On the L^p boundedness of the non-centered Gaussian Hardy-Littlewood Maximal Function

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Abstract: The purpose of this paper is to prove the $L^p(\mathcal{R}^n, d\gamma)$ boundedness, for $p > 1$, of the non-centered Hardy-Littlewood maximal operator associated with the Gaussian measure $d\gamma = e^{-|x|^2}dx$.

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Let $d\gamma = e^{-|x|^2}dx$ be a Gaussian measure in Euclidean space $\mathcal{R}^n$. We consider the non-centered maximal function defined by

$$\mathcal{M}f(x) \quad = \quad \sup_{x \in B} \frac{1}{\gamma(B)} \int_B |f| \, d\gamma,$$

where the supremum is taken over all balls B in $\mathcal{R}^n$ containing x. P. Sjögren [2] proved that $\mathcal{M}$ is not of weak type (1,1) with respect to $d\gamma$. A more general result was obtained by A. Vargas [3], who characterized those radial and strictly positive measures for which the corresponding maximal operator is of weak type (1,1). However, these papers leave open the question of the $L^p(d\gamma)$ boundedness of $\mathcal{M}$ for $p > 1$ and $n > 1$.
The main result in this paper is

Theorem 1 *$\mathcal{M}$ is a bounded operator on $L^p(d\gamma)$ for $p > 1$, that is, there exists a constant $C = C(n,p)$ such that for $f \in L^p(d\gamma)$,*

$$\|\mathcal{M}f\|_{L^p(d\gamma)} \leq C\|f\|_{L^p(d\gamma)}.$$

We denote $S_r^{n-1} = \{x \in \mathcal{R}^n : |x| = r\}$ and $S^{n-1} = S_1^{n-1}$, and write $d\sigma$ for the area measure on S^{n-1}. The spherical maximal function

$$\mathcal{M}^e f(h) = \sup_{R>0} \frac{1}{\sigma(|z'-h| \leq R)} \int_{|z'-h|\leq R} |f(z')| \, d\sigma(z'), \qquad h \in S^{n-1},$$

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is bounded on $L^p(d\sigma)$. We extend $\mathcal{M}^e$ to functions defined in $\mathcal{R}^n$ by using polar coordinates $x = \rho x'$ with $x' \in S^{n-1}$ and applying $\mathcal{M}^e$ in the x' variable. Then $\mathcal{M}^e$ is bounded on $L^p(d\gamma)$.
In order to prove Theorem 1, we need the following technical lemma, proved later.

Lemma 1 *Let B be a closed ball in $\mathcal{R}^n$ of radius r. Denote by q the point of B whose distance to the origin is minimal. Assume that $|q| \geq 1$ and that $r \geq 1/|q|$. Then for all $x,\ y \in B$*

$$\gamma(B) \geq C\,\frac{e^{-|q|^2}}{|q|}\left(1\ \wedge\ \frac{|y-x|^2}{(|q|(|x|\vee|y|-|q|))}\right)^{\frac{n-1}{2}}. \tag{1}$$

Here and in the sequel, we write C for various positive finite constants and denote $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$.

Proof of Theorem 1: We assume that $n \geq 2$, since the case $n = 1$ is well known, see e.g. [2]. Take $0 \leq f \in L^p(d\gamma)$ and $x \in \mathcal{R}^n$. For any ball B containing x, we must estimate the average $\mathcal{A}f(B) = \dfrac{1}{\gamma(B)}\displaystyle\int_B f\,d\gamma$. Let r and q be defined as in Lemma 1.
We first consider small balls B, and denote by $\mathcal{M}_0 f(x)$ the supremum of $\mathcal{A}f(B)$ taken only over balls B containing x and verifying $r < 1 \wedge |q|^{-1}$. Split $\mathcal{R}^n$ into rings $R_k = \{x : \sqrt{k-1} \leq |x| < \sqrt{k}\}$, $k = 1, 2, \ldots$. The width of R_k is no larger than $1/\sqrt{k}$, and so the Gaussian density is of constant order of magnitude in each R_k. Using Lebesgue measure arguments, one can easily estimate the $L^p(d\gamma)$ norm of $\mathcal{M}_0 f$ in R_k in terms of the $L^p(d\gamma)$ norm of f in $\cup\{R_{k'} : |k'-k| \leq C\}$. This takes care of small balls.
Consider now balls B with $r \geq 1 \wedge |q|^{-1}$. Observe to begin with that the case $|q| < 2$ is simple, since then $\gamma(B) \geq C$ and thus

$$\mathcal{A}f(B) \leq C \int f d\gamma \leq C \parallel f \parallel_{L^p(d\gamma)} .$$

The corresponding part of $\mathcal{M}f$ thus satisfies the $L^p(d\gamma)$ estimate.
It remains to consider $\tilde{\mathcal{M}}f(x) = \sup \mathcal{A}f(B)$, the supremum taken over balls B containing x and with the property that $r \geq |q|^{-1}$ and $|q| \geq 2$. Let B be such a ball, and observe that it satisfies the hypotheses of Lemma 1.
For each $\rho \geq 1$ such that S^{n-1}_ρ intersects B, let $y_\rho \in S^{n-1}_\rho \cap \partial B$ be such that $|y_\rho - x| = \sup_{z \in B \cap S^{n-1}_\rho} |z - x|$. Write $x' = x/|x|$.

For each $z' \in S^{n-1}$ such that $\rho z' \in B$ we have

$$\begin{aligned} |x'-z'| &= \frac{1}{\rho}|\rho x' - \rho z'| \\ &\leq \frac{1}{\rho}\left[|x-\rho z'| + |\rho - |x||\right] \\ &\leq \frac{2}{\rho}|y_\rho - x|; \end{aligned} \qquad (2)$$

and trivially $|x'-z'| \leq 2$.
Because of (2) and the definition of $\mathcal{M}^e$,

$$\begin{aligned} \mathcal{A}f(B) &= \int_{|q|}^{|q|+2r} \frac{1}{\gamma(B)} \int_{S^{n-1}} \chi_B(\rho z') f(\rho z') d\sigma(z')\ \rho^{n-1} e^{-\rho^2}\ d\rho \\ &\leq \int_{|q|}^{|q|+2r} \frac{1}{\gamma(B)} \int_{|z'-x'|\leq 2\left(1\wedge \frac{|y_\rho - x|}{\rho}\right)} f(\rho z') d\sigma(z')\ \rho^{n-1} e^{-\rho^2}\ d\rho \\ &\leq C \int_{|q|}^{|q|+2r} \frac{\left\{1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{n-1}\right\}}{\gamma(B)} \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2}\ d\rho \\ &\leq C \int_{|q|}^{|q|+2r} |q| e^{|q|^2} \left\{1 \vee \left(\frac{|q|(\rho \vee |x| - |q|)}{|x-y_\rho|^2}\right)^{\frac{n-1}{2}}\right\} \left\{1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{n-1}\right\} \\ &\qquad \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2}\ d\rho, \end{aligned} \qquad (3)$$

where we applied Lemma 1 with $y = y_\rho$ to get the last inequality.
Write $M = \rho \vee |x|$ and $m = \rho \wedge |x|$, so that $|q| \leq m \leq M$.

Lemma 2 *For $|q| < \rho < |q| + 2r$ and some C,*

$$e^{|q|^2} \left\{1 \vee \left(\frac{|q|(M-|q|)}{|x-y_\rho|^2}\right)^{\frac{n-1}{2}}\right\} \left\{1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{n-1}\right\} \leq C e^{m^2} \left(\frac{1}{m^2} \vee \frac{M-m}{m}\right)^{\frac{n-1}{2}}.$$

Assuming this lemma for the moment, we conclude from (3) that

$$\mathcal{A}f(B) \leq C \int_1^\infty m\, e^{m^2} \left(\frac{1}{m^2} \vee \frac{M-m}{m}\right)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2}\ d\rho.$$

We split this integral into five integrals taken over the following intervals

$$\begin{aligned} I_1 &= \left[1, \frac{|x|}{2}\right], \quad I_2 = \left(\frac{|x|}{2}, |x| - \frac{1}{|x|}\right], \quad I_3 = \left(|x| - \frac{1}{|x|}, |x| + \frac{1}{|x|}\right], \\ I_4 &= \left(|x| + \frac{1}{|x|}, \frac{5}{4}|x|\right], \quad I_5 = \left(\frac{5}{4}|x|, +\infty\right). \end{aligned}$$

Let for $i = 1, ..., 5$

$$\mathcal{M}_i f(x) = \int_{I_i} m\, e^{m^2} \left(\frac{1}{m^2} \vee \frac{M-m}{m}\right)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2}\, d\rho.$$

Then $\tilde{\mathcal{M}} f \leq C \sum_1^5 \mathcal{M}_i f$.

Bound for $\mathcal{M}_1 f(x)$.
One finds that

$$\mathcal{M}_1 f(x) \leq |x|^n \int_1^{\frac{|x|}{2}} \mathcal{M}^e f(\rho x')\, d\rho.$$

Hölder's inequality and the $L^p(d\sigma)$ boundedness of $\mathcal{M}^e$ imply

$$\| \mathcal{M}_1 f \|^p_{L^p(d\gamma)} \leq \int_1^{+\infty} \int_{S^{n-1}} \left(s^n \int_1^{\frac{s}{2}} \mathcal{M}^e f(\rho x') d\rho \right)^p d\sigma(x') s^{n-1} e^{-s^2} ds$$

$$\leq \int_1^{+\infty} \int_{S^{n-1}} s^{np} \int_1^{\frac{s}{2}} |\mathcal{M}^e f(\rho x')|^p \rho^{n-1} e^{-\rho^2} d\rho \left(\int_1^{\frac{s}{2}} \rho^{-(n-1)\frac{p'}{p}} e^{\frac{p'}{p}\rho^2} d\rho \right)^{\frac{p}{p'}} d\sigma(x') s^{n-1} e^{-s^2} ds$$

$$\leq \left(\int_1^{+\infty} s^C e^{-\frac{3}{4}s^2} ds \right) \| f \|^p_{L^p(d\gamma)} \leq C \| f \|^p_{L^p(d\gamma)} .$$

Bound for $\mathcal{M}_2 f(x)$.
Making the change of variable $\rho = |x| - \dfrac{t}{|x|}$, we get

$$\begin{aligned} \mathcal{M}_2 f(x) &\leq C|x|^{\frac{n+1}{2}} \int_{|x|/2}^{|x|-\frac{1}{|x|}} (|x| - \rho)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x')\, d\rho \\ &\leq C \int_1^{|x|^2/2} t^{\frac{n-1}{2}} \mathcal{M}^e f\left((|x| - \frac{t}{|x|}) x' \right) dt. \end{aligned}$$

¿From Minkowski's integral inequality and the $L^p(d\sigma)$ boundedness of $\mathcal{M}^e$, we obtain

$$\begin{aligned} \|\mathcal{M}_2 f\|_{L^p(d\gamma)} &\le C\int_1^{+\infty} t^{\frac{n-1}{2}} \left\| \mathcal{M}^e f\left((|x| - \frac{t}{|x|})x'\right) \chi_{\{1\le t\le \frac{|x|^2}{2}\}} \right\|_{L^p(d\gamma)} dt \\ &= C\int_1^{+\infty} t^{\frac{n-1}{2}} \left[\int_{S^{n-1}}\int_{\sqrt{2t}}^{+\infty} f\left((s-\frac{t}{s})x'\right)^p s^{n-1}e^{-s^2} ds d\sigma(x')\right]^{\frac{1}{p}} dt. \end{aligned}$$

We now make the change of variables $s \to \rho = s - t/s$, observing that $s \le 2\rho$ and $-s^2 = -\rho^2 - 2t + t^2/s^2 \le -\rho^2 - 3t/2$ and $d\rho/ds \ge 1$. Thus

$$\begin{aligned} \|\mathcal{M}_2 f\|_{L^p(d\gamma)} &\le C\int_1^{+\infty} t^{\frac{n-1}{2}} \left[\int_{S^{n-1}}\int_{\sqrt{t/2}}^{+\infty} |f(\rho x')|^p \rho^{n-1} e^{-\rho^2} e^{-3t/2} d\rho d\sigma(x')\right]^{\frac{1}{p}} dt \\ &\le C\, \|f\|_{L^p(d\gamma)} \left(\int_1^{+\infty} t^{\frac{n-1}{2}} e^{-\frac{3t}{2p}} dt\right) \le C\, \|f\|_{L^p(d\gamma)}. \end{aligned}$$

Bound for $\mathcal{M}_3 f(x)$.
Let $d\mu = \rho^{n-1}e^{-\rho^2}\, d\rho$ in $\mathcal{R}_+$. We have

$$\begin{aligned} \mathcal{M}_3 f(x) &\le C|x|\int_{|x|-1/|x|}^{|x|+1/|x|} \mathcal{M}^e f(\rho x') d\rho \\ &\le C(\mu(|x|-1/|x|, |x|+1/|x|))^{-1} \int_{|x|-1/|x|}^{|x|+1/|x|} \mathcal{M}^e f(\rho x') d\mu(\rho). \end{aligned}$$

Let $\mathcal{M}^\mu$ denote the one-dimensional centered maximal operator defined in terms of μ, acting in the ρ variable. Then

$$\mathcal{M}_3 f(x) \le C\mathcal{M}^\mu \mathcal{M}^e f(|x|x').$$

But $\mathcal{M}^\mu$ is known to be bounded on $L^p(d\mu)$, see [1] or [2]. The $L^p(d\gamma)$ boundedness of $\mathcal{M}_3$ follows.

Bound for $\mathcal{M}_4 f(x)$.
Making the change of variable $\rho = |x| + \dfrac{t}{|x|}$, we have

$$\begin{aligned} \mathcal{M}_4 f(x) &\le C|x|^{\frac{n+1}{2}} e^{|x|^2} \int_{|x|+\frac{1}{|x|}}^{\frac{5}{4}|x|} (\rho - |x|)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x')\, e^{-\rho^2}\, d\rho \\ &\le C\int_1^{\frac{|x|^2}{4}} t^{\frac{n-1}{2}} \mathcal{M}^e f\left((|x| + \frac{t}{|x|})x'\right) e^{-2t}\, e^{-\frac{t^2}{|x|^2}} dt. \end{aligned}$$

Minkowski's integral inequality implies

$$\|\mathcal{M}_4 f\|_{L^p(d\gamma)} \leq C \int_1^{+\infty} t^{\frac{n-1}{2}} \left\| \mathcal{M}^e f\left((|x| + \frac{t}{|x|})x' \right) e^{-\frac{t^2}{|x|^2}} \chi_{\{1\leq t \leq \frac{|x|^2}{4}\}} \right\|_{L^p(d\gamma)} e^{-2t} dt.$$

But $\mathcal{M}^e$ is bounded on $L^p(d\sigma)$, so that

$$\begin{aligned} &\|\mathcal{M}^e f((|x| + \frac{t}{|x|})x')\, e^{-\frac{t^2}{|x|^2}} \chi_{\{1\leq t\leq \frac{|x|^2}{4}\}}\|^p_{L^p(d\gamma)} \\ \leq\ & C \int_{2\sqrt{t}}^{\infty} \int_{S^{n-1}} |f((s + \frac{t}{s})x')e^{-\frac{t^2}{s^2}}|^p d\sigma(x') s^{n-1} e^{-s^2} ds. \end{aligned}$$

Almost as in the case of $\mathcal{M}_2$, we make the change of variable $\rho = s + t/s$ and observe that $s \leq \rho$ and $-s^2 = -\rho^2 + 2t + t^2/s^2$ and $d\rho/ds \geq 1/2$. Since $e^{-pt^2/s^2} e^{t^2/s^2} < 1$, it follows that the above double integral is at most

$$C \int_{S^{n-1}} \int_1^{+\infty} |f(\rho x')|^p \rho^{n-1} e^{-\rho^2}\, d\rho d\sigma(x')\, e^{2t} \leq C\|f\|^p_{L^p(d\gamma)} e^{2t}.$$

Thus

$$\|\mathcal{M}_4 f\|_{L^p(d\gamma)} \ \leq\ C \int_1^{+\infty} t^{\frac{n-1}{2}} \|f\|_{L^p(d\gamma)} e^{\frac{2t}{p}} e^{-2t} dt \ \leq\ C\|f\|_{L^p(d\gamma)}.$$

Bound for $\mathcal{M}_5 f(x)$.

Observe that

$$\mathcal{M}_5 f(x) \ \leq\ |x|^{\frac{3-n}{2}} e^{|x|^2} \int_{\frac{5}{4}|x|}^{+\infty} \mathcal{M}^e f(\rho x')\, \rho^{\frac{n-1}{2}} \rho^{n-1} e^{-\rho^2} d\rho.$$

We take the L^p norm and then apply Hölder's inequality, getting

$$\|\mathcal{M}_5 f\|^p_{L^p(d\gamma)} \leq \int_1^{+\infty} \int_{S^{n-1}} \frac{e^{ps^2}}{s^{p\frac{n-3}{2}}} \left(\int_{\frac{5s}{4}}^{+\infty} \mathcal{M}^e f(\rho x')\, \rho^{\frac{3(n-1)}{2}} e^{-\rho^2} d\rho \right)^p d\sigma(x') s^{n-1} e^{-s^2} ds$$

$$\begin{aligned} &\leq \int_1^{+\infty} \int_{S^{n-1}} \frac{e^{ps^2}}{s^{p\frac{n-3}{2}}} \int_0^{+\infty} |\mathcal{M}^e f(\rho x')|^p \rho^{n-1} e^{-\rho^2} d\rho \left(\int_{\frac{5s}{4}}^{+\infty} \rho^{(\frac{p'}{2}+1)(n-1)} e^{-\rho^2} d\rho \right)^{\frac{p}{p'}} d\sigma(x') s^{n-1} e^{-s^2} ds \\ &\leq \|f\|^p_{L^p(d\gamma)} \left(\int_1^{+\infty} s^C e^{(p-1)s^2} e^{-(p-1)(\frac{5}{4}s)^2} ds \right) \\ &\leq C\|f\|^p_{L^p(d\gamma)}. \end{aligned}$$

To finish the proof of Theorem 1, it now only remains to prove the two lemmas.

Proof of Lemma 1.
Consider the hyperplane orthogonal to q whose distance from the origin is $|q|+t$, with $1/(2|q|) < t < 1/|q|$. Its intersection with B is an $(n-1)$-dimensional ball whose radius is at least $C\sqrt{rt} \geq C\sqrt{r/|q|}$. Integrating the Gaussian density first along this $(n-1)$-dimensional ball and then in t, we get

$$\gamma(B) \geq \int_{1/(2|q|)}^{1/|q|} e^{-(|q|+t)^2} dt \int_{|v|<C\sqrt{r/|q|}} e^{-|v|^2} dv,$$

where v is an $(n-1)$-dimensional variable. The inner integral here is at least $C\min(1, (r/|q|)^{(n-1)/2})$ and $e^{-(|q|+t)^2} \geq Ce^{-|q|^2}$ for these t; therefore

$$\gamma(B) \geq C\frac{e^{-|q|^2}}{|q|}\left(1 \wedge \left(\frac{r}{|q|}\right)^{\frac{n-1}{2}}\right). \tag{4}$$

To estimate r from below, we let z be the center of B and w the projection of x onto the line passing through 0, q and z. Write $h = |x-w|$ and $a = |w-q|$. Applying the Pythagoras Theorem twice, we get

$$|x-z|^2 - (r-a)^2 = h^2 = |x-q|^2 - a^2.$$

Since $|x-z| \leq r$, we conclude that $2ar \geq |x-q|^2$. Clearly $a \leq |x| - |q|$ so that

$$r \geq \frac{|x-q|^2}{2(|x|-|q|)} \geq \frac{|x-q|^2}{2(|x| \vee |y| - |q|)}.$$

Since x and y are arbitrary points of B, the same argument also implies

$$r \geq \frac{|y-q|^2}{2(|x| \vee |y| - |q|)}.$$

From the triangle inequality we conclude that $2|x-q| \vee |y-q| \geq |x-y|$ and so

$$r \geq \frac{|x-y|^2}{8(|x| \vee |y| - |q|)}.$$

Combining this with (4), we obtain the inequality of Lemma 1. The proof is complete.

Proof of Lemma 2.
We write LHS for the left-hand side of the inequality to be proved. Assume first that
$$\left(\frac{|q|(M-|q|)}{|x-y_\rho|^2}\right)^{\frac{n-1}{2}} \le 1. \tag{5}$$
Then LHS $\le e^{|q|^2}(|x-y_\rho|/\rho)^{n-1}$. The angles at q of the triangles $0qx$ and $0qy_\rho$ are obtuse, so that $|x|^2 \ge |q|^2 + |x-q|^2$ and $|y_\rho|^2 \ge |q|^2 + |y_\rho - q|^2$. But $|x-y_\rho| \le |x-q| + |y_\rho - q|$, and this implies

$$|x-y_\rho|^2 \le 4\max(|x-q|^2,\, |y_\rho-q|^2) \le 4\max(|x|^2-|q|^2,\, |y_\rho|^2-|q|^2) = 4(M^2-|q|^2).$$

If $|x| \le 2\rho$, this last quantity is at most $16\rho(M-|q|)$, and then
$$\text{LHS} \le Ce^{|q|^2}\left(\frac{M-|q|}{\rho}\right)^{\frac{n-1}{2}}. \tag{6}$$

In the contrary case $|x| > 2\rho$, we simply observe that LHS $\le Ce^{|q|^2}$ whereas the right-hand side is at least Ce^{m^2}. This case of the lemma is thus trivial.
Assume now that (5) is false. Then
$$\text{LHS} \le e^{|q|^2}\frac{(|q|(M-|q|))^{\frac{n-1}{2}}}{\rho^{n-1}}$$
and we arrive again at (6).
It thus only remains to see that (6) implies Lemma 2. This would follow from the estimate
$$e^{|q|^2-m^2}(M-|q|)^{\frac{n-1}{2}} \le C((1/m)\vee(M-m))^{\frac{n-1}{2}}. \tag{7}$$
To prove (7), we use the fact that
$$(M-|q|)^{\frac{n-1}{2}} \le C\left((M-m)^{\frac{n-1}{2}} + (m-|q|)^{\frac{n-1}{2}}\right)$$
and when $m-|q| > 1/m$ also
$$e^{|q|^2-m^2} = e^{-(m-|q|)(m+|q|)} \le \frac{C}{(m-|q|)^{\frac{n-1}{2}}m^{\frac{n-1}{2}}}.$$
Now (7) and Lemma 2 follow.

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