ON THE POSSIBILITY OF KINETIC FLOW THROUGH FRACTAL HOLES IN A SURFACE.

Alexei Heintz

Department of Mathematics, University of Göteborg and
Chalmers University of Technology,
41296 Göteborg, Sweden

Abstract

The paper considers initial boundary value problems for the nonlinear Boltzmann equation with large L^1 initial data and for the linearised Bolzmann equation with L^2 initial data. Particular problems under analysis describe flows in a domain consisting of two subdomains connected through a joint part Γ of the boundary which can have a complicated fractal geometry.

The presence of interaction between these two parts of such a flow and the possibility to reach a uniform stationary state when time goes to infinity depends on initial data and on the Hausdorff dimension of the contact set Γ.

Geometrical conditions for the existence of such a global stabilisation for the nonlinear Boltzmann equation and for the linearised Boltzmann equation are studied and found to be different. Renormalised L^1 solutions can in general penetrate Γ only if its 2-dimensional Hausdorff measure is positive. Solutions to the linearised Boltzmann equation in L^2 can go through much thinner "holes" that have Hausdorff dimension larger than 1.

1 Introduction

The initial boundary value problems for the Boltzmann equation were considered for more than 30 years. Solutions to the linearised Boltzmann equation and solutions to the nonlinear Boltzmann equation close to a global equilibrium were first studied by C. Cersignani, J. P. Guiraud, S. Ukai, K. Asano, N. Maslova, A. Heintz and by other, see references in [7], [17].

The existence of solutions for problems with large L^1 initial data was considered in a DiPerna-Lions setting in several papers by O. Hamdache [10], L. Arkeryd and C. Cercignani [2], L. Arkeryd, N. B. Maslova [3], L. Arkeryd, A. Heintz [6]. S. Mischler has recently announced [19] an important result that reflection boundary conditions for renormalised solutions to the Boltzmann equation are satisfied with equality.

The long time asymptotics of solutions to the nonlinear Boltzmann equation and the stabilisation to a uniform Maxwellian for initial boundary problems were considered in papers by L. Desvillettes [8], L. Arkeryd, A. Nouri [4], L. Arkeryd and A. Heintz [6].

For strongly nonlinear problems the convergence to the stationary equilibrium state is shown [5], [6] in the case when the temperature on the boundary is constant and the reflection on the boundary is of the Maxwell diffuse type.

Corresponding questions for the linearised Boltzmann equation are much less involved and in contrast with the strongly nonlinear case are usually considered in L^2 space, see [7], [17], [23] and references therein.

In the above mentioned papers the spatial domain where the flow takes place is assumed to be an open connected set.

Questions concerning boundary value problems for kinetic equations in domains with irregular boundaries were considered by A. Heintz in papers [12]-[15] and by L. Arkeryd and A. Heintz in [6]. In these papers the boundary is supposed to have finite two dimensional Hausdorff measure and satisfies a cone condition.

The behaviour of kinetic equations in domains with "fat" fractal boundaries ("fat" meaning that the Hausdorff dimension is larger than 2) is still unclear, because the classical kinetic boundary conditions use the notion of normal on the boundary which is generally not defined in any reasonable sense for such boundaries.

It is interesting to investigate instead a complementary problem of the behaviour of solutions to kinetic equations in the case when the gas is inside a domain which has "thin" fractal subsets.

In the present paper we consider initial boundary problems for kinetic equations in a domain consisting of two distinct Lipschitz subdomains such that intersection Γ of their boundaries is a fractal set. This "contact set" is a kind of fractal hole that connects two subsets of the flow. We investigate here how large the Hausdorff dimension of this "hole" must be to guarantee that solutions to a kinetic equation in one part influence solutions in another part. In particular we establish conditions when both solutions tend to the same uniform Maxwellian equilibrium function in both parts of the flow as time tends to infinity.

The physical idea behind the constructions in this problem can be illustrated by the following example. Consider a unit cube divided in the middle into two equal parts by a square with a set Γ of holes. One can choose a sequence Γ_n of these holes such that each Γ_n is the n-th step in the classical construction of a fractal Cantor set or of the Sierpinski carpet in the plane. One can pick a sequence of sets Γ_n that converge to a fractal set Γ with arbitrary Hausdorff dimension between zero and two [20].

We suppose that the boundary conditions on the surface of the cube and on both sides of the square are of the Maxwell diffuse type with constant temperature. Results from [6] imply that for each Γ_n a non-stationary renormalised or mild L^1 solution to the initial boundary value problem for the nonlinear Boltzmann equation converges to an equilibrium distribution. It is a spatially uniform Maxwell distribution $const \cdot M(v)$.

A natural question is how fast this convergence can be if with t tending to infinity, the holes Γ_n tend to a fractal set. In one case, for any fixed $\varepsilon > 0$ there exists a time moment T_ε such that the difference between the solution and the equilibrium state becomes less then ε uniformly with respect to n after the time T_ε. One can say in this case that a gas governed by the Boltzmann kinetic equation can penetrate these fractal holes. In another case it might take an infinitely long time to get close to equilibrium when the set Γ_n of holes tends to a fractal set. It means that in this case the gas cannot go through fractal holes.

Such kind of problems were considered for elliptic and parabolic equations in the paper [22] by Zgikov where also a useful in this context notion of p-connected domains was introduced. It is established there that solutions to the heat conduction equation with Neumann boundary conditions tend to a constant limit solution as $t \to \infty$ for arbitrary initial data from L^1. Expressing the result in [22] in physical terms, it is shown there that heat can penetrate holes with the Hausdorff dimension larger than one.

It is *apriori* not evident what kind of behaviour a rarefied gas has with respect to fractal holes in a membrane. Locally on short distances there are almost no collisions between molecules. The fine fractal geometry of holes means that this local behaviour of the gas governs its interaction with the boundary close to the holes.

Physical reasons imply that such a flow can interact via small holes only if their Hausdorff dimension is equal to two or equivalently, if their area is nonzero.

A mathematical answer to this question in two particular cases is the main result of the present paper. It differs from what intuition says and also differs from the case with the heat conduction. The behaviour depends on the regularity of the initial data and the corresponding regularity of solutions. Strong L^2 solutions to the linearised Boltzmann equation converge to a global spatially uniform Maxwellian if the holes have Hausdorff dimension larger than one. Renormalised L^1 solutions to the nonlinear Boltzmann equation converge to such a global equilibrium only if the set of holes has Hausdorff dimension equal to two.

Our analysis uses results from [22] on p - connected domains but their application to kinetic equations depends on specific issues.

2 Equations and boundary conditions.

We consider a set Ω from R^3 consisting of two disjoint open domains Ω_1 and Ω_2 having Lipschitz boundaries $\partial\Omega_1$ and $\partial\Omega_2$ and also including a "contact" set Γ which belongs to the intersection $\partial\Omega_1 \cap \partial\Omega_2$. Initial boundary value problems for kinetic equations with reflection boundary conditions on $(\partial\Omega_1 \cup \partial\Omega_2)\backslash\Gamma$ are considered. We are interested in how large the contact set Γ should be to let the part of the solution in Ω_1 influence the solution in Ω_2.

Let $f(t,x,v)$ be the mass density distribution function for molecules in the phase space $\Omega \times R^3$ at time $t > 0$, where x is in Ω, and v is in R^3.

One of the equations under analysis is the nonlinear Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \quad t \in (0,T), \quad x \in \Omega, \ v \in R^3. \tag{2.1}$$

The conservation of mass, momentum and energy of molecules in collisions and natural symmetries of collisions imply that [7]

$$\int_{R^3} Q(f,f)\{1, v_i, |v|^2\} dv = 0, \tag{2.2}$$

and the equation

$$Q(m,m) = 0 \tag{2.3}$$

has a unique solution

$$m(t,x,v) = \rho(t,x) \exp\{-a(t,x)|v - u(t,x)|^2\}, \tag{2.4}$$

where $\rho(t,x)$ is the mass density of the gas, $u(t,x)$ is the macroscopic velocity, and $a(t,x)$ is proportional to the mean internal energy density of the gas at the time t in the point x.

The function $M = M(v) = \left(\frac{1}{2\pi}\right)^{3/2} e^{-|v|^2/2}$, is an equilibrium distribution function corresponding to the macroscopic velocity $u(x,t) = 0$ the density $\rho(t,x) = 1$ and constant temperature.

The linearisation $f = M + M^{1/2}\tilde{f}$ of (2.1) gives the linearised Boltzmann equation for the deviation $\tilde{f}$ of the distribution function from the equilibrium M. We use for simplicity the notation f instead of $\tilde{f}$ also in the formulation of the linearised equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = L(f), \quad t \in (0,T), \quad x \in \Omega,\ v \in R^3. \tag{2.5}$$

The collision operators $Q(f,f)$ and $L(f)$ have the following structure:

$$Q(f,f) = \int_{R^3} \int_{|w|=1} B(v - v_*, w)\ f' f'_* - f\, f_*\ dv_*\ dw, \tag{2.6}$$

$$M^{1/2} \cdot L(f) = Q(M^{1/2} f, M) + Q(M^{1/2} f, M) \tag{2.7}$$

with $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$, where $t > 0$, and x, w, v, v', v_*, $v'_* \in R^3$. The velocities after collision v', v'_* are functions of the velocities v, v_* before collision and of $w \in \{w \in R^3 : |w| = 1\}$:

$$v' = v - w\,(w, v - v_*), \qquad v'_* = v_* + w\,(w, v - v_*). \tag{2.8}$$

The kernel $B(|v - v^*|, w)$ in the collision operators $Q(f,f)$ and $L(f)$ is a function depending on the physical model of collisions between molecules. We assume that for $A(z) = \int_{|w|=1} B(|z|, w) dw$ the following estimate is valid: $A(z) < C\,|z|^{2-\varepsilon}$, $0 < \varepsilon \leq 2$. This requirement is satisfied for hard potentials with an angular cut-off [7]. In this case operators $Q(f,f)$ and L can be splitted up to a sum of positive and negative part, gain term and loss term:

$$\begin{aligned} Q(f,f) = Q^+(f,f) - Q^-(f,f), &\qquad Q^-(f,f) = \nu(f)\ \cdot f, \\ Q^+(f,f) &= \int_{R^3} \int_{|w|=1} B(v - v_*, w)\ f'\, f'_*\ dv_*\ dw, \\ \nu(f) &= \int_{R^3} \int_{|w|=1} B(v - v_*, w)\ f_*\ dv_*\, dw, \end{aligned} \tag{2.9}$$

and correspondingly

$$L(f) = K(f) - \lambda(v) \cdot f. \tag{2.10}$$

The following properties for the operator L in $L^2(R^3)$ take place. The operator is defined for functions f such that $\lambda^{1/2} f \in L^2(R^3)$.

$$L\psi_j = 0, \quad j = 1,2,3,4; \quad (\psi_j, \psi_i)_{L_2(R^3)} = \delta_{i,j}; \qquad (2.11)$$

$$\psi_0 = M^{1/2};\ \psi_j = M^{1/2} v_j,\ j = 1,2,3;\ \psi_4 = M^{1/2} 6^{-1/2}(|v|^2 - 3). \qquad (2.12)$$

L is symmetrical and not positive in $L_2(R^3)$. If P_0 is the projection in $L_2(R^3)$ to the subspace builded on ψ_i, $i = 0, ...4$ and P is the projection to the orthogonal subspace then

$$(Lf, f)_{L_2(R^3)} \leq -k\|Pf\|^2_{L_2(R^3)}, \quad k > 0. \qquad (2.13)$$

Projections in $L^2(R^3)$ to ψ_j, $j = 0,1,2,3,4$ are deviations of density, velocity and energy from the their equilibrium values corresponding to $M(v)$. Operator K is symmetric and compact in the space $L_2(R^3)$, $\lambda(v)$ is a continuous function such that $\lambda_0 \leq \lambda(v) < \lambda_1(1 + |v|)$, $\quad \lambda_0, \lambda_1 > 0$.

The initial conditions are

$$f(0, x, v) = F_0(x, v), \quad x \in \Omega,\ v \in R^3, \qquad (2.14)$$

Boundary conditions on $\partial\Omega \backslash \Gamma$ are of the Maxwell diffuse type corresponding to the equilibrium distribution $M(v)$. In the case of the full nonlinear Boltzmann equation (2.1) they are

$$f^+(t, x, v) = R(f) = \sqrt{2\pi} M(v) \int_{v' \cdot n(x) < 0} f^-(t, x, v')\, |v' \cdot n(x)|\, dv', \qquad (2.15)$$
$$t > 0, \quad x \in \partial\Omega \backslash \Gamma,\ v \in R^3.$$

with relation $\int_{v' \cdot n(x) > 0} \sqrt{2\pi} M(v) |v \cdot n(x)| dv = 1$ giving mass conservation for collisions of particles with the boundary.

Boundary conditions of this type corresponding to the linearised equation (2.5) are

$$f^+(t, x, v) = \sqrt{2\pi} M^{1/2}(v) \int_{v' \cdot n(x) < 0} M^{1/2}(v') f^-(t, x, v')\, |v' \cdot n(x)|\, dv', \quad (2.16)$$
$$t > 0, \quad x \in \partial\Omega \backslash \Gamma,\ v \in R^3.$$

The outgoing and ingoing distribution functions f^- and f^+ on the boundary are

$$f^-(t, x, v) = \begin{cases} f(t, x, v), & \text{if } x \in \partial\Omega, \quad v \cdot n(x) < 0 \\ 0, & \text{if } x \in \partial\Omega, \quad v \cdot n(x) \geq 0 \end{cases},$$

$$f^+(t, x, v) = f(t, x, v) - f^-(t, x, v) \qquad \text{if } x \in \partial\Omega \backslash \Gamma\ .$$

Here $n(x)$ denotes the unit inward normal to $\partial\Omega\backslash\Gamma$ at the point x. In the case of Lipshitz domains Ω_1 Ω_2 it exists almost everywhere and a cone condition is satisfied for Ω_1 and Ω_2.

In the case when both directions of the normal $n(x)$ in the point x, are inward with respect to Ω we set two boundary conditions of the type as above at both sides of $\partial\Omega$ in such a point x.

The product of linearised collision operator $L(f)$ with $\lambda(v)^{-1}$ acting on on an L^2 function is an L^2 function. It makes possible considering traces of solutions and boundary conditions almost everywhere on the boundary. The corresponding analysis of existence, uniqueness and stability for solutions in the case of smooth domains is given in [17], [23]. The case with irregular domains was investigated in [12] and further generalised in [14]

The lack of summability of the nonlinear collision term $Q(f,f)$ implies that in the nonlinear case one needs a special analysis of traces. We use below the following notations: $D=(0,T)\times\Omega\times R^3$; $V=\Omega\times R^3$;

$\Sigma^+=(0,T)\times(\partial\Omega\backslash\Gamma)\times R^3$ with $v\cdot n(x)\geq 0$; $\Sigma^-=(0,T)\times(\partial\Omega\backslash\Gamma)\times R^3$, with $v\cdot n(x)\leq 0$; $\mathfrak{r}=(t,x,v)\in D$.

Define forward stay time as $t^-(\mathfrak{r})=inf\,(\{s>0:(t+s,x+sv,v)\in\Sigma^-\})$ and a related quantity $s^-(\mathfrak{r})=min\{T-t,t^-(\mathfrak{r})\}$ with $\mathfrak{r}=(t,x,v)$. Introduce a parametrisation of the distribution function

$$f^{\#}(t,\mathfrak{r}_b)=f(t_b+t,x_b+tv,v),\qquad \mathfrak{r}_b=(t_b,x_b,v)\in D,$$

for $0\leq t<t^-(\mathfrak{r}_b)$, and zero otherwise.

For mild solutions f to the nonlinear Boltzmann equation discussed later, it is convenient [6] to obtain traces $f^{\pm}$ on $\Sigma^{\pm}$ as a limit of mean values along segments of collisionless trajectories inside the cylinder $[0,T]\times\Omega$:

$$f^{\pm}(\mathfrak{r})=\lim_{\varepsilon_0\to 0}\frac{1}{\varepsilon_0}\int_0^{\varepsilon_0}f^{\sharp}(\pm\varepsilon\cdot s^{\mp},\mathfrak{r})d\varepsilon \tag{2.17}$$

The analysis of an integral form of the nonlinear Boltzmann equation [6] shows that such traces of solution satisfy boundary conditions with inequality

$$f^+\geq Rf^-. \tag{2.18}$$

Equality takes place if the collision term $Q(f,f)$ is integrable.

Definition 2.1 *$f\in L^1(D)$ is a mild solution of the problem (2.1), (2.14), if f has the following properties: $f\geq 0$, $Q^{\pm}(f,f)^{\sharp}\in L^1([0,s^-])$ and*

$$f^{\sharp}(\tau_2,\mathfrak{r})-f^{\sharp}(\tau_1,\mathfrak{r})=\int_{\tau_1}^{\tau_2}Q(f,f)^{\sharp}(\tau,\mathfrak{r})d\tau \tag{2.19}$$

for $0\leq\tau_1,\tau_2\leq s^-(\mathfrak{r})$ for σ - almost all $\mathfrak{r}\in\Sigma^+$ and for $\mathcal{L}^{2n}$ - almost all $(x,v)\in V$, where $\mathfrak{r}=(0,x,v)$. The traces 2.17 of the solution f exist and satisfy (2.18) for σ - almost all $\mathfrak{r}\in\Sigma$.

Exponential, renormalised solutions and solutions in iterated integral form [3] are defined similarly. The equivalence relations for this case are proved in [14] and use arguments from [9], [3].

3 Notion of p-connected domains.

Studying the questions we have declared in the Introduction needs adequate analytical tools for characterisation of geometrical those properties of the joint set Γ that influence the asymptotic properties of solutions.

In the context of potential theory and elliptic equations the notion of capacity corresponding to a given equation or a given class of functions serves as a tool that helps to classify such subsets that influence the behaviour of solutions [1], [18].

In the paper [22] a useful notion of p-connected domains was introduced for purposes similar to ours. Let Ω be a bounded domain and $W^{1,p}(\Omega)$ be a closure of $C^\infty(\bar{\Omega}) = \{u|_{\bar{\Omega}}, u \in C^\infty(R^3)\}$ in the Sobolev norm $\int_\Omega (|u|^p + |\nabla u|^p)^{1/p} dx$.

Let $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_1 are disjoint domains and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. Let $W^{1,p}(\Omega)$ be a closure in the Sobolev $W^{1,p}$ norm of the set of functions u,on $\bar{\Omega}$ such that $u \in C^\infty(R^3)$, $i = 1, 2$.

Definition 3.1 *Two disjoint domains Ω_1 and Ω_2 are p-connected (through the contact set Γ) if*

$$u \in W^{1,p}(\Omega), \quad \nabla u = 0 \quad \text{a.e. on} \quad \Omega \Longrightarrow \quad u = const \quad \text{a.e. on} \quad \Omega.$$

The following useful criteria for p-connectedness were proved in [22]. They use the p-capacity $c_p(\Gamma)$ of the contact set Γ with respect to a sphere B such that $\Gamma \subset B$.

$$c_p(\Gamma) = \inf\left\{ \int_B |\nabla\phi|^p dx, \quad \phi \in C_0^\infty(B), \quad \phi = 1 \quad \text{near} \quad \Gamma \right\} \tag{3.1}$$

We use below the following result from [22].

Theorem 3.1 *Let Ω_1 and Ω_2 be Lipschitz domains. Ω_1 and Ω_2 are p-connected if and only if the p -capacity $c_p(\Gamma)$ of the contact set Γ is not zero.*

A well known connection between Hausdorff measure H^α and p-capacity implies that $\inf(p)$ such that Ω_1 and Ω_2 are p-connected is equal to $3 - \dim(\Gamma)$. In the case when $p = 1$ Ω_1 and Ω_2 are 1-connected if and only if $H^2(\Gamma) > 0$.

One can find such and similar results and useful references in [1], [18].

4 Convergence to equilibrium. Linear case.

We consider here the long time asymptotic of solutions to the linearised Boltzmann equation on a set Ω consisting of two disjoint Lipschitz parts Ω_1 and Ω_2 and the contact set Γ.

We formulate results in terms using Hausdorff dimension instead of capacity because it is geometrically more tractable. We denote by $\dim(\Gamma)$ the Hausdorff dimension of Γ.

Theorem 4.1 *Let $F_0 \in L^2(\Omega \times R^3)$.*

Then there is a unique solution $f(t,x,v) \in C((0,\infty), L^2(\Omega \times R^3)$ to the problem (2.5), (2.14), (2.16).

If the contact set Γ between Ω_1 and Ω_2 has the Hausdorff dimension $\dim(\Gamma)$ *larger then 1 then $f(t,x,v)$ tends to $CM^{1/2}(v)$ in $L^2(\Omega \times R^3)$ when $t \to \infty$, where $C = \int_{\Omega_1 \cup \Omega_2} F_0(x,v) M^{1/2} dxdv$ is the initial deviation of the mass of the gas in the whole Ω.*

If $dim(\Gamma) < 1$, then the limit stationary states in Ω_1 and Ω_2 can be different.

Proof. A proof for existence of a unique weak solution f in $L^2(\Omega \times R^3)$ for an open connected set Ω with regular boundary is given in [23], [11], [17]. The case when the set Ω is open, connected, and has irregular boundary $\partial\Omega$ with $\dim(\partial\Omega) = 2$ and satisfying a cone condition was studied [12], [13].

The existence part of the proof in [12] is valid in the present case. It is based on the statement that the operator $-v\nabla_x f + Lf$ with boundary conditions (2.16) is dissipative is the space $L^2(\Omega \times R^3)$ and generates a strongly continuous semigroup of operators which in turn gives a weak solution to the initial boundary value problem. The integral form of the problem gives the regularity properties of solutions.

The asymptotic properties of solutions when $t \to \infty$ depend on the spectrum of operator $-v\nabla_x f + Lf$ with the corresponding boundary conditions. The case with smooth boundaries is classical, see [17], [11], [23]. The case with irregular domains was studied in [12]. There is a a half plane Re $z > -r_0$, $r_0 > 0$ in the complex z - plane such that there are no eigenvalues to $-v\nabla_x f + Lf$ except zero. The asymptotic of non-stationary solutions depends on the multiplicity of this zero eigenvalue.

Multiplying the equation for the corresponding eigensolutions $f(x,v)$

$$-v\nabla_x f + Lf = 0 \tag{4.1}$$

by f, integrating over $\Omega \times R^3$ by parts and taking into account boundary conditions and (2.13) one gets that $Pf = 0$ and therefore $f = \sum_{i=0}^{4} a_i(x)\psi_i(v)$ with $a_i \in L_2(\Omega)$. In the linearised problems f having this form is a substitution for the local Maxwellian distribution in the nonlinear case.

The boundary conditions imply that $a_i = 0$ on $\partial\Omega\backslash\Gamma$ for $i = 1, 2, 3, 4$. The linear collision operator L acting on f gives $Lf = 0$, and (4.1) implies that

$$\|\lambda^{-1/2} \cdot v\nabla_x \sum_{i=0}^{4} a_i(x)\psi_i(v)\|^2_{L_2(\Omega\times R^3)} = 0$$

Therefore $\dfrac{\partial a_i(x)}{\partial x_m} \in L_2(\Omega)$, and for all $v \in R^3$

$$\sum_{i,j=0}^{4} \sum_{l,m=1}^{3} \psi_i\psi_j v_l v_m \int_\Omega \frac{\partial a_i(x)}{\partial x_l} \cdot \frac{\partial a_j(x)}{\partial x_m} dx = 0 \tag{4.2}$$

Setting $v = |v|e_i$ and $v = |v|(e_i + e_j)$ into the last equation with e_i - vectors of the orthonormalised basis in R^3, we get

$$\frac{\partial a_0(x)}{\partial x_m} = \frac{\partial a_4(x)}{\partial x_m} = 0, \quad \frac{\partial a_i(x)}{\partial x_m} + \frac{\partial a_m(x)}{\partial x_i} \quad (i, m = 1, 2, 3) \tag{4.3}$$

almost everywhere in Ω.

The last equations together with boundary conditions relations $a_i = 0$ on $\partial\Omega\backslash\Gamma$ for $i = 1, 2, 3, 4$ imply that $a_4 = a_1 = a_2 = a_3 = 0$ almost everywhere in Ω.

If the contact set Γ has Hausdorff dimension larger than 1 then Ω_1 and Ω_2 are 2-connected [22].

Then a_0 must be constant in the whole Ω and the zero eigenvalue is simple with the eigenfunction $a_0\psi_0$. For the non-stationary problem it implies that non-stationary solutions converge to $a_0\psi_0$ when $t \to \infty$.

The mass conservation laws for collisions in the volume and for reflections from the boundary imply that $a_0|\Omega| = \int_{\Omega\times R^3} F_0(x, v)dxdv$.

If the contact set Γ has the Hausdorff dimension less then 1 then Ω_1 and Ω_2 are not 2-connected and a_0 can attain two different constant values C_1 and C_2 in Ω_1 and Ω_2.

Then the limit solution is $C_1\psi_0$ in Ω_1 and $C_2\psi_0$ in Ω_2.

5 Convergence to equilibrium. Non-linear case.

In this section we study the asymptotic behaviour of non-stationary solutions to the nonlinear Boltzmann equation on a set $\Omega = \Omega_1 \cup \Omega_2$ consisting disjoint domains Ω_1 and Ω_2 with Lipschitz boundaries and having a contact set $\Gamma = \partial\Omega_1 \cap \partial\Omega_1$.

We recall the following existence result from [6] where it is proved for a domain Ω with even more general boundaries.

Theorem 5.1 *Assume that* $F_0(1+v^2) \in L^1(\Omega \times R^3)$, $F_0 \ln F_0 \in L^1(\Omega \times R^3)$, $F_0 \geq 0$. *Then there exists a mild solution to (2.1), (2.14), (2.15) in the sense of Definition 2.19 satisfying*

$$f \in C\left([0,T], L^1(\Omega \times R^n)\right), \quad f \geq 0, \quad \int_V f(t)dvdx = \int_V f_0 dvdx,$$

$$(1+v^2)\, f^{\pm} \in L^1(\Sigma^{\pm}), \tag{5.1}$$

and uniformly with respect to t *satisfying*

$$\sup_{t \leq T} \left[\|f \ln f\|_{L^1(V)} + \|(1+v^2)f\|_{L^1(V)}\right] + \|e(f)\|_{L^1(D)} \leq C. \tag{5.2}$$

Let $\rho_0 = \int_{\Omega \times R^3} F_0\, dxdv/|\Omega|$ be the initial mean density of the gas in Ω. We assume here that the kernel B in the collision operator is nowhere vanishing. The main result of the section is the following theorem.

Theorem 5.2 *Let* f *be a solution in the sense of Theorem 5.1.*

If $\dim(\Gamma) = 2$ *for the contact set* Γ *between* Ω_1 *and* Ω_2 *then* f *converges strongly in* $L^1(\Omega \times R^3)$ *to* $\rho_0 M(v)$.

If $\dim(\Gamma) < 2$ *then* f *converges strongly in* $L_1(\Omega \times R^3)$ *to a function that is equal to* $\rho_1 M(v)$ *in* Ω_1 *and* $\rho_2 M(v)$ *in* Ω_2 *with some in general different constants* ρ_1 *and* ρ_2 *such that* $\rho_1|\Omega_1| + \rho_2|\Omega_1| = \rho_0|\Omega|$.

Proof. It is enough to show that for every sequence t_j tending to infinity there is a subsequence t_{j_k} such that $f_{j_k}(t,x,v) = f(t + t_{j_k}, x, v)$ converges in $L^1(D)$ to cM for all $T > 0$.

The weak $L^1(D)$ convergence of a subsequence follows from (5.2). The argumentation as in [5] or [16] gives that the limit is of strong L^1 type and that it satisfies the Boltzmann equation in the mild sense. The limit is a local Maxwellian $m(t,x,v)$,

$$m(t,x,v) = \rho(t,x)\exp\{-a(t,x)|v - u(t,x)|\} \tag{5.3}$$

$$m(t,x,v)(1+|v|^2) \in L_1(\Omega \times R^3), \tag{5.4}$$

since the collision kernel is nowhere vanishing.

The function $m(t,x,v)$ satisfies the equation

$$\frac{\partial}{\partial t} m + v\nabla_x m = 0$$

because $Q(m,m) = 0$.

Having boundary conditions satisfied with inequality

$$f_{j_k}^{+} \geq R f_{j_k}^{-} \tag{5.5}$$

on the boundary, we get that the limit m also satisfies the same inequality

$$m^{+} \geq R m^{-} \tag{5.6}$$

The collision term $Q(m, m)$ being zero for m, implies that the inflow of mass on $\partial\Omega$ over the time interval $[0, T]$ is equal to the corresponding outflow, and that m satisfies boundary conditions with equality for a.a. (t, x) on $[0, T] \times \partial\Omega$, [4], [6].

The uniqueness up to a multiplier dependent only on (t, x) for the Maxwellian distribution satisfying the boundary conditions (2.15) implies that the velocity u is equal to zero $u(t, x) = 0$ and the temperature corresponding to the Maxwellian m is constant and is equal to the temperature of the boundary corresponding to the distribution $M(v)$.

Functions $\rho(x, t)$, $u(x, t)$ and $a(x, t)$ satisfy the following classical macroscopic conservation equations [7]

$$\frac{\partial \rho}{\partial t} + u \nabla_x \rho = 0$$

$$2\nabla_x \rho + 2\rho a \frac{\partial u}{\partial t} + 2\rho a\, u \nabla_x u = 0$$

that together with $u = 0$ and $a = const$ imply that $\nabla_x \rho = 0$ almost everywhere in Ω. We have that $\rho \in L^1(\Omega)$. The further properties of ρ depend on $\dim(\Gamma)$. The density ρ is constant in the whole Ω if $\dim(\Gamma) = 2$. The mass conservation for the Boltzmann equation and the boundary conditions give the normalisation for m.

When $\dim(\Gamma) < 2$ domains Ω_1 and Ω_2 are not 1-connected [22]. It means that densities can be different constants in Ω_1 and Ω_2 in the limit stationary solution.

6 Conclusions

The asymptotic with $t \to \infty$ behaviour of solutions to the linearised and the nonlinear Boltzmann equations on a set consisting of two disjoint domains and a fractal contact set is investigated. It depends essentially on regularity of solutions and on the the Hausdorff dimension of the contact set that can be interpreted as holes in a boundary surface between these domains.

The relatively simple analysis of the problem here is based on the independent of the boundary conditions convergence of non-stationary solutions to a locally equilibrium distribution having simple form and smooth with respect to the velocity variable v. The advantage of that is that the space dependence of this limit is governed by the classical macroscopic equations for density, temperature, and velocity of the gas. These equations together with the boundary conditions imply that the gradient $\nabla_x \rho$ of the density is equal to zero almost everywhere, temperature is constant and velocity is equal to zero. Different in linear and nonlinear case integrability properties of these macroscopic variables cause the difference in asymptotic properties of solutions.

A difficult open problem is the analysis of similar questions in the case when the temperature of the boundary is not constant and a possible stationary solution has more complicated structure.

Acknowledgements.

The author wants to thank Leif Arkeryd for his interest to the problem considered in this work and for helpful discussions. The research was supported by the TMR-Network grant no. ERB FMRX CT97 0157.

References

[1] R. Adams, L. Hedberg, *Function spaces and potential theory* , Springer, 1996.

[2] L. Arkeryd, C. Cercignani, *A Global Existence Theorem for the Initial-Boundary-Value Problem for the Boltzmann equation when the boundaries are not isothermal*, Arch. Rat. Mech. Anal. 125, (1993), 271-287.

[3] L. Arkeryd, N. B. Maslova, *Boundary value problems and diffuse reflection*, J. Stat. Phys., 77 (1994), 1051-1077.

[4] L. Arkeryd, A. Nouri, *Asymptotics of the Boltzmann equation with diffuse reflection boundary conditions,* Monatsheft für Mathematik, 123 (1997), 285-298. Nice, 1994.

[5] L. Arkeryd, *Some examples of NSA methods in kinetic theory* , Lecture Notes in Mathematics, 1551, Springer Verlag, 1993.

[6] L. Arkeryd, A. Heintz *On the solvability and asymptotics of the Boltzmann equation in irregular domains.* Comm. Partial Differential Equations 22 (1997), no. 11-12, 2129–2152.

[7] C. Cercignani, R. Illner, M. Pulvirenti, *The mathematical theory of dilute gases.* Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994. 347 pp.

[8] L. Desvillettes, *Convergence to equilibrium in large time for Boltzmann and B.G.K. equations*, Arch. Rat. Mech. Anal. 110, (1990), 73-91.

[9] R. J. DiPerna, P. L. Lions, *On the Cauchy problem for Boltzmann equation: Global existence and weak stability*, Ann Math., 130 (1989), 321-366.

[10] K. Hamdache, *Weak solutions of the Boltzmann equations* , Arch. Rat. Mech. Anal. 119, (1992), 309-353.

[11] A. Heintz (Geints) On the solution of initial - boundary value problems for the nonlinear Boltzmann equation in a bounded domain. in Rarefied Gas Aerodynamics, issue 11. Leningrad (1983), 166 - 174. (in Russian)

[12] A. Heintz, *Boundary value problems for nonlinear Boltzmann equation in domains with irregular boundaries*, Ph.D.Thesis (1986), Leningrad State University.

[13] A. Heintz, *On solutions of boundary value problems for the Boltzmann equation in domains with irregular boundaries*, in Statistical Mechanics , Numerical Methods in Kinetic Theory of Gases, Novosibirsk, 1986, 148-154.

[14] A. Heintz, *Initial boundary value problems in irregular domains for nonlinear kinetic equations of Boltzmann type.* Transport theory and Statistical Physics, 28, (1999), no 2, 30 pp.

[15] A. Heintz, *Initial boundary value problems for the Enskog equation in irregular domains.* J. of Statistical Physics, 90 (1998), no. 3-4, 663–695.

[16] P.L. Lions, *Compactness in Boltzmann equation via Fourier integral operators and applications*, J. Math. Kyoto Univ. 34 (1994), 391-427.

[17] N.B. Maslova, *Nonlinear evolution equations. Kinetic approach*, World Scientific, Singapore, 1993.

[18] V.G. Mazja *Sobolev spaces.* Springer-Verlag, Berlin-New York, 1985, 486 pp.

[19] S. Mischler *On weak-weak convergence and applications to the initial boundary value problem for kinetic equation.* Preprint, 1999.

[20] F. Morgan, *Geometric measure theory. A beginner's guide. Second edition.* Academic Press, Inc., San Diego, CA, 1995, 175 pp.

[21] R. Pettersson, *On weak and strong convergence to equilibrium for solutions to the linear Boltzmann equation* , J. Stat. Phys. 72, (1993), 355-380.

[22] V.V. Zgikov *Connectedness and averaging. Examples of fractal conductivity.* Matem. Sbornik, 187 (1996), no. 8, 3–40.

[23] S. Ukai, *Solutions of the Boltzmann equation*, in Patterns and Waves-Qualitative Analysis of Nonlinear Differential equations, H.Fujita, J.L.Lions, Papanicolau, and H.B.Keller, eds., vol. 18, Kinokuniya, 1986, 37-96.