BEREZIN TRANSFORM ON REAL BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. Let $\mathbb{D}$ be a bounded symmetric domain in a complex vector space $V_{\mathbb{C}}$ with a real form V and $D = G/K$ be a real bounded symmetric domain in a real vector space V. We construct the Berezin kernel and consider the Berezin transform on the L^2-space on D. The corresponding representation of G is then unitarily equivalent to the restriction to G of a scalar holomorphic discrete series of holomorphic functions on $\mathbb{D}$ and is also called the canonical representation. We find the spectral symbol of the Berezin transform under the irreducible decomposition of the L^2-space.

INTRODUCTION

The main purpose of the present paper is to calculate the spectral symbol of the Berezin transform on real bounded symmetric domains. To explain our results and motivations we let $\mathbb{D}$ be the unit disk in the complex plane with the Lebesgue measure $dm(z)$. We consider the weighted Bergman space H^ν ($\nu > 1$) of holomorphic functions on $\mathbb{D}$ square integrable with respect to the measure $(1 - |z|^2)^{\nu-2}dm(z)$. It has up to some positive constant the reproducing kernel $K_w(z) = K(z, w) = (1 - z\bar{w})^\nu$. Moreover the group $G_c = SU(1, 1)$ of fractional transformations of $\mathbb{D}$ acts on the space H^ν via $f(z) \mapsto f(gz)g'(z)^{\frac{\nu}{2}}$ and it forms an irreducible unitary (projective) representation. Consider the subgroup $SO(1, 1)$ consisting of transformations of the form $z \mapsto \frac{az+b}{bz+a}$ with $a, b \in \mathbb{R}$ and $a^2 - b^2 = 1$. Thus it is of interests to study the irreducible decomposition of the weighted Bergman space under the subgroup $G = SO(1, 1)$. For that purpose we consider the unit interval $D = \mathbb{D} \cap \mathbb{R} = (-1, 1)$ as a trivial symmetric space $G/K = SO(1, 1)/\{\pm 1\}$ and the restriction of holomorphic functions in H^ν to the interval D. More precisely, consider $R : H^\nu \to C^\infty(D)$,

$$Rf(x) = f(x)(1 - x^2)^{\frac{\nu}{2}}, \quad x \in D.$$

Let $L^2(D, d\mu_0)$ be the L^2 space on D with the $SO(1, 1)$-invariant measure $d\mu_0(x) = \frac{dx}{(1-x^2)}$, whose decomposition under $SO(1, 1)$ can be done via the Mellin transform (see below). The restriction R is a bounded operator from H^ν into the space $L^2(D, d\mu_0)$ with dense image, and intertwines the respective actions on $SO(1, 1)$, see Section 7. To get a unitary intertwining operator we consider the polar decomposition of

R, $R = |R|U$. Thus $U = |R|^{-1}R$ is a unitary intertwining operator from H^ν onto $L^2(D, d\mu_0)$, with $|R|^2 = RR^*$ given by

$$B_\sigma f(x) = \frac{\Gamma(\nu)}{2^{\nu-1}\Gamma(\frac{\nu}{2})^2} \int_D \frac{(1-x^2)^\sigma(1-y^2)^\sigma}{(1-xy)^{2\sigma}} d\mu_0(x)$$

which we call the Berezin tranform on D.

Now the $L^2(D, d\mu_0)$, after performing the Cayley transform $x \mapsto \frac{1-x}{1+x}$ mapping D to the half line $(0, \infty)$, can be decomposed under $SO(1,1)$ via the Mellin transform,

$$f \in L^2(D) \mapsto \hat{f}(\lambda) = \int_0^\infty f(x)e_\lambda(x)\frac{dx}{x},$$

with $e_\lambda(x) = x^\lambda$ and $\lambda \in i\mathbb{R}$. On the half real line it becomes

$$B_\nu f(x) = \frac{\Gamma(\nu)}{\Gamma(\frac{\nu}{2})^2} \int_0^\infty \frac{x^{\frac{\sigma}{2}}y^{\frac{\sigma}{2}}}{(x+y)^\sigma}\frac{dx}{x}$$

on the space $L^2(\mathbb{R}^+, \frac{dx}{x})$. B_ν is then a function the self-adjoint operator $L = ix\frac{d}{dx}$. In terms of the eigenfunction $e_\lambda(x)$, $B_\nu e_\lambda = b_\nu(\lambda)e_\lambda$,

$$b_\nu(\lambda) = \frac{\Gamma(\frac{\nu}{2}+\lambda)\Gamma(\frac{\nu}{2}-\lambda)}{\Gamma(\frac{\nu}{2})^2};$$

$b_\nu(\lambda)$ is obtained by calculating the integral $B_\nu e_\lambda(x)$ at $x = 1$. Thus $B_\nu = f(L)$ with $f(s) = b(-is)$.

One can formulate the above problem for any bounded symmetric domain, where the operator L is replaced by a system of generators. The intertwining operator R for a general bounded symmetric domain was introduced earlier in [12]. The main purpose of the present paper is to find the symbol b_σ. Moreover we prove that the Berezin transform defines bounded operator on the L^p-space ($1 \le p \le \infty$) on D with the invariant measure when ν is in certain interval, and thus justify the operator theoretic meaning of the symbol. In particular this implies that the intertwining operator from the analytic continuation of the holomorphic discrete series to the L^2-space on D. The exact interval of ν depends on different root system. For type BC and C the interval of ν is so that H^ν is a discrete series of $\mathbb{D}$; however for other types of domains the range of ν is large than the holomorphic discrete series, but still above the reducible points.

Before explaining our methods of calculations we make some general remarks and put our results in perspectives. The problem of finding the irreducible decomposition of a unitary representation (π, G) of a Lie group G under a subgroup $H \subset G$ has been studied for a long time, both in mathematical physics and in representation theory, and is called the "Branching rule". We can also put the Berezin transform on the unit disk on $\mathbb{D}$, originally introduced by Berezin [2] for any complex Kähler manifold, in this context. Consider the tensor product $H^\nu \otimes \overline{H^\nu}$ of the weighted Bergman

space on $\mathbb{D}$ with its conjugate realized as functions $f(z, w)$ holomorphic in z and anti-holomorphic in w. It forms an unitary representation of the group $G \times G$. Then G can be considered as the subgroup $\{(g, g) \in G \times G; g \in G\}$ of $G \times G$. To find the irreducible decomposition of the tensor product under G we consider also the restriction map of function $f(z, w)$ to its diagonal $f(z, z)$; more precisely $Rf(z) = f(z, z)(1 - |z|^2)^\nu$. In this way we also obtain the Berezin tranform on $\mathbb{D}$, $B_\nu = RR^*$,

$$(0.1) \qquad B_\nu f(z) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^\nu (1 - |w|^2)^\nu}{|(1 - z\bar{w})^\nu|^2} \frac{dm(z)}{(1 - |z|^2)^2}.$$

See [13]. The symbol of the Berezin transform for a general bounded symmetric domain was found by Unterberger and Upmeier [17]. The spectral symbol of the Berezin tranform then gives the spectral decomposition of the tensor product $H^\nu \otimes \overline{H^\nu}$ of G for large ν. (See also [21] where the tensor product $H^\nu \otimes \overline{H^\kappa}$ for different ν and κ is considered and the corresponding Berezin symbol is calculated.) The exact formula plays a decisive role in decomposing the tensor product for small ν. Interestingly there appears discrete parts in the decomposition; cf [14] for the case of unit disk $\mathbb{D} = SU(1, 1)/U(1)$ and [6] for the matrix ball $SU(p, q)/S(U(p) \times U(q))$.

We proceed to explain the methods of our calculations. We will use the idea of Unterberger and Upmeier [17], applying the theory of Jordan triples and Gindikin Gamma function. Let $V = V_2 \oplus V_1$ be the Peirce decomposition of V with respect to a maximal tripotent e. The Jordan algebra V_2 has further a Cartan decomposition $V_2 = A \oplus B$. The Harish-Chandra e_λ-function has then a rather explicit form, in terms of the determinant function on V_2; see (2.6). The symbol $b_\nu(\underline{\lambda})$ is then an integral on the Siegel domain. The integral reduces further to one integration on V_1, one on B and one on the symmetric cone Ω in A. In the case of the complex Siegel domains, $B = iA$ and the integration on B can be obtained directly by the Laplace transform on Ω. However for real domains B is not very much related to A, and we calculate the integral by using a method of Shimura [16], writing the integral as one on the cone Ω in A; see Proposition 3.1.

The symbols of the Berezin transform have also been calculated previously by Neretin [10] for classical domains, and by van Dijk and Hille [18] for classical rank one domains. As will be seen in the paper, the integrals involved are generalization of the classical Beta-integral. For certain special cases they had been also studied earlier by Hua [7].

The Berezin transfrom formally defines a positive operator, namely $(B_\nu f, f) \geq 0$ for all $f \in C^\infty(\mathbb{D})$. Thus the problem of finding the irreducible decomposition of the tensor product under G can also be formulated as finding the decomposition defined by the positive definite kernel $\frac{(1 - |z|^2)^\nu (1 - |w|^2)^\nu}{|(1 - z\bar{w})^\nu|^2}$. In this context it is also called canonical representation, introduced and named by Vershik, Gel'fand and Graev [20].

Finally we remark that even though our main goal is the symbol of the Berezin transform, some results in this paper are of independent interests. In particular our Propositions 3.1 and 4.1 generalize the formula (2.13) in [16]. The exact value of those integrals is critical in studying the analytic continuation of Bessel functions and Eisenstein series; see loc. cit.. We hope to study applications of our results to Bessel functions in future publications.

The paper is organized as follows. In Section 1 we introduce real bounded symmetric domains and the Berezin transform. In Section 2 we consider the Harish-Chandra e-function defined abstractly in terms of the Iwasawa decomposition $G = NAK$. We give explicit formulas Lie algebra $\mathfrak{n}$ of positive root vectors, and thus give a formula of the e-function in terms of the conical functions. The symbol of the Berezin transform is calculated in Sections 3, 4 and 5 for the different root systems. In Section 6 we prove the L^p-bounded properties of the Berezin transform, therefore justifying the operator theoretic meaning of the symbol b_ν of the Berezin transform.

After this paper was finished the author received a preprint [19] by van Dijk and M. Pevzner, where they also found the symbol for real tube domains by slightly different methods.

For the reader's convenience we list the main symbols used in the paper.

1. $\mathbb{D} = G_c/K_c$, an irreducible bounded symmetric domain in a complex vector space $V_{\mathbb{C}} = \mathbb{C}^n$, V a real form of $V_{\mathbb{C}}$ and $D = \mathbb{D} \cap V$ a real bounded symmetric domain in V.

2. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{a} \subset p$ a maximal abelian subspace of $\mathfrak{p}$.

3. $D(z, \bar{w})v = \{z\bar{w}v\}$, the Jordan triple product;

4. $B(z, w) = I - D(z, \bar{w}) + Q(z)Q(\bar{w})$, the Bergman operator;

5. (z, w), normalized scalar on V so that a minimal tripotent has norm 1.

6. p the genus of domain $\mathbb{D}$ and p_0 the genus of D (see Definition 2.2)

7. $h(z, w)$ an irreducible polynomial on $V_{\mathbb{C}} \times \overline{V_{\mathbb{C}}}$, and $\det B(z, w) = h(z, w)^p$.

8. $V = V_2 \oplus V_1$ Peirce decomposition with respect to a maximal tripotent e, $V_2 = A \oplus B$ the Cartan decomposition of V_2 with respect to the involution $z \mapsto Q(e)\bar{z}$;

9. Δ the determinant function of V_2 and δ the determinant function of A.

10. $\mathcal{S}$ the Siegel domain realization of D

1. REAL BOUNDED SYMMETRIC DOMAINS AND RESTRICTION OF HOLOMORPHIC FUNCTIONS

We recall first some preliminary results on real bounded symmetric domains and fix notation. Our presentation is mainly based on Loos [9].

Let $\mathbb{D} = G_c/K_c$ be an irreducible bounded symmetric domain in a complex vector space $V_{\mathbb{C}} = \mathbb{C}^n$. The space V has then a Jordan triple structure. We denote $\{x\bar{y}z\}$ the Jordan triple product. Let V be a real form of $V_{\mathbb{C}}$ and τ be the conjugation with respect to V. Suppose $\tau(\mathbb{D}) = \mathbb{D}$, namely, τ fixes the bounded symmetric domain. The the real form $D = \mathbb{D} \cap V$ will be called a *real bounded symmetric domain*. In this case the triple product $D(x, \bar{y})z = \{x\bar{y}z\}$ restricted on V defines also a triple product on V. A complete list of real bounded symmetric domains D is given in [9]. As a Riemannian symmetric space, $D = G/K$, where G is the subgroup of G_c consisting biholomorphic transformations of $\mathbb{D}$ which keep D invariant. The coset space G_c/G is called a causal symmetric space a complete list of the pairs (G_c, G) can be found in e.g. [11], [5] and [6].

We describe briefly some algebraic and geometric structures of the domain D.

Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. The Lie algebra of G_c will be realized as completely integrable holomorphic vector fields on $\mathbb{D}$. The elements in $\mathfrak{g}$ can then be written as $\mathfrak{g} = \{\xi_v(z) = v - Q(z)\bar{v}; v \in V\}$ where $Q(z)\bar{v} = \frac{1}{2}\{z\bar{v}z\}$ is the quadratic operator.

We fix a K-invariant inner product $(\cdot, \cdot)$ on V so that a minimal tripotent has norm 1, and let $\langle \cdot, \cdot \rangle$ be the corresponding Hermitian inner product on $V_{\mathbb{C}}$. (A minimal tripotent of $V_{\mathbb{C}}$ has norm 2 if the root system is of Type BC or C, otherwise it is of norm 1; see below.) We let dz be the corresponding Lebesgue measure.

Denote further $B(x, \bar{y})$ the Bergman operator

$$B(x, y) = 1 - D(x, \bar{y}) + Q(x)Q(\bar{y}).$$

The Bergman kernel of $\mathbb{D}$ is up to a constant $h(z, w)^{-p}$ where $h(z, w)$ is an irreducible polynomial and p is the genus of $\mathbb{D}$. For simplicity we write $h(z) = h(z, z)$. The Bergman metric of $\mathbb{D}$ is

$$\langle B(z, z)^{-1}u, v \rangle$$

The domain D is a totally real and totally geodesic submanifold of $\mathbb{D}$ and its Riemannian metric is

$$\langle B(z, z)^{-1}u, v \rangle;$$

with this metric it is also a Riemannian symmetric space, $D = G/K$. The G-invariant measure on D is

$$h(z)^{-\frac{p}{2}}dm(z),$$

and G acts unitarily on $L^2(D, h^{-\frac{p}{2}}dm)$ via change of variables.

Consider the weighted Bergman space $H^\nu(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ so that

$$\int_{\mathbb{D}} |f(z)|^2 h(z)^\nu \frac{dm(z)}{h(z)^p}.$$

The group G_c acts unitarily on $H^\nu(\mathbb{D})$ via the following

$$\pi_\nu f(z) = J_{g^{-1}}(z)^{\frac{\nu}{p}} f(g^{-1}z),$$

and it forms a projective representation of G_c. Let R be the restriction map $R : H^\nu \to C^\infty(D)$ by

(1.1) $$Rf(x) = f(x)h(x)^{\frac{\nu}{2}}.$$

Consider the restriction of the group action π_ν of G_c to its subgroup G. Then R is an G-intertwining map, as one can easily checks from the transformation properties of $h(x, x)$. Consider its formal conjugate operator R^* from $L^2(D, d\mu_0)$ to H^ν and form the operator R^*R on $L^2(D, \omega)$. It is of the form, up to a constant,

$$B_\nu = c_\nu \int_D \frac{h(z)^{\frac{\nu}{2}} h(w)^{\frac{\nu}{2}}}{h(z, w)^\nu} f(w) \frac{dm(z)}{h(z)^{\frac{p}{2}}}.$$

where the constant c_ν is a constant normalized so that $B_\nu 1 = 1$.

We note that the kernel $h(z, w)$ actually is positive, so there is no ambiguity about its power $h(z, w)^\nu$. Indeed by the transformation rule of $h(z, w)$ on $\mathbb{D} \times \mathbb{D}$

$$h(gz, gw) = J_g(z)^{\frac{1}{p}} h(z, w) \overline{J_g(z)^{\frac{1}{p}}}$$

for $g \in G_c$ and $(z, w) \in \mathbb{D} \times \mathbb{D}$, and that $h(0, w) = 1$ we know that $h(z, w)$ is not vanishing on $\mathbb{D} \times \mathbb{D}$. Moreover $h(z, w)$ on $D \times D$ is a real-valued functions. Thus being a continuous non-vanishing real-valued function on $D \times D$, it must be positive. Consequently we can write B_ν as

$$B_\nu f(z) = c_\nu \int_D \frac{h(z, z)^{\frac{\nu}{2}} h(w, w)^{\frac{\nu}{2}}}{|h(z, w)^{\frac{\nu}{2}}|^2} f(w) \frac{dm(z)}{h(z)^{\frac{p}{2}}}.$$

2. Siegel domain realization of real bounded symmetric domains and the Harish-Chandra e_λ-functions

The complex bounded symmetric domain $\mathbb{D}$ can be also realized as a Siegel domain in $V_\mathbb{C}$ via Cayley transform. The Cayley transform in the same time maps D into a unbounded domain in V, which will be its Siegel domain realization. The advantage with the unbounded realization is that the Harish-Chandra functions e_λ have explicit form.

Let $\{e_1, \cdots, e_r\}$ be a frame of V and $e = e_1 + \cdots + e_r$; e is then a maximal tripotent of V and $V_\mathbb{C}$. Let

(2.1) $$V = V_2 \oplus V_1$$

be the Peirce decomposition of V with respect to e, where

$$V_j = \{z \in V; D(e,e)z = 2z\}, \quad j = 1, 2.$$

The space V_2 is a Jordan algebra with the product $z \circ w = \frac{1}{2}\{z\bar{e}w\}$ with unit e. Denote x^{-1} the inverse of $x \in V_2$ with respect to this product. The Cayley transform is defined by

$$\gamma_e(x) = (e + x_2) \circ (e - x_2)^{-1} \oplus \sqrt{2}\{(e - x_2)^{-1}\bar{e}x_1\}.$$

To describe the image of γ_e we consider the involution $Q(e) : x \mapsto x^* = Q(e)\bar{x}$. Let A and B be the eigenspaces of the involution with eigenvalues 1 and -1, respectively. The subspace A is a Euclidean Jordan algebra. We will write $\Re(x)$ the A-part of an element $x \in V_2$. Let Ω be the cone of positive elements in A. Denote $F(x_1, x_1) = D(e, \bar{e})x_1$. It maps V_1 into A. The image of D under γ_e is the unbounded domain

$$\mathcal{S} = \left\{x = x_2 \oplus x_1 \in V = V_2 \oplus V_1; x_2 + x_2^* - \frac{1}{2}F(x_1, x_1) \in \Omega\right\}$$

We observe that the differential of γ_e at $z = 0$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

under the decomposition of $V = V_2 \oplus V_1$.

We will denote the conjugation of the group G also by G. Thus $\mathcal{S} = G/K$ with $e \in \mathcal{S}$ being the base point. We will hereafter fix this realization of G/K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. The elements $D(e_j, e_j)$, $j = 1, \dots, r$ span then a maximal subspace $\mathfrak{a}$ of $\mathfrak{p}$ of dimension r. Let $\gamma_j \in \mathfrak{a}^*$ be the linear functional on $\mathfrak{a}$ defined by

$$\gamma_j(D(e_k, e_k)) = 2\delta_{j,k},$$

where $\delta_{j,k}$ is the Kronecker symbol. Then the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of the following type:

(A_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\gamma_j, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(B_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\frac{\gamma_j}{2}, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(C_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\gamma_j, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(BC_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\gamma_j, \pm\frac{\gamma_j}{2}, \frac{\gamma_j \pm \gamma_k}{2}\}$;
(D_r): $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\frac{\gamma_j \pm \gamma_k}{2}\}$.

We arrange an ordering of the roots so that

$$\gamma_1 < \gamma_2 < \cdots < \gamma_r.$$

For type C and BC we further let $\gamma_1 > 0$. The root space decomposition can be explicitly done in terms of the Peirce decomposition of V. We present below formulas for the positive root spaces, which willbe sufficient for our purposes. For the complex

bounded symmetric domain $\mathbb{D}$ and its Siegel domain realization, the corresponding root spaces are calculated in [13] and [17].

Let

$$V = \sum_{1 \leq j \leq k} V_{jk} \oplus \sum_{j=1}^{r} V_{0j}$$

be the joint Peirce decomposition with respect to $e_1, \cdots, e_r$, and

$$V_{jk} = A_{jk} \oplus B_{jk}$$

the Cartan decomposition of V_{jk}, $1 \leq j \leq k$ under the involution $z \mapsto Q(e)\bar{z}$. With the notation in (2.1) we have

$$V_2 = \sum_{1 \leq k \leq j} V_{kj}, \quad V_1 = \sum_{j=1}^{r} V_{0j}$$

and $V_2 = A \oplus B$,

$$A = \sum_{1 \leq k \leq j} A_{kj}, \quad B = \sum_{1 \leq j \leq k} B_{kj}.$$

We give the explicit formulas for the positive root spaces, in the bounded realization, it is essentially in [9] Propositions 11.18 and 9.19.

Proposition 2.1. *The positive root spaces are given by*

$$\mathfrak{g}_{\gamma_j} = B_{j,j},$$

$$\mathfrak{g}_{\frac{\gamma_j}{2}} = \{v + D(e, \bar{v}); v \in V_{0,j}\},$$

$$\mathfrak{g}_{\frac{\gamma_j - \gamma_k}{2}} = \{D(v, e_k); v \in A_{kj}\}, \quad j > k$$

$$\mathfrak{g}_{\frac{\gamma_j + \gamma_k}{2}} = B_{kj}, \quad j > k.$$

The type D_2 is somewhat special and will be treated separately in Section 7. For all other types we define the following invariants

(2.2) $$\iota = \dim V_{jj}, \ a = \dim A_{jk}, \ b = \dim V_{j0},$$

where $0 \neq j \neq k$. They are independent of j, k and choice of the frame. Observe that $\dim A_{jj} = 1$ and $\dim B_{jj} = \iota - 1$; $\dim B_{jk} = \dim A_{jk} = a$ for type BC, C and D_r $(r \geq 3)$, and $\dim B_{jk} = 0$ for type A. Also $\iota = 1$ for type B_r and D_r $(r \geq 3)$. The dimension of A, B, V_2 can then be calculated in terms of those invariants

(2.3) $$n_A = \dim A = r + \frac{a}{2}r(r-1), n_B = \dim B = r(\iota - 1) + \frac{a}{2}r(r-1)$$

and $n_2 = \dim V_2 = n_A + n_B$. Recall that the genus p of the complex bounded symmetric domain D is defined

$$p = p(\mathbb{D}) = \frac{1}{r(\mathbb{D})}(2n_2 + n_1)$$

where $r(\mathbb{D})$ is the rank of $\mathbb{D}$.

Definition 2.2. The *genus* p_0 of D is defined by

$$p_0 = p(D) = \frac{1}{2r}(2n_2 + n_1).$$

Note that the genus $p_0 = p(D) = \frac{1}{2}p(\mathbb{D})$ if it is of type A, B or D, and $p(D) = p(\mathbb{D})$ if it is of type BC or C. (The rank of the complex domain $\mathbb{D}$ is $2r$ if D is of type BC or C.)

For $\underline{\lambda} \in (\mathfrak{a}^*)^{\mathbb{C}}$ we write

$$\underline{\lambda} = \sum_{j=1}^{r} \lambda_j \gamma_j$$

and we will identify $\underline{\lambda}$ with its coordinates $(\lambda_1, \ldots, \lambda_r)$.

We let $\underline{\rho} = \sum_{j=1}^{r} \rho_j \gamma_j$ be the half sum of positive roots. Then

$$(2.4) \qquad \rho_j = \frac{1}{2}((\iota - 1 + a(j-1) + \frac{b}{2}), \quad 1 \le j \le r$$

for type B, BC, C, D_r $(r \ge 3)$, and

$$(2.5) \qquad \rho_j = \frac{a}{4}(2j - (r+1))$$

for type A.

Let $G = NAK$ be the corresponding Iwawasa decomposition of G. The Harish-Chandra $e_{\underline{\lambda}}$ on $\mathcal{S} = G/K$ is defined by

$$e_{\underline{\lambda}}(z) = e^{(\underline{\lambda} + \rho)A(g)}$$

where $g \in G$ is such that $z = gK$ and $A(g)$ is the $\mathfrak{a}$-part of $g \in G$ in its Iwasawa decomposition: $g = n \exp(A(g))k$.

We will need an explicit formula for the function $e_{\underline{\lambda}}$. For that purpose we introduce the conical functions. Let Ω the symmetric cone of positive elements in A. Then the Harish-Chandra e-function on Ω can be expressed in terms of the conical functions on A. So let $\delta(x)$ be the determinant function of the Jordan algebra A. Put $e^{(j)} = e_1 + \cdots + e_r$ and consider the Peirce decomposition of A with respect to $e^{(j)}$, $A = A_2(e^{(j)}) \oplus A_1(e^{(j)}) \oplus A_0(e^{(j)})$. Let δ_j be the determinant function for the Jordan algebra $A_2(e^{(j)})$ with identity $e^{(j)}$ and extend it to a polynomial function to A via the projection from A onto $A_2(e^{(j)})$. For $\underline{\alpha} = (\alpha_1, \cdots \alpha_r) \in \mathbb{C}^r$, we denote

$$\delta_{\underline{\alpha}}(x) = \delta_1(x)^{\alpha_1 - \alpha_2} \cdots \delta_{r-1}^{\alpha_{r-1} - \alpha_r}(x)\delta_r(x)^{\alpha_r}, \quad x \in \Omega.$$

The $\delta_{\underline{\alpha}}$ is the Harish-Chandra e-function on Ω. To find the e-function on $\mathcal{S}$ we let Δ be the determinant function of Jordan algebra V_2. On $\Omega \subset A \subset V_2$, $\Delta = \delta$ and has degree r if $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type A_r, B_r or D_r; $\Delta = \delta^2$ and is of degree $2r$ for type BC_r and C_r; see e.g. [8]. We define similarly the functions $\Delta_{\underline{\alpha}}$ on V_2.

Lemma 2.3. *The Harish-Chandra e_λ-function on $\mathcal{S}$ is given by*

(2.6)
$$e_{\underline{\lambda}}(z) = \Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1)) = \Delta_{\underline{\alpha}}(\frac{1}{2}(z_2 + z_2^*) - \frac{1}{2}F(z_1, z_1)), \quad z = z_2 \oplus z_1 \in \mathcal{S}$$

with

(2.7)
$$\underline{\alpha} = \underline{\lambda} + \underline{\rho}$$

if $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type A, B, or D, and

(2.8)
$$\underline{\alpha} = \frac{1}{2}(\underline{\lambda} + \underline{\rho})$$

for type BC or C.

Proof. Indeed the function $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$ and e_λ coincides on $A \cdot e = \exp(\mathfrak{a}) \cdot e$. To prove that they are the same we only need to know that $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$ is invariant under N. By Proposition 3.1 it is clear that the function is invariant under the subgroups $\exp(\mathfrak{g}_{\gamma_j})$ and $\exp(\mathfrak{g}_{\frac{\gamma_j + \gamma_k}{2}})$, since $\Re(z_2) - \frac{1}{2}F(z_1, z_1)$ does not depend on B-part of z_2. We consider now $\mathfrak{g}_{\frac{\gamma_j - \gamma_k}{2}}$, $j > k$. In view of Proposition 2.1, we see that the elements in the space, when considered as elements acting on $V = V_2 \oplus V_1 = A \oplus B \oplus V_1$, annhilate V_1 keeps the subspace A invariant. When restricted to A, the elements are in the Lie algebra of the corresponding Iwasawa N-group of the symmetric cone Ω. and thus $\exp(\mathfrak{g}_{\frac{\gamma_j - \gamma_k}{2}})$. keeps the conical function $\Delta_\alpha(x)$ invariant (see [3], Chapter XI. Section 2). Consequently $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$ is invariant under $\exp(\mathfrak{g}_{\frac{\gamma_j - \gamma_k}{2}})$. Finally consider an element $v + D(e, \bar{v}) \in \mathfrak{g}_{\frac{\gamma_j}{2}}$, $v \in V_{0j}$. It follows from [9], Lemma 10.7 (and its proof) that

$$\exp(v + D(e, \bar{v})) = \exp(v + \frac{1}{2}F(v, v)) \exp(D(e, \bar{v}))$$

where the element $\exp(v + \frac{1}{2}F(v, v))$ acts on vectors in V via translation, and that

$$\exp(D(e, \bar{v}))(z_2 \oplus z_1) = (z_2 + F(z_1, v)) \oplus z_1.$$

Thus

$$\exp(v + D(e, \bar{v}))(z_2 \oplus z_1) = (z_2 + F(z_1, v) + \frac{1}{2}F(v, v)) \oplus (z_1 + v).$$

From this formula we see that the action $\exp(v + D(e, \bar{v}))$ on $(z_2 \oplus z_1)$ keeps $\Re(z_2) - \frac{1}{2}F(z_1, z_1)$, and consequently $\Delta_{\underline{\alpha}}(\Re(z_2) - \frac{1}{2}F(z_1, z_1))$, invariant. This completes the proof. $\square$

The Berezin transform on $\mathcal{S}$ is then of the form

$$B_\nu f(z) = C_\nu \int_{\mathcal{S}} f(w) \frac{\Delta(\Re z_2 - \frac{1}{2}F(z_1, z_1))^{\frac{\nu}{2}} \Delta(\Re w_2 - \frac{1}{2}F(w_1, w_1))^{\frac{\nu}{2}}}{(\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1)))^\nu} d\mu_0(w)$$

where

$$d\mu_0(w) = \frac{dm(w)}{\Delta(\Re w_2 - \frac{1}{2}F(w_1, w_1))^{\frac{p}{2}}},$$

is the G-invariant measure on $\mathcal{S}$. Here C_ν is a constant normalized so that $B_\nu 1 = 1$, its exact value will be given in the following sections.

As will be proved in Section 7, the operator B_ν is a positive bounded operator on $L^2(\mathcal{S}, d\mu_0)$ when ν is in certain interval. The the Harish-Chandra e-function $e_{\underline{\lambda}}$ is an eigenfunction of B_ν with eigenvalue $b_\nu(\underline{\lambda})$, is the sense of spectral calculus. Namely

$$(2.9) \qquad\qquad B_\nu e_{\underline{\lambda}}(z) = b_\nu(\underline{\lambda}) e_{\underline{\lambda}}(z).$$

Taking $z = e$ the base point we get formally $b_\nu(\underline{\lambda}) = B_\nu e_{\underline{\lambda}}(e)$. Denote

$$(2.10) \qquad I(\nu, \underline{\alpha}) = \int_{\mathcal{S}} e_{\underline{\lambda}}(w) \frac{\Delta(\Re w_2 - \frac{1}{2}F(w_1, w_1))^{\frac{\nu}{2}}}{(\Delta(\frac{1}{2}(e + w_2^*)))^\nu} d\mu_0(w).$$

Here $\underline{\alpha}$ and $\underline{\lambda}$ are related as in (2.7) and (2.8). Then $b_\nu(\underline{\lambda}) = C_\nu I(\nu, \underline{\alpha})$. So our main goal is the calculation of the integral $I(\nu, \underline{\alpha})$. Note that the normalization constant $C_\nu = I(\nu, 0)^{-1}$, and will be found at the same time.

We will prove that the integral is convergent for $\underline{\lambda}$ in certain domain in $(\mathfrak{a}^*)^{\mathbb{C}}$. However the function $e_{\underline{\lambda}}$ is N-invariant and transforms under A by the character $e^{\underline{\lambda}}$ and NA acts transitively on $\mathcal{S}$. Thus if the integral (2.10) is absolutely convergent, that is if the integral defining the Berezin transform $B_\nu e_{\underline{\lambda}}(z)$ at $z = e$ is absolutely convergent, then by a simple change of variable argument we know that the integral $B_\nu e_{\underline{\lambda}}(z)$ is absolutely convergent for all $z \in \mathcal{S}$, see [1].

We recall finally the Gindikin Famma function on symmetric cone; see e.g. [3], Chapter VII. Consider the Laplace transform of the function $\delta_{\underline{\alpha}}(x)\delta(x)^{-\frac{n_A}{r}}dx$,

$$\int_\Omega e^{-(u, x)} \delta_{\underline{\alpha}}(x)\delta(x)^{-\frac{n_A}{r}} dx,$$

for $u \in \Omega$. The integral is convergent where the integral converges if and only if $\Re(\alpha_j) > (j-1)a/2$ for $j = 1, 2, \cdots, r$. Moreover,

$$(2.11) \qquad \int_\Omega e^{-(u, x)} \delta_{\underline{\alpha}}(x)\delta(x)^{-\frac{n_A}{r}} dx = \Gamma_\Omega(\underline{\alpha}) \delta_{\underline{\alpha}^*}^*(x),$$

where

$$\Gamma_\Omega(\underline{\alpha}) = (2\pi)^{(n_A - r)/2} \prod_{j=1}^r \Gamma(\alpha_j - (j-1)a/2).$$

and $\underline{\alpha}^* = (-\alpha_r, \ldots, -\alpha_1)$ and $\delta_{\underline{\beta}}^*(x)$ for $\underline{\beta} = (\beta_1, \ldots, \beta_r)$ is the conical function defined via Peirce decomposition with respect to $e_r, e_r + e_{r-1}, \ldots, e_r + e_{r-1} + \cdots + e_1$.

3. Calculations of the symbol: Type BC and C

The type C root systems can be viewed as special cases of the type BC with the root multiplicity of $\frac{\gamma_j}{2}$ being 0 (and the other types can be viewed as degenerate cases). So we consider the type BC. Notice that the genus $p_0 = \iota + a(r-1) + \frac{b}{2}$.

The determinant function Δ of V_2 when restricted to the Jordan subalgebra A is the square of the determinant function of that of A. To differ them we write $\delta(z)$ the determinant function of A. Thus

$$\Delta(x) = \delta(x)^2, \quad x \in A$$

We present first some integral formulas.

Proposition 3.1. *Suppose $\beta > \iota - 2 + \frac{1}{2}(1 + a(r-1))$. The following integral formula holds*

$$I_B(\nu) = \int_B \frac{d\eta}{\Delta(e - \eta)^\nu} = 4^{n_A - r\beta} \sqrt{\pi}^{r(\iota - 2) + 2n_A} \frac{\Gamma_\Omega(2\nu - (\iota - 2) - \frac{n_A}{r})}{\Gamma_\Omega(\nu)\Gamma_\Omega(\nu - \frac{\iota - 2}{2})}$$

and the integral is absolutely convergent.

Notice that

$$
(3.1) \quad
\begin{aligned}
\Delta(e - \eta)^2 &= \Delta(e - \eta)\Delta(e + \eta) = \Delta(e - \eta^2) = \Delta(e + \eta \circ \eta^*) \\
&= \Delta(e + \eta \circ \eta^*) = \delta(e + \eta \circ \eta^*)^2
\end{aligned}
$$

The integral $I_B(\nu)$ is now

$$(3.2) \qquad I_B(\nu) = \int_B \frac{d\eta}{\delta(e + \eta \circ \eta^*)^\nu}$$

To prove the integral formula we need the following result; see [3], Chapter VII, Ex. 5.

Lemma 3.2. *Suppose $\beta > \frac{1}{2}(1 + a(r-1))$. The following integral formula holds*

$$I_A(\beta) = \int_A \frac{d\eta}{|\delta(e + it)|^{2\beta}} = 4^{n_A - r\beta} \pi^{n_A} \frac{\Gamma_\Omega(2\beta - \frac{n_A}{r})}{\Gamma_A(\beta)^2}$$

and the integral is absolutely convergent.

The integrals $I_A(\beta)$ and $I_B(\nu)$ above can also be written as an integration on the symmetric cone Ω in A; by comparing the two formulas we will then be able to calculate $I_B(\nu)$.

Indeed, we write $|\delta(e + it)|^2 = \delta(e + it)\overline{\delta(e + it)} = \delta(e + it)\delta(e - it) = \delta(e + t^2)$, and use the Gindikin Gamma function formula (2.11)

$$\delta(e + t^2)^{-\beta} = \frac{1}{\Gamma_\Omega(\beta)} \int_\Omega e^{-(e + t^2, u)} \delta(u)^{\beta - \frac{n_A}{r}} du;$$

$$
(3.3) \quad
\begin{aligned}
I_A(\beta) &= \int_A \frac{d\eta}{|\delta(e + it)|^{2\beta}} = \frac{1}{\Gamma_A(2\beta)} \int_A \int_\Omega e^{-(e + t^2, u)} \delta(u)^{\beta - \frac{n_A}{r}} \, du \, dt \\
&= \frac{1}{\Gamma_A(2\beta)} \int_\Omega \int_A e^{-(eu)} e^{-(t^2, u)} \delta(u)^{\beta - \frac{n_A}{r}} \, dt \, du.
\end{aligned}
$$

Namely

$$(3.4) \qquad I_A(\beta) = \frac{1}{\Gamma_\Omega(2\beta)} \int_\Omega G_A(u) e^{-(e,u)} \delta(u)^{\frac{\beta}{2} - \frac{n_A}{r}} dt du.$$

with

$$G_A(u) = \int_A e^{-(t^2, u)} dt, \quad u \in \Omega$$

being a Gaussian type integral. Let $u = \sum_{j=1}^r u_j c_j$ be the Peirce decomposition of u with c_j being minimal orthogonal tripotents, and $u_j > 0$ since $u \in \Omega$. Let $A = \sum A_{jk}$ be the joint Peirce decomposition and write $t = \sum t_{jk}$. Then we find that, by the Peirce rule (see [9], Theorem 3.15, Corollary 3.16)

$$(t^2, u) = \sum_{j<k} \frac{u_j + u_k}{2} (t_{jk}, t_{jk}) + \sum_j u_j (t_{jj}, t_{jj})$$

Thus

$$
\begin{aligned}
G_A(u) &= \prod_{j=1}^r \int_{A_{jj}} e^{-u_j(t,t)} \prod_{j<k} \int_{A_{jk}} e^{-\frac{1}{2}(u_j+u_k)(t,t)} \\
&= \sqrt{\pi}^{r + \frac{a}{2} r(r-1)} \left(\prod_{j=1}^r u_j \right)^{-1} \left(\prod_{j<k} \frac{u_j + u_k}{2} \right)^{-a}.
\end{aligned}
$$
(3.5)

We start now to prove Proposition 3.1

Proof. Using (3.2) and performing the same caluclation as above (with A replaced by B) we get

$$(3.6) \qquad I_B(\nu) = \frac{1}{\Gamma_\Omega(\nu)} \int_\Omega G_B(u) e^{-(e,u)} \delta(u)^{\nu - \frac{n_A}{r}} dt du,$$

with

$$G_B(u) = \int_B e^{(t^2, u)} dt = \int_B e^{-(tot^*, u)} dt$$

Consider as in the above the Peirce decomposition of u and B. We can then calculate $G_B(u)$ and obtain

$$(3.7) \qquad G_B(u) = \sqrt{\pi}^{r(\iota - 1) + \frac{a}{2} r(r-1)} \left(\prod_{j=1}^r u_j \right)^{-(\iota-1)} \left(\prod_{j<k} \frac{u_j + u_k}{2} \right)^{-a}.$$

Therefore, comparing $G_B(u)$ with $G_A(u)$ given in (3.5) it follows that

$$G_B(u) = \sqrt{\pi}^{r(\iota-2)} \left(\prod_{j=1}^r u_j \right)^{-(\iota-2)} G_A(u) = \sqrt{\pi}^{r(\iota-2)} \delta(u)^{-(\iota-2)} G_A(u).$$

Subsituting this into (3.6) we find

$$I_B(\nu) = \sqrt{\pi}^{r(\iota-2)} \frac{1}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(e,u)} \delta(u)^{\nu - \frac{n_A}{r}} \delta(u)^{-(\iota-2)} G_A(u) du.$$

Comparing this with (3.4) we find that the integral $I_B(\nu)$ is of the same type as $I_A(\beta)$ with $\beta = \nu - (\iota - 2)$; more precisely

$$I_B(\nu) = \sqrt{\pi}^{r(\iota-2)} \frac{\Gamma_\Omega(\nu - (\iota - 2))}{\Gamma_\Omega(\nu)} I_A(\nu - (\iota - 2))$$

which again by Lemma 3.2, is

$$I_B(\nu) = 4^{n_A - r\beta} \sqrt{\pi}^{r(\iota-2)+2n_A} \frac{\Gamma_\Omega(2\nu - (\iota - 2) - \frac{n_A}{r})}{\Gamma_\Omega(\nu)\Gamma_\Omega(\nu - \frac{\iota-2}{2})}$$

$\square$

We can now calculate the integral $I(\nu, \underline{\alpha})$.

Proposition 3.3. *Suppose $\nu > p_0 - 1 = p - 1$. The integral $I(\nu, \underline{\alpha})$ is absolutely convergent if*

$$-\frac{1}{2}((\nu - p_0 + 1) + \frac{a}{2}(r - j)) < \alpha_j < \frac{1}{2}(\nu - \frac{a}{2}(r - j)), \quad 1 \le j \le r$$

and its value is given by

$$I(\nu, \underline{\alpha}) = 2^{2r\nu} \sqrt{2\pi}^{rb} I_B(\nu) \frac{\Gamma_\Omega(\nu - p_0 + \frac{n_A}{r} + 2\underline{\alpha})\Gamma_\Omega(\nu + p_0 + 1 + \frac{n_2}{r} - \frac{b}{2} + 2\underline{\alpha}^*)}{\Gamma_\Omega(2\nu - \frac{n_B}{r})}$$

Proof. We write $w = w_2 + w_1$ and $w_2 = x + y$ according to the Peirce decomposition of $V = V_2 \oplus V_1$ and the Cartan decomposition $V_2 = A \oplus B$. We perform the change of variables $\xi = x - \frac{1}{2}F(w_1, w_1)$, Thus $\mathcal{S}$ is parametrized by the product $\Omega \times B \times V_1$. In terms of the new coordinates, $w_2 = x + y = \xi + \frac{1}{2}F(w_1, w_1) + y$ and $w_2^* = \xi + \frac{1}{2}F(w_1, w_1) - y$. The integral can be rewritten as

$$2^{2r\nu} \int_{\Omega \times V_1 \times B} \Delta_{\underline{\alpha}}(\xi) \frac{\Delta(\xi)^{\frac{\nu}{2} - \frac{p}{2}}}{\Delta(e + \xi + \frac{1}{2}F(w_1, w_1) - y)^\nu} dy \, dw_1 \, d\xi.$$

(Here the constant $2^{2r\nu}$ appears because Δ is of degree $2r$, and $\Delta(\frac{e+w_2^*}{2}) = 2^{-2r}\Delta(e + w_2^*)$.) We consider first the integral with respect to y. Denote temporarily $u = e + \xi + \frac{1}{2}F(w_1, w_1) \in \Omega$ and note that

(3.8)
$$\Delta(e + \xi + \frac{1}{2}F(w_1, w_1) - y)$$
$$= \Delta(u - y) = \Delta(P(u^{\frac{1}{2}})e - P(u^{\frac{1}{2}})(P(u^{-\frac{1}{2}})(y))$$
$$= \Delta(u)\Delta(e - P(u^{-\frac{1}{2}})(y)) = \delta(u)^2 \Delta(e - P(u^{-\frac{1}{2}})(y)).$$

Change variables $\eta = P(u^{-\frac{1}{2}})(y)$. Then $dm(\eta) = \delta(u)^{-\frac{n_2-n_A}{r}} dy$ since $\dim B = n_2 - n_A$. The integration with respect to y becomes,

$$\delta(u)^{\frac{n_2-n_A}{r} - 2\nu} \int_B \Delta(e - \eta)^\nu d\eta = \delta(u)^{\frac{n_2-n_A}{r} - 2\nu} I_B(\nu)$$

and $I_B(\nu)$ is given by Propostion 3.1.

We consider now the integral with respect to w_1, which appears in u. We have

$$\delta(u)^{\frac{n_2-n_A}{r}-2\nu} = \frac{1}{\Gamma(2\nu - \frac{n_2-n_A}{r})} \int_\Omega e^{-(u,\zeta)} \delta(\zeta)^{2\nu - \frac{n_2}{r}} d\zeta$$

and $(u,\zeta) = (e + \xi, \zeta) + \frac{1}{2}(F(w_1, w_1), \zeta)$. Thus

$$\int_{V_1} \delta(u)^{\frac{n_2-n_A}{r}-2\nu} dw_1$$

$$(3.9) \qquad = \frac{1}{\Gamma(2\nu - \frac{n_2-n_A}{r})} \int_{V_1} \int_\Omega e^{-(e+\xi,\zeta)} e^{-\frac{1}{2}(F(w_1,w_1),\zeta)} \delta(\zeta)^{2\nu - \frac{n_2}{r}} d\zeta dw_1$$

$$= \frac{1}{\Gamma(2\nu - \frac{n_2}{r})} \int_\Omega e^{-(e+\xi,\zeta)} \delta(\zeta)^{2\nu - \frac{n_2-n_A}{r}} \left(\int_{V_1} e^{-\frac{1}{2}(F(w_1,w_1),\zeta)} dw_1 \right) d\zeta,$$

and the change of the order of integration will be justified on our way of calculations. Writing $(F(w_1, w_1), \zeta) = (F(w_1, w_1), P(\zeta^{\frac{1}{2}})e) = (P(\zeta^{\frac{1}{2}})F(w_1, w_1), e) = (F(\zeta^{\frac{1}{2}} \circ w_1, \zeta^{\frac{1}{2}} \circ w_1, e) = $ by [9], Proposition 10.11. Here $\zeta^{\frac{1}{2}} \circ w_1 = \frac{1}{2}D(\zeta^{\frac{1}{2}}, e)w_1$ is the Jordan product on V. Performing change of variables $v = \zeta^{\frac{1}{2}} \circ w_1$, we have $dv = \delta(\zeta)^{\frac{b}{2}} dw_1$ and

$$\int_{V_1} e^{-\frac{1}{2}(F(w_1,w_1),\zeta)} dw_1 = \delta(\zeta)^{-\frac{b}{2}} \int_{V_1} e^{-\frac{1}{2}(F(v,v),e)} dv;$$

the last integral is of Gaussian type and can be easily evaluated, observing $\frac{1}{2}(F(v,v), e) = \frac{1}{2}(D(e,v)v, e) = \frac{1}{2}(v, D(v,e)e) = \frac{1}{2}(v, D(e,e)v) = \frac{1}{2}(v,v)$, so that

$$\int_{V_1} e^{-\frac{1}{2}(F(v,v),e)} dv = \sqrt{2\pi}^{rb}$$

Continuing the formula (3.9), we find it is, disregarding the Gamma factor,

$$\sqrt{2\pi}^{rb} \int_\Omega e^{-(e+\xi,\zeta)} \delta(\zeta)^{2\nu - \frac{n_2}{r} - \frac{b}{2}} d\zeta,$$

which is integrable if

$$2\nu - \frac{n_2}{r} - \frac{b}{2} > \frac{a}{2}(r-1),$$

namely if $\nu > \frac{n_2}{+} \frac{b}{2} + \frac{a}{2}(r-1)$ which is satisfied by our assumption of ν. This consequently justifies the change of order of integration in (3.9).

Our integral in question is

$$2^{r\nu} \sqrt{2\pi}^{rb} \frac{1}{\Gamma(2\nu - \frac{n_2-n_A}{r})} I_B(\nu) \int_\Omega \int_\Omega e^{-(e,\zeta)} e^{-(\xi,\zeta)} \Delta_{\underline{\alpha}}(\xi) \Delta(\xi)^{\frac{\nu}{2} - \frac{p}{2}} \delta(\zeta)^{2\nu - \frac{n_2-n_A}{r} - \frac{b}{2}} d\xi d\zeta.$$

We calculate the integral with respect to ξ. Recall the formula (2.11) and that $\Delta(\xi) = \delta(\xi)^2$,

$$\int_\Omega e^{-(\xi,\zeta)} \Delta_{\underline{\alpha}}(\xi) \Delta(\xi)^{\frac{\nu}{2} - \frac{p}{2}} d\xi = \Gamma_\Omega(2\underline{\alpha} + \nu - p + \frac{n_A}{r}) \delta^*_{2\underline{\alpha}^* - \nu + p - \frac{n_A}{r}}(\zeta).$$

which is absolutely convergent if

$$(3.10) \qquad 2\underline{\alpha}_j + \nu - p + \frac{n_A}{r} > \frac{a}{2}(j-1), \quad j = 1, \ldots, r.$$

The remaining integral, disregarding the constant, is then

$$\int_\Omega e^{-(e,\zeta)} \delta^*_{2\underline{\alpha}^* - \nu + p - \frac{n_A}{r}}(\zeta)\delta(\zeta)^{2\nu - \frac{n_2}{r} - \frac{b}{2}} = \Gamma_\Omega(2\underline{\alpha}^* + \nu + p - \frac{n_2}{r} - \frac{b}{2}),$$

But since $p = \frac{n_2}{r} + \frac{b}{2}$, the above is $\Gamma_\Omega(2\underline{\alpha}^* + \nu)$. Moreover the above integral is absolutely convergent if

$$(3.11) \qquad\qquad 2(\underline{\alpha}^*)_j - \nu < \frac{a}{2}(j-1), \quad j = 1, \dots, r$$

The inequalites (3.10) and (3.11) combined give our condition $\underline{\alpha}$. This finishes the proof. $\qquad\qquad\square$

4. Calculations of the symbol: Types B_r $(r \geq 1)$ and D_r $(r \geq 3)$

In this section we calculate the integral $I(\nu, \underline{\alpha})$ for type B_r $(r \geq 1)$ and D_r $(r \geq 3)$. The type D_r $(r \geq 3)$ can be viewed as a degenerate case of type B_r with the root multiplicity of $\frac{\gamma_j}{2}$ being 0. The method of calculation is basically the same as in the previous section. So we will only indicate the necessary changes needed in the calculations.

Proposition 3.1 in this case takes the following form; we remark that, as we did for the function $h(z, w)$ in Section 2, the function $\Delta(e - \eta)$ is positive on B.

Proposition 4.1. *Suppose $\nu > a(r - 1) - 1$. The integral*

$$I_B(\nu) = \int_B \frac{d\eta}{\Delta(e - \eta)^\nu}$$

is absolutely convergent and its values is given by

$$I_B(\nu) = \sqrt{\pi}^{n_2} 4^{n_A - r(\frac{\nu}{2} + \frac{1}{2})} \frac{\Gamma_\Omega(\nu + 1 - \frac{n_A}{r})}{\Gamma_\Omega(\frac{\nu}{2})\Gamma_\Omega(\frac{\nu}{2} + \frac{1}{2})}$$

The value of the integral $I(\nu, \underline{\alpha})$ is given in the following; we note that in comparision with Types BC and C, the convergence region of ν improves to $\nu > p - 2$. (The discrete series are for $\nu > p - 1$.)

Proposition 4.2. *Suppose $\nu > 2p_0 - 2 = p - 2$, that is $\frac{\nu}{2} > p_0 - 1 = \frac{p}{2} - 1$. The integral $I(\nu, \underline{\alpha})$ is absolutely convergent if*

$$-\frac{1}{2}((\nu - 2p_0 + 2) + a(r - j)) < \alpha_j < \frac{1}{2}(\nu - a(r - j)), \quad 1 \leq j \leq r$$

and its values is given by

$$I_S(\nu, \underline{\alpha}) = \sqrt{2\pi}^{rb} I_B(\nu) \frac{\Gamma_\Omega(\underline{\alpha} + \frac{\nu}{2} - \frac{p}{2} + \frac{n_A}{r})\Gamma_\Omega(\underline{\alpha}^* + \frac{\nu}{2})}{\Gamma_\Omega(\frac{\nu}{2} - \frac{n_B}{r})}$$

Example 4.3. We consider the simple case of the unit disk D in $\mathbb{R}^2$, considered as a real bounded symmetric domain in the unit ball $\mathbb{D}$ of $\mathbb{C}^2$. It is of type B_1 with roots $\pm\frac{\gamma}{2}$. The ball $\mathbb{D}$ is a realization of the Hermitian symmetric space $SU(2,1)/S(U(2) \times U(1))$, and the real disk of $SO(2,1)/SO(2)$. The genus p of $\mathbb{D}$ is now 3 and the

weighted Bergman space H^ν considered in Section 2 has now reproducing kernel $(1 - \langle z, w \rangle)^{-\nu}$ with $\nu > p - 1 = 2$. The Berezin transform on D is now

$$(4.1) \qquad B_\nu f(y) = \frac{\nu - 1}{2\pi} \int_D \frac{(1 - \|x\|^2)^{\frac{\nu}{2}} (1 - \|y\|^2)^{\frac{\nu}{2}}}{(1 - (x, y))^\nu} \frac{dx}{(1 - \|x\|^2)^{\frac{3}{2}}},$$

where $x = (x_1, x_2)$ and $(x, y) = x_1 y_1 + x_2 y_2$. However the unit disk $D = SO(2, 1)/SO(2)$ in $\mathbb{R}^2$ itself has a complex structure defined by the Lie algebra of $SO(2)$, and in this realization it *is not* the standart complex structure on $\mathbb{R}^2$ defined by $(u, v) \mapsto (-v, u)$. However those two structures *are equivalent*; denoting the the unit disk in $\mathbb{C} = \mathbb{R}^2$ with the standard complex structure by $D_\mathbb{C}$, a biholomorphic mapping from $D_\mathbb{C}$ onto $D = SO(2, 1)/SO(2)$ is given by the Hua transformation

$$z = u + iv \mapsto \frac{2}{1 + |z|^2}(u, v).$$

See [4], Chapter X, Exercise D2. The corresponding Berezin transform on $D_\mathbb{C}$, obtained from (4.1) by conjugating the transformation, is then

$$B_\nu f(z) = \frac{2(\nu - 1)}{\pi} \int_{D_\mathbb{C}} \frac{(1 - |z|^2)^\nu (1 - |w|^2)^\nu}{|(1 - z\bar{w})^\nu|^2} \left(1 + \left|\frac{z - w}{1 - z\bar{w}}\right|^2\right)^{-\nu} \frac{dm(z)}{(1 - |z|^2)^2}.$$

The symbol of B_ν in this case is, with $\lambda\gamma$ identified with λ

$$b_\nu(\lambda) = \frac{\Gamma(\frac{\nu}{2} - \frac{1}{4} + \lambda)\Gamma(\frac{\nu}{2} - \frac{1}{4} - \lambda)}{\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu}{2} - \frac{1}{2})}.$$

It is interesting to see that the Berezin transform so obtained is quite close to the classical Berezin transform (0.1), whose symbol, after normalization, is

$$\frac{\Gamma(\nu - \frac{1}{2} + \frac{\lambda}{2})\Gamma(\nu - \frac{1}{2} - \frac{\lambda}{2})}{\Gamma(\nu)\Gamma(\nu - 1)};$$

see [17] and [15].

5. Calculations of the symbol: Type A and D_2

The integral $I(\nu, \underline{\alpha})$ for Type A is very simple. The real Siegel domain $\mathcal{S} = \Omega$, and our integral is

$$2^{r\nu} \int_\mathcal{S} \Delta_\alpha(w) \frac{\Delta(w)^{\frac{\nu}{2}}}{\Delta(e + w)^\nu} \frac{dw}{\Delta(w)^{\frac{p}{2}}}$$

which is Beta-type integral. The result is now

Proposition 5.1. *Suppose $\nu > p - 2 = 2p_0 - 2$. The integral $I(\nu, \underline{\alpha})$ is absolutely convergent if*

$$\frac{1}{2}(\nu - a(j - 1)) < \alpha_j < \frac{1}{2}(\nu - a(r - j)), \quad 1 \le j \le r$$

and its value is given by

$$2^{r\nu} \frac{\Gamma_\Omega(\frac{\nu}{2} + \underline{\alpha})\Gamma_\Omega(\frac{\nu}{2} + \underline{\alpha}^*)}{\Gamma_\Omega(\nu)}$$

We now consider type D_2, the Jordan triple is denoted by $IV_n^{\mathbb{R},m}$ ($2 \le m \le [\frac{n}{2}]$) in [9]. So we introduce the necessary notation. Let $V = \mathbb{R}^n$, and consider the quadratic operator $Q(x)$ on V defined by

$$Q(x)y = q(x,y)x - q(x)y$$

where $q(x) = x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_n^2$ and $q(x,y)$ is its polarization $q(x,y) = q(x+y) - q(x) - q(y)$. The involution $\bar{x}$ of x is $\bar{x} = (x_1, \cdots, x_m, -x_{m+1}, \cdots, -x_n)$ Let $e_1 = \frac{1}{2}(1,0,\cdots,0,1)$ and $e_2 = \frac{1}{2}(1,0,\cdots,0,\ldots,-1)$ and $e = e_1 \oplus e_2$. Then e_1 and e_2 form a frame of tripotents and e is a maximal tripotent. The joint Peirce decomposition of V is $V = V_{11} \oplus V_{22} \oplus V_{12}$ with $V_{jj} = \mathbb{R}e_j$, $j = 1,2$. The Cartan involution $x \mapsto Q(e)\bar{x}$ gives a decomposition of V, $V = A \oplus B$ with $A = V_{11} \oplus V_{22} \oplus A_{12}$,

$$A_{12} = \{x \in V; x_n = x_1 = x_2 = \cdots = x_m = 0\}$$

and

$$B = B_{12} = \{x \in V; x_1 = x_{m+1} = \cdots = x_n = 0\},$$

$\dim A_{12} = n - m - 1$, $\dim B_{12} = m - 1$. The determinant function $\Delta(x) = x_1^2 + \cdots + x_n^2$. In terms of the notation in Section 2, the positive roots are $\frac{\gamma_2 + \gamma_1}{2}$, $\frac{\gamma_2 - \gamma_1}{2}$ with root multiplicities $m - 1$ and $n - m - 1$ respectively, and

$$\underline{\rho} = \frac{1}{4}((2m - n)\gamma_1 + (n - 2)\gamma_2).$$

The symmetric cone in A consists of all elements $x = (x_1, 0, \ldots, 0, x_{m+1}, \ldots, x_n)$ so that $q(x) = x_1^2 - x_{m+1}^2 - \cdots - x_n^2 > 0$ and $x_1 > 0$, namely it is the forward light cone.

Observe that with our normalization of the Jordan triple system, we have $\|x\|^2 = 2(x_1^2 + \cdots + x_n^2)$. Correspondingly $dx = \sqrt{2}^n dx_1 \ldots dx_n$. Also the genus $p = n$ The integral $I(\nu, \underline{\alpha})$ is, after the change of variables as in Section 3,

$$I(\nu, \underline{\lambda}) = \int_\Omega \Delta_\alpha(\xi) \Delta(\xi)^{\frac{\nu}{2} - \frac{n}{2}} \left(\int_B \frac{1}{\Delta(e + \xi - y)^\nu} dy \right) d\xi$$

The integration on B can be evaluated directly. Observe that $\Delta(x - y) = \Delta(x) + \Delta(y)$ if $x + y \in A \oplus B$. Performing change of variable $\eta = \Delta(e + \xi)^{-\frac{1}{2}} y$ we have

$$\int_B \frac{1}{\Delta(e + \xi - y)^\nu} dy = \Delta(e + \xi)^{-\nu + \frac{m-1}{2}} \int_B \frac{1}{\Delta(e - \eta)^\nu} d\eta$$

$$= \sqrt{2}^{m-1} \Delta(e + \xi)^{-\nu + \frac{m-1}{2}} \int_{\mathbb{R}^{m-1}} \frac{1}{(1 + \eta_2^2 + \cdots + \eta_m^2)^\nu} d\eta_2 \ldots d\eta_m$$

$$= \sqrt{2\pi}^{m-1} \frac{\Gamma(\nu - \frac{m-1}{2})}{\Gamma(\nu)} \Delta(e + \xi)^{-\nu + \frac{m-1}{2}}$$

Eventually we find that

$$
\begin{aligned}
I(\nu,\underline{\lambda}) &= \sqrt{2\pi}^{m-1}\frac{\Gamma(\nu-\frac{m-1}{2})}{\Gamma(\nu)}\int_{\Omega}\Delta_{\alpha}(\xi)\Delta(\xi)^{\frac{\nu}{2}-\frac{n}{2}}\Delta(e+\xi)^{-\nu+\frac{m-1}{2}} \\
&= \sqrt{2\pi}^{m-1}\frac{\Gamma(\nu-\frac{m-1}{2})}{\Gamma(\nu)}\frac{\Gamma_{\Omega}(\alpha+\frac{\nu}{2}-\frac{n}{2}+\frac{n-m+1}{2})\Gamma_{\Omega}(\alpha+\frac{\nu}{2}+\frac{n}{2}-\frac{n-m+1}{2})}{\Gamma_{\Omega}(\nu-\frac{m-1}{2})}
\end{aligned}
$$

(5.1)

where the Gamma function Γ_{Ω} is this case is

$$
\Gamma_{\Omega}(\beta) = \sqrt{2\pi}^{n-m-1}\prod_{j=1}^{2}\Gamma(\beta_j - \frac{n-m-1}{2}(j-1)).
$$

Proposition 5.2. *Suppose $\nu > p - 2 = 2p_0 - 2$,*

$$
-\frac{1}{2}(\nu-m+1) < \alpha_1 < \frac{1}{2}(\nu-n+2m+2)
$$

and

$$
-\frac{1}{2}(\nu-n) < \alpha_2 < \frac{1}{2}(\nu+m-1).
$$

The integral $I(\nu,\underline{\alpha})$ is absolutely convergent and its value is given by

$$
I(\nu,\underline{\lambda}) = \sqrt{2\pi}^{m-1}\frac{\Gamma(\nu-\frac{m-1}{2})}{\Gamma(\nu)}\frac{\Gamma_{\Omega}(\alpha+\frac{\nu}{2}-\frac{m-1}{2})\Gamma_{\Omega}(\underline{\alpha}^{*}+\frac{\nu}{2}+\frac{m-1}{2})}{\Gamma_{\Omega}(\nu-\frac{m-1}{2})}
$$

6. The L^p-bounded properties of the Berezin transform

We let $C_{\nu} = I(\nu,0)^{-1}$, which is the normalization constant. Denote

(6.1) $$b_{\nu}(\underline{\lambda}) = C_{\nu}I(\nu,\underline{\alpha}).$$

Recall the definition of the genus p_0 of D. We see that, after simplifying,

$$
b_{\nu}(\underline{\lambda}) = \frac{\prod_{j=1}^{r}\Gamma(\nu-\frac{p_0-1}{2}+\lambda_j)\Gamma(\nu-\frac{p_0-1}{2}-\lambda_j)}{\prod_{j=1}^{r}\Gamma(\nu-\frac{p_0-1}{2}+\rho_j)\Gamma(\nu-\frac{p_0-1}{2}-\rho_j)}
$$

if it is of type BC or C,

$$
b_{\nu}(\underline{\lambda}) = \frac{\prod_{j=1}^{r}\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}+\lambda_j)\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}-\lambda_j)}{\prod_{j=1}^{r}\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}+\rho_j)\Gamma(\frac{\nu}{2}-\frac{p_0-1}{2}-\rho_j)}
$$

for other types except D_2. So that $b_{\nu}(\underline{\lambda})$ has a rather general form in terms of p_0. For type D_2 it is

$$
b_{\nu}(\underline{\lambda}) = \frac{\prod_{j=1}^{2}\Gamma(\frac{\nu}{2}-\frac{n-2}{4}+\lambda_j)\Gamma(\frac{\nu}{2}-\frac{n-2m}{4}-\lambda_j)}{\prod_{j=1}^{2}\Gamma(\frac{\nu}{2}-\frac{n-2}{4}+\rho_j)\Gamma(\frac{\nu}{2}-\frac{n-2m}{4}-\rho_j)}
$$

Proposition 6.1. *Suppose $\nu > p - 1 = 2p_0 - 1$ for type BC or C and $\frac{\nu}{2} > p_0 - 1$ for other types. The Berezin transform B_{ν} defines a bounded positive operator on $L^p(D, d\mu_0)$ for all $1 \le p \le \infty$.*

We recall that the functions e_λ transform under a character of NA. Observe that by change of variables in the formula $B_\nu e_\lambda(z) = b_\nu(\underline{\lambda})e_\lambda(z)$ for $z = e$ one obtains

$$B_\nu e_\lambda(z) = b_\nu(\underline{\lambda})e_\lambda(z)$$

for all $z \in \mathcal{S}$; see [1].

In particular taking $\underline{\lambda} + \underline{\rho} = 0$ we get

$$(6.2) \qquad C_\nu \int_{\mathcal{S}} \frac{\Delta(\Re z)^{\frac{\nu}{2}}\Delta(\Re w)^{\frac{\nu}{2}}}{\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1))^\nu}d\mu_0(w) = 1$$

Proof. We use interpolation. For $p = 1$ we have, in view of (6.2),

$$\|B_\nu f\|_{L^1} \le C_\nu \int_{\mathcal{S}} \int_{\mathcal{S}} |f(w)| \frac{\Delta(\Re z)^{\frac{\nu}{2}}\Delta(\Re w^{\frac{\nu}{2}}}{\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1))^\nu}d\mu_0(w)d\mu_0(z)$$

$$(6.3) \qquad = \int_{\mathcal{S}} |f(w)| \left(C_\nu \int_{\mathcal{S}} \frac{\Delta(\Re z)^{\frac{\nu}{2}}\Delta(\Re w^{\frac{\nu}{2}}}{\Delta(\frac{1}{2}(z_2 + w_2^*) - \frac{1}{2}F(z_1, w_1))^\nu}d\mu_0(z) \right) d\mu_0(w)$$

$$= \int_{\mathcal{S}} |f(w)|d\mu_0(w) = \|f\|_{L^1},$$

that is, B_ν is L^1-bounded. However B_ν is a formally self-adjoint operator thus it is L^∞ bounded. By interpolation we see that it is bounded on all L^p, for $1 \le p \le \infty$. $\quad\square$

We summarize our main results in the following

Theorem 6.2. *Suppose $\nu > p - 1 = 2p_0 - 1$ for type BC or C and $\frac{\nu}{2} > p_0 - 1$ for other types. The Berezin transform B_ν is a positive bounded operator on $L^2(D, d\mu_0)$ and its spectral symbol is given by $b_\nu(\underline{\lambda})$ under the decomposition of L^2-space into irreducible representations of G.*

Remark 6.3. Recall the definition (1.1) of the restriction operator R. The operator $B_\nu = RR^*$ is a bounded operator, which in turn implies that R is bounded when ν satisfies the condition above.

There are various implications of our results. We plan to persue them in the future.

REFERENCES

1. J. Arazy and G. Zhang, *L^q-estimates of spherical functions and mean-value property*, Integral Equations and Operator Theory, **23** (1995), 123–144.
2. F. A. Berezin, *General concept of quantization*, Commun. Math. Phys. **40** (1975), 153–174.
3. J. Faraut and A. Koranyi, *Analysis on symmetric cones*, Oxford University Press, Oxford, 1994.
4. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, London, 1978.
5. J. Hilgert and G. Ólafsson, *Causal symmetric spaces, geometry and harmonic analysis*, Perspectives in Mathematics, vol. 18, Academic Press, 1996.
6. S. C. Hille, *Canonical representations*, Ph.D. thesis, Leiden University, 1999.
7. L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Amer. Math. Soc., Providence, Rhode Island, 1963.

8. B. Kostant and S. Sahi, *Jordan algebras and Capelli identities*, Invent. Math. **112** (1993), 657–664.

9. O. Loos, *Bounded symmetric domains and Jordan pairs*, University of California, Irvine, 1977.

10. Yu. Neretin, *Matrix analogs of the integral $B(\alpha, \rho - \alpha)$ and Plancherel formula for Berezin kernel representations*, (1999), preprint, Math.RT/9905045.

11. G. Ólafsson, *Causal symmetric spaces*, Mathematica Gottingensis **15** (1990).

12. G. Ólafsson and B. Ørsted, *Generalizations of the Bargmann transform*, Lie theory and its applications in physics. Proceedings of the international workshop, Clausthal, Germany, August 14-17, 1995. (H.-D.Doebner et al, ed.), World Scientific, Singapore, 1996, pp. 3–14.

13. B. Ørsted and G. Zhang, *Weyl quantization and tensor products of Fock and Bergman spaces*, Indiana Univ. Math. J. **43** (1994), 551–582.

14. ———, *L^2-versions of the Howe correspondence I*, Math. Scand. **80** (1997), 125–160.

15. J. Peetre, *Berezin transform and ha-plitz opertors*, J. Oper. Theory **24** (1990), 165–168.

16. G. Shimura, *Generalized bessel functions on symmetric spaces*, J. reine angew. Math. **509** (1999), 35–66.

17. A. Unterberger and H. Upmeier, *The Berezin transform and invariant differential operators*, Comm. Math. Phys. **164** (1994), 563–597.

18. G. van Dijk and S. C. Hille, *Canonical rrepresentations related to Hyperbolic spaces*, J. Funct. Anal. **147** (1997), 109–139.

19. G. van Dijk and M. Pevzner, *Berezin kernels and tube domains*, (1999), preprint,.

20. A. M. Vershik, I.M. Gel'fand, and M.I. Graev, *Representations of the group $SL(2, \mathbf{R})$ where $\mathbf{R}$ is a ring of functions*, Uspekhi Mat. Nauk **28** (1973), no. 5, 83–128.

21. G. Zhang, *Berezin transform on line bundles over hermitian symmetric spaces*, J. Lie Theory, to appear.

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