On the stationary Boltzmann equation in $\mathbb{R}^n$

Leif Arkeryd*

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a strictly convex domain with C^1 boundary and inward normal $\vec{n}(x)$. Consider in Ω the stationary, non-linear Boltzmann equation for hard and soft forces with Grad's angular cut-off,

$$v\nabla_x F(x,v) = Q(F,F)(x,v),\ x \in \Omega, v \in \mathbb{R}^n. \tag{1.1}$$

Solutions $F \in L^1_+(\Omega \times \mathbb{R}^n)$ are understood in renormalized sense, or an equivalent form (mild, exponential, iterated integral, etc., cf [7], [1]). Constants are denoted by c.

Given a total mass $\mathcal{M}$, solutions are sought with $\int F dx dv = \mathcal{M}$ and with a given indata profile on the boundary $\partial\Omega$

$$F(x,v) = \frac{1}{c} f_b(x,v),\ x \in \partial\Omega,\ v \cdot \vec{n}(x) > 0. \tag{1.2}$$

The constant $c > 0$ of the indata profile and the total mass $\mathcal{M} > 0$ are not independent.

The collision operator Q is the classical nonlinear Boltzmann collision operator,

$$\begin{aligned} Q(f,f) &= \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\omega B(\omega, |v-v_*|)(f'f'_* - ff_*) = \\ &= Q^+(f,f) - Q^-(f,f) = Q^+(f,f) - f\nu(f), \end{aligned}$$

where $Q^+ - Q^-$ is the usual splitting into gain and loss terms, S^{n-1} the unit sphere in $\mathbb{R}^n$, and

$$\begin{aligned} &f_* = f(x,v_*),\ f' = f(x,v'),\ f'_* = f(x,v'_*), \\ &v' = v - [(v-v_*)\cdot\omega]\omega,\ v'_* = v_* + [(v-v_*)\cdot\omega]\omega. \end{aligned}$$

For simplicity the kernel B is taken as $B(\omega, |v-v_*|) = b(\omega)|v-v_*|^\beta$, with $-n < \beta < 2$ and $b \in L^1(S^{n-1})$ with a strictly positive lower bound. Let $\eta > 0$ be

*Department of Mathematics, Chalmers University, S-41296 Gothenburg, Sweden.

given (fixed and small) and let $\chi_\eta(v, v_*, \omega)$ be the characteristic function of the subset of $\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}$ for which

$$|v| \geq \eta,\ |v_*| \geq \eta,\ |v'| \geq \eta,\ |v'_*| \geq \eta,$$

and set $B_\eta = B \cdot \chi_\eta$. The weight function

$$\psi(v) = (1 + |v|)^{max(0,\beta)}$$

is used throughout. Define $\mathcal{M}_0, \mathcal{M}_2$, and $\mathcal{M}_\beta$ for $0 < \beta < 2$ by

$$\int_\Omega dx \int_{\mathbb{R}^n} F(x,v)dv = \mathcal{M}_0, \quad \int_\Omega dx \int_{\mathbb{R}^n} v^2 F(x,v)dv = \mathcal{M}_2,$$
$$\int_\Omega dx \int_{\mathbb{R}^n} (1+ \mid v \mid)^\beta F(x,v)dv = \mathcal{M}_\beta.$$

Theorem 1. *Suppose* $f_b > 0$ *and* $\int_{x \in \partial\Omega} dx \int_{v \cdot \vec{n}(x) > 0} v \cdot \vec{n}(x) f_b(x,v) dv = 1$,

$$\int_{x \in \partial\Omega} dx \int_{v \cdot \vec{n}(x) > 0} [v \cdot \vec{n}(x)(1 + v^2 + \log^+ f_b(x,v)) + 1] f_b(x,v) dv < \infty.$$

Consider the problem (1.1-2) with collision kernel B_η, *and boundary value* f_b. *The equation (1.1) has a family of solutions* $(F_m)_{m>0}$ *satisfying (1.2) with* $c = c_m > 0$, *and with the property* $\mathcal{M}_j(m) > 0$, *together with* $\lim_{m\to 0} \mathcal{M}_j(m) = 0$ *for* $j = 0, 2$,
$\lim_{m\to\infty} \mathcal{M}_2(m) = \infty, \quad \lim_{m\to\infty} \mathcal{M}_0(m) = \infty$ *when* $-n < \beta \leq 0$,
and $\lim_{m\to\infty} \mathcal{M}_\beta(m) = \infty$ *when* $0 < \beta < 2$.

In the present proof, the parameter m of the theorem has the value of the ψ-moment of certain approximations (cf. (3.2) below). Those approximations are then split into two pieces, one 'decoupling' in the limit, and the other defining the solution F of the theorem, this latter possibly having a smaller ψ-moment than m.

In the close to equilibrium case, there are a number of results concerning the non-linear stationary Boltzmann equation in $\mathbb{R}^n$, [9], [10], [11], [17] and others. Here general techniques such as contraction mappings can be utilized. Stationary problems in small domains can be solved in a similar way, [15], [13]. The unique solvability of interior, stationary problems for the Boltzmann equation at large Knudsen numbers is established in [14]. Existence results far from equilibrium for the stationary Povzner equation for bounded domains in $\mathbb{R}^n$ are obtained in [4]. In the slab case, results on boundary value problems with large indata for the BGK equation are presented in [16], and for the Boltzmann equation in a measure sense in [2] and others, and in an L^1-sense in [5-6].

In the Povzner and 1D Boltzmann papers [4-6] the entropy dissipation term is used to obtain weak L^1 compactness, necessary for applying the techniques from the time-dependent case. However, for (1.1-2) in several space dimensions, the compactness properties from the Povzner and 1D Boltzmann cases are no longer available. Instead, a careful analysis of the entropy dissipation term reveals that the approximations split into a bounded and a singular component - with, in the limit, the latter 'decoupling' and the remaining component by itself satisfying (1.1-2) in the sense of Theorem 1. The decoupled component may carry part of, but not all the original ψ-moment. However, that latter possibility (i.e. the whole ψ-moment disappearing) cannot be excluded by the arguments of the present paper, when the cut-off factor χ_η in Q is dropped. In this paper the study of the decoupling uses nonstandard analysis, which has a number of advantages in terms of simplicity and available techniques. Similarly, the author has earlier used a nonstandard approach to get initial insights into kinetic mechanisms (later usually followed by detailed standard studies of the problems). In connection with the present paper, work has also started on a second, standard proof.

In the following sections, we seek to present a reasonably coherent sketch of how the proof goes. The first step is standard and uses the same type of initial approximation that was introduced in [4-6]. But quite different arguments are required to conclude the proof. After the removal of a number of parameters from the initial approximation by standard arguments, the remaining approximation ((2.4) below) is taken, via transfer, infinitesimally close to its measure limit. A splitting is introduced for this approximation. The analysis of the splitting makes extensive use of the entropy dissipation control. This is where nonstandard analysis is brought in to carry the proof through. A relevant form of the averaging lemma connects nonstandard averages to corresponding standard ones in a limit at Lebesgue points. Finally, the gain and the loss terms of the nonstandard approximation in iterated integral form are shown to be infinitesimally close to those of the standard candidate for solution.

For clarity of exposition the proof is presented for $\mathbb{R}^3$.

2. A first approximation

The proof of Theorem 1 starts from an approximation for (1.1-2) of the same type as in [4-6], namely

$$\alpha cF(x,v) + cv\cdot\nabla_x F(x,v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega \chi^r \chi^{pn}$$

$$B_\mu \Big[\frac{cF}{1+\frac{cFc'}{j}}(x,v'(\frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v'_*) - cF(x,v)\frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v_*)\Big], (x,v)\in\Omega\times\mathbb{R}^3,$$

$$cF(x,v) = f_b(x,v)\wedge j, \quad x\in\partial\Omega, v\cdot\vec{n}(x) > 0. \qquad (2.1)$$

In (2.1) the notations are as follows. We assume in this first step that $b\in C^\infty$, that $|v-v_*|^\beta$ is replaced by a C^∞ approximation of $\max(\frac{1}{\mu},\min(\mu,|v-v_*|^\beta))$.

The functions $\chi^r(v, v_*, \omega)$ and $\chi^{pn}(v, v_*, \omega)$ are taken invariant with respect to the collision transformation $J(v, v_*, \omega) = (v', v'_*, -\omega)$, invariant under an exchange of v and v_*, with

$$\begin{aligned} \chi^r, \chi^{pn} \in C^\infty, \quad 0 \le \chi^r, \chi^{pn} \le 1, \\ \chi^r(v, v_*, \omega) = 1 \text{ if } |v| \ge r,\ |v_*| \ge r,\ |v'| \ge r,\ |v'_*| \ge r, \\ \chi^r(v, v_*, \omega) = 0 \text{ if } |v| \le \frac{r}{2}, \text{ or } |v_*| \le \frac{r}{2}, \text{ or } |v'| \le \frac{r}{2}, \text{ or } |v'_*| \le \frac{r}{2}, \\ \chi^{pn}(v, v_*, \omega) = 1 \text{ if } v^2 + v_*^2 \le \frac{n^2}{2}, \frac{1}{p} \le \left| \frac{v - v_*}{|v - v_*|} \cdot \omega \right| \le 1 - \frac{1}{p}, |\ v - v_*\ | \ge \frac{1}{p}, \\ \chi^{pn}(v, v_*, \omega) = 0 \text{ if } v^2 + v_*^2 \ge n^2, \text{ or } \\ |\frac{v - v_*}{|v - v_*|} \cdot \omega| \le \frac{1}{2p}, \text{ or } |\frac{v - v_*}{|v - v_*|} \cdot \omega| > 1 - \frac{1}{2p}, \text{ or } |\ v - v_*\ | \le \frac{1}{2p}. \end{aligned}$$

Moreover, $r > 0$, $0 < \alpha < 1$, $\rho, p, n, j, \mu \in \mathbb{N}^+$, where $\mathbb{N}^+$ denotes the set of strictly positive integers. The functions φ_ρ are mollifiers in x, defined by $\varphi_\rho(x) = \rho\varphi(\rho x), 0 \le \varphi \in C_0^\infty(\mathbb{R}^3), \varphi = 0$ for $|x| \ge 1, \int \varphi(x)dx = 1$.

Define the map T by $T(f, c') = (F, \frac{1}{c})$ on $K \times \mathbb{R}_+$, where

$$K := \{f \in L^1_+(\Omega \times \mathbb{R}^3); \int \psi_\mu(v) f(x, v) dx dv = 1\},$$

and cF solves (2.1) with c so chosen that $F \in K$. Here $\psi_\mu(v) = \min(\mu, \psi(v))$. E.g. by a monotone iteration scheme applied to (2.1) it is easy to see that T is well defined. The uniqueness in L^1_+ of (2.1) follows by considering the difference between an arbitrary non-negative solution and the iterated one. By the iteration scheme any such difference is non-negative, and so uniqueness follows from Green's formula.

Characteristics in $\Omega \times \mathbb{R}^3$ are of the type $\{(x + sv, v); s \in \mathbb{R}, x + sv \in \Omega\}$, and in Ω , for $\gamma \in S^2$ given, of the type $\{x + s\gamma; s \in \mathbb{R}, x + s\gamma \in \Omega\}$. There is a lower bound $c_0 > 0$ for c, only depending on f_b but not on $c' \ge 0$. Namely, from the exponential form of the problem obtained by integration along characteristics,

$$cF = f_b e^{-\int \nu} + c \int e^{-\int \nu} Q^+,$$

it follows that F is bounded from below by the ingoing value f_b along the corresponding characteristic times the negative exponential of a collision frequency integral along this characteristic. And so

$$c = \int c\psi_\mu(v) F(x, v) dx dv \ge c_0,$$

where $c_0 > 0$ is a certain integral of f_b.

Following the line of proof in [4-6] one shows that the map T is continuous and compact on $K \times [0, \frac{1}{c_0}]$ with the strong L^1 topology for K. Green's formula gives for $\alpha > 0$ that the solution cF of (2.1) for $c' = 0$ has finite mass, hence $\frac{1}{c} > 0$. So by the Schauder fixed point theorem, there is a function $f \in K$ and $c' \in (0, \frac{1}{c_0}]$ with $c' = \frac{1}{c}$ and

$$\alpha f(x,v) + v\nabla_x f(x,v) = \int \chi^r \chi^{pn} B_\mu \Big(\frac{f}{1+\frac{cfc'}{j}}(x,v') \frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v'_*)$$
$$-f(x,v)\frac{f*\varphi_\rho}{1+\frac{f*\varphi_\rho}{j}}(x,v))dv_* d\omega, \quad (x,v) \in \Omega \times \mathbb{R}^3,$$
$$cf(x,v) = f_b(x,v) \wedge j, \quad x \in \partial\Omega, v \cdot \vec{n}(x) > 0,$$

with

$$c \geq c_0, \int \psi_\mu(v) f(x,v) dx dv = 1.$$

Again following the proof in [4-6], we can pass to the limit when $\rho \to \infty$ using a strong L^1 compactness argument. For each $j \in \mathbb{N}^+$ the solution f^j satisfies $f^j \leq c' j^2$. By Green's formula $\alpha c \int f^j(x,v) dx dv \leq c_b$ with c_b depending on f_b but not on j. By computations similar to [4-5], Green's formula for $f^j \log \frac{f^j}{1+\frac{f^j}{j}}$ gives that $\alpha \int f^j \log f^j \leq C_b$ with C_b depending on f_b but not on j. And so the weak L^1 limit when $j \to \infty$ follows as in the time-dependent case, giving a solution f to

$$\alpha f(x,v) + v\nabla_x f(x,v) = \int \chi^r \chi^{pn} B_\mu (f(x,v') f(x,v'_*) -$$
$$-f(x,v) f(x,v_*)) dv_* d\omega, (x,v) \in \Omega \times \mathbb{R}^3,$$
$$cf(x,v) = f_b(x,v), x \in \partial\Omega, v \cdot \vec{n}(x) > 0, \tag{2.2}$$

with

$$c_0 \leq c \leq \frac{c_b}{\alpha}, \quad \text{and} \quad \int \psi_\mu(v) f(x,v) dx dv = 1.$$

By Green's formula, the solution f of (2.2) satisfies $c \int v^2 f(x,v) dx dv \leq c_1$ for α small enough, and with c_1 only depending on f_b, but not on p, n, μ, α. Write $Q(f,f) - \alpha f = Q^+(f,f) - \nu(f) f$, where Q^+ is the gain term and the collision frequency ν includes the parameter α. Then the exponential form implies

$$f_b e^{-\int \nu(t)} \leq cf \leq e^{\int \nu(f)} c f_{out}, \tag{2.3}$$

along characteristics, where f_{out} is taken on the outgoing boundary along the characteristic. This in turn implies that $c \int f(x,v) dx dv \geq c_2$, with the constant c_2 only depending on f_b, but not on p, n, μ, α. It follows that

$$c_1^{-1} \int v^2 f(x,v) dx dv \leq c^{-1} \leq c_2^{-1} \int f(x,v) dx dv,$$

and that $c^{-1}, \int f(x,v)dxdv, \int v^2 f(x,v)dxdv$ have strictly positive lower and upper bounds independent of p, n, μ (but depending on f_b and with the upper bounds independent of α).

Also by Green's formula for $f \log f$, it follows that $\alpha \int f \log f dxdv \leq C_b$ with C_b depending on f_b but not on p, n, μ. The previous weak L^1 limit first introduced in the time-dependent case, can now be repeated with respect to the approximations of B and to the parameters p, n, μ, giving a solution to

$$\begin{aligned} &\alpha f(x,v) + v\nabla_x f(x,v) = \int \chi^r B(f(x,v')f(x,v'_*) - \\ &f(x,v)f(x,v_*))dv_* d\omega, (x,v) \in \Omega \times \mathbb{R}^3, \\ &cf(x,v) = f_b(x,v), x \in \partial\Omega, v \cdot \vec{n}(x) > 0, \end{aligned} \tag{2.4}$$

with $\int \psi(v) f(x,v)dxdv = 1$, as well as strictly positive upper and lower a priori bounds of $c^{-1}, \int f(x,v)dxdv, \int v^2 f(x,v)dxdv$. The bounds depend on f_b, and the upper bound is independent of α. In the same way, letting χ^r converge to χ_η for $r = \eta$, the result of (2.4) holds with χ_η instead of χ^r. Also for some finite constant c_e depending on f_b but not on α,

$$\int \chi_\eta B(f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*} dxdvdv_* d\omega \leq c_e. \tag{2.5}$$

The previous limit procedure cannot be invoked to remove the term αf in (2.4), since we have no α-independent entropy estimate but only the entropy dissipation estimate (2.5). Instead the subject of the rest of the paper is a study of the α-limit with the help of (2.5).

3. Splitting the limit

For the rest of the paper the results (2.4-5) will be employed after transfer to a nonstandard context with $0 < \alpha \approx 0$ fixed, and with χ_η for $0 < \eta \in \mathbb{R}_+$ fixed. The values of c_e and the bounds for c^{-1}, $\int f(x,v)dxdv$, $\int v^2 f(x,v)dxdv$ remain the same. The convention will be followed of not putting stars on standard objects in the nonstandard context, except for clarity. We refer to [12] for all notations used from nonstandard analysis. In particular, "st" denotes the standard part of a hyperreal number.

The collision frequency ν still includes the parameter α. Define the standard measure σ on $\Omega \times \mathbb{R}^3$ by

$$C_c(\bar{\Omega} \times \mathbb{R}^3) \ni \varphi \curvearrowright \mathrm{st} \int_{^*\Omega \times \mathbb{R}^3} f^* \varphi^* dxdv =: \int_{\Omega \times \mathbb{R}^3} \varphi d\sigma(x,v).$$

Split σ into a Lebesgue absolutely continuous and a Lebesgue singular component, $d\sigma = F_s dxdv + d\sigma_s$. For $\lambda \in \mathbb{R}_+$, $st \int_{f \leq \lambda} \psi(v) f dxdv \leq 1$, and the value of this integral increases with λ. Recalling (2.3), there is a limit $\mathcal{M}'$ when $\lambda \to \infty$ with $0 < \mathcal{M}' \leq 1$. Hence, by spillover, for some $\lambda \in {}^*\mathbb{R}_+^\infty$,

$$\int_{f \leq \lambda} \psi(v) f(x,v) dx dv - \mathcal{M}' \approx 0.$$

For $z \in \mathbb{R}_+$, set $a_1(z) = \log z, a_i(z) = \max(1, \log a_{i-1}(z)), g(z) = (a_i(z))_{i \in \mathbb{N}^+}$. Then ${}^*g(\lambda)$ has $a_i(\lambda) \in {}^*\mathbb{R}_+^\infty$ for $i \in \mathbb{N}^+$, and so by spillover for all $i \leq i_0$ for some $i_0 \in {}^*\mathbb{N}^\infty$. Set $n' = a_{i_0}(\lambda)$. By (2.3), for $|v| \leq n'$ along all characteristics outside some set $S_{n'}$ of infinitesimal measure, ${n'}^{-1} \leq f \leq n'$. A slightly larger internal infinitesimal set $S_{n'}$ will now be constructed in a particular way and used to define a subsequent splitting $f = f_c + f_s$ related to $d\sigma = F_s dx dv + d\sigma_s$.

Given $\gamma \in {}^*S^2$, let π_γ denote a plane in ${}^*\mathbb{R}^3$ orthogonal to γ. Let Ω_γ be the projection of Ω into π_γ. By Green's formula,

$$\int_{x \in \partial\Omega} dx \int_{v \cdot \vec{n}(x) < 0} |v \cdot \vec{n}(x)| f(x,v) dv \leq \frac{1}{c} \int_{x \in \partial\Omega} dx \int_{v \cdot \vec{n}(x) > 0} v \cdot \vec{n}(x) f_b(x,v) dv \leq c_b,$$

where c_b is finite. It follows that after

i) taking into $S_{n'}$ the union of all characteristics $\zeta_{xv} := \{(x + t\gamma, v); v = \gamma|v|, x + t\gamma \in {}^*\Omega\}$ for a suitable internal infinitesimal set of $\gamma \in^* S^2$;

ii) for all other $\gamma \in {}^*S^2$, taking into $S_{n'}$ for a suitable γ-dependent internal infinitestimal set of $x \in \Omega_\gamma$, all characteristics ζ_{xv} for all $v = \gamma|v|$;

iii) for all $\gamma \in {}^*S^2$ not removed in i), for all $x \in \Omega_\gamma$ not removed in ii), taking into $S_{n'}$ for a suitable internal infinitesimal set of $|v|$, the characteristics ζ_{xv} with $v = \gamma|v|$;

then at the outgoing boundary point (x,v) of each remaining characteristic, $f(x,v) \leq n'$.

After taking into $S_{n'}$ an analogous, internal, infinitesimal union of of characteristics, on the ingoing boundary point (x,v) of each remaining characteristic $f_b(x,v) \geq {n'}^{-1}$. Given $\gamma \in {}^*S^2$, if $x \in {}^*\partial\Omega$ and $|\gamma \cdot \vec{n}(x)| \leq {n'}^{-1/8}$, then let $\zeta_{xv} \subset S_{n'}$ for all v parallel to γ. For $|v| \leq n'$ take ζ_{xv} into $S_{n'}$ if

$$\int_{\zeta_{xv}} ds \int_{|v_*| \geq n'} B_\eta f_* dv_* \geq n'^{-1}.$$

Recall that

$$\int \psi(v) f(x,v) dx dv = 1.$$

Then for $0 \leq \beta < 2$ and $\gamma \in {}^*S^2$, after taking for a suitable internal, infinitesimal set of $x \in \Omega_\gamma$, all characteristics ζ_{xv} with v parallel to γ, into $S_{n'}$, it holds for all other $x \in \Omega_\gamma$, and near-standard $v = \gamma|v|$ that

$$\int ds \int \chi_\eta |v - v_*|^\beta f(x + sv, v_*) dv_* \int b(\omega) d\omega \leq n'.$$

Since $\int_{n' \leq f \leq \lambda} \psi(v) f dx dv \approx 0$, it also holds given $\gamma \in {}^*S^2$, that outside a suitable internal, infinitesimal set in Ω_γ,

$$\int ds \int_{n' \leq f \leq \lambda} \chi_\eta |v - v|_*^\beta f(x + sv, v_*) dv_* \int b(\omega) d\omega \approx 0.$$

All characteristics parallel to γ, through x from the above infinitesimal set in Ω_γ, are taken into $S_{n'}$. Finally let $\zeta_{xv} \subset S_{n'}$, if $\zeta_{x(-v)} \subset S_{n'}$.

Thus, given $\gamma \in {}^*S^2$ and $x \in {}^*\partial\Omega$, either for all $|v| \leq n'$, the characteristic $\zeta_{x\gamma|v|} \subset S_{n'}$ or this holds for at most an infinitesimal set of such $|v|$. Also ${n'}^{-1} e^{-n'} \leq f \leq n' e^{n'}$ holds along all characteristics with $|v| \leq n'$ not in $S_{n'}$. To obtain the same structure for $-3 < \beta < 0$, due to the singularity in $|v - v_*|^\beta$, the three steps i)-iii) above may be performed once again.

Set $f_s(x, v) = f(x, v)$ for $(x, v) \in \zeta_{xv}$ if $\zeta_{xv} \subset S_{n'}$ or $|v| > n'$, $f_s(x, v) = 0$ otherwise, $f_\lambda(x, v) = f(x, v)$ for $f(x, v) \geq \lambda, f_\lambda(x, v) = 0$ otherwise, and $f_c(x, v) = f(x, v) - f_s(x, v)$. Then

$$f_s - f_\lambda \geq 0, \int \psi(v)(f_s - f_\lambda) dx dv \approx 0. \tag{3.1}$$

Obviously f_c only contributes to F_s (not to σ_s). Define $F(x, v) \leq F_s(x, v)$ by

$$C_c(\bar{\Omega} \times \mathbb{R}^3) \ni \varphi \curvearrowright st \int_{{}^*\Omega \times \mathbb{R}^3} f_c^* \varphi^* dx dv =: \int_{\Omega \times \mathbb{R}^3} F \varphi dx dv).$$

The study of f_s is by (3.1) reduced to a study of f_λ. It immediately follows from an a priori estimate of the outgoing entropy flow that the integral of f_λ is infinitesimal when restricted to the set of those characteristics where $\int \nu ds < n'$.

Consider the set O_ρ^c of those $x \in \Omega$ for which $\int \psi(v) f_\lambda dv < \frac{1}{\rho}$. For $\rho \in {}^*\mathbb{N}^\infty$ it follows that $\int_{O_\rho^c} dx \int \psi(v) f_\lambda dv \approx 0$. Given $i, \rho \in \mathbb{N}^+$ and $x \in O_\rho$, suppose the set of $\gamma \in S^2$ with $\zeta_{xv} \cap S_{n'} = \phi$ for all v parallel to γ (except possibly an internal infinitesimal subset) has measure $\geq \frac{4\pi}{i}$. The set of such x is denoted O_ρ^i.

By construction ${n'}^{-1} e^{-n'} \leq f(x + s\gamma, \gamma|v|) \leq n' e^{n'}$ along the corresponding characteristics.

Lemma 1. *For $i, \rho \in \mathbb{N}$ it holds that*

$$\eta_i := \int_{x \in O_\rho^i} dx \int \psi(v) f_\lambda dv \approx 0.$$

This is proved, using local estimates from the entropy dissipation integral. An essential ingredient is that the range of $f(x, \cdot)$ when $x \in O^i_\rho$, for a large part of velocity space has $f \leq n'$, whereas obviously $f \geq \lambda$ for the domain of integration of f_λ. A number of geometrically different cases have to be considered separately.

Given $\rho \in \mathbb{N}^+$, by spillover, the lemma holds for some $i_0 \in {}^*\mathbb{N}^\infty$ with $\eta_{i_0} \approx 0$. Given $\gamma \in S^2$, for $x \in \Omega_\gamma$, if $\int_{x+s\gamma \in O^{i_0}_\rho} ds \int \psi(v) f_\lambda(x + s\gamma, v_*) dv_* > \eta^{1/2}_{i_0}$, then for all v parallel to γ, move to $S_{n'}$ the characteristic ζ_{xv} and change the definition of f_c and f_m accordingly. Obviously for $0 \leq \beta < 2$ along characteristics not in (the new) $S_{n'}$, $f_\lambda(x, .)$ for all $x \in O^{i_0}_\rho$ only contributes infinitesimally to the integral of the collision frequency along the characteristic. For $-3 < \beta < 0$, the same follows after moving as before, for an infinitesimal set of v, all characteristics into $S_{n'}$.

Finally consider $x \in O_\rho \setminus O^{i_0}_\rho$.

Lemma 2. *Define A_γ as the (internal) set of those $x \in O_\rho \setminus O^{i_0}_\rho$ with at most an infinitesimal set of characteristics with $|v| \leq n'$ through x in direction γ belonging to $S_{n'}$. There then exists an (internal) infinitesimal subset I_ρ of S^2, such that given $\gamma \in S^2 \setminus I_\rho$,*

$$\int_{A_\gamma} dx \int \psi(v) f_\lambda(x, v) dv \approx 0.$$

The proof is reduced via approximation to a problem where the result is a simple consequence of the fact that, when an infinite sum of positive terms takes a finite value, then at most a countable number of the terms are noninfinitesimal. Here those terms correspond to an infinitesimal set of directions $\gamma \in S^2$, contained in an internal infinitesimal set of directions.

The lemma holds for all $\rho \in \mathbb{N}^+$, and so for all $\rho \leq \rho_0$ for some $\rho_0 \in {}^*\mathbb{N}^\infty$. For $\rho = \rho_0$, all characteristics with $\gamma \in I_{\rho_0}$ are moved into $S_{n'}$, and f_c, f_s are changed accordingly.

For $\gamma \in S^2 \setminus I_{\rho_0}$, there is an internal infinitesimal set J_γ of $x \in \Omega_\gamma$, so that for all other $x \in \Omega_\gamma$, it holds that

$$\int_{x+s\gamma \in A_\gamma} ds \int \psi(v) f_\lambda(x + s\gamma, v) dv \approx 0.$$

All characteristics in the direction γ through $x \in J_\gamma$ are moved into $S_{n'}$ and f_c, f_s are changed accordingly.

It follows that $S_{n'}$ consists of characteristics of the following types. For an infinitesimal set of $\gamma \in S^2$, all characteristics in direction γ are in $S_{n'}$. For all other $\gamma \in S^2$, the set of $x \in O_{\rho_0}$ that for all $|v|$ belong to a characteristic for $S_{n'}$ in direction γ, has infinitesimal projection into Ω_γ. For such γ the integral $\int_A dx \int \psi(v) f_\lambda(x, v) dv \approx 0$, where A is the set of those $x \in O_{\rho_0}$ for which (x, v) with $v = \gamma|v|$ outside an infinitesimal set of $|v| \leq n'$, belong to characteristics parallel to γ and not in $S_{n'}$. It follows that $\int \nu(f)ds \approx \int \nu(f_c)ds$ along characteristics with $f_c \not\equiv 0$.

The previous discussion for

$$\int \psi(v)f(x,v)dxdv = 1$$

can be repeated with an arbitrary finite $m > 0$ in place of 1, giving corresponding $f, f_c, f_s, f_\lambda, F$ and σ_s. For

$$\int \psi(v)f(x,v)dxdv = m$$

set,

$$\int \psi(v)F(x,v)dxdv = \mathcal{M}_\beta(m), \int F(x,v)dxdv = \mathcal{M}_0(m), \int v^2F(x,v)dxdv = \mathcal{M}_2(m). \quad (3.2)$$

Let c_b denote positive constants depending on f_b but not on m.

Lemma 3. *$\mathcal{M}_j(m) > 0$ for $m > 0$, $\lim_{m\to 0} \mathcal{M}_j(m) = 0$ for $j = 0, 2$, $\lim_{m\to\infty} \mathcal{M}_2(m) = \infty$, $\lim_{m\to\infty} \mathcal{M}_0(m) = \infty$ for $-n < \beta \leq 0$, $\lim_{m\to\infty} \mathcal{M}_\beta(m) = \infty$ for $0 < \beta < 2$.*

Proof. The property $\lim\limits_{m\to 0} \mathcal{M}_0(m) = 0$ follows from $\int F(x,v)dxdv = st\int f_c(x,v)dxdv \leq \int \psi(v)f(x,v)dxdv = m$. Denote by c_m the constant c of (2.4) when $\int \psi f = m$. Using Green's formula on (2.4), it follows that $c_m \int v^2 f dxdv \leq C_{b1}$ with C_{b1} finite and independent of $0 < \alpha < \alpha_0$ for some α_0 with $st\ \alpha_0 > 0$.

Moreover,

$$\int_\Omega dx \int_{\mathbb{R}^3} v^2 f dv \geq \frac{\eta^2}{2^{\beta'}} \int_\Omega dx \int_{\mathbb{R}^3} \psi(v)f(x,v)dv = \frac{m\eta^2}{2^{\beta'}},$$

with $\beta' = max(\beta, 0)$, and so

$$\mathcal{M}_2(m) = \int v^2F(x,v)dxdv \leq st\int v^2 f_c(x,v)dxdv \leq \int v^2 f(x,v)dxdv \leq \frac{C_{b1}}{c_m}, \quad (3.3)$$

$$mc_m = c_m \int_\Omega dx \int_{\mathbb{R}^3} \psi(v)f(x,v)dv \leq \frac{2^{\beta'}}{\eta^2} c_m \int v^2 f(x,v)dxdv \leq \frac{2^{\beta'}}{\eta^2} C_{b1},$$

$$\frac{1}{c_m} \geq \frac{m\eta^2}{2^{\beta'}C_{b1}}. \quad (3.4)$$

Given $d \in \mathbb{R}, 0 < d < \infty$, it holds that

$$\int_\Omega dx \int_{|v|\leq d} \nu(f)dv \leq mC(d) < \infty,$$

since

$$\int_\Omega dx \int_{\mathbb{R}^3} \psi(v) f(x,v) dv = m.$$

Using (2.3), it now follows that

$$m \geq \int_{|v|\leq d} f_c(x,v) dx dv \geq c_m^{-1} c(m),$$

where st $c(m) \geq$ st $c > 0$ for $m \leq m_0$, m_0 finite. Hence, also using (2.3), for m finite it holds that

$$\mathcal{M}_0(m) = \int_\Omega dx \int_{\mathbb{R}^3} F(x,v) dv > \text{st} \int_\Omega dx \int_{|v|\leq 100} f_c(x,v) dv > 0,$$

and also using (3.3),

$$0 \leq \lim_{m\to 0} \mathcal{M}_2(m) \leq \lim_{m\to 0} \frac{C_{b1}}{c_m} = 0.$$

But F satisfies (1.1-2) (as will be proved in next section), hence also (2.3). Suppose $(\mathcal{M}_{\beta'}(m))$ is bounded for some sequence $m \to \infty$. Then it follows from (2.3) for F, that $\mathcal{M}_0(m) \geq \frac{c_b}{c_m}$, and so by (3.4) that $\lim \mathcal{M}_0(m) = \infty$. This contradicts the hypothesis, and so $\lim_{m\to\infty} \mathcal{M}_{\beta'}(m) = \infty, \lim_{m\to\infty} \mathcal{M}_2(m) = \infty$. □

4. End of proof of Theorem 1

The part of Theorem 1 concerned with estimates for the $\mathcal{M}$-moments is contained in Lemma 3 of the previous section. It remains to show that F solves the boundary value problem (1.1-2). This is here carried out for $m = 1$. Test functions are functions $\varphi \in L^\infty(\Omega \times \mathbb{R}^3)$ with $v\nabla_x \varphi$ in $L^\infty(\Omega \times \mathbb{R}^3)$ and continuous along characteristics, with supp φ compact, and $\varphi(x,v) = 0$ for $x \in \partial\Omega$ with $v \cdot \vec{n}(x) < 0$.

Lemma 4. *For test functions φ,*

$$\int dx | \int {f_c}^* \varphi dv - {}^* \int F \varphi dv | \approx 0. \qquad (4.1)$$

Proof. The proof of (4.1) uses a nonstandard consequence of the following inequality from the proof [8] of the averaging lemma

$$\int\int \frac{|\tilde{g}(x) - \tilde{g}(y)|^2}{|x-y|^4} dx dy \leq C \|g\|_{L^2} \|v \nabla_x g\|_{L^2}, \qquad (4.2)$$

where

$$\tilde{g}(x) = \int g(x, v)dv.$$

For $i \in \mathbb{N}^+$ define f_n^i along characteristics where $|v| \leq i$ and

$$i^{-1} \leq f_c(x, v) \leq i$$

by

$$\begin{aligned} &v\nabla_x f_n^i = \bar{Q}^+ - \bar{\nu} f_n^i,\ x \in {}^*\Omega, v \in {}^*\mathbb{R}^3 \\ &c f_n^i(x, v) = f_b(x, v) \wedge i,\ x \in \partial\Omega, v \cdot \vec{n}(x) > 0. \end{aligned}$$

Here $\bar{\nu} = \nu \wedge j$ and the integrand of $\bar{Q}^+$ is the integrand of Q^+ multiplied with the characteristic function of the set of (x, v, v_*, ω) such that

$$|v|, |v_*| \leq i, \quad f' f'_* \leq i f f_*, \quad f_* \leq j.$$

Since $0 \leq f_n^i \leq f_c \leq i$ is finitely bounded together with $Q^+, \bar{\nu}$, obviously $v\nabla_x f_n^i, f_n^i \in {}^*L^2(\Omega \times \mathbb{R}^3)$, and $f_u^i := f_c - f_n^i$ satisfies (for a suitable sequence (j_i)) $\lim_{i\to\infty} \text{st} \int f_u^i dx dv = 0$.

To prove the lemma it is enough to prove (4.1) for f_n^i and a corresponding standard Lebesgue function F_n. Set $\tilde{f}_n = \int f_n^i dv$. For $0 < s \in \mathbb{R}_+$

$$\begin{aligned} &\text{st} \int \left| \frac{3}{4\pi s^3} \int_{|h|\leq s} \tilde{f}_n(x+h)dh - \tilde{f}_n(x) \right| dx \leq \\ &\leq \frac{3}{4\pi s^3} \ \text{st} \int_{|h|\leq s} dh \int_\Omega dx |\tilde{f}_n(x+h) - \tilde{f}_n(x)| \leq \\ &\leq C s^{1/2} (\text{st} \int \frac{|\tilde{f}_n(x+h) - \tilde{f}_n(x)|^2}{h^4} dh dx)^{1/2} \leq C s^{1/2} \to 0, \end{aligned}$$

when $s \to 0$.

But f_n^i defines a standard Lebesgue integrable function F_n by

$$\varphi \curvearrowright {}^0 \int f_n^i \, {}^*\varphi dx dv := \int F_n \varphi dx dv.$$

Set $\tilde{F}_n = \int F_n dv$. For ${}^o s > 0$, $\int_{|h|\leq s} \tilde{f}_n(x+h)dh$ only changes infinitesimally, when x changes infinitesimally. Moreover, for $x \in \Omega$ a Lebesgue point of $\tilde{F}_n$,

$$\frac{3}{4\pi s^3} \quad \text{st} \int_{|h|\leq s} \tilde{f}_n(x+h)dh \to \tilde{F}_n(x), 0 < {}^o s \to 0.$$

Hence st $\tilde{f}_n(x) = \tilde{F}_n \circ \text{st}(x)$, Loeb a.e. $x \in {}^*\Omega$.

Analogously, for every test function φ,

$$\mathrm{st}\int f_n^i(x,v)\,{}^*\varphi(x,v)dv = \int F_n({}^o x,v)\varphi({}^o x,v)dv, \text{Loeb a.e. } x\in {}^*\Omega.$$

This concludes the proof of (4.1). □

¿From here a proof can be given that F is a L^1 solution to the problem (1.1-2) in iterated integral form.

Proof. Let χ_i be the characteristic function of the set of characteristics along which $F\leq i$. It is enough to prove that for any test function φ, the product $\chi_i\varphi=\varphi_i$ satisfies a weak form of the problem

$$\int_{\Omega\times\mathbb{R}^3} Fv\cdot\nabla_x\varphi_i dxdv + \int_{\mathbb{R}^3\times\Omega\times\mathbb{R}^3\times S^2}\chi_\eta B\varphi_i(F'F'_*-FF_*)$$
$$dvdxdv_*d\omega = \frac{1}{{}^o c}\int_{\partial\Omega}dx\int_{v\cdot\vec{n}(x)>0} v\cdot\vec{n}(x)\varphi_i f_b dv. \qquad (4.3)$$

We let χ_k denote the characteristic function of those characteristics along which $f_c\not\equiv 0, f\leq k$. Then f satisfies a corresponding nonstandard version of (4.3) for the test function $\chi_k\,{}^*\varphi_i$ and with the extra term $\alpha\int f\chi_k\,{}^*\varphi_i dxdv\approx 0$. For $k\in {}^*\mathbb{R}_+^\infty$ with $k\leq n'$, the first integral in (4.3) for F is infinitesimally close to the corresponding nonstandard one for f. The same holds for the boundary term. So it remains to study the gain and the loss term.

An analogous result will now be proved for those terms, when $k\in {}^*\mathbb{N}^\infty$ is sufficiently small. For convenience we write B for $\chi_\eta B$. For the loss term it holds for $k\in\mathbb{N}$ that

$$\int {}^*\varphi_i\chi_k Bff_*dxdvdv_*d\omega\approx\int {}^*\varphi_i\chi_k Bf_cf_{c*}dxdvdv_*d\omega, \qquad (4.4)$$

since supp $\chi_k f\subset$ supp $f_c, \chi_k f\leq k$. By spillover, this holds for $k\leq k_0$ for some $n'\geq k_0\in {}^*\mathbb{N}^\infty$. By Lemma 4, k_0 may be chosen so that, moreover, (4.4) is infinitesimally close to

$$\int {}^*\varphi_i\chi_k Bf_c\,{}^*F_*dxdvdv_*d\omega\approx\int {}^*\varphi_i B\,{}^*F\,{}^*F_*dxdvdv_*d\omega =$$
$$\int \varphi_j BFF_*dxdvdv_*d\omega.$$

The desired result is thus proved for the loss term

$$\int {}^*\varphi_i\chi_k Bff_*dxdvdv_*d\omega\approx\int\varphi_i BFF_*dxdvdv_*d\omega.$$

For the gain term, we start with an estimate from below, and then give an estimate from above. Together they imply equality for the gain term. For $k_0 \geq k \in {}^*\mathbb{N}^\infty$ it holds that

$$\int {}^*\varphi_i \chi_k B f' f'_* dx dv dv_* d\omega = \int {}^*\varphi'_i \chi'_k B f f_* dx dv dv_* d\omega$$
$$\geq \int {}^*\varphi'_i \chi'_k B f_c f_{c*} dx dv dv_* d\omega \tag{4.5}$$

Set $f_{cR} = f_c \wedge R$, and define F_R by

$$\varphi \curvearrowright \mathrm{st} \int f_{cR} {}^*\varphi dx dv := \int F_R \varphi dx dv.$$

Using (4.1), for $R \in \mathbb{R}$, the last member in (4.5) can be estimated from below by

$$\int {}^*\varphi'_i \chi'_k B f_{cR} f_{c*} dx dv dv_* d\omega$$
$$\approx \int {}^*\varphi'_i \chi'_k B f_{cR} {}^*F_* dx dv dv_* d\omega$$
$$\approx \int {}^*\varphi'_i B^* F^*_R F_* dx dv dv_* d\omega$$
$$= \int \varphi'_i B F_R F_* dx dv dv_* d\omega$$
$$\to \int \varphi'_i B F F_* dx dv dv_* d\omega \tag{4.6}$$

when $R \to \infty$. For the opposite inequality, let χ_h be defined in the same way as χ_k above. Then, for $h \in \mathbb{N}$,

$$\chi_h f = \chi_h f_c, \int \chi_h f_* dv_* dx \approx \int \chi_h f_{c*} dv_* dx,$$

and so using (4.1)

$$\int {}^*\varphi'_i \chi'_k \chi_h B f f_* dx dv dv_* d\omega$$
$$\approx \int {}^*\varphi'_i \chi'_k \chi_h B f_c f_{c*} dx dv dv_* d\omega$$
$$\lessapprox \int {}^*\varphi'_i \chi_h B f_c {}^*F_* dx dv dv_* d\omega$$
$$\lessapprox \int {}^*\varphi'_i B {}^*F {}^*F_* dx dv dv_* d\omega$$
$$= \int \varphi'_i B F F_* dx dv dv_* d\omega. \tag{4.7}$$

In (4.7) a comparison between gain and loss integrand, together with the entropy dissipation estimate, was invoked to handle large velocities. For k finite fixed, for h sufficiently large and finite, the first member of (4.7) is arbitrarily close to the same term without the χ_h factor. And so the inequality between the first and the last member in (4.7) holds without χ_h, i.e., for $k \in \mathbb{N}^+$

$$\int {}^*\varphi_i \chi'_k B f f_* dx dv dv_* d\omega \lessapprox \int \varphi'_i B F F_* dx dv dv_* d\omega.$$

By spillover, this also holds for $k \leq k_0$ for some $k_0 \in {}^*\mathbb{N}^\infty$. Together with (4.6) this implies

$$\int {}^*\varphi_i \chi_k B f' f'_* dx dv dv_* d\omega \approx \int \varphi^i B F' F'_* dx dv dv_* d\omega.$$

It follows that F satisfies (4.3). □

Remark. The previous discussion can be carried through also in the case of B without the χ_η-factor. But in that case, the present proof does not seem to rule out the possibility that $\mu = \sigma_s$, $F \equiv 0$, and $\frac{1}{c} f_b \equiv 0$.

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