Monge-Ampère currents over pseudoconcave spaces

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1 Introduction.

This paper is an attempt to understand growth of Monge-Ampère masses along pseudoconcave directions in a complex manifold.

This problem arises in differential geometry when studying compactification of complete Kähler manifolds under certain curvature conditions (see *e.g.* articles of Mok-Zhong [27], Nadel-Tsuji [28], Siu-Yau [35]).

In complex analysis, bounds on Monge-Ampère masses of a closed positive current near a pluripolar set implies an extension of this current through the set (see *e.g.* works of El Mir [26], Sibony [33], Skoda [37]).

In this direction, the L^2-Riemann-Roch inequality of Nadel-Tsuji (see [28]) implies that a complete Kähler Hodge metric on a pseudoconcave manifold is of finite volume.

Our first result is obtained in the framework of pluripotential theory. Let M be a complex manifold, $\dim M = n \geq 2$, and let ω be a closed positive $(1,1)-$current. Assume that ω admits local locally bounded potentials. To each open subset U of M is associated an extremal admissible function φ^*, which is defined on a suitable pseudoconvex hull U_1 of U. It satisfies the Monge-Ampère equation $(\omega + dd^c\varphi^*)^n = 0$ on $U_1 \setminus \bar{U}$, as current of order zero. We deduce the following comparison between measures (we work in the relative topology of U_1).

Theorem. In the above situation, let X be a connected component of $U_1 \setminus \bar{U}$ which has a compact boundary. Assume that $\{\varphi^* \leq c\} \cap X$ is relatively compact in U_1 for any $c \in \mathbb{R}$. Then

$$\int_{\bar{X}} \omega^n \leq \int_{\partial X} (\omega + dd^c\varphi^*)^n < +\infty \, .$$

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Here, to check the hypothesis we restrict ourself to domains on projective manifolds. It allows us to obtain a complex analytic treatment of the problem. Related methods appear already in [12, 29]. For a more differential-geometric point of view, we refer to papers cited below.

We obtain the following applications. Let V be a projective manifold, $\dim V = n \geq 2$, and let H be a complex hypersurface in V such that $V \setminus H$ is pseudoconcave in the sense of Andreotti (see Definition 7.1). Let $X \subset\subset M$ open neighbourhoods of H. Then the following Hartogs' theorem for currents holds.

Theorem. In the above situation, let ω a closed positive $(1,1)-$current defined on $M \setminus H$ which admits local locally bounded potentials. Then

$$\int_{\bar{X}\setminus H} \omega^n < +\infty ,$$

and ω^k extends through H as a closed positive currents, $k = 1, \ldots, n$.

If $X = V$ and ω is a smooth complete Hodge Kähler metric on $V \setminus H$, then the above result is a variation of the L^2-Riemann-Roch inequality of Nadel-Tsuji (see [28]). In general, the difficulty in establishing the above finiteness estimate is that neither pseudoconcavity nor completeness assumptions are made on M itself. To overreach it, we used line bundle convexity and algebraic properties of pseudoconcave spaces (see [16]).

Next, we try to derive similar estimate for more singular closed positive currents. We are able to work with currents (on spread domains W over V) such as pullback $\psi^*\omega_{FS}$, where $\psi : W \to \mathbb{P}^N$ is a meromorphic map from W to a projective space and ω_{FS} is a Fubiny-Study form on it.

Our technique is to produce, by mean of the L^2 theory of ideals (see Skoda [36]), positive currents ω_k linked to $\psi^*\omega_{FS}$ but with Lelong number globally shifted by $-k$ (see Demailly [13] for other methods in the compact case). These currents are pluricomplete (see Def. 6.7). This is a convexity condition on ω_k and A_k, the non-smooth locus of ω_k, which allows to work on $M = W \setminus A_k$.

As an application, we deduce that global Hartogs' extension phenomena occur in projective manifolds for meromorphic maps.

Theorem. Let U be an open subset of the projective manifold V such that $V \setminus \bar{U}$ is a pseudoconcave domain in the sense of Andreotti. Assume $\overset{\circ}{\bar{U}} = U$. Then any meromorphic map $\psi : W(\partial U) \to \mathbb{P}^N$ define on a neighbourhood of ∂U extends as a meromorphic map to U.

These results give some understanding of global and compact singularities for meromorphic maps or currents. We note that there exists hypersurfaces H as above which may not be blow down to lower dimensional spaces. Hence, even for meromorphic maps, the situation may not be reduced to local extension results similar to results of Ivashkovich [23]. Moreover, note that non compact complex singularities of strict positive dimension are already local essential singularities for Monge-Ampère currents.

Many interesting points are not yet explored or stated. In particular, in the inequality above, an estimation of the measure $(\omega + dd^c\varphi^*)^n \ 1_{\partial X}$ is needed and also the extension properties of currents through a divisor H with pseudoconcave complement. We will return to these problems later.

The paper is organised as follow.

In section 2, we recall background on pluripotential theory and Monge-Ampère currents following works of Bedford-Taylor [7, 9].

In section 3, we introduce the class $P_\omega(M)$ associated to a closed positive $(1,1)-$current ω on M (see also [8, 21]). We define Monge-Ampère currents such as $d\varphi \wedge d^c\varphi \wedge (\omega + dd^c\varphi)^k$ and state a basic comparison lemma.

In section 4, we introduce various almost well known pseudoconvex hulls and study elementary properties of them. In particular, we recall that the subset where a family $\mathcal{F} \subset P_\omega(M)$ is locally bounded from above is an open locally pseudoconvex subset in M. For an open subset U of M, we defined two kind of extremal functions on a suitable pseudoconvex hull of U. We study their properties by mean of techniques in pluripotential theory (see [6, 7, 9]). When ω is a Chern current of a line bundle $E \to M$, we give an interpretation of these extremal functions in term of hulls of holomorphy in E^*.

In section 5, we prove the above inequality and make some comments.

In section 6, we study a hypersurface Z in a pseudoconvex spread domain W over a projective manifold $V = (V, \mathcal{O}(1))$ by mean of L^2 techniques. We prove the existence of $l_1 \in \mathbb{N}$ (which depends only of the canonical bundle of V) such that $\mathcal{O}(kl_1) \otimes [Z]$ is spanned by its global sections away of $\{p \in W \ : \nu_p(Z) \geq k+1\}$, where $\nu_p(Z)$ is the multiplicity of Z at p. We introduce then the notion of pluricomplete currents and, essentially, construct the currents ω_k above.

The last section is devoted to applications of the preceding results for Hartogs' extension properties in projective manifolds for meromorphic maps and Monge-Ampère currents.

The starting point of this paper is the classical result that a hull of holomorphy in the trivial bundle over a domain in $\mathbb{C}^n$ is a geometric counterpart of a complex Monge-Ampère equation in that domain (see Bremermann [10]).

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2 Quasi-continuous functions and the class $\mathcal{G}(M)$.

We recall some definitions which appear in [7, 9].
Let M be a complex manifold, $\dim M = n$. If an open subset U of M is biholomorphic to an open subset $U' \subset \mathbb{C}^n$, via a biholomorphic map $h : U \to U'$, then, for any subset E of U, the relative capacity $C(E, U)$ is well defined and equal to $C(h(E), U')$. A subset F in M is pluripolar if F read in any open chart is pluripolar.

Definition 2.1 *Let Ω be an open subset of $\mathbb{C}^n$.*

(1) A function $f : \Omega \to \{-\infty, +\infty\}$ is said to be quasi-continuous if, for any $\epsilon > 0$, there exists an open subset $\mathcal{O}$ of Ω with $C(\mathcal{O}, \Omega) < \epsilon$ s.t. f is continuous on $\Omega \setminus \mathcal{O}$.

(2) A function $f : M \to \{-\infty, +\infty\}$ is said to be quasi-continuous if f read in any open chart of M is quasi-continuous.

(3) A sequence $\{f_j\}_{j\in\mathbb{N}}$ of functions is said to converge quasi-everywhere to a function f on M, if there exists a pluripolar set F in M such that $f_j \to f$ on $M \setminus F$.

(4) A sequence $\{f_j\}_{j\in\mathbb{N}}$ of Borel functions on Ω is said to converge quasi-uniformly to f, if it is uniformly bounded, it converges almost everywhere to f, and, for any $\epsilon > 0$, there exits an open subset $\mathcal{O}$ of Ω such that $C(\mathcal{O}, \Omega) \leq \epsilon$ and $f_j \to f$ uniformly on $\Omega \setminus \mathcal{O}$.

(5) A sequence $\{f_j\}_{j\in\mathbb{N}}$ of Borel functions on M is said to converge locally quasi-uniformly to f, if there exits a covering of M by coordinate charts $\{U_\alpha\}$, such that, the sequence $\{f_j\}_{j\in\mathbb{N}}$, read in these coordinate charts, converge quasi-uniformly to f.

Quasi-continuous functions form an algebra which, according to [6], Theorem 3.5, contains plurisubharmonic functions. Note that if f is quasi-continuous on M, then for any continuous function $\chi : \mathbb{R} \to \mathbb{R}$, $\chi(f)$ is quasi-continuous on M. The following was shown in [6].

Lemma 2.2 *Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence of plurisubharmonic functions which converge monotonically almost everywhere to the plurisubharmonic function φ. Then the convergence is locally quasi-uniform.*

Definition 2.3 ([9]) *We denote by $\mathcal{G}(M)$ the class of currents on M which locally are represented by currents in the exterior algebra generated by*

- *smooth forms,*
- *locally bounded plurisubharmonic functions,*
- *du, $d^c u$, $dd^c u$ where u is a locally bounded plurisubharmonic function.*

That is of the form

$$T = \chi P(\tau_1, \dots , \tau_l)\delta u_1 \wedge \dots \wedge \delta u_r \wedge dd^c w_1 \wedge \dots \wedge dd^c w_r \tag{2.1}$$

where χ is a smooth form, P is a polynomial, the τ_i, u_i, w_i are locally bounded plurisubharmonic functions, and each occurrence of δ is either d or d^c.

Note, that these currents, for non smooth functions, are defined as limit of smooth ones, where each occurrence of the plurisubharmonic functions is replaced by smooth plurisubharmonic functions which converge, monotonically, to the given plurisubharmonic functions. This definition is justified by the fact that, for smooth function (and so, a posteriori, for non smooth ones) the above currents put small mass on set of small capacity (see [9], Lemma 2.2).

It follows from [9], that currents in the class $\mathcal{G}(M)$ are of order 0, that the expression (2.1) is continuous under monotone, uniformly bounded, almost everywhere convergence in the plurisubharmonic functions τ_i, u_i, w_i and in the weak topology on the space of currents of order 0.
We state in a weak form Theorem 2.6 of [9].

Theorem 2.4 *Let $T_j, j \in \mathbb{N}$ and T_∞ be currents in $\mathcal{G}(M)$ which are locally of the form*

$$\sigma_0^{(j)}\delta\sigma_1^{(j)} \wedge \dots \wedge \delta\sigma_q^{(j)} \wedge dd^c\sigma_{q+1}^{(j)} \wedge \dots \wedge dd^c\sigma_r^{(j)} \tag{2.2}$$

where, each occurrence of δ denotes either the operator d or the operator d^c, $\sigma_k^{(j)} = u_k^{(j)} - v_k^{(j)}$, the $u_k^{(j)}$ and $v_k^{(j)}$, $j \in \mathbb{N} \cup \{\infty\}$, are locally bounded plurisubharmonic functions such that

$$u_k^{(j)} \underset{k\to+\infty}{\to} u_\infty^{(j)} , \tag{2.3}$$

$$v_k^{(j)} \underset{k\to+\infty}{\to} v_\infty^{(j)} , \tag{2.4}$$

and the convergence is monotone in k. If $\{\varphi_j\}_{j\in\mathbb{N}}$ is a sequence of quasi-continuous functions which converges locally quasi-uniformly to the quasi-continuous function φ then

$$\lim_{j\to+\infty}\varphi_j T_j = \varphi T_\infty$$

as currents of order 0.

3 The class $P_\omega(M)$.

Let M be a complex manifold, $\dim M = n$. Let ω be a closed positive $(1,1)-$current on M. It is known (see [20], p.387) that ω admits local potentials. For an open subset X biholomorphic to an open Euclidean ball in $\mathbb{C}^n$, there exists $a \in \mathrm{PSH}(X)$ such that $dd^c a = \omega$. We will refer to this situation by saying that a is a local potential for ω on X. In this paper, we make the following assumption.

$$\text{The current } \omega \text{ admits local potentials which are locally bounded.} \qquad (3.1)$$

Hence we assume that there exists $a \in \mathrm{PSH}(X) \cap L^\infty(X, \mathrm{loc})$ such that $dd^c a = \omega$.

Definition 3.1 *A measurable function $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ belongs to $P_\omega(M)$ if there exists an open covering $\mathcal{W} = \{W_i\}_{i\in A}$ by subsets biholomorphic to Euclidean balls in $\mathbb{C}^n$, and local potentials $a_i \in \mathrm{PSH}(W_i) \cap L^\infty(W_i, \mathrm{loc})$, such that $a_i + \varphi$ is plurisubharmonic, i.e. $a_i + \varphi$ is upper semi-continuous and its restriction to any complex line in W_i is a subharmonic function.*

We note that $\varphi \in P_\omega(M)$ if and only if, for any open subset W biholomorphic to an open Euclidean ball, any local potential $a \in \mathrm{PSH}(W) \cap L^\infty(W, \mathrm{loc})$, the function $a + \varphi$ is plurisubharmonic. For, two local potentials for ω differ locally on there common definition set, by a pluriharmonic function. Moreover, any function which belong to $P_\omega(M)$ is quasi-continuous. For if a is a local potential for ω on an open set U, and $\varphi \in P_\omega(U)$, then a and $a + \varphi$ are plurisubharmonic. Hence $\varphi = a + \varphi - a$ is quasi-continuous.

Upper regularization with respect to ω.

Definition 3.2 *(1) A function $\varphi : M \to [-\infty, +\infty[$ will be said upper semi-continuous with respect to ω, if, for any $p \in M$, there exists an open neighbourhood W of p, a local locally bounded potential $a \in \mathrm{PSH}(W) \cap L^\infty(W, \mathrm{loc})$*

for ω, such that $a+\varphi$ is upper semicontinuous on W. A function h on M will be said lower semicontinuous with respect to ω if $-h$ is upper semicontinuous with respect to ω.

(2) Let $\varphi : M \to [-\infty, +\infty[$ be a function which is locally bounded from above. Define φ^*, the upper regularization of φ with respect to ω, as follow. If a is a local locally bounded potential for ω on an open subset W, then

$$\varphi^* = (a+\varphi)^* - a \tag{3.2}$$

where $(a+\varphi)^*$ stands for the usual upper regularization of $a+\varphi$ on W in the classical topology $(a+\varphi)^*(p) = \limsup_{z\to p}(a+\varphi)(z)$.

We note that these definitions are well posed since two local potentials for ω differ locally by a pluriharmonic function.

With this notion of upper regularization w.r.t ω, we will have, *e.g.*, classical stability properties of $P_\omega(M)$ with respect to upper envelope (see Lemma 4.5). Note that Choquet's lemma is valid.

Lemma 3.3 *Let $\{u_\alpha\}_{\alpha\in A}$ be a family of real valued functions on a complex manifold M. Assume that $a+u_\alpha$ is upper semicontinuous for any local potential a of ω and any $\alpha \in A$. Assume this family is locally bounded from above on M. Then there exist a countable subset $B \subset A$ such that $(\sup_{\alpha\in A} u_\alpha)^* = (\sup_{\alpha\in B} u_\alpha)^*$ (upper regularization w.r.t. ω).*

Proof. Let $\{W_i\}_{i\in\mathbb{N}}$ be a countable open cover of M, such that on W_i, ω admits a local potential $a_i \in \mathrm{PSH}(W_i) \cap L^\infty(W_i, \mathrm{loc})$. Applying Choquet's lemma (see [25, 24]) to each family of upper semicontinuous functions $\{a_i + u_\alpha\}_{\alpha\in A}$ on W_i, there exists B_i, a countable subset of A, such that $(\sup_{\alpha\in A}(a_i+u_\alpha))^* = (\sup_{\alpha\in B_i}(a_i+u_\alpha))^*$. Let $B = \bigcup_{i\in\mathbb{N}} B_i$. Then B is countable and $(\sup_{\alpha\in B}(a+u_\alpha))^* - a = (\sup_{\alpha\in A}(a+u_\alpha))^* - a$ for any local potential for ω. By definition, $(\sup_{\alpha\in A} u_\alpha)^* = (\sup_{\alpha\in B} u_\alpha)^*$.

Definition of Currents of order 0.

Let ω_i, $1 \le i \le r$, be closed positive $(1,1)-$currents which satisfy the condition (3.1). From Theorem 2.4, if $\varphi_i \in P_{\omega_i}(M) \cap L^\infty(M, \mathrm{loc})$ then expression of the form

$$T = \delta\varphi_1 \wedge \ldots \wedge \delta\varphi_k \wedge (\omega_{k+1} + dd^c\varphi_{k+1}) \wedge \ldots \wedge (\omega_r + dd^c\varphi_r), \tag{3.3}$$

where δ is either d or d^c, defined a current which belongs to the class $\mathcal{G}(M)$. T is the unique current which is locally equal to

$$T = \delta\left((a_1+\varphi_1)-a_1\right)\wedge\ldots\wedge\delta\left((a_k+\varphi_k)-a_k\right)\wedge \\ dd^c(a_{k+1}+\varphi_{k+1})\wedge\ldots\wedge dd^c(a_r+\varphi_r), \quad (3.4)$$

where a_i denotes a local locally bounded potential for ω_i, $1\le i\le r$.

For these currents, usual calculus rules are satisfied. We state in particular the following lemma.

Lemma 3.4 *Let $\varphi \in P_\omega(M)\cap L^\infty(M,\mathrm{loc})$, $\chi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. Then for any $\theta\in\mathcal{C}_0^\infty(M)$, the following algebraic identity holds*

$$\int \theta\chi(\varphi)\left(\omega+dd^c\varphi\right)^n = \int\theta\chi(\varphi)\omega^n \\ -\int (d\theta)\chi(\varphi)d^c\varphi P(\varphi) - \int\theta\chi'(\varphi)d\varphi\wedge d^c\varphi P(\varphi), \quad (3.5)$$

where

$$P(\varphi) = \sum_{\substack{\alpha+\beta=n-1\\ \alpha,\beta\ge 0}} \left(\omega+dd^c\varphi\right)^\alpha\omega^\beta. \quad (3.6)$$

Proof. First, it is enough to check the above formula locally. Assume, $\mathrm{supp}\theta \subset\subset B$, where B is the Euclidean unit ball in $\mathbb{C}^n$, $\omega = dd^c a$, with $a\in\mathrm{PSH}(B)\cap L^\infty(B,\mathrm{loc})$, so that $a+\varphi\in\mathrm{PSH}(B)\cap L^\infty(B,\mathrm{loc})$. Let $(a+\varphi)_\epsilon$, a_ϵ, $1>\epsilon>0$, be family of smooth plurisubharmonic functions defined on B, which decrease, as $\epsilon\to 0$, to $a+\varphi$ and a respectively on an open neighbourhood $W\subset\subset B$ of $\mathrm{supp}\theta$.
Let $M = \|(a+\varphi)_1\|_{W,\infty}+\|a_1\|_{W,\infty}+\|a+\varphi\|_{W,\infty}+\|a\|_{W,\infty} < +\infty$.

From [6], Theorem 7.2, for any $\eta>0$, there exists Ω, an open subset of W, such that $C(W,\Omega)<\eta$, and the above convergences are uniform on $W\setminus\Omega$. Define $\psi_\epsilon = (a+\varphi)_\epsilon - a_\epsilon$, then

$$\|\chi(\psi_\epsilon)-\chi(\varphi)\|_{W\setminus\Omega,\infty} \le (\max_{[-M,M]}|\chi'|)\|\psi_\epsilon-\varphi\|_{W\setminus\Omega,\infty} \underset{\epsilon\to 0}{\to} 0\,. \quad (3.7)$$

Since the ψ_ϵ and φ are uniformly bounded on W, for any $\chi\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, $\chi(\psi_\epsilon)$ converge quasi-uniformly on W to $\chi(\varphi)$. But for smooth functions

$$\int\theta\chi(\psi_\epsilon)\left(dd^c(a_\epsilon+\psi_\epsilon)\right)^n = \\ \int\theta\chi(\psi_\epsilon)\left(dd^c a_\epsilon\right)^n + \int\theta\chi(\psi_\epsilon)dd^c\psi_\epsilon P(\psi_\epsilon)\,, \quad (3.8)$$

where

$$P(\psi_\epsilon) = \sum_{\substack{\alpha+\beta=n-1 \\ \alpha,\beta\geq 0}} (dd^c(a_\epsilon + \psi_\epsilon))^\alpha \, (dd^c a_\epsilon)^\beta \,. \tag{3.9}$$

Hence,

$$\begin{aligned} \int \theta\chi(\psi_\epsilon) \, (dd^c(a_\epsilon + \psi_\epsilon))^n &= \int \theta\chi(\psi_\epsilon) \, (dd^c a_\epsilon)^n \\ &- \int (d\theta)\chi(\psi_\epsilon) d^c\psi_\epsilon P(\psi_\epsilon) - \int \theta\chi'(\psi_\epsilon) d\psi_\epsilon d^c\psi_\epsilon P(\psi_\epsilon). \end{aligned} \tag{3.10}$$

Letting $\epsilon \to 0$, we have that $\chi(\psi_\epsilon)$ and $\chi'(\psi_\epsilon)$ converge quasi-uniformly to $\chi(\varphi)$ and $\chi'(\varphi)$ respectively, on W. By definition, $d^c\psi_\epsilon P(\psi_\epsilon)$ converges to $d^c\varphi P(\varphi)$, $(dd^c(a_\epsilon + \psi_\epsilon))^n$ converges to $(\omega + dd^c\varphi)^n$ and $(dd^c a_\epsilon)^n$ converges to ω^n. Since the hypothesis of Theorem 2.4 are satisfied, we obtain the formula (3.5) above.

We state next our basic lemma.

Lemma 3.5 *Let M be a complex manifold, and let X be an open subset of M with compact boundary. Let ω be a closed positive $(1,1)-$current which admits local locally bounded potentials. Let $\varphi \in P_\omega(X) \cap L^\infty(X, \text{loc})$ such that*

(1) there exists a neighbourhood W of ∂X, with $\varphi_{|W\cap X} \geq 0$,

(2) $\limsup_{z\to\partial X} \varphi = 0$,

(3) $\forall p \in X,\ \{\varphi \leq \varphi(p)\} \subset\subset M$.

Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a positive smooth decreasing function. Then

$$+\infty \geq \int_{\bar{X}} \chi(\tilde{\varphi})(\omega + dd^c\tilde{\varphi})^n \geq \int_{\bar{X}} \chi(\tilde{\varphi})\omega^n$$

where $\tilde{\varphi}$ denotes the extension by 0 of φ to M.

Proof. First, we note that $\tilde{\varphi}$ belongs to $P_\omega(M) \cap L^\infty(M, \text{loc})$. For

$$\tilde{\varphi} = \begin{cases} \varphi & z \in X \setminus W \\ \max(\varphi, 0) = \varphi & z \in X \cap W \\ 0 & z \in M \setminus X \end{cases} .$$

Hence, for any local locally bounded potential a of ω on an open charts W', we have

$$a+\tilde{\varphi}=\begin{cases} a+\varphi & z\in(X\setminus W)\cap W' \\ \max(a+\varphi,a)=a+\varphi & z\in(X\cap W)\cap W' \\ a & z\in(M\setminus X)\cap W' \end{cases}$$

which is a plurisubharmonic function in W' (see [24], p.69).
Hence, we will assume that $\varphi\in P_\omega(M)\cap L^\infty(M,\mathrm{loc})$ and that it vanishes on $M\setminus X$. Let W_1 be a relatively compact open neighbourhood of ∂X. Let θ be a smooth positive function with supp$\theta\subset X\cup W_1$, $\theta\equiv 1$ on a neighbourhood of $\bar{X}$.
Assume first that $c_0=\sup_X\varphi$ is not a maximum. Then, for any $c\in\mathbb{R}$, $c<c_0\ \Rightarrow\{\varphi\le c\}\cap X\subset\subset M$. If c_0 is a maximum, then $\bar{X}$ is a compact subset in M. Hence, it's enough to prove the Lemma under the following technical assumption.

(3') There exists an increasing sequence $\{\chi_k\}_{k\in\mathbb{N}}$ of smooth positive decreasing functions such that supp$\chi_k(\varphi)\cap X$ is a relatively compact subset in M and $\lim\limits_{k\to+\infty}\chi_k(\varphi)=\chi(\varphi)$ on M.

Since supp $\theta\chi_k(\varphi)$ is a compact set in M, Lemma 3.4 gives

$$\int\theta\chi_k(\varphi)\,(\omega+dd^c\varphi)^n=\int\theta\chi_k(\varphi)\omega^n$$
$$-\int(d\theta)\chi_k(\varphi)d^c\varphi P(\varphi)-\int\theta\chi_k'(\varphi)d\varphi\wedge d^c\varphi P(\varphi)\ , \tag{3.11}$$

where

$$P(\varphi)=\sum_{\substack{\alpha+\beta=n-1\\ \alpha,\beta\ge 0}}(\omega+dd^c\varphi)^\alpha\,\omega^\beta. \tag{3.12}$$

Note that $d\varphi\wedge d^c\varphi P(\varphi)$ is a positive current on M. But χ_k' is negative, hence $-\int\theta\chi_k'(\varphi)d\varphi\wedge d^c\varphi P(\varphi)\ge 0$.

Since φ is vanishing on a neighbourhood of supp $d\theta$, the second term of the right hand side vanishes. Hence

$$\int\theta\chi_k(\varphi)(\omega+dd^c\varphi)^n\ge\int\theta\chi_k(\varphi)\omega^n\ . \tag{3.13}$$

The above integrals being finite, letting first θ decreasing to the characteristic function of $\bar{X}$; and then $k\to+\infty$, as the sequence χ_k is increasing in k, we get the result.

We give the following example. Let $M = \mathbb{C}^n$, $X = B(1)$, where $B(1)$ is the unit ball, and let $\omega = dd^c\|z\|^2$ the standard Kähler metric. Then $1 - \|z\|^2$ belongs to $P_\omega(X)$ and satisfies the conditions of the above lemma. Its extension by zero is $\widetilde{(1-\|z\|^2)} = \max(0, 1-\|z\|^2)$ so that $\|z\|^2 + \widetilde{(1-\|z\|^2)} = \max(\|z\|^2, 1)$. Lemma 3.5, for $\chi = 1$, says

$$\int_{\partial B(1)} (dd^c \max(1, \|z\|^2))^n \geq \int_{B(1)} (dd^c\|z\|^2)^n \ ,$$

which is in fact an equality.

4 Pseudoconvex hulls.

Let M be a complex manifold, $\dim_{\mathbb{C}} M = n \geq 2$, and let M_1 be an open subset of M.

We recall that M_1 is said to be *locally pseudoconvex in* M, if there exists an open cover $\mathcal{W}$ of M by Stein open subsets W such that $M_1 \cap W$ is Stein, for any $W \in \mathcal{W}$.

It follows, from Oka's theorem [30] and Docquiert-Grauert's theorem [19], that M_1 is locally pseudoconvex in M if and only if, for any Stein open subset X of M, $M_1 \cap X$ is a Stein manifold.
The above definition being local, any connected component of the interior of an intersection of a family of locally pseudoconvex open subsets of M is a locally pseudoconvex open subset of M.

Definition 4.1 *Let U be an open subset of M. Then there exists $\hat{U}$, the smallest locally pseudoconvex open set in M which contains U. We say that $\hat{U}$ is the pseudoconvex hull of U in M.*

We list elementary properties of pseudoconvex hulls.

Lemma 4.2 *i. let $f : M \to Y$ be a holomorphic map between complex manifolds, Y_1 an open subset of Y such that $U \subset f^{-1}(Y_1)$ then $\hat{U} \subset f^{-1}(\hat{Y_1})$.*
ii. Let a group G acting, on the right, on M by holomorphic transformations, such that $\forall g \in G$, $Ug = U$, then $\forall g \in G$, $\hat{U}g = \hat{U}$.

Although the properties of the boundary of $\hat{U}$ in M, if it is non empty, may be expressed, away of $\bar{U}$, in the language of uniform algebras (see [22, 7]), we will only state the following elementary lemma.

Lemma 4.3 *Let $(W', (z))$ be a holomorphic charts, with W' a relatively compact Stein open set of $M \setminus \bar{U}$. Then, for any open relatively compact subset W in W', and any polynomial P in the complex coordinates (z),*

$$\max_{\bar{W}\cap\partial\hat{U}} P = \max_{\partial W\cap\partial\hat{U}} P\ .$$

Proof. We argue by contradiction, and prove that if the above condition is not satisfied, we may push a hypersurface in $\hat{U}$ which is disjoint from U. Denote $K = \partial\hat{U}$. Assume there exists a polynomial P such that $||P||_{K\cap\bar{W}} = |P(z_0)| = 1$ for some $z_0 \in K \cap W$ and $||P||_{K\cap\partial W} < 1$.

$K \cap \partial W$ being compact, there exists $0 < \epsilon < 3^{-1}d(z_0, \partial W)$ s.t. $|P| < 1$ on $S_\epsilon = \{z \in \bar{W}, d(z, K \cap \partial W) < \epsilon\}$. Let $W_{2^{-1}\epsilon} = \{z \in W,\ d(z, \partial W) > 2^{-1}\epsilon\}$, and let $A_k = \{z \in W,\ P(z) = 1 + \frac{1}{k}\}$, $k \in \mathbb{N}^*$. A_k is an algebraic hypersurface in $W \setminus K \cup S_\epsilon$,

and $\bigcup_1^{+\infty} A_k \cap \partial W_{2^{-1}\epsilon} \subset\subset W \setminus K \cup S_\epsilon$.

There exists $\alpha_0 > 0$ s.t. $\bigcup_1^{+\infty}(A_k + B_{\mathbb{C}^n}(0, \alpha_0)) \cap \partial W_\epsilon \subset\subset W \setminus K \cup S_\epsilon$.

$W' \cap \hat{U}$ being a Stein open set and $\overline{\bigcup_1^{+\infty} A_k} \ni z_0$, there exists a sequence of integers $k_1, k_2, \ldots$, and irreducible component C_{k_i} of A_{k_i} such that

$C_{k_i} \cap \bar{W}_\epsilon \subset \bar{W}_\epsilon \setminus \bar{\hat{U}}$ and $\lim_{i\to+\infty} d(z_0, C_{k_i}) = 0$.

Hence $(C_{k_i} + B_{\mathbb{C}^n}(0, \alpha_0)) \cap \partial W_\epsilon \subset \partial W_\epsilon \setminus \bar{\hat{U}}$. $\overline{\bigcup_1^{+\infty} A_k \cap W_\epsilon}$ is a compact subset of $\overline{W_\epsilon} \setminus S_\epsilon$, hence there exists $\alpha_0 > \alpha_1 > 0$ such that

$\bigcup_1^{+\infty}(C_{k_i} + B_{\mathbb{C}^n}(0, \alpha)) \cap W_\epsilon \subset\subset W \setminus S_\epsilon$. Take i big enough such that

$d(z_0, C_{k_i}) < 2^{-1}\alpha_1$, take $z_1 \in C_{k_i} \cap B(z_0, 2^{-1}\alpha_1)$, $z_2 \in \hat{U} \cap B(z_0, 2^{-1}\alpha_1)$.
Then $(C_{k_i} + \overrightarrow{z_1z_2}) \cap W_\epsilon \cap \hat{U}$ is non empty and
$(C_{k_i} + \overrightarrow{z_1z_2}) \cap \partial(W_\epsilon \cap \hat{U}) \subset \partial\hat{U} \cap W_\epsilon$,
since $\partial(W_\epsilon \cap \hat{U}) \subset (\partial W_\epsilon \cap \hat{U}) \cup (\partial\hat{U} \cap \bar{W}_\epsilon)$ and
$\partial\hat{U} \cap \bar{W}_\epsilon = (\partial\hat{U} \cap W_\epsilon) \cup (\partial\hat{U} \cap \partial W_\epsilon)$.

In particular, $H = (C_{k_i} + \overrightarrow{z_1z_2}) \cap W_\epsilon \cap \hat{U}$ is a hypersurface in $\hat{U}$ which does not intersect U. However $\hat{U} \setminus H$ is locally pseudoconvex, contains U and is strictly smaller than $\hat{U}$, which is a contradiction.

Remark. The proof of the above lemma shows that, if $\dim M = 2$, then, for any open Stein subset of $M\backslash\bar{U}$, $W\backslash\partial\hat{U}$ is Stein. Hence $\partial\hat{U}$ is a pseudoconcave set in the sense of Oka in $M \setminus \bar{U}$ (see [32], p. 88).

Lemma 4.4 *Let W be an open set in M and let K be a compact subset in W. Then the pseudoconvex hull of $(\hat{U} \cap (W \setminus K)) \cup (W \cap U)$ is $W \cap \hat{U}$.*

We may define other kinds of pseudoconvex hulls with respect to the class $P_\omega(M)$ which are constructed by the following procedure.

Lemma 4.5 *Let $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subset P_\omega(M)$. Then, the open set*

$$X = \{p \in M \ : \varphi = \sup_{\alpha\in\Lambda} \varphi_\alpha \textit{ is locally bounded from above at } p\}$$

is locally pseudoconvex in M. Further, on X, φ^* the upper regularization of φ w.r.t. ω belongs to $P_\omega(X)$.

Proof. Let W be an open subset of X such that ω admits on W a local potential $a \in \mathrm{PSH}(W) \cap L^\infty(W, \mathrm{loc})$. For any $\alpha \in \Lambda$, $a + \psi_\alpha \in \mathrm{PSH}(W) \cap L^\infty(W, \mathrm{loc})$. So does $(a + \varphi)^* = (\sup_{\alpha\in\Lambda} a + \varphi_\alpha)^*$, where $*$ stands for the classical upper regularization (see [25], Theorem 5). Hence $\varphi^* = (a + \varphi)^* - a$ belongs to $P_\omega(W)$. Next, we prove that X is locally pseudoconvex in M.

Let $h : \Delta^n \to M$ be a biholomorphic map from a neighbourhood of the $n-$dimensional unit polydisc to M ($\dim M = n \geq 2$). Denote

$\mathrm{H} = \{p \in \Delta^n \ : \frac{1}{2} < |z_n(p)| < 1\} \cup \{p \in \Delta^n \ : \ z_1(p) = \ldots = z_{n-1}(p) = 0, \ |z_n(p)| < 1\}$. Let $a \in \mathrm{PSH}(W) \cap L^\infty(W, \mathrm{loc})$ a potential for ω on W, a neighbourhood of $h(\Delta^n)$. Assume $h(\mathrm{H}) \subset\subset X$. By construction, $(a + \varphi_\alpha)(p) \leq \max_{h(\bar{H})}(a + \varphi)^*$ for any $p \in h(\bar{H})$ and $\alpha \in \Lambda$. Since the holomorphic hull of any neighbourhood of H contained Δ^n, we have $(a + \varphi_\alpha)(p) \leq \max_{h(\bar{H})}(a + \varphi)^*$ for $p \in h(\Delta^n)$ and $\alpha \in \Lambda$. Hence, by definition, $h(\Delta^n) \subset X$ since $a \in L^\infty(W, \mathrm{loc})$.

We will use the following lemma, which gives a property of the preceding hulls, not realized for general pseudoconvex open subsets.

Lemma 4.6 *Let M be a complex manifold, and let ω be a closed positive $(1, 1)-$current in M. Assume there exists an analytic subset B in M such that ω admits local locally bounded potentials on $M \setminus B$. Let $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subset P_\omega(M \setminus B)$. And let X denote the open set in $M \setminus B$ where this family is locally bounded from above. Then, the interior of $X \cup B$ in M is a locally pseudoconvex open subset in M.*

Proof. The lemma is local, hence we assume M is the unit ball. Let $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ be a set of plurisubharmonic functions on $M\setminus B$, and let U be the maximal open subset of $M\setminus B$ for which this family is locally uniformly bounded from above. Let φ be the upper envelope of this family, and let φ^* denote its upper regularization, which is a plurisubharmonic function in U.

Write $B = B_1\cup B_2$, with $\mathrm{codim}B_1 = 1$ and $\mathrm{codim}B_2 \geq 2$. First, we prove that, in $M\setminus B_1$, the interior U' of $U\cup B_2$ is locally pseudoconvex. Let $h : (H,\Delta^n) \to M\setminus B_1$ be a Hartogs' frame (as defined in the above proof) such that $h(H) \subset\subset U'$ and $h(\Delta^n) \subset\subset M\setminus B_1$. Since $\mathrm{codim}B_2 \geq 2$, each plurisubharmonic function φ' in $M\setminus B$ admits a plurisubharmonic extension, which we denote $\tilde{\varphi}'$, to $M\setminus B_1$. $\tilde{\varphi}'$ satisfies that for any relatively compact open subset X in $M\setminus B_1$, $\sup_X \tilde{\varphi} = \sup_{X\setminus B_2}\varphi'$. This fact applies to φ^*. Hence for any α, $\max_{\bar{h}(H)}\tilde{\varphi}^* \geq \sup_{h(H)}\tilde{\varphi}_\alpha \geq \sup_{h(\Delta^n)}\tilde{\varphi}_\alpha$. In particular, any point of $h(\Delta^n)\setminus B_2$ belongs to U, so $h(\Delta^n)\subset U'$.

Next, by using the disc characterisation of pseudoconvexity, it is classical that if X is an open pseudoconvex subset in $M\setminus B_1$, then the interior of $X\cup B_1$ is pseudoconvex, when B_1 is a complex hypersurface.

Remark. In particular, this set X may be thought as a kind of bimeromorphic invariant.

The following lemma gives a key property of the class $P_\omega(M)$.

Lemma 4.7 *Let W be an open subset of M biholomorphic to the unit ball in $\mathbb{C}^n$. Let $D\subset\subset W$ be a strongly pseudoconvex open subset of W. Then, for any $\psi\in P_\omega(M)$, there exists a unique function $\tilde{\psi} = T_D(\psi)\in P_\omega(M)$ such that $\tilde{\psi} = \psi$ on $M\setminus D$ and $(\omega + dd^c\tilde{\varphi})^n = 0$ on D. Further $\tilde{\psi}\geq\psi$.*

Proof. Let $a\in \mathrm{PSH}(W)\cap L^\infty(W,\mathrm{loc})$ be a potential for ω on W. From [6], Proposition 9.1, a unique function $\widetilde{a+\psi}$ exists such that

$$(dd^c(\widetilde{a+\varphi}))^n = 0 \qquad \text{on } D$$

$$\widetilde{a+\varphi} = a+\varphi \qquad \text{on } W\setminus D$$

and $\widetilde{a+\varphi}\geq a+\varphi$ on W. Note that $\widetilde{a+\psi} - a = \psi$ on $W\setminus D$ and we define

$$\tilde{\psi} = \begin{cases} \max(\psi, \widetilde{a+\psi} - a) = \widetilde{a+\psi} - a & z\in W \\ \psi & z\in M\setminus W \end{cases}$$

Definition 4.8 *A family $\Lambda \subset P_\omega(M)$ is stable with respect to an open subset $U \subset M$, if, any point $p \in M \setminus \bar{U}$ admits a pair of open neighbourhoods (W, D) as in Lemma 4.7, with $W \subset M \setminus \bar{U}$, such that, for all $u \in \Lambda$, the function $T_D(u)$, deduced from u by the above Lemma 4.7, belongs to Λ.*

Fix an open subset U of M and a family $\Lambda \subset P_\omega(M)$ which is stable with respect to U and the max operation. Assume that the open subset X where Λ is locally bounded from above contains U. Denote $\varphi^* = (\sup_{\psi \in \Lambda} \psi)^*$, the upper regularization (w.r.t. ω) of the upper envelope of this family, which belongs to $P_\omega(X)$.

Lemma 4.9 *Under the above hypothesis, the positive measure $(\omega + dd^c\varphi^*)^n$ has support in $\bar{U}$.*

Proof. Notice that there exists an increasing sequence $\{u_j\}_{j\in\mathbb{N}} \subset \Lambda$ such that $(\lim_{j\to+\infty} u_j)^* = \varphi^*$. Indeed, Λ being locally bounded from above on X, from Choquet's Lemma 3.3, there exists a countable subset $\{\psi_n\}_{n\in\mathbb{N}} \subset \Lambda$ such that $\varphi^* = (\sup_{n\in\mathbb{N}} \psi_n)^*$. Define $u_j = \max_{p\leq j} \psi_p$. By hypothesis $u_j \in \Lambda$. Moreover $\sup_{n\in\mathbb{N}} \psi_n = \lim_{j\to+\infty} u_j$. Let (W, D) be open neighbourhoods of $x \in X \setminus \bar{U}$ as in Definition 4.8. Replacing each u_j by $\tilde{u}_j = T_D(u_j) \in \Lambda$ as in Lemma 4.7, we have

the sequence $\{\tilde{u}_j\}_{j\in\mathbb{N}}$ is increasing, since $\tilde{u}_j$ may be obtain by a Perron method,

it increases to φ^* outside a pluripolar set, since $\varphi^* = (\lim_{j\to+\infty} \tilde{u}_j)^*$ and the negligible set $\left\{(\lim_{j\to+\infty} \tilde{u}_j) < (\lim_{j\to+\infty} \tilde{u}_j)^*\right\}$ is pluripolar.

Applying [6], Theorem 7.4, we see that $(\omega + dd^c\tilde{u}_j)^n$ is weak$^*-$ convergent to $(\omega + dd^c\varphi^*)^n$. Since the former measures are vanishing on D, the later is vanishing on D. Since this property is valid for any such pair (W, D), with $W \cap \bar{U} = \emptyset$, the assertion is proved.

4.10 Some extremal functions.

Let ω be a closed positive $(1,1)-$current on the complex manifold M. Assume that ω admits local locally bounded potentials near every point in M (see (3.1)).

Definition 4.11 *Let U be a domain in M, and let h be a function on U which is locally bounded and lower semicontinuous w.r.t. ω. Define*

$$X(h,\omega) = \{p \in M \ : \varphi = \sup_{\psi \in P_\omega(M,U,h)} \psi \text{ is locally bounded from above at } p\ \} \ ,$$

where $P_\omega(M,U,h) = \{\varphi \in P_\omega(M)$ *such that* $\varphi_{|U} \leq h\}$.

Let φ^* *be the upper regularization of* φ *(w.r.t.* ω*) in* $X(h,\omega)$ *and call it the extremal function associated to* U*,* ω *and* h*. Define* $U(h,\omega)$ *to be the connected component of* $X(h,\omega)$ *which contains* U*.*

By assumption, $P_\omega(M,U,h)$ is locally bounded from above on U, hence $X(h,\omega)$ contains U. When $h=0$, and M is a pseudoconvex domain in $\mathbb{C}^n$, we obtain the usual hull of holomorphy of U with respect to M. For M a projective manifold, $h=0$, our hull is similar to the hull introduced in [21]. We refer to this article for further properties when this hull is assumed to be compact in some locally pseudoconvex domain.

We note that this family is stable by the the max operation and with respect to U (see Definition 4.8). Hence the extremal function $\varphi^* \in P_\omega(U(h,\omega))$ satisfies the interesting property quoted in Lemma 4.9. Moreover, in U, we have $(\omega + dd^c\varphi^*)^n = 0$ on the open subset $\{\varphi^* < h\}$ (see [6], Corollary 9.2).

Another interesting hull is obtain by the following balayage procedure.

Definition 4.12 *Let* U *be a domain in* M *and* $\psi \in P_\omega(M)$*. Fix* $\mathcal{D} = \{D_i\}_{i\in\mathbb{N}}$ *an open cover of* $M \setminus \bar{U}$ *by open strongly pseudoconvex subsets* D_i*, which are relatively compact in complex holomorphic charts* $f_i : W_i \to B_{\mathbb{C}^n}(0,1)$*. Assume that each* D_i *is repeated infinitely often in the sequence* $\mathcal{D}$*. Define by induction,* $\psi_{-1} = \psi$*, and* $\psi_i = T_{D_i}(\psi_{i-1})$*, for* $i \in \mathbb{N}$*. Let* $X(\psi)$ *denote the open subset where the family* $\{\psi_i\}_{i\in\mathbb{N}}$ *is locally bounded from above, and let* $U(\psi)$ *be the connected component of* $X(\psi)$ *which contains* U*. Define* $B(\psi) = (\sup_{i\in\mathbb{N}} \psi_i)^*$*, which belongs to* $P_\omega(U(\psi))$*.*

Note that this family is an increasing sequence w.r.t. $i \in \mathbb{N}$. By construction, it is stable with respect to U. Hence $B(\psi)$ satisfies $(\omega + dd^cB(\psi))^n = 0$ on $U(\psi) \setminus \bar{U}$. Moreover, $B(\psi) \geq \psi$ on $U(\psi)$. Although $B(\psi)$ depends in general of the cover chosen, we will not indicate this dependence.

Remark.

i. Assume ω is strictly positive in a neighbourhood of a point $p \in M$, *e.g.* there exists an open set W in M which is biholomorphic, through the holomorphic map h, to the open unit ball in $\mathbb{C}^n$, and s.t. $\omega \geq ch^*(\omega_e)$, where $c > 0$ and ω_e is the usual Kähler metric on $\mathbb{C}^n$. Let $\theta \in \mathcal{C}^\infty(M)$ with compact support in W. Then $\frac{1}{N}\theta \in P_\omega(M)$ for $N \in \mathbb{N}$ large enough. In particular, if (M,ω_0) is a Kähler manifold, $P_{\omega+\omega_0}(M)$ and $P_{\omega+\omega_0}(M,U,0)$ are non empty sets.

ii. Under the above hypothesis (i). Apply Lemma 4.7 to the open set W (for some strongly pseudoconvex $D \subset\subset W$) and the function ψ which is identically equal to zero on M. We obtain a function $\tilde{\psi}$ which is zero on $M \setminus D$, strictly positive on D, and satisfies $(\omega + dd^c\tilde{\psi})^n = 0$ on D.

iii. In particular, the balayage procedure, as defined in Definition 4.12, when applied to the zero function, gives a positive function which is strictly positive in points where ω is strictly positive.

iv. Let X be a relatively compact domain in M with smooth boundary. Assume for simplicity that ω is smooth and strictly positive. Then applying the Green formula for $\bar{X}$ with respect to the Kähler metric ω (see [5]), we see that a family $\mathcal{F} \subset P_\omega(M)$ is locally bounded from above in X if it is bounded for the L^1 norm induced on ∂X.

4.13 The case of a Chern class.

In this section, we interpret our results when ω is a Chern current of a line bundle. First, note that a closed positive $(1,1)-$current ω is the Chern current of a hermitian line bundle L over a complex manifold M if it lies in $H^2(M,\mathbb{Z})$ via the De-Rham isomorphism.
Let $(E,h) \to M$ be a complex hermitian line bundle with positive (singular) metric curvature. Denote $\pi : E^* \to M$ the bundle map from E^* to M, the dual line bundle of E, and denote $|\zeta|^2$ the norm of $\zeta \in E^*$ induced by h. In a local trivialization, $t_W : E^*_{|W} \simeq W \times \mathbb{C}$, $|\zeta|^2 = a_{W,t}(\pi(\zeta))|l \circ t(\zeta)|^2$ where $a_{W,t}$ is a logarithmic plurisubharmonic function in W, with $dd^c \ln(a_{W,t}) = iC(E,h)$. Here $|l \circ t(\zeta)|$ is the complex modulus of the image of ζ by the natural projection $E^*_W \to \mathbb{C}$.
Let A be a subset of M. Denote $T_A(\alpha) = \{\zeta \in E^*_{|A},\ |\zeta| < \alpha\}$, and denote $T_A = T_A(1)$. In this section we will study more closely $\widehat{T_U}$, the pseudoconvex hull of T_U in the complex manifold E^*.

Lemma 4.14 *$\widehat{T_U}$ is a disqued pseudoconvex subset of E^*.*

Proof. Here we are concerned with the action of $\mathbb{C}^*$, in the fibre of E, $(\lambda,\zeta) \to \lambda.\zeta$. Let λ be a non zero complex number, $\lambda T_U \subset \lambda\widehat{T_U}$, hence $\widehat{\lambda T_U} \subset \lambda\widehat{T_U}$. But $T_U \subset \lambda^{-1}\widehat{\lambda T_U}$, hence $\lambda\widehat{T_U} \subset \widehat{\lambda T_U}$. So $\lambda\widehat{T_U} = \widehat{\lambda T_U}$. This is a classical result that if W is a pseudoconvex domain in $\mathbb{C}^n$, H an irreducible hypersurface in W and K a compact subset in W, with $H \cap K$ non void, then the pseudoconvex hull of $(W \setminus H) \cup K$ is W. Hence $\widehat{T_U}$ contains $0.\widehat{T_U}$ since it contains $0.T_U$. As for any complex number λ in the unit disc, $\lambda T_U \subset \widehat{T_U}$, we have $\lambda\widehat{T_U} \subset \widehat{T_U}$.

Since $\widehat{T_U} \subset \pi^{-1}(\hat{U})$ and $0.\widehat{T_U} \simeq \hat{U}$, from the above lemma, we see that $\widehat{T_U}$ is a twisted pseudoconvex Hartogs' domain over $\hat{U}$. Moreover $\widehat{T_U} \subset T_M(1)$. In particular, when we assume that $iC(E)$ admits local locally bounded potentials, there exists an u.s.c (w.r.t. $iC(E)$) function $\varphi \in P_{iC(E)}(\hat{U})$ such that $\widehat{T_U} = \{\zeta \in E^*,\ \ln|\zeta|^2 + \varphi < 0\}$. Indeed, let $t_1 : E^*_{|W_1} \simeq W_1 \times \mathbb{C}$ be a local trivialization of E^* over the open subset W_1. Since t_1 is a morphism of vector bundle, $t_1(\widehat{T_{U|W_1}})$ is a Hartogs' locally pseudoconvex domain with base W_1. Assume that W_1 is biholomorphic to an open ball in $\mathbb{C}^n$ and write $t_1(\widehat{T_{U|W_1}}) = \{(z,p) \in \mathbb{C} \times W_1,\ \ln|z|^2 + \psi_1(P) < 0\}$ with ψ_1 a plurisubharmonic function in W_1. Define $\varphi = \psi_1 - \ln a_{W_1,t_1}$. Let $t_2 : E^*_{|W_2} \simeq W_2 \times \mathbb{C}$ be another trivialization of E^*, with W_2 biholomorphic to an open ball in the complex Euclidean space, and let ψ_2 the corresponding function defined as above. On $W_1 \cap W_2$ (if non void), $t_1 \circ t_2^{-1} : (W_2 \cap W_1) \times \mathbb{C} \simeq (W_1 \cap W_2) \times \mathbb{C}$ is an isomorphism of holomorphic line bundles with transition function $g_{1,2}$. We have $\ln|g_{1,2}(p).z|^2 + \ln a_{W_1,t_1}(p) = \ln|z|^2 + \ln a_{W_2,t_2}(p)$ and $\ln|g_{1,2}(p).z|^2 + \psi_1(p) = \ln|z|^2 + \psi_2(p)$ hence $\psi_1 - \ln a_{W_1,t_1} = \psi_2 - \ln a_{W_2,t_2}$. Hence φ is well defined on $\hat{U}$. By construction $iC(E) + dd^c\varphi \geq 0$ on $\hat{U}$.

Note that φ is maximal in the following sense.

Let W be an open set in $\hat{U} \setminus \bar{U}$, and let $\psi \in P_{iC(E)}(W)$. If W' is a relatively compact open subset of W and if $\liminf\limits_{z \to \partial W'} \varphi(z) - \psi(z) \geq 0$ then $\varphi \geq \psi$ in G.

For the function

$$\varphi' = \begin{cases} \max(\varphi, \psi) & z \in W' \\ \varphi & z \in W \setminus W' \end{cases}$$

belongs to $P_{iC(E)}(\hat{U})$ and is zero on U. Hence $\{\ \zeta \in E^*_{\hat{U}}\ : \ln|\zeta|^2 + \varphi' \circ \pi(\zeta) < 0\}$ is pseudoconvex, contains T_U, hence contains $\widehat{T_U}$. So $\varphi' = \varphi$.

In particular, we have the following property for φ.

Lemma 4.15 *Assume that $iC(E)$ admits local potentials which belongs to* $\mathrm{PSH}(M) \cap L^\infty(M, \mathrm{loc})$. *Then, the positive measure $(iC(E) + dd^c\varphi)^n$ as support in $\bar{U}$, the closure of U in $\hat{U}$.*

Proof. Let D, W be domains as in Lemma 4.7 with $W \cap \bar{U} = \emptyset$. Since φ is maximal in the sense as above, $T_D(\varphi) = \varphi$. However, $(\omega + T_D(\varphi))^n$ vanishes on D, by construction.

Next, we look to the class $P_{iC(E)}(M, U, 0)$ and to $U(0, \omega)$.

Lemma 4.16 *Let $T_U(0,0)$ denote the hull of T_U with respect to globally defined plurisubharmonic functions on E^* (see section 4.10). Then $T_U(0,0)$ is* *a disqued subset over $U(0,\omega)$ which contains the image of $U(0,\omega)$ by the null section. Moreover $T_U(0,0) = \{\zeta \in E^*,\ \ln|\zeta|^2 + \varphi^*(\pi(\zeta)) < 0\}$, where φ^* is the extremal function associated with U and ω (see section 4.10).*

Proof. By definition $T_U(0,0) \subset \{\zeta \in E^*,\ \ln|\zeta|^2 + \varphi^*(\pi(\zeta)) < 0\} = A$. To prove the equality, we argue by contradiction. Let $\zeta_0 \in A \setminus T_U(0,0)$. A being open, there exists a neighbourhood W of ζ_0 in A, a non constant plurisubharmonic function ψ on E^*, such that $\{\psi < 0\}$ contains T_U but does not contains W. ψ being upper semicontinuous, $\{\psi \geq 0\}$ is the closure of $\{\psi > 0\}$. Hence there exists $\zeta_1 \in W \cap \{\psi > 0\}$. Let us replace ψ by $\psi' = \log|\zeta|^2 + N\psi$. Then $\{\psi' < 0\}$ contains T_U and for N large enough, still not contains ζ_1. That is $T_U \subset \{\psi' < 0\} \cap A \underset{\neq}{\subset} A$. Hence,

$T_U \subset \bigcap\limits_{\theta \in [\,0,\,2\pi]} e^{i\theta}\{\psi' < 0\} \cap A \underset{\neq}{\subset} A$. However $\bigcap\limits_{\theta \in [\,0,\,2\pi]} e^{i\theta}\{\psi' < 0\}$ is a twisted Hartogs' pseudoconvex domain over M, *i.e.*, it contains the image of M by the zero section of E (this is why we change ψ). It contains T_U, hence, it is defined by a function $\varphi' \in P_{iC(E)}(M, U, 0)$.

5 Bounds of Monge-Ampère masses.

Recall that if M is a complex manifold, a non relatively compact connected component of $M \setminus K$ where K is a compact set in M, is called an end of M. Let ω be a closed positive $(1,1)-$current on M, which admits local locally bounded potentials. Let $\mathcal{F} \subset P_\omega(M)$, and let $X(\mathcal{F})$ denote the open subset in M where this family is locally bounded from above.

Definition 5.1 *An end of $X(\mathcal{F})$ will be called a pseudoconcave end with respect to $\mathcal{F}$.*

Note that $X(\mathcal{F})$ may be not connected. We may prove finiteness theorem for Monge-Ampère integrals on pseudoconcave ends with respect to $\mathcal{F}$, if $\mathcal{F}$ is rich enough. To avoid generality, we will restrict ourself to the following situation. Let M be a complex manifold, let U be an open subset of M, and let $\mathcal{F} = P_\omega(M, U, 0)$. Let $U(0,\omega)$ as defined in section 4.10. Working in the relative topology of $U(0,\omega)$, we assume that $U(0,\omega) \setminus \bar{U}$ admits a connected component X with compact boundary. That is X is a pseudoconcave end with respect to $P_\omega(U, M, 0)$, if it is non relatively compact.

Let φ^* be the extremal function associated with $U(0,\omega)$. Recall that φ^* is everywhere positive and restricted to U is identically vanishing. We make the following assumption

For any $p \in X$, $\{\varphi^* \leq \varphi(p)^*\} \cap \bar{X}$ is a relatively compact subset of $U(0,\omega)$.

Let $M_1 = U \cup \bar{X}$. We have $\partial_{M_1} X = \partial_{U(0,\omega)} X$. Let $X_\epsilon = \{z \in M_1 \ : \ d(z,X) < \epsilon\}$. For ϵ small enough, this open subset has a relatively compact boundary in M_1, and φ^* satisfies all hypothesis of the Lemma 3.5. Hence, from it, we obtain (for small ϵ)

$$+\infty > \int_{\bar{X}_\epsilon} \chi(\varphi^*)(\omega + dd^c\varphi^*)^n \geq \int_{\bar{X}_\epsilon} \chi(\varphi^*)\, \omega^n$$

for any positive smooth decreasing function $\chi : \mathbb{R} \to \mathbb{R}^+$.

The integrals are finite since on $\bar{X}_\epsilon$, the positive measure $(\omega + dd^c\varphi^*)^n$ has support on $\bar{X}_\epsilon \cap \bar{U}$, which is a compact set. Letting ϵ going to zero, we obtain the following Proposition (we work in the topology of $U(0,\omega)$).

Proposition 5.2 *Let $U(0,\omega)$ be as above and let X be a connected component of $U(0,\omega) \setminus \bar{U}$ with compact boundary. Let φ^* be the extremal function associated with $U(0,\omega)$. Assume that $\{\varphi^* \leq \varphi(p)^*\} \cap \bar{X}$ is a relatively compact subset of $U(0,\omega)$ for every $p \in X$. Then, for any positive decreasing smooth function $\chi : \mathbb{R} \to \mathbb{R}^+$, we have*

$$\int_{\bar{X}} \chi(\varphi^*)\omega^n \ \ \leq \ \ \int_{\partial X} \chi(\varphi^*)(\omega + dd^c\varphi^*)^n < +\infty\,. \tag{5.1}$$

In particular

$$\int_{\bar{X}} \omega^n \ < +\infty\,. \tag{5.2}$$

Remark.

i. Let M be a complex manifold and let ω be a closed positive $(1,1)-$current. Assume that $\varphi \in P_\omega(M)$ is exhaustive and satisfies the Monge-Ampère equation $(\omega + dd^c\varphi)^n = 0$. Then Lemma 3.4 implies that $\omega^n = 0$.

ii. We want to point out that the conditions we impose on pseudoconcave end to obtain finiteness of Monge-Ampère masses is global in nature.

First, we recall results obtained by N. Sibony in [33].

Recall that a closed subset L of a domain U in $\mathbb{C}^n$ satisfies the condition (C), if, for any point $x \in U$, there exists a strictly pseudoconvex neighbourhood $U_1 \subset\subset U$ of x, a smooth plurisubharmonic function ϕ on U such that $\phi(x) > 1$ and $\phi \leq 0$ on a neighbourhood V of $\partial U_1 \cap L$. For a closed subset L which satisfies this condition N. Sibony proved the following theorem.

Theorem Let u be a plurisubharmonic function on U which is of class $\mathcal{C}^2$ on $U \setminus L$. Then for any compact subset K in U, $\displaystyle\int_{K\setminus L}(dd^c u)^n < +\infty$.

We notice that, by definition of the condition (C), L is not a pseudoconcave set in the sense of Oka, that is, for a Stein domain B in U, s.t. $B\cap L \neq \emptyset$, $B \setminus L$ is not a Stein domain. In particular, any point $x \in L$ belongs to the pseudoconvex hull of a relatively compact open subset W in $U \setminus L$, $(*)$. Condition $(*)$ is weaker than condition (C). Indeed, let $\mathbb{C}^n$ equipped with coordinates $z_1, \ldots, z_n$. Consider the line $Z = \{z_2 = \ldots z_n = 0\}$ and denote $B(r)$ the ball of radius r. If $n \geq 3$, then the hull of $B(1) \cap \{\sum_{i=2}^n |z_i|^2 > \frac{1}{2}\}$ contains a neighbourhood of the origin. But, Z being complex analytic does not satisfy condition (C).

iii. However, for a closed set L, the above condition $(*)$ is far from being sufficient to have finiteness of Monge-Ampère masses near x. Indeed, B. Shiffman and B. A. Taylor constructed a plurisubharmonic function ψ in the unit ball, which is smooth on $B(1) \setminus Z$, such that $\int_{B(r)\setminus Z}(dd^c\psi)^n$ is infinite for $0 < r < 1$ (see [34]). Hence the global condition to be a pseudoconcave end in order to obtain finiteness result on Monge-Ampère masses may not be removed, even locally, without other hypothesis.

For compact singularities in the unit ball, we obtain the following well known fact (see *e.g.* [33]).

Corollary 5.3 *Let $u \in \mathrm{PSH}(B(1))$, such that its polar set $L = \{u = -\infty\}$ is a compact subset of $B(\frac{1}{2})$, and u is locally bounded on $B \setminus L$.*
Then $\displaystyle\int_{B(\frac{1}{2})\setminus L}(dd^c u)^n < +\infty.$

Proof. We work in $M = B(1)\setminus L$. The pseudoconvex hull of $U = B(1)\setminus \bar{B}(\frac{1}{2})$ is M. Now, $-u \in P_\omega(M)$, where $\omega = dd^c u$, and this function satisfies that $\{-u < c\} \cap \bar{B}(\frac{1}{2})$ is relatively compact in M for any $c \in \mathbb{R}$. So does $-u - C$ for some constant, chosen such that $-u - C$ is negative on a neighbourhood of $\partial B(\frac{1}{2})$. Let φ^* be the extremal function associated to ω and U. But $\varphi^* \geq -u - C$, hence from Proposition 5.2,

$$\int_{\bar{B}(\frac{1}{2})\setminus L}\omega^n \leq \int_{\partial B(\frac{1}{2})}(\omega + dd^c\varphi^*)^n < +\infty\ .$$

6 Pluricomplete currents.

As we have seen, we may obtain bounds of Monge-Ampère masses, associated to a closed positive $(1,1)-$currents ω, on pseudoconcave ends, under the assumptions that ω admits local locally bounded potential and that an extremal function associated to it is exhaustive along these ends. In this section, we explain a way to deal with a more singular current ω on a manifold M which satisfies the first condition on $M \setminus B$, where B is an analytic subset in M. If B may be written as intersection of hypersurfaces (*e.g.* an indeterminacy set of a meromorphic map with value in a projective manifold), we are able to construct a function $\varphi \in P_\omega(M \setminus B)$ which goes to $+\infty$ near B. Hence, under suitable pseudoconcavity conditions, we will be able to prove a bounds on Monge-Ampère masses of ω. To avoid numerous hypothesis, we will essentially restrict ourself to spread manifolds over a projective manifold.

6.1 Spread spaces and distance to the boundary.

Definition 6.2 *Let (M, ω_0) be a Kähler manifold. A complex manifold $\pi : U \to M$ is spread over M if the map π is a local biholomorphism. We said that $\pi : U \to M$ is locally pseudoconvex over M (with respect to π), if there exists an open covering $\mathcal{W}$ of M by Stein open subsets $W \in \mathcal{W}$ such that $\pi^{-1}(W)$ is a Stein manifold for any $W \in \mathcal{W}$.*

We say that $\pi : U \to M$ is a domain over M, if U is connected. Examples of spreading are
the canonical injection $i : U \hookrightarrow M$ of an open subset U of M,
a covering map $\pi : U \to M$ of M,
the restriction $\pi_{|U'} : U' \to M$ of a covering map $\pi : U \to M$ to an open subset.
In the first case, $i : U \hookrightarrow M$ is locally pseudoconvex over M if and only if U is a locally pseudoconvex open subset of M.

We recall a generalisation of Oka's theorem concerning the plurisubharmonicity of $-\log d_{\partial U}$, for U a pseudoconvex domain in $\mathbb{C}^n$. First, we recall the notion of boundary distance for a spread space. Let $\pi : U \to M$ be a spread space. We still denote ω_0 the pullback by π of a Kähler metric ω_0 on M. For $Q \in U$, let $d_{\partial U}(Q) = \sup\{r > 0, \text{ s.t. } \exp_Q : B(0,r) \to U \text{ is defined}\}$. Hence $d_{\partial U}(Q)$ is the infinitum of length of geodesics (parameterised by arc-length) emanating from Q which are outside any compact subset of U in finite time. This function is either identically ∞ or Lipschitzian.

Theorem 6.3 (Takeuchi [38]) *Let (M, ω_0) be a Kähler manifold and K a compact subset in M. Then, there exists real constants $\delta > 0$ and α, such* *that, for any locally pseudoconvex spread domain $\pi : U \to M$, subject to the condition $\pi(U) \subset K$, the function $-\log d_{\partial U}$ (if U admits some boundary points over M) satisfies $dd^c - \log d_{\partial U} \geq -\alpha\omega_0$ for any point p in U such that $d_{\partial U}(p) < \delta$.*

Although, we will not really used it, we want to point out that α (and δ) *does not* depend on U, that is the default of plurisubharmonicity is a differential one and depends only on bounds on the curvature tensor of ω_0.

As an application, we will state some spannedness theorems. We restrict ourself to the case of a spread manifold over a projective one's.
We fix notations.
Let V be a projective manifold of dimension $n \geq 2$. Denote $\mathcal{O}(1)$ the line bundle over V which gives the projective embedding of V. If $\pi : U \to V$ is a spreading, we still denote $\mathcal{O}(l)$ for the pullbacks $\pi^*\mathcal{O}(l)$ of $\mathcal{O}(l)$ by π. Also we still denote ω_0 for the pullback by π of a Kähler metric ω_0 on V. If s is a section of some line bundle on a manifold M, we denote $\mathrm{ord}_p s$ its vanishing order at a point p, if Y is a complex hypersurface in M, we denote $\mathrm{mult}_p Y$ its multiplicity at p. For a divisor D on M, we denote $\nu_p(D)$ its multiplicity at p. If s is a meromorphic section of a line bundle over M, we denote (s) its divisor and Z_s its zero set.

Theorem 6.4 *Let $(V, \mathcal{O}(1))$ be a projective manifold. Then there exists $l_1 \in \mathbb{N}$, such for any $l \geq l_1$, for any locally pseudoconvex domain $\pi : U \to V$ over the projective manifold V, any hypersurface $Y \hookrightarrow U$, and any $p \in U$, there exists an $\tilde{s} \in H^0(U, \mathcal{O}(l) \otimes [Y])$ of minimal growth (in a sense precised during the proof of the theorem) and such that $\mathrm{ord}_p \tilde{s} \leq \mathrm{mult}_p Y - 1$.*

Proof. We will only give the main arguments of the proof, since similar methods appears already in great details in [4, 31] for the univalent case and in [17] in the above case.
Since V is compact and $\mathcal{O}(1)$ is strictly positive, there exists a real number β such that $i\mathrm{Ricci}(\omega_0) \geq -i\beta C(\mathcal{O}(1))$. Let $l_0 = \mathrm{Ent}(1 + n + \beta) + 1$, where $\mathrm{Ent}(r)$ denotes the integer part of a real number r.

Let δ and α denote the real constants which appear in the above Theorem 6.3. Let $\frac{1}{4} \geq \epsilon_0 > 0$ such that $4\alpha\epsilon_0 < 1$. Let $l_1 = \mathrm{Ent}(\max(4\alpha\epsilon_0 + 1 + n - 1 + \beta, 1 + n)) + 1 \geq l_0$. Let $l \geq l_0$.

First, note that there exists a finite number of square integrable holomorphic sections of $\mathcal{O}(l)$ over U which give an immersion of U in some projective space, see [18]. Hence, if $p \notin Y$, one of those sections satisfies our requirements. Now, let $p \in Y$.

Let $t_1, ..., t_n$ be sections of $\mathcal{O}(1)$ which give local coordinates centred in $\pi(p)$ and denote by the same letter their pullback by π. Let W be some small open neighbourhood of p in U, biholomorphic by π to some coordinate open set. Let s_1 be a smooth section of $\mathcal{O}(l+1)$ with compact support in W, holomorphic and non zero in a neighbourhood of p.

Let $k = \epsilon + n - 1$, with $0 < \epsilon \leq \epsilon_0$. For $l \geq l_0$, we solve the $\bar{\partial}-$equation $\bar{\partial} s_1 = \bar{\partial} s_2$ with weight $\exp -(k+1)\log \|t\|^2$ by L^2 methods (see [11]).

Hence the holomorphic section $s_3 = s_1 - s_2$ on U, is non-vanishing at p. Moreover, from the L^2 estimates, we deduce

$$I = \int_U \frac{|s_1 - s_2|^2}{\|t\|^{2(k+1)}} e^{-(-4\epsilon \log \min(\delta, d_{\partial_{U \setminus Y}}))} dV_{\omega_0} < +\infty \quad ,$$

since $\|t\|^2(p) = (|t_1|^2 + ... + |t_n|^2)(p) \geq C_1 d^2_{\partial_{U \setminus Y}}(p)$ in a neighbourhood of p. Hence for $l \geq l_1$, from Skoda [36], the section $(s_3)_{|U}$ is in the range of the morphism induced on global sections by the bundle morphism

$$\mathcal{O}(l) \otimes \mathbb{C}^n \to \mathcal{O}(l+1), (h_1, ...h_n) \mapsto \sum_{i=1}^{i=n} h_i t_i.$$

There exits $h_1, ..., h_n \in H^0(U \setminus Y, \mathcal{O}(l))$ such that

$$s_3 = \sum_{i=1}^{i=n} h_i t_i \tag{6.1}$$

$$I = \int_U \frac{\|h\|^2}{\|t\|^{2k}} e^{-(-4\epsilon \log \min(\delta, d_{\partial_{U \setminus Y}}))} dV_{\omega_0} < +\infty \; . \tag{6.2}$$

From the growth condition, the sections $h_1, \ldots, h_n$ define sections $\tilde{s}_i$ of $H^0(U, \mathcal{O}(l) \otimes [Y])$. Let f be a minimal local equation of Y at p and write $h_i = \frac{g_i}{f}$. Then, $f s_3 = \sum_{i=1}^{i=n} g_i t_i$. Hence, $s_3(p) \neq 0$, one of the g_i's has a vanishing order lower than $\mathrm{ord}_p f - 1 = \mathrm{mult}_p Y - 1$. Next the sections g_i globalize as sections $\tilde{s}_i$ of $H^0(U, \mathcal{O}(l) \otimes [Y])$, and one of them satisfies our requirements.

Remark. Since V is compact, $\max_V \|t\|^{2k}$ exists, hence

$$\int_{U \setminus Y} \|h\|^2 e^{-(-4\epsilon \log \min(\delta, d_{\partial_{U \setminus Y}}))} dV_{\omega_0} \quad \leq \quad \max_V \|t\|^{2k} I \tag{6.3}$$

So, rescaling the sections h_i by a linear factor, we may assume that the right hand side is lower than one.

Corollary 6.5 *Under the hypothesis of Theorem 6.4, let $l \geq l_1$. Let $E \to U$ be a line bundle, and let $s \in H^0(U,E) \setminus \{0\}$. Then, for any $k \in \mathbb{N}$ and any $p \in U$, there exists $\check{s} \in H^0(U, E \otimes \mathcal{O}(kl))$ such that $\nu_p((\check{s} = 0)) \leq (\nu_p(s = 0) - k)^+$.*

Proof. First, we prove the corollary for $k = 1$. If the point p does not belong to Z_s, since $\mathcal{O}(l)$ is very ample, the corollary is true. Assume $p \in Z_s$ and let $Y_1, \ldots, Y_r$ be its global irreducible (reduced) components which contain p. Write $Y' = Y_1 \cup \ldots \cup Y_r$. Let $t_1, \ldots, t_r$ be minimal local equations at p for $Y_1, \ldots, Y_r$ respectively, so that $\text{mult}_p Y' = \text{ord}_p t_1 + \ldots + \text{ord}_p t_r$. Let $\tilde{s}' \in H^0(U, \mathcal{O}(l) \otimes [Y'])$ a section as in the Theorem 6.4 and let us write s' for the corresponding meromorphic section of $\mathcal{O}(l)$ over U. We may assume that the polar divisor of s' is $Y_1 + \ldots + Y_{r'}$, with $r' \leq r$. By hypothesis, there exists strictly positive integers $n_1, \ldots, n_r$, such that $s = t_1^{n_1} \ldots t_r^{n_r} e$ where $e \in E_p$ is a local non vanishing germ at p. In the same way, $s' = \dfrac{g}{t_1 \ldots t_r} e'$ where $e' \in \mathcal{O}(l)_p$ is a local non vanishing germ at p, and $\text{ord}_p g \leq \text{mult}_p Y' - 1$. Hence, $\check{s} = s' \otimes s \in H^0(U, E \otimes \mathcal{O}(l))$ and $s' \otimes s = g t_1^{n_1 - 1} \ldots t_r^{n_r - 1} e' \otimes e$. So $\text{ord}_p s' \otimes s \leq \text{mult}_p(Y') - 1 + \text{ord}_p(t_1^{n_1-1} \ldots t_r^{n_r-1}) = \text{ord}_p s - 1$.

Next, assume the corollary is true for some integer $k \geq 1$. Let $\check{s}_k$ denote the corresponding section of $E \otimes \mathcal{O}(kl)$. We apply the step $k = 1$ to $E \otimes \mathcal{O}(kl)$ and $\check{s}_k$ to conclude.

Remark. If we apply this corollary to the line bundle $[D]$, where D is an effective divisor, and to its canonical section, we see that $\mathcal{O}(kl_1) \otimes [D]$ is globally generated outside the analytic subset $\{p \in U \,;\, \nu_p(D) > k\}$.

6.6 Pluricomplete currents.

Definition 6.7 *A closed positive $(1,1)-$current on a complex manifold M will be called pluricomplete current if there exists a closed set L on M such that ω admits local locally bounded potentials on $M \setminus L$, a function $\varphi \in P_\omega(M \setminus L)$ with $\liminf\limits_{M \setminus L \ni p' \to L} \varphi = +\infty$.*

If $\mathbb{P}^k$ is a projective space, we will denote ω_{FS} its Fubiny-Study form without indication of the dimension.

Lemma 6.8 *Let $E \to M$ be a line bundle, with smooth hermitian metric and positive Chern curvature ω_0. Let $s_0, \ldots, s_k \in H^0(M,E) \setminus \{0\}$ be holomorphic sections of E. Let A denote their common zeros locus in M. Let ψ be the associated meromorphic map from M to $\mathbb{P}^k$, given in homogeneous* *coordinate by* $p \to [s_i(p)]_{0\le i\le k}$ *. Then, the function* $p \to -\log \|s\|^2(p)$ *belongs to* $P_{\psi^*\omega_{FS}+\omega_0}(M \setminus A)$ *and satisfies* $\liminf_{M\setminus A \ni p' \to A} \psi = +\infty$.

We give the following example over a projective manifold. We keep notations of the preceding paragraph.

Proposition 6.9 *Let* $U \to V$ *be a locally pseudoconvex domain over* V *and* $E \to U$ *a line bundle over* U*. Let* $s_0, \dots, s_N \in H^0(U, E) \setminus \{0\}$ *and denote* $B = \bigcap_{0\le i\le N} Z_{s_i}$ *their common zero locus. Let* e_α, $0 \le \alpha \le N'$, *be global sections of* $\mathcal{O}(l)$, $l \ge l_1$, *without common zeros. Let* $\psi : U \to \mathbb{P}^{(N+1)(N'+1)-1}$ *be the meromorphic map given in homogeneous coordinate by* $p \mapsto [e_\alpha s_i]_{\alpha,i}(p)$, *which is holomorphic on* $U \setminus B$*. Considers the closed positive* $(1,1)-$*current* $\omega = \psi^*\omega_{FS}$*. Then, there exists* $\varphi \in P_\omega(U \setminus B)$ *with* $\liminf_{U\setminus B \ni z \to B} \varphi(z) = +\infty$.

Proof. Denote B_2 the indeterminacy of ψ. Hence $B = B_1 \cup B_2$ with B_1 an hypersurface and codim$B_2 \ge 2$. ψ is holomorphic on $U \setminus B_2$. The associated bundle morphism $U \times \mathbb{C}^{(N+1)(N'+1)} \to \mathcal{O}(l) \otimes E$ gives an induced hermitian singular metric on $\mathcal{O}(l) \otimes E$ whose curvature is $\omega = \psi^*\omega_{FS}$. By construction, it is smooth on $U \setminus B$. To prove the proposition, it's enough to prove the following claim.

For any $z_0 \in U \setminus B$, there exists real strictly positive constants C_{z_0} and ϵ_{z_0} such that, for any $p \in B$, there exists $\varphi_p \in P_\omega(U \setminus B)$, with

$$\liminf_{U\setminus B \ni z \to p} \varphi_p(z) = +\infty \tag{6.4}$$

$$\forall\, p \in B, \quad \sup_{B(z_0, \epsilon_{z_0})} \varphi_p \le C_{z_0}, \tag{6.5}$$

where $B(z_0, 2\epsilon_{z_0})$ is a ball in a complex analytic chart centred at z_0 and disjoint from B.

Indeed, if this claim is proved then, $\varphi = (\sup_{p\in B} \varphi_p)^*$ will be well defined on $U \setminus B$ due to (6.5). It belongs to $P_\omega(U \setminus B)$ and satisfies $\liminf_{U\setminus B \ni z \to B} \varphi = +\infty$. First, we construct the function $\varphi_p \in P_\omega(U \setminus B)$, $p \in B$. Note Y_i the zero set of the section s_i, $i = 0, \dots, N$. Recall that for each integer $0 \le i \le N$, $p \in Y_i$. From Theorem 6.4 and the remark which follows it, we may construct section $\tilde\beta_i^k \in H^0(U, \mathcal{O}(l) \otimes [Y_i])$, $k = 1, \dots, n$, subject to the following conditions

$$s_p = \sum_{k=1}^{n} \beta_i^k t_k \tag{6.6}$$

$$\int_{U\setminus Y_i} \|\beta_i\|^2 e^{-(-2\log\min(\delta, d_{\partial_{U\setminus Y_i}}))} dV_{\omega_0} \quad \le \quad 1 \tag{6.7}$$

where, $s_p \in H^0(U, \mathcal{O}(l_1+1))$ is non vanishing at p, the section $t_1, \dots, t_n$ belongs to $H^0(V, \mathcal{O}(1))$. We denote by the same symbols their pullback by π and they give coordinates centred at p. Moreover, we consider the sections $\tilde{\beta}_i^k$ as meromorphic sections β_i^k of $\mathcal{O}(l)$ over U, and $\|\beta_i\|^2 = \sum_{k=1}^n |\beta_i^k|^2$. We note that $\beta_i^k \otimes s_i \in H^0(U, \mathcal{O}(l) \otimes E)$ and define

$$\varphi_p = \log \left(\sum_{\substack{1 \le k \le n \\ 0 \le i \le N}} |\beta_i^k \otimes s_i|^2 \right) , \tag{6.8}$$

where the norm of the sections is the induced one. Hence, $\varphi_p \in P_\omega(U \setminus B)$.

To show that $\liminf\limits_{U \setminus B \ni z \to p} \varphi_p = +\infty$, we have outside B

$$\begin{aligned} \sum_{k,\, i} |\beta_i^k \otimes s_i|^2 &= \frac{\sum_{k,\, i} |\beta_i^k \otimes s_i|^2 \cdot \sum_k |t_k|^2}{\sum_k |t_k|^2} && (6.9) \\ &\ge \frac{\sum_i |\sum_k \beta_i^k t_k s_i|^2}{\sum_k |t_k|^2} && (6.10) \\ &= \frac{\sum_i |s_p \otimes s_i|^2}{\sum_k |t_k|^2} && (6.11) \end{aligned}$$

where the sum is over $1 \le k \le n$ and $0 \le i \le N$. Line (6.11) is due to (6.6). Assume $e_0(p) \neq 0$. Recall that $s_p \in H^0(U, \mathcal{O}(l+1))$, hence write locally $s_p = s'_p \otimes e_0$. Next, in each charts $e_0 s_i \neq 0$, $0 \le i \le N$, say $e_0 s_0 \neq 0$, we have

$$\begin{aligned} \sum_{0 \le i \le N} |s_p \otimes s_i|^2 &= |s'_p|^2 \frac{\sum_i |\frac{e_0 s_i}{e_0 s_0}|^2}{\sum_{\alpha,\, i} |\frac{e_\alpha s_i}{e_0 s_0}|^2} && (6.12) \\ &= |s'_p|^2 \frac{\sum_i |\frac{s_i}{s_0}|^2}{\sum_\alpha |\frac{e_\alpha}{e_0}|^2 \cdot \sum_i |\frac{s_i}{s_0}|^2} && (6.13) \\ &= \frac{|s'_p|^2}{\sum_\alpha |\frac{e_\alpha}{e_0}|^2} && (6.14) \end{aligned}$$

The last expression is strictly positive at P, say greater than equal to $2c > 0$, does not depend on i, so

$$\varphi_p \ge -\log(\|t\|^2) + \log c \tag{6.15}$$

in a neighbourhood of p.

Next, we prove the uniform bound in the φ_p. Let $z_0 \in U \setminus B$, and let W be an open chart centred at z_0. Denote $B(z_0, \epsilon)$, $\epsilon > 0$, the induced ball in W, and assume $B(z_0, 1) \subset\subset W$. Let $\frac{1}{2} > \epsilon > 0$, such that $B(z_0, 2\epsilon) \subset\subset U \setminus B$ and such that, say, e_0 is non vanishing on $\bar{B}(z_0, 2\epsilon)$. Let t be a holomorphic section of E, on $B(z_0, 1)$, non vanishing there. Then

$$\sum_{k,i} |\beta_i^k s_i|^2 = \frac{\sum_{i,k} |\frac{\beta_i^k s_i}{e_0 t}|^2}{\sum_{\alpha,i} |\frac{e_\alpha s_i}{e_0 t}|^2} \tag{6.16}$$

Here, only the β_i^k, $1 \leq k \leq n$, $0 \leq i \leq N$, depend on $p \in B$. In the left hand side, the norm symbol represents the induced hermitian metric, in the right hand side it represents a modulus of a holomorphic function. Let $m = \max_{\bar{B}(z_0,\epsilon)} \sum_{k,i} |\beta_i^k s_i|^2 (< +\infty)$, $0 < m_1 = \min_{\bar{B}(z_0,\epsilon)} \sum_{\alpha,i} |\frac{e_\alpha s_i}{e_0 t}|^2$, and $0 < m_2 = \min_{\bar{B}(z_0,2\epsilon)} |e_0|^2$. Then

$$\begin{aligned} m &\leq \frac{1}{m_1} \max_{\bar{B}(z_0,\epsilon)} \sum_{i,k} |\frac{\beta_i^k s_i}{e_0 t}|^2 && (6.17) \\ &\leq \frac{C(\epsilon, n)}{m_1} \sum_{i,k} \int_{B(z_0,2\epsilon)} |\frac{\beta_i^k s_i}{e_0 t}|^2 dV_{\omega_e} && (6.18) \\ &\leq \frac{C(\epsilon, n)}{m_1} \sum_{i} \int_{B(z_0,2\epsilon)\setminus Y_i} \left(\sum_k |\frac{\beta_i^k}{e_0}|^2 \right) |\frac{s_i}{t}|^2 dV_{\omega_e} && (6.19) \\ &\leq \frac{C(\epsilon, n)}{m_1} \sum_{i} \int_{B(z_0,2\epsilon)\setminus Y_i} \frac{\|\beta_i\|^2}{|e_0|^2} \gamma_i \times |\frac{s_i}{t}|^2 \frac{1}{\gamma_i} dV_{\omega_e} && (6.20) \end{aligned}$$

with $\gamma_i = \min(\delta, d_{U\setminus Y_i})$ and ω_e is the usual Kähler metric on $\mathbb{C}^n$. Next, there exists a constant A such that $|\dfrac{s_i}{t}|^2 \dfrac{1}{\gamma_i} \leq A$ on $B(z, 2\epsilon) \setminus Y_i$ for any i, since $|\frac{s_i}{t}|^2$ is lipchitzian and vanishes on Y_i. Hence

$$m \leq \frac{C(\epsilon, n)}{m_1.m_2} C'(\epsilon) A \times (N+1) \tag{6.21}$$

where $C'(\epsilon)$ bounds the ratio of the Euclidean volume form and the Kähler one and $N+1$ appear since the vector $(\beta_i^1, \dots, \beta_i^n)$ belongs to the unit ball of $L^2(U \setminus Y_i, \gamma_i dV_{\omega_0})$ by (6.7).

Corollary 6.10 *Let $U \to V$ be a locally pseudoconvex domain over the projective manifold V,* $\dim V \geq 2$*. Let Y be an effective divisor on U. Then $[Y] \otimes \mathcal{O}(kl_1)$ is spanned by its global sections outside $E_{k+1}(Y) = \{p \in U : \nu_p(Y) \geq k+1\}$. If $k \geq 1$, it admits a singular hermitian metric of positive curvature, which is smooth away from $E_k(Y)$ and is a pluricomplete positive current in U.*

Proof. The first assertion is the content of Corollary 6.5 (in particular $[Y] \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric with a positive Chern current which are smooth away from $E_{k+1}(Y)$). Next, let $k \geq 1$. By a Baire argument, we select $N+1 \geq\geq n+1$ sections in $H^0(U, [Y] \otimes \mathcal{O}((k-1)l_1))$, which together span $[Y] \otimes \mathcal{O}((k-1)l_1)$ away from $B \subset E_k(Y)$. When applying Proposition 6.9 to this set of sections, we obtain a singular metric for the line bundle $[Y] \otimes \mathcal{O}(kl_1)$, which is smooth away from B, and is pluricomplete.

Remark.

i. In the construction of Proposition 6.9, we may select the sections e_α such that the holomorphic map given by them is biholomorphic onto its image (see [18]). In particular, the current $\psi^*\omega_{FS}$ obtained is strictly positive. Moreover, adding some pullback by π of elements in $H^0(V, \mathcal{O}(l_1))$, we may always assume that $\psi^*\omega_{FS} \geq C\omega_0$, where C is a strictly positive constant.

ii. Let D denote the fixed part (as a divisor) of the linear system generated by $s_0, \dots, s_N$. Then $\psi^*\mathcal{O}(1)_{\mathbb{P}(N+1)(N'+1)-1} \simeq E \otimes \mathcal{O}(l_1) \otimes [-D]$ over $U \setminus B_2$. Via this isomorphism, the sections give a hermitian metric on $E \otimes \mathcal{O}(l_1) \otimes [-D]$ which is smooth away from B_2. Let $Y_{i,k}$ denote the polar divisor of β_i^k. Then $\beta_i^k s_i$ defines a global section of $E \otimes \mathcal{O}(l_1) \otimes [-D] \otimes [Y_{i,k}]$. This may be a short way to understand that the function φ_p is non bounded near p. However, we need the full precision of equation (6.6) to deduce that $\liminf_{U \setminus B \ni z \to p} \varphi_p(z) = +\infty$.

iii. Let Y be an effective divisor on U. In general the set $\{p \in U : \nu(p) \geq k\}$ is not of codimension at least two. However, let U_1 be a relatively compact domain in U. Let Y_2 denote the largest divisor such that $Y \geq Y_2$ and $\mathrm{supp}Y_2 \cap U_1 = \emptyset$. Denote $Y = Y_1 + Y_2$. Since $U \setminus \mathrm{supp}Y_2$ is pseudoconvex in U, the function $-\epsilon \log(\min(\delta, d_{\partial U \setminus \mathrm{supp}Y_2}))$ belongs to $P_{\omega_0}(U \setminus \mathrm{supp}Y_2)$, for $\epsilon > 0$ small enough. In particular, hulls of U_1 with respect to some current $\omega \geq \omega_0$, will not contain $\mathrm{supp}Y_2$. Then, there exists $k_0 \in \mathbb{N}$ such that $\{p \in U : \nu_p(Y_1) \geq k\}$ is at least of codimension two for $k \geq k_0$.

iv. Let ω be a closed positive $(1,1)-$current on a complex manifold M. Assume that it admits local locally bounded potentials on $M \setminus B$, where B is an analytic subset of M. Assume that for any $p \in B$, there exists a function

$\varphi_p \in P_\omega(M \setminus B)$ such that $\liminf_{M \setminus B \ni z \to p \in B} \varphi_p = +\infty$. For any relatively compact open subset U in $M \setminus B$, let U_1 denote the interior of $U(0,\omega) \cup B$, which is locally pseudoconvex in M (see section 4). Then by definition of $U(0,\omega)$, there exists $\varphi \in P_\omega(U_1 \setminus B)$ such that $\liminf_{U_1 \setminus B \ni z \to p \in B} \varphi = +\infty$.

v. Let $E \to U$ be a line bundle which admits a singular metric with a positive current curvature. Let $\mathcal{I}$ denote its Nadel multiplier ideal sheaf (see [14] for a definition). Using standard L^2 methods (see [15], prop. 4.2.1 in the compact case), we see that $E \otimes \mathcal{O}(l_0) \otimes \mathcal{I}$ is spanned by its global sections. Hence, let us assume that $E \otimes \mathcal{I}$ is spanned by its global sections. Let s be a global section. To each $p \in Z_s$, we may associated the meromorphic sections β^k of $\mathcal{O}(l_1)$, which are holomorphic on $U \setminus Z_s$ (*i.e.* associated to sections $\tilde{\beta}^k \in H^0(U, \mathcal{O}(l_1) \otimes [Z_s])$ and which satisfies the usual ideal relation (6.6)). We obtain then new sections $\beta^k \otimes s \in H^0(U, \mathcal{O}(l_1) \otimes E)$. Making this procedure for any $s \in H^0(U, E \otimes \mathcal{I})$ and any $p \in Z_s$, we obtain a set of global section G_1 of $\mathcal{O}(l_1) \otimes E$. Let $\mathcal{I}_1$ denote the coherent ideal sheaf it generates. Then $\mathcal{I} = \mathcal{I}_0 \subset \mathcal{I}_1$. Working with G_1 as before, we obtain a set G_2 of global section of $\mathcal{O}(2l_1) \otimes E$ which defines an ideal sheaf $\mathcal{I}_2$, and so on. This procedure gives a sequence of coherent ideal sheafs $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \ldots$. By Notherian properties, this sequence become locally stationary equal to the structure sheaf $\mathcal{O}$ (as was shown). For a point $p \in U$, we may define $m(p)$ to be the least integer such that $(\mathcal{I}_k)_p = \mathcal{O}_p$ for any $k \geq m(p)$. By construction the set $M_l = \{p \in U \ : \ m(p) > l\}$ are analytic subsets in U. Summing up our results, the line bundle $E \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric with a positive Chern current which are smooth away from M_k (due to spannedness). If $k \geq 1$, the line bundle $E \otimes \mathcal{O}(kl_1)$ admits a singular hermitian metric, with a Chern current ω_k, which are smooth on $U \setminus M_{k-1}$ and ω_k is pluricomplete. There exists $\varphi \in P_{\omega_k}(U \setminus M_{k-1})$ with $\liminf_{U \setminus M_{k-1} \ni z \to p \in M_{k-1}} \varphi = +\infty$.

7 Some Hartogs' phenomenon in projective manifolds.

We recall the definition of pseudoconcave manifold in the sense of Andreotti [2] and some properties of them.

Definition 7.1 *Let X be a normal complex space of pure dimension $n \geq 2$. For $V \subset U$ open subsets of X, we define the hull of V in U by*

$$\widehat{V}_U = \{x \in U \ : |f(x)| \leq \sup_V |f|, \forall f \in \mathcal{O}(U)\} .$$

An open subset $Y \subset X$ is said to be pseudoconcave at the boundary point $P \in \partial_X Y$ (topological boundary) if there exists $\{W_\alpha\}_\alpha$, an open basis of P in X, s.t. P is an interior point of $\widehat{W_\alpha \cap Y}_{W_\alpha}$. X is said to be pseudoconcave in the sense of Andreotti, if there exists Y, an open relatively compact subset of X, which is pseudoconcave in each of its boundary point.

Remark. No boundary condition on X is assumed.

In [16], the following proposition, which may be thought as a geometric version of the Hartogs' theorem, is proved.

Proposition 7.2 *Let Ω be an open pseudoconcave subset of the projective manifold V. Assume that Ω is locally pseudoconvex in V, then $\partial_V \Omega$, the topological boundary of Ω in V, is a compact hypersurface. Hence if Ω' is a pseudoconcave open subset of the projective manifold V, then $V \setminus \Omega'$ contains a maximal compact hypersurface H (which may be empty). Moreover, if* $\dim_{\mathbb{C}} V = 2$, *then H may be blow down onto points.*

Notice that for $\dim V \geq 3$, there exists example of hypersurface H as in the proposition below (that is $V \setminus H$ is a pseudoconcave domain in the sense of Andreotti), such that no irreducible component of H may be blow down. Indeed, let V be a projective manifold of dimension $n \geq 2$, and let $(L, h) \to V$ be a hermitian line bundle with curvature form ω. Assume ω has one strictly positive eigenvalue and another one strictly negative. Then, the real hypersurface, in $L \hookrightarrow \mathbb{P}(L \oplus \mathbb{C})$, given as $\{\zeta \in L \ : \ h(\zeta) = 1\}$ is pseudoconcave, but the zero section (or the hyperplan to infinity) does not contract to a lower dimensional analytic set in general.

The purpose of the section is to prove the existence of a Hartogs' phenomena in projective manifolds for meromorphic maps with value in a projective manifold (see Corollary 7.6 below). However before to state it, we want to prove an extension theorem for currents which implies, in the projective case, a result of Nadel-Tsuji [28]. In our opinion it explains the main arguments of the Hartogs' Theorem below. For meromorphic maps (or more singular integral currents), there is an indeterminacy locus. We have explained, in the last section, a way to deal with if the target space is a projective manifold. We hope to return to this problem for a general meromorphic map later.

Theorem 7.3 *Let $V = (V, \mathcal{O}(1))$ be a projective manifold,* $\dim V \geq 2$. *Let H be a hypersurface in V such that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let U be an open neighbourhood of H in V. Let ω be a $(1,1)-$closed positive current on $U \setminus H$ which admits local locally bounded potentials. Then*

$$\int_{K \setminus H} \omega^n < +\infty \,, \tag{7.1}$$

for any compact set K in U. Moreover, if $1 \leq k \leq n$ then ω^k extends as a closed positive currents through H.

Proof. We may assume that U does not intersect Y, the subset which gives the pseudoconcavity condition on $V \setminus H$ (see Definition 7.1). Let U_1 be a relatively compact subset in U which contains $H \cup K$. From proposition 7.2, write $H' = H \cup H_1$ for the maximal compact hypersurface contained in U_1. Next, we may assume that K is a compact subset in U_1 which contains a neighbourhood of H' and that $\overline{\mathring{K}} = K$. Let ω_0 be the Chern curvature of the line bundle $\mathcal{O}(1)$, and denote $\omega_1 = \omega + \omega_0$. Let X be the open subset of $U \setminus H'$ where the family $P_{\omega_1}(U \setminus H', U_1 \setminus K, 0)$ is locally bounded from above (see 4.10). From Lemma 4.5, X is locally pseudoconvex in $U \setminus H'$ and contains $U_1 \setminus K$. Note that $(V \setminus K) \cup X$ is locally pseudoconvex in V. Since it contains Y, it is pseudoconcave in the sense of Andreotti. Hence, from proposition 7.2, $(V \setminus K) \cup X = V \setminus H'$, for H' is the maximal compact hypersurface in K. From Takeuchi's theorem 6.3, there exists $\delta, \epsilon > 0$ and a constant C, such that $\psi_1 = -\epsilon \log(\min(\delta, d_{\partial V \setminus H'})) - C$ belongs to $P_{\omega_1}(U \setminus H', U_1 \setminus K, 0)$, since $\omega_1 \geq \omega_0$. In particular, the extremal function φ^* associated to $P_{\omega_1}(U \setminus H', U_1 \setminus K, 0)$, being greater than ψ_1, satisfies $\{\varphi^* \leq c\} \cap K$ is a relatively compact subset of $K \setminus H'$ for any $c \in \mathbb{R}$. We are in the situation describe in Proposition 5.2, hence

$$+\infty > \int_{\partial K} (\omega_1 + dd^c \varphi^*)^n \geq \int_{K \setminus H'} (\omega + \omega_0)^n \,. \tag{7.2}$$

We deduce that the closed positive currents ω^k, $k = 1, \ldots, n$, have finite trace measure near H. Hence they extend as closed positive currents through H (see *e.g.* [33, 37]).

Corollary 7.4 *Let H be a hypersurface in a projective manifold V, $\dim V \geq 2$. Assume that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let U be a neighbourhood of H. Let $f : U \setminus H \to M$ be a holomorphic map into the compact Kähler manifold (M, ω_1). Then f extends as a meromorphic map through H.*

Proof. Theorem 7.3 applied to $\omega = f^*\omega_1 + \omega_0$, implies that the graph of h is of finite volume near $H \times M$. Hence it extends through it.

We state the following extension theorem for currents defined by divisors, which will yield as a corollary a Hartogs' extension theorem.

Theorem 7.5 *Let V be a projective manifold,* $\dim V \geq 2$. *Let H be a compact complex hypersurface in V. Assume that $V \setminus H$ is pseudoconcave in the sense of Andreotti. Let U be an open subset of V which contains H. Let $\pi : W_1 \to V$ be a locally pseudoconvex spread domain over V which contains $U \setminus H$. Then any complex hypersurface Z of W_1 extends through H.*

Proof. Denote $\mathcal{O}(1)$ the line bundle which gives the projective embedding of V. As usual, we will denote by the same symbols pullbacks by π of the line bundle $\mathcal{O}(l)$, $l \in \mathbb{N}$, and of ω_0, the Chern curvature of $\mathcal{O}(1)$.

In the following, we assume that H is not a subset of W_1. Shrinking U if necessary, we may assume that H is the maximal compact hypersurface in U (according to Proposition 7.2), that ∂U the topological boundary of U in W_1 is relatively compact in W_1 and that U does not intersect Y (the relatively compact open subset in $V \setminus H$ which gives the pseudoconcavity condition on $V \setminus H$, see Definition 7.1). Let $Z \hookrightarrow W_1$ a (reduced) complex hypersurface in W_1. Let X be a relatively compact open neighbourhood of ∂U in W_1. We may assume $\mathring{\bar{X}} = X$. Let $m = \max_{p \in \bar{X}} \mathrm{mult}_p Z$. From Corollary 6.10 (see the proof of the second assertion), sections $s_0, \ldots, s_r \in H^0(W_1, \mathcal{O}((m+1)l_1) \otimes [Z])$ exist such that

- the meromorphic map ψ, from W_1 to $\mathbb{P}^r$, given by $z \to [s_i(z)]_{0 \leq i \leq r}$ has base points B contained in $E_{m+1}(Z) = \{z \in W_1,\ \mathrm{mult}_z Z \geq m+1\}$,
- the current $\omega = \psi^*(\omega_{FS})$ is strictly positive, and pluricomplete in W_1.

Moreover, by adding a non trivial section of $\mathcal{O}((m+1)l_1) \simeq \mathcal{O}((m+1)l_1) \otimes [Z] \otimes [-Z]$, we may assume s_0 is vanishing on Z. Let $\hat{X}$ denote the pseudoconvex hull of X in W_1. Then $\hat{X}$ contains $U \setminus H$. For, $(V \setminus U) \cup (X \cap U)$ is a locally pseudoconvex domain which is pseudoconcave and H is the maximal compact hypersurface in U, see Proposition 7.2. Next, let $X(0, \omega + \omega_0)$ the pseudoconvex hull of X in $W_1 \setminus B$ with respect to $\omega + \omega_0$ (as defined in section 4.10). We claim that $X(0, \omega + \omega_0) \cap U = U \setminus (H \cup B)$. Indeed, by Lemma 4.6, X' the interior of $X(0, \omega + \omega_0) \cup B$ is a pseudoconvex subset in W_1 which contains X. Hence X' contains $\hat{X}$. From the description of $\hat{X}$, we deduce $X(0, \omega + \omega_0) \cap U = U \setminus (H \cup B)$. In particular, those connected components of $\hat{X} \setminus \bar{X}$ which meet U are pseudoconcave ends (with respect to $P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$).

Next, we look to the extremal function $\varphi^* \in P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$ associated to X. We claim that $U \cap \{\varphi^* < t\} \subset\subset \bar{U} \setminus (H \cup B)$, for all $t \in \mathbb{R}$.

Since $\omega + \omega_0 \geq \omega_0$, from Theorem 6.3, there exists strictly positive constants δ, ϵ (small enough), and C (large enough), such that the function $\varphi_1 = \big(-\epsilon \log \min(\delta, d_{\partial V \setminus H}) - C\big)^+$ belongs to $P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$. Recall that to show that ω is pluricomplete on W_1, we have constructed the function $\varphi_2' \in P_\omega(W_1 \setminus B)$ in Proposition 6.9, which satisfies $\liminf_{W_1 \setminus B \ni z \to B} \varphi_2'(z) = +\infty$. It gives us a function $\varphi_2 = \left(\varphi_2' - \max_{\bar{X}} \varphi_2'\right)^+$ which belongs to $P_\omega(W_1 \setminus B, X, 0)$ and satisfies $\liminf_{W_1 \setminus B \ni p \to B} \varphi_2 = +\infty$, since $E_{m+1}(Z) \cap \bar{X} = \emptyset$. Hence $\max(\varphi_1, \varphi_2)$ belongs to $P_{\omega+\omega_0}(W_1 \setminus B, X, 0)$ and satisfies the exhausting condition required above. So does φ^*.

From Proposition 5.2, we obtain

$$\int_{U \setminus (X \cup B \cup H)} (\omega + \omega_0)^n \;\leq\; \int_{\partial X \cap U} (\omega + \omega_0 + dd^c \varphi)^n < +\infty \,. \qquad (7.3)$$

In particular, all the Chern numbers $\int_{U \setminus (X \cup B \cup H)} \omega^k \omega_0^{n-k}$ are finite. Hence the graph of the meromorphic map ψ is of finite volume near $H \times \mathbb{P}^1$. So ψ extends through H. Hence $Z \subset Z_{s_0}$ extends through H.

We obtain the following Hartogs' Theorem type which strengthened results in [16].

Corollary 7.6 (Hartogs' Kugelsatz) *Let U be an open subset of the projective manifold V, $\dim V \geq 2$. Assume that $V \setminus \bar{U}$ is a connected pseudoconcave open subset of V, and assume $\mathring{\bar{U}} = U$. Let H denote the maximal compact hypersurface in U, and let $F \to V$ be a holomorphic vector bundle over V. Then any meromorphic section s of F defined on a neighbourhood of the boundary of U extends to a meromorphic section of F on U. Moreover, any holomorphic section s of F extends to a meromorphic section on U which is holomorphic on $U \setminus H$.*

Proof. The first part of the proof appears already in [16]. There the arguments needed are developed. Hence, we will only sketch this first part.

α) We recall the meaning of the topological condition. As in [3], from the exact sequence

$$0 \to H^0(V, \mathcal{O}) \to H^0(V \setminus U, \mathcal{O}) \to H^1_{\mathrm{comp}}(U, \mathcal{O}) \to H^1(V, \mathcal{O}) \to H^1(V \setminus U, \mathcal{O}) \,,$$

and since $V \setminus U = \overline{V \setminus \bar{U}}$, we deduce that the last arrow is injective. As $\mathbb{C} = H^0(V, \mathcal{O}) = H^0(V \setminus \bar{U}, \mathcal{O})$, we deduce that each connected component of U has a connected topological boundary. We may assume U connected with connected topological boundary.

Next, we recall that it is enough to work with meromorphic functions and fix some notations. Let W be a connected neighbourhood of the topological boundary of U. Let W_1 denote the domain of holomorphic existence of any holomorphic section on W of any holomorphic vector bundle over V. Since over open ball in V, any holomorphic vector bundle is trivial, $W_1 \to V$ is locally pseudoconvex. From [17], $W_1 \to V$ is the domain of holomorphic existence of the algebra $\bigoplus_{n\in\mathbb{N}} H^0(W, \mathcal{O}(n))$. Let W_2 denote the hull of meromorphy of W with respect to any meromorphic section on W of any holomorphic vector bundle over V (see [17]). Any meromorphic section of F on W defines a meromorphic map from W to $\mathbb{P}(F \bigoplus \mathbb{C})$. For any such vector bundle F, $\mathbb{P}(F \bigoplus \mathbb{C})$ is projective, hence $W_2 \to V$ is the meromorphic hull of W.

Using Proposition 7.2, we proved the following theorem (see [16]).

Theorem Under the hypothesis of the theorem, let $A \to V$ be a locally pseudoconvex domain over V which contains W. Then A contains $U \setminus H$ and W_1.

Hence

$$W \cup (U \setminus H) \hookrightarrow W_1 \hookrightarrow W_2\,. \tag{7.4}$$

If H is the empty set the corollary is proved.

β) Assume H is non void. It is enough to prove that, if $\pi : W_1 \to V$ is a locally pseudoconvex domain over V, which admits a section along $U \setminus H$, then any meromorphic function in W_1 extends meromorphically through H. We will prove that its graph, in $W_1 \times \mathbb{P}^1$ extends through $H \times \mathbb{P}^1$ (see however remark below). First, note that $H \times \mathbb{P}^1$ is a hypersurface in $V \times \mathbb{P}^1$ s.t. $(V \setminus H) \times \mathbb{P}^1$ is pseudoconcave in the sense of Andreotti. Indeed let Y denote the open subset in $V \setminus \bar{U}$ which gives the pseudoconcavity condition (see Definition 7.1). Then $Y \times \mathbb{P}^1$ has a pseudoconcave boundary in the sense of Andreotti. Let $p \in \partial Y$, $W_1 \subset W_2$ open neighbourhood of p in V such that $W_1 \subset \widehat{Y \cap W_2}_{W_2}$. Let Δ denote the unit disc in $\mathbb{C}$. For any $h \in \mathcal{O}(W_2 \times \Delta)$, and any point $(z,t) \in W_1 \times \Delta$, we have $|h(z,t)| \le \sup_{W_2 \cap Y} |h(\,.\,,t)| \le \sup_{(W_2\cap Y)\times\Delta} |h|$. Hence $W_1 \times \Delta$ is in the hull of $(W_2 \cap Y) \times \Delta$ with respect to $W_2 \times \Delta$ (see Definition 7.1). Next, we notice that $W_1 \times \mathbb{P}^1 \to V \times \mathbb{P}^1$ is a locally pseudoconvex domain over $V \times \mathbb{P}^1$ and that it contains $(U \setminus H) \times \mathbb{P}^1$. Hence, from Theorem 7.5, we conclude the proof.

Remark.

i. Another way of proving the corollary goes as follow.

In the above situation, any hypersurface of W_1 extends through H (this is the main point). Hence, any meromorphic function f on W_1 satisfies that any of its level set extends through H. So we may find a point $p \in H$, which admits a neighbourhood W_p in V such that $W_1 \setminus H$ does not meet the polar set, the zero set of f nor its level set $\{f = 1\}$. By shrinking W_p if necessary, in suitable coordinates on W_p, we may write, $W_p = (H \cap W_p) \times \Delta$, where Δ is the unit disc in $\mathbb{C}$. We apply then the big Picard's theorem (see [1]) to the holomorphic function $f_{|W_p}$ restricted on each slice $\{p'\} \times (\Delta \setminus \{0\})$, $p' \in H \cap W_p$. These restrictions are holomorphic functions on $\Delta \setminus \{0\}$, which omit two values, hence they extend to Δ. By Hartogs-Levi theorem, our meromorphic function extends to $(U \setminus H) \cup W_p$ and by the Thullen extension theorem, it extends through each irreducible component of H which meet W_p.

ii. We used that $(V \setminus H) \times \mathbb{P}^1$ is pseudoconcave in the sense of Andreotti. However, pseudoconvex hulls behave functorialy under fibre product. The last corollary still holds under the technical assumption that the pseudoconvex hull of a neighbourhood of ∂U contains $U \setminus H$.

iii. We know, using results of S. Ivashkovich [23] and result from [17] that, in the above situation, if $f : W(\partial U) \to M$ is a meromorphic map from a neighbourhood $W(\partial U)$ of U to a complex compact Kähler manifold (M, ω_1), then f extends meromorphically to $U \setminus H$. However, we do not know at that time if $\omega_0 + f^*\omega_1$ is a pluricomplete current.

8 *

References

[1] L. V. Ahlfors. *Conformal invariants. Topics in geometric function theory.* McGraw-Hill series in higher mathematics. McGraw-Hill Book Conpany, 1973.

[2] A. Andreotti. Théorème de dépendance algébrique sur les espaces complexes pseudo-concaves. *Bull. Soc. Math. France*, 91: p.1–38, 1963.

[3] A. Andreotti and G. Tomassini. *Some remarks on pseudoconcave manifolds*, pages 85–104. Essays in topology and related topics. Mémoires dédiées à G. de Rham. Springer, 1970.

[4] S. Asserda. The levi problem on projective manifolds. *Math. Zeit.*, 219: p. 631–636, 1995.

[5] T. Aubin. *Nonlinear analysis on Manifolds. Monge-Ampere equations*, volume 252 of *Grund. der math. Wissenschaften*. Springer-Verlag, 1982.

[6] E. Bedford and B.A. Taylor. A new capacity for plurisubharmonic functions. *Acta Mathematica*, 149: p. 1–40, 1982.

[7] E. Bedford and B.A. Taylor. Fine topology, Šilov boundary, and $(dd^c)^n$. *Journal of Functional analysis*, 72: p.225–251, 1987.

[8] E. Bedford and B.A. Taylor. Plurisubharmonic functions with logarithmic singularities. *Ann. Inst. Fourier*, 38(4): p.133–171, 1988.

[9] E. Bedford and B.A. Taylor. Uniqueness for the complex Monge-Ampere equation for functions of logarithmic growth. *Indiana U. Math. J.*, 38: p.455–469, 1989.

[10] H.J. Bremermann. On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains. Characterization of Shilov boundaries. *Trans. of the Ame. Math. Soc.*, 91: p. 246–276, 1959.

[11] J. P. Demailly. Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semipositif au dessus d'une variété kählérienne complète. *Ann. scient. Éc. Norm. Sup.*, 15(4): p. 457–511, (1982).

[12] J.P. Demailly. Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines. *Mémoires. de la Soc. Math. de France*, 19: p.123, 1985.

[13] J.P. Demailly. Regularization of closed positive currents and intersection theory. *J. Algebraic Geometry*, 1: p. 361–409, 1992.

[14] J.P. Demailly. Théorie de Hodge L^2 et théorèmes d'annulation. In *Introduction à la théorie de Hodge*, volume 3 of *Panoramas et Synthèses*. Soc. Math. de France, 1996.

[15] J.P. Demailly. Méthodes L^2 et résultats effectifs en géométrie algébrique. *Séminaire Bourbaki*, n° 852:32 pages, 1998–99. 51ème année.

[16] P. Dingoyan. Un phénomène de Hartogs dans les variétés projectives. To appear in Math. Zeith.

[17] P. Dingoyan. Un théorème d'Oka-Levi pour les domaines étalés au dessus de variétés projectives. To appear in Bulletin des Sciences Mathématiques.

[18] P. Dingoyan. Fonctions méromorphes sur un ouvert localement pseudoconvexe étalé au dessus d'une variété projective. *C. R. Acad. Sci. Paris*, 324(1): p. 817–822, 1997.

[19] K. Docquier and H. Grauert. Levisches problem und Rungerscher Satz für Teilgebiete Steincher Mannigfaltigkeiten. *Math. Ann.*, 140: p. 94–123, 1960.

[20] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley Classics Library. John Wiley and Sons, inc, 1 edition, 1994.

[21] V. Guedj. Approximation of currents on complex manifolds. *Preprint of Université Paris-Sud*, (97-80), 1997.

[22] A. Huckleberry and W. Stoll. On the thickness of the Shilov boundary. *Math. Annalen*, 207: p. 213–231, 1974.

[23] S. Ivashkovich. The hartogs-type extension theorem for the meromorphic maps into compact kähler manifolds. *Invent. Math.*, 109: p. 47–54, 1992.

[24] M. Klimek. *Pluripotential theory.* London Mathematical Society Monographs, new series. Oxford University Press, 1991.

[25] P. Lelong. *Plurisubharmonic functions and positive differential forms.* Notes on math. and its appl. Gordon and Breach Science Publishers, 1969.

[26] H. El Mir. Sur le prolongement des courants positifs fermés de masse finie. *C. R. Acad. Sc. Paris*, 264: p. 181–184, 1982.

[27] N. Mok and J.-Q. Zhong. Compactifying complete Kähler-Einstein manifolds of finite topological type and bounded curvature. *Ann. of Math.*, 129: p. 427–470, 1986.

[28] A. Nadel and H. Tsuji. Compactification of complete Kähler manifolds of negative curvature. *J. Diff. Geo.*, 28: p. 503–512, 1988.

[29] T. Napier and M. Ramachandran. The L^2 method, weak lefschetz theorems, and the topology of kähler manifolds. *Preprint.*

[30] K. Oka. Sur les fonctions de plusieurs variables IX. Domaine fini sans point critique intérieur. *Jap. Jour. Math.*, 23: p. 97–155, 1953.

[31] K. R. Pinney. Line bundle convexity of pseudoconvex domains in complex manifolds. *Math. Zeit.*, 206: p. 605–605, 1991.

[32] A. Sadullaev. Plurisubharmonic functions. In *Several complex variables II, E.M.S.*, volume 8. Springer Verlag.

[33] N. Sibony. Quelques problemes de prolongements de courants en analyse complexe. *Duke Math. J.*, 52: p. 157–197, 1985.

[34] Y. T. Siu. Extension of meromorphic maps into Kähler manifolds. *Ann. Math.*, 102: p. 421–462, 1975.

[35] Y.-T. Siu and S.-T. Yau. Compactification of negatively curved complete Kähler manifolds of finite volume. In *Seminar on differential Geometry*, volume 102 of *Ann. of Math. Studies*. Princeton University Press, Princeton, NJ, 1982.

[36] H. Skoda. Morphismes surjectifs et fibrés linéaires semi-positifs. In *Séminaire Pierre Lelong-Henri Skoda*, volume 822 of *Lecture Notes in Mathematics*. Springer Verlag, 1978–79.

[37] H. Skoda. Prolongement des courants, positifs, fermés, de masse finie. *Invent. Math.*, 66: p. 361–376, 1982.

[38] A. Takeuchi. Domaines pseudoconvexes sur les variétés kählériennes. *J. Math. Kyoto Univ.*, 6: p. 323–357, 1967.