/(s-2)}, \quad t \ge 0.$$

The conclusion that can be drawn from this is that

$$\|f^n(\cdot,t)\|_{L^1_s} \le W_{E,H,s}(t), \tag{21}$$

where

$$W_{E,H,s}(t) = \left[\frac{a_s}{\bar{a}_s(1-\exp(-\frac{\beta}{s-2}a_s\,t)}\right]^{(s-2)/\beta}, \qquad t > 0\,. \tag{22}$$

Through a and $\bar{a}$, this estimate depends on the entropy and on the energy of the functions (but these are uniform in time and in for all f^n in the sequence), and on the moment s. For this construction we use the fact that all moments are bounded for the initial functions f_0^n, but in the final estimate, the dependence on the initial data disappears, except through the entropy and the energy. The construction here has been made only for moments of even integer order, but by interpolation one directly obtains the same estimate for all real $s > 2$. Note, however, that if one needs optimal values of the involved constants (as $s \to 2$), then a more precise Povzner inequality is needed [Lu2].

Theorem 3._Let the kernel $B(v-v_*,\omega)$ satisfy (3) and (5) for $0 < \beta \le 1$. Then the Boltzmann equation (1) has a weak solution that conserves mass, momentum and energy. Moreover, for all $s > 2$ we have the estimate_

$$\|f(\cdot,t)\|_{L^1_s} \le W_s(t; E(f_0), H(f_0)) \qquad t > 0\,.$$

4. Solutions with continuously increasing energy

We begin this section by proving that equation (1) allows for solutions for which the energy makes an arbitrary number of jumps (but always increasing). This is an extension of the result from [W3] to weak solutions of the non-cutoff Boltzmann equation. Then we go on to proving that the energy may increase at an arbitrary rate, in a way to be formulated precisely in Theorem 5.

Theorem 4. _Let the collision kernel $B(v-v_*,\omega)$ satisfy (3) and (5) with $0 < \beta \le 1$, and let the initial data satisfy (7). Then for any positive integer N, and any set of times_

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N\ ,$$

and any set of numbers ("energy jumps")

$$0 \le e_1 \le e_2 \le \cdots \le e_N \le e_{N+1} ,$$

there is a weak solution to the Boltzmann equation (1) with initial data f_0 *such that mass and momentum are conserved, the entropy is non-increasing,*

$$H(f(\cdot,t)) \quad \le \quad H(f_0) ,$$

and such that the energy jumps at the given times t_k

$$E(f(\cdot,t)) \quad = \quad \begin{cases} E(f_0) & \textit{for} \quad t=0 \\ E(f_0)+e_k & \textit{for} \quad t_{k-1}<t\le t_k \\ E(f_0)+e_{N+1} & \textit{for} \quad t_N<t\ . \end{cases}$$

Proof: For any $n \ge 1$, let

$$\psi_n(v) \quad = \quad \frac{1}{4\pi n}|v|^{-(5+\frac{1}{n})}\chi_{\{|v|\ge 1\}}, \qquad\qquad f_0^n(v)=f_0(v)+e_1\,\psi_n(v).$$

Then for all $\alpha<2$, $\lim_{n\to\infty}\|f_0^n-f_0\|_{L^1_\alpha}=0$, and

$$\begin{aligned} E(f_0^n) \quad &= \quad E(f_0)+e_1, \\ \sup_{n\ge 1}\int_{\mathbf{R}^3} f_0^n(v)(1+v|^2+|\log f_0^n(v)|)dv \quad &< \quad \infty. \end{aligned} \tag{23}$$

By Theorem 3, there exist weak solutions f^n of (1) with $f^n|_{t=0}=f_0^n$ satisfying:

1. f^n conserve the mass, momentum and energy;
2. $H(f^n(\cdot,t)) \le H(f_0^n)$;
3. for any $s>2$,

$$\|f^n(\cdot,t)\|_{L^1_s} \quad \le \quad W_s(t\,;E(f_0^n)\ ,H(f_0^n)), \qquad \forall t>0, \quad \forall n\ge 1. \tag{24}$$

By weak stability, there exist a subsequence $\{f^{n_j}\}_{j=1}^{\infty}$ and a weak solution f_1 of (1) having the same bounds as the initial data, (7) and conserving the mass and momentum with initial datum $f_1|_{t=0} = f_0$, such that $f^{n_j}(\cdot,t) \rightharpoonup f_1(\cdot,t)$ $(j\to\infty)$ weakly in $L^1(\mathbf{R}^3)$, $\forall t\ge 0$. Because $\lim_{n\to\infty}H(f_0^n)=H(f_0)$, it follows from the weak lower semi-continuity of the entropy that f_1 also satisfies $H(f_1(\cdot,t))\le H(f_0),\ t\ge 0$. Moreover, by weak convergence and because of the indentity (23), we obtain from eq. (24) that

$$\|f_1(\cdot,t)\|_{L^1_s} \quad \le \quad W_s(t\,;E(f_0)+e_1,H(f_0)), \qquad \forall t>0, \qquad s>2. \tag{25}$$

Next by the equality (23), the moment estimate (24) and by the weak convergence, it follows that

$$E(f_1(\cdot,t)) \quad = \quad E(f_0)+e_1, \qquad \forall t>0\,. \tag{26}$$

Replacing $f(v,0)$ and e_1 by $f(v,t_1)$ and $e_2 - e_1$, respectively, and using the same argument, we see that there is a weak solution f_2 on $[t_1,\infty)$ with the initial datum $f_2(v,t_1) = f_1(v,t_1)$ and conserving the mass and momentum and satisfying $H(f_2(\cdot,t)) \le H(f_1(\cdot,t_1))$, $(t > t_1)$, such that

$$E(f_2(\cdot,t)) \quad = \quad E(f_1(\cdot,t_1)) + e_2 - e_1 = E(f_0) + e_2 \qquad t > t_1$$

and, for all $s > 2$,

$$\|f_2(\cdot,t)\|_{L^1_s} \ \le \ W_s(t-t_1;\ E(f_1(\cdot,t_1)), H(f_1(\cdot,t_1)), \qquad t > t_1.$$

Now for the solutions f_k $(k = 1,2,\cdots,N+1)$, we define

$$f(\cdot,t) \quad = \quad f_k(\cdot,t), \qquad t \in [t_{k-1},t_k], \qquad k = 1,2,\cdots,N; \tag{27}$$

Then by the integral form (6) of weak solutions of eq. (1), it is easily verified that f is a desired weak solution having all properties stated in the theorem.

Remark: Note that this construction does not set any effective limit as to what energy could be reached in a finite time. Take a sequence $t_n \to T$ and a sequence $e_n \to \infty$. Then the above construction can be carried out for any for any set $t_1 < t_2 \cdots < t_n$ and corresponding energies. Then one can go on for larger and larger n.

The estimate in Theorem 3 is not so useful when the interval lengths are allowed to shrink, and this poses a problem when trying to construct a solution with continuously increasing energy. Hence we construct here a solution for which the energy is a Cantor function, where the energy is constant in many intervals of t. Before formulating the theorem, we first recall the construction of a general symmetric Cantor set (see e.g. [Be]); the set we are looking for here has finite measure, and hence the corresponding Cantor function is well behaved as compared to the usual example used in the theory of integration. The only difference between the set we use here and the usual Cantor set, is that rather than removing "middle thirds", we remove smaller and smaller fractions of the remaining intervals: at step n, the fraction removed is $(1-2\xi_n)$, where ξ_n tends to $1/2$ sufficiently fast to guarantee that the remaining set has positive measure. Apart from that, this Cantor set has all the usual properties: it is perfect, totally disconnected and so on.

Let $\xi \in [0,1)$ and $\{\xi_n\}_{n=1}^{\infty} \subset (0,1/2)$ satisfy

$$\lim_{n\to\infty} 2^n \xi_1 \xi_2 \cdots \xi_n \quad = \quad \xi.$$

First the half-open interval $]0,1]$ is subdivided as

$$(0,1] \ = \ J_1^{(0)} \quad = \quad J_1^{(1)} \cup I_1^{(1)} \cup J_2^{(1)}$$

where

$$J_1^{(1)} = (0\,,\ \xi_1],\quad I_1^{(1)} = (\xi_1\,,\ 1-\xi_1],\quad J_2^{(1)} = (1-\xi_1\,,\ 1].$$

The removed part, $I_1^{(1)}$ has measure $\mu(I_1^{(1)}) = 1 - 2\xi_1$. We denote by $\mu(A)$ the Lebesgue–measure of a set $A \subset \mathbf{R}$. Again , let

$$J_1^{(1)} = J_1^{(2)} \cup I_1^{(2)} \cup J_2^{(2)},\quad J_2^{(1)} = J_3^{(2)} \cup I_2^{(2)} \cup J_4^{(2)}$$

where $I_1^{(2)}$ and $I_2^{(2)}$ are middle subintervals of $J_1^{(1)}$ and $J_2^{(1)}$ respectively. Each of these middle intervals have measure $\xi_1(1-2\xi_2)$, and so

$$\mu\big(I_1^{(2)} \cup I_2^{(2)}\big)/\mu\big(J_1^{(1)} \cup J_2^{(1)} J_2^{(1)}\big) \quad = \quad (1-2\xi_2)$$

and (from left to right)

$$]0,1] = J_1^{(2)} \cup I_1^{(2)} \cup J_2^{(2)} \cup I_1^{(1)} \cup J_3^{(2)} \cup I_2^{(2)} \cup J_4^{(2)}.$$

Inductively, each set J_k^n is decomposed as

$$J_k^n \quad = \quad J_{2k-1}^{n+1} \cup I_k^{n+1} \cup J_{2k}^{n+1},$$

where all three sets are left-open, right-closed, and where

$$\mu(I_k^{n+1})/\mu(J_k^n) \quad = \quad (1-2\xi_n)\,.$$

Next, at each level we let

$$\begin{aligned} I^{(n)} &= \cup_{k=1}^{2^{n-1}} I_k^{(n)}\,, \\ J^{(n)} &= J^{(n-1)} \setminus I^{(n)} \qquad \text{and} \\ C_\xi &= \cap_{n=1}^{\infty} J^{(n)}\,. \end{aligned}$$

It follows directly from this construction that

$$\mu(C_\xi) = \lim_{n\to\infty} \mu(J^{(n)}) = \lim_{n\to\infty} \sum_{i=1}^{2^n} \mu(J_i^{(n)}) = \lim_{n\to\infty} 2^n \xi_1 \xi_2 \cdots \xi_n = \xi$$

and that

$$\mu\Big(\cup_{n=N}^{\infty} I^{(n)}\Big) \quad \to \quad 0 \qquad \text{as} \qquad N \to \infty.$$

The set C_ξ is the Cantor set we are looking for; this is a set of positive measure (i.e., $\xi > 0$), and the corresponding Cantor function (which we shall denote by Θ) is defined by

$$\Theta(x) \quad \equiv \quad \xi^{-1} \int_{-\infty}^{x} 1\!\!1_{C_\xi}(y)\, dy \tag{28}$$

This Cantor function Θ is (Lipschitz–) continuous and satisfies

$$\Theta(x) \quad = \quad (2k-1)2^{-n} \qquad \text{for all} \qquad x \in I_k^{(n)} \qquad 1 \leq k \leq 2^{n-1} \qquad n \geq 1.$$

Here $1\!\!1_A$ is the characteristic function of the set A. We also note that any non-decreasing function $\tilde{\Theta}$ that agrees with Θ on all intervals $I_k^{(n)}$ must be continuous; this is a direct consequence of the fact that C_ξ is totally disconnected, and that the set $\{(2k-1)2^{-n}\}$ is dense in $(0,1)$.

Theorem 5. *Let the collision kernel $B(v-v_*,\omega)$ satisfy (3) and (5) with $0<\beta\leq 1$, and assume that f_0 satisfies (7). Take any $T>0$, and let Φ be a continuously increasing function on $[0,\infty)$ with $\Phi(0)=0$. There exists a weak solution f of the Boltzmann equation (1) with $f|_{t=0}=f_0$ for which the mass and momentum are conserved, $H(f(\cdot,t))\leq H(f_0)$, and such that the energy grows as*

$$E(f(\cdot,t)) \quad = \quad E(f_0)+\Phi(\Theta_T(t)), \qquad t\in[0,\infty)\,. \tag{29}$$

where $\Theta_T(t)=T\Theta(T^{-1}t)$, and where Θ is the function defined in (28)

Proof: For convenience, in the following we denote $E(f(t))=E(f(\cdot,t))$, and without loss generality, we may suppose $T=1$. Let $I_k^{(m)}=(l_k^{(m)},r_k^{(m)}]\,,J_i^{(n)}$ be the left-open and right-closed intervals constructed in above.

Using Theorem 3 we may, for each $n\geq 1$ construct a weak solution to eq. (1) with $f^n|_{t=0}=f_0$, such that mass and momentum are conserved, and such that $H(f^n(\cdot,t))\leq H(f_0)$, $t\geq 0$, and such that

$$\begin{aligned} E(f^n(t)) = E(f_0) + &\sum_{m=1}^{n}\sum_{k=1}^{2^{m-1}} \Phi(\Theta(r_k^{(m)}))\chi_{I_k^{(m)}}(t) \\ &+\sum_{k=1}^{2^n}\Phi(\Theta(r_k^{(n+1)}))1\!\!1_{J_k^{(n)}}(t)+\Phi(1)1\!\!1_{(1\,,\,\infty)}(t) \qquad t\geq 0 \end{aligned} \tag{30}$$

On each interval $I_k^{(m)}$, $J_i^{(n)}$ and $(1,\infty)$, f^n satisfies moment estimates as in Theorem 3. For instance, for any $s>2$, $1\leq k\leq 2^{m-1}$, $m\geq 1$ and any $n\geq m$,

$$\|f^n(\cdot,t)\|_{L^1_s}\leq W_s(t-l_k^{(m)})\,, \qquad t\in I_k^{(m)}=(l_k^{(m)},r_k^{(m)}] \tag{31}$$

and

$$\|f^n(\cdot,t)\|_{L^1_s}\leq W_s(t-1), \qquad t>1\,. \tag{32}$$

The function $W_s(\tau)$ is defined as i eq. (22). Since there are uniform bounds on the energy and on the entropy for all the involved functions, we may safely omit the indices E and H. The same uniform bounds imply that a subsequence may be extracted that converges weakly to a weak solution f of (1) with $f|_{t=0}=f_0$

By (8) and (9), the $H(f(\cdot,t)) \leq H(f_0)$ and the energy $E(f(t))$ is non-decreasing on $[0,\infty)$ because of Theorem 2.

Now we prove that the solution f satisfies (29). We first prove that

$$E(f(t)) \quad = \quad E(f_0) + \Phi(\Theta(t)), \qquad t \in \cup_{m=1}^{\infty} \cup_{k=1}^{2^{m-1}} I_k^{(m)} \cup (1,\infty). \quad (33)$$

Let $t \in I_k^{(m)}$. Since Θ is constant on $I_k^{(m)}$, we have $\Theta(t) = \Theta(r_k^{(m)})$. Then by (30), for any $n_j > m$, $E(f^{n_j}(t)) = E(f_0) + \Phi(\Theta(t))$. Since f^{n_j} converges weakly to f, $E(f(t)) \leq E(f_0) + \Phi(\Theta(t))$. On the other hand, for $s > 2$, the estimate (31) implies that for all $M > 0$,

$$\begin{aligned}\int_{\mathbf{R}^3} f^{n_j}(v,t)|v|^2 \, 1\!\!1_{\{|v|\leq M\}} \, dv &= \left[E(f^{n_j}(t)) - \int_{\mathbf{R}^3} f^{n_j}(v,t)|v|^2 \, 1\!\!1_{\{|v|>M\}} \, dv \right] \\ &\geq E(f_0) + \Phi(\Theta(t)) - \frac{1}{M^{s-2}} \|f^{n_j}(\cdot,t)\|_{L^1_s} \\ &\geq E(f_0) + \Phi(\Theta(t)) - \frac{1}{M^{s-2}} W_s(t - l_k^{(m)}).\end{aligned}$$

Thus first letting $j \to \infty$ and then letting $M \to \infty$ we obtain $E(f(t)) \geq E(f_0) + \Phi(\Theta(t))$. Therefore $E(f(t)) = E(f_0) + \Phi(\Theta(t)),\ t \in I_k^{(m)}$. With the same argument (using (31)), we also have $E(f(t)) = E(f_0) + \Phi(\Theta(t)),\ t > 1$. This proves (33), and hence Theorem 5, at least for this set of t. But the energy of any weak solution is non-decreasing, and hence, by the comment just before the statement of Theorem 5, the energy must take the correct values for all t.

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