Similarity problem for certain martingale uniform algebras*

S. V. Kislyakov

The long standing question about similarity to a contraction of every polynomially bounded operator was recently settled by a counterexample in Pisier's paper [Pi1]. For the history of the problem, the reader may refer to the same paper. We also mention [D-P] and Ki2] for other expositions.

In different terms, the result of Pisier can be stated as follows: there exists a bounded but not completely bounded (the definition is given below) homomorphism $\varphi : C_A \to B(H)$, where $C_A = \{f \in C(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0\}$ is the disk algebra, and $B(H)$ is the algebra of all bounded operators on Hilbert space. It is a natural to ask whether C_A can be replaced here by other proper uniform algebras. This question was mentioned in [Pi2].

We remind the reader that a *uniform algebra* is a closed subalgebra A of $C(X)$ that contains the constant functions and separates the points of X; A is said to be *proper* if $A \neq C(X)$. A bounded linear map $T : A \to B(H)$ is *completely bounded* if there is a constant C such that for every n, every matrix $\{\varphi_{ij}\}_{1\le i,j\le n}$ with entries in A, and every vectors $x_1, \dots, x_n, y_1, \dots, y_n \in H$ we have

$$\|\sum_{i,j}\langle T(\varphi_{ij})x_j, y_i\rangle\| \le C \sup_t \|\{\varphi_{ij}(t)\}\|_{M_n}(\sum_j \|x_j\|^2)^{1/2}(\sum_i \|y_i\|^2)^{1/2},$$

where M_n is the space of $(n \times n)$-matrices endowed with the norm of $B(\ell^2_n)$. (Throughout, angular brackets stand for the scalar product in a relevant Hilbert space, which is sometimes indicated explicitly, like this: $\langle\cdot,\cdot\rangle_K$). The best possible constant C is denoted by $\|T\|_{cb}$.

The conjecture that every proper uniform algebra admits a bounded but not completely bounded homomorphism to $B(H)$ looks quite natural. However, this has not yet been proved or disproved. The situation resembles that of the Glicksberg question before it was solved in the positive in [Ki1]. Glicksberg asked in 1964 if every proper uniform algebra is uncomplemented in $C(X)$. It turned out finally that this can be proved by transferring an argument suitable for the disk algebra to the abstract setting; however, in most of the specific examples, the uncomplementedness had been apparent prior to [Ki1].

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To transfer Pisier's counterexample to a uniform algebra A, it would suffice, e.g., that the disk algebra be completely isomorphic to a quotient algebra of A, and again, in many specific cases this can indeed be ensured. For instance, if $U \subset \mathbb{C}^n$ is a bounded strinctly pseudoconvex domain with smooth boundary, and A is the algebra of all functions analytic in U and continuous in $\bar{U}$, then Theorem 5.12 in [Kh] can be used to construct such a quotient. That theorem is about bounded analytic extension to U of bounded functions originally defined and holomorphic on a section of U by some analytic manifold; the complete isomorphism mentioned above follows from the fact that extension is done by a linear operator.

However, for an abstract uniform algebra A a similar statement is unavailable. Moreover, any reasonable approach seems to require the existence of certain traces of complex analytic structure on an abstract algebra A; the problem is that these must be produced basically from "nothing". In [Ki1] a statement of this sort was really found for the setting of the Glicksberg question, but that statement is insufficient in our situation, though enabled Pisier to prove a partial result (see [Pi2, Theorem 5.1]).

The purpose of this note is to present another partial result. In it the condition imposed on a uniform algebra A is quite mild: A must only possess a nontrivial bounded point derivation. The drawback is that a bounded but not completely bounded homomorphism to $B(H)$ will be constructed not on A itself, but on a certain *martingale extension* of A.

The most well-known example of a martingale extension is the algegra of Hardy martingales. Consider the infinite product $\mathbb{T}^{\mathbb{N}}$. This space has a natural filtration of σ-algebras: the n-th σ-algebra consists of the sets depending on the first n coordinates only. A martingale adapted to this filtration is called a *Hardy martingale* if for each n the conditional expectation of it relative to the n-th algebra is analytic (i.e., it belongs to H^1) in the variable z_n. The Hardy martingales with limit function continuous on $\mathbb{T}^{\mathbb{N}}$ constitute a uniform algebra $\mathcal{H}$; it is natural to refer to $\mathcal{H}$ as to a martingale extension of C_A.

In an unpublished preliminary version of [Pi1], as an intermediate step towards the counterexample for the disk algebra, it was shown that $\mathcal{H}$ admits a bounded but not completely bounded homomorphism to $B(H)$). (In the final publication, the disk algebra result was proved directly, which made the statement for $\mathcal{H}$ trivial.) It turns out that this martingale construction of Pisier can be carried over to an abstract setting. Some modifications of Pisier's arguments are needed, but they are really slight, so that the proof of the main result (Theorem 1 below) is not entirely new. Nor does the statement tell us too much in fact, but it gives some hope in further attempts to resolve the problem for an *abstract* uniform algebra under *minimal* assumptions.

Also, Theorem 1 suggests that it would probably be interesting to study the problem in question for Cole-type uniform algebras, i.e., proper algebras for which every point of the maximal ideal space is a peak point (see [Br], [Ba]). Such an algebra has only zero bounded point derivations.

Now, we pass to formal presentation. Let A be a uniform algebra on a compact space X, and let μ be a probability measure on X multiplicative on A. Consider the product space $(X^\infty, \mu^\infty) = (X, \mu)^{\mathbb{N}}$ with the natural filtration $\{\mathcal{A}_n\}$, i.e., $\mathcal{A}_n$ is the σ-algebra of sets depending on the first n coordinates only, $n = 1, 2, \ldots$.

The *martingale extension* Mart (A, μ) of A relative to μ is the space of all $f \in C(X^\infty)$ such that for each n the function $\mathbb{E}_n f \stackrel{\text{det}}{=} (\mathbb{E}f|\mathcal{A}_n)$ is in A as a function of the n-th variable if the values of the preceding variables are fixed. We also put $d_n f = \mathbb{E}_n f - \mathbb{E}_{n-1} f$.

The multiplicativity of μ on $\mathcal{A}$ immediately implies that $\mathbb{E}_n(fg) = (\mathbb{E}_n f)(\mathbb{E}_n g)$ if $f, g \in$ Mart (A, μ) (it is convenient to verify this first for functions depending on a finite number of variables, and then to pass to the limit). Thus, Mart(A, μ) is a uniform algebra on X^∞.

Now consider a linear multiplicative functional φ on A (φ may or may not be related to the above measure μ). A linear functional ψ on A is called a *φ-derivation* if $\psi(fg) = \varphi(f)\psi(g) + \psi(f)\varphi(g)$, $f, g \in A$. The main theorem will say that if for some φ the algebra A has a nonzero bounded φ-derivation, then some martingale extension of A admits a bounded but not completely bounded homomorphism to $B(H)$. In fact, we shall give a more detailed statement, for which some preparations are needed.

On the disk algebra C_A, evaluation at the center of this disk is a multiplicative linear functional (we denote it by φ), and the mapping $\psi : f \mapsto f'(0)$ is a bounded (norm 1) φ-derivation. Next, if m is normalized Lebesgue measure on $\mathbb{T}$, then

$$\varphi(f) = \int_{\mathbb{T}} f(z)dm(z),\ \psi(f) = \int_{\mathbb{T}} f(z)\bar{z}dm(z), \quad f \in C_A.$$

We claim that in the abstract setting the situation is similar.

If ψ is a bounded nonzero φ-derivation on an arbitrary uniform algebra $A \subset C(X)$, it is easily seen that ψ varnishes on the constant functions; so, the restriction of ψ to the ideal $I = \{f \in A : \varphi(f) = 0\}$ is also nonzero. There is no loss of generality in assuming that $\|\psi|_I\| = 1$.

Lemma 1. *Under the above assumptions, there is a representing measure ν for φ and a sequence $\{f_n\} \subset I$, $\|f_n\| \le 1$, such that*

1^o $f_n \to F \in L^\infty(\nu)$ *a.e. with respect to* ν;

2^o $|F| = 1$ *a.e. and* $F\nu \perp A$;

3^o $\psi(g) = \int_X g\bar{F}d\nu$, $g \in A$.

This fact seems to be well known. We present the proof for the sake of completeness. Let a measure η represent a norm 1 extension of $\psi|_I$ to $C(X)$. We choose a sequence $\{f_n\} \subset I, |f_n| \leq 1$, such that $\int f_n d\eta \to 1$. Let F be a weak limit point of this sequence in $L^2(|\eta|)$. There is no loss of generality in assuming that $f_n \to F$ in $L^2(|\eta|)$ and a.e. relative to η. We have

$$1 = \int F d\eta = |\int F d\eta| \leq \int d|\eta| = 1,$$

whence $F\eta \stackrel{\text{def}}{=} \nu \geq 0$ and $|F| = 1 \quad \nu$-a.e.

If $g \in A$, then $gf_n \in I$; therefore

$$\begin{aligned}\int g d\nu = \int gF d\eta = \lim_{n\to\infty} \int g f_n d\eta = \lim \psi(gf_n) \\ = \lim_{n\to\infty} \varphi(g)\psi(f_n) = \varphi(g),\end{aligned}$$

i.e., ν represents φ. Hence, for every $g \in A$ we have

$$\int gF d\nu = \lim_{n\to\infty} \int g f_n d\nu = \lim_{n\to\infty} \varphi(g)\varphi(f_n) = 0,$$

i.e., $F\nu \perp A$. In particular, $\int F d\nu = 0$. By conjugation, $\int \bar{F} d\nu = 0$, i.e., the measure $\bar{F}\nu$ (which is none other than η) is orthogonal to the constants. Combined with the fact that $\eta = \bar{F}\nu$ represents $\psi|_I$ (by construction), this yields 3^o. □

A measure ν as in Lemma 1 will be called a *point derivation measure.*

Theorem 1. *Let A be a uniform algebra on a compact space X, and let ν be a point derivation measure for A. Then there exists a bounded but not completely bounded homomorphism from $A_1 = \mathrm{Mart}(A, \nu^\infty)$ to $B(H)$.*

Proof. We extend the functionals φ and ψ mentioned in Lemma 1 to $L^1(\nu)$ in accordance with the formulas $\varphi(g) = \int_X g d\nu$ and $\psi(g) = \int_X g\bar{F} d\nu$. Next, let φ_n and ψ_n be the operators that act in the n-th variable on functions defined on X^∞ as φ and ψ, respectively.

We denote by K an auxiliary infinite-dimensional Hilbert space, and put

$$\begin{aligned}H^2(K, \nu^\infty) = \mathrm{clos}_{L^2(K,\nu^\infty)} K \otimes A_1, \\ \bar{H}^2(K, \nu^\infty) = \mathrm{clos}_{L^2(K,\nu^\infty)} K \otimes \bar{A}_1\end{aligned}$$

(on the right in the latter formula, the bar stands for complex conjugation).

Let $Q : L^2(K, \nu^\infty) \to \bar{H}^2(K, \nu^\infty)$ be the orthogonal projection. For $f \in H^\infty(\nu^\infty)$ (the latter space is the w^*-closure of A_1 in $L^\infty(\nu^\infty)$), we define the operators

$$M_f : H^2(K, \nu^\infty) \to H^2(K, \nu^\infty),\ M_f g = fg,$$

and

$$\sigma_f : \bar{H}^2(K,\nu^\infty) \to \bar{H}^2(K,\nu^\infty),\ \sigma_f(g) = Q(fg).$$

Clearly, $M_{f_1}M_{f_2} = M_{f_1f_2}$. We claim that also $\sigma_{f_1}\sigma_{f_2} = \sigma_{f_1f_2}$. For this, we observe that

$$h \perp \bar{H}^2(K,\nu^\infty),\ f \in H^\infty(\nu^\infty) \implies fh \perp \bar{H}^2(K,\nu^\infty). \tag{1}$$

(Here and below the orthogonality relation is understood in terms of the Hilbert space sesquilinear duality, and not in the bilinear sense as in the proof of Lemma 1.) Indeed, $h \perp \bar{H}^2(K,\nu^\infty)$ if and only if for every $u = \sum x_i \otimes \bar{u}_i \in K \otimes \bar{A}_1$ we have

$$0 = \int \langle h, \sum x_i \otimes \bar{u}_i\rangle_K d\nu^\infty = \int \sum u_i \langle h, x_i\rangle_K d\nu^\infty. \tag{2}$$

But $fu_i \in H^\infty(\nu^\infty)$ because $u_i \in A_1$; substituting these products for the u_i in (2), we obtain (1).

From (1) we deduce the formula

$$Q(fg) = Q(fQg), \quad f \in H^\infty(\nu^\infty),\ g \in L^2(K,\nu^\infty). \tag{3}$$

The relation $\sigma_{f_1}\sigma_{f_2} = \sigma_{f_1f_2}$ is a consequence of (3).

Now (as in [Pi1], [D-P], [Ki2]) we invoke a sequence $\{V_n\}_{n\geq 1}$ of operators on K satisfying the *canonical anticommutation relations*:

$$V_nV_k + V_kV_n = 0, \quad V_nV_k^* + V_k^*V_n = \delta_{nk}I,$$
$$1 \leq n,\ k < \infty.$$

(Not to expand the reference list, we remark that a construction of such operators was outlined in [D-P] or [Ki2].) It is well known and easily seen that $\|\Sigma\alpha_iV_i\| = (\Sigma|\alpha_i|^2)^{1/2}$ for every complex scalars α_i. We fix a uniformly bounded sequence $\eta = \{\eta_n\}_{n\geq 0}$ of measurable scalar functions on X^∞ such that η_n depends on the first n coordinates only (accordingly, $\eta_0 =$ const). For every $f \in H^\infty(\nu^\infty)$, we define an operator $D_f^\eta : H^2(K,\nu^\infty) \to \bar{H}^2(K,\nu^\infty)$ by the formula

$$D_f^\eta(u) = Q(\sum_{n\geq 1} \eta_{n-1}\psi_n(\mathbb{E}_n f)V_n u),$$

or, equivalently,

$$\langle D_f^\eta(u), v\rangle = \int \sum_{n\geq 1} \eta_{n-1}\psi_n(\mathbb{E}_n f)\langle V_n u, v\rangle_K d\nu^\infty, \qquad u \in K \otimes A_1, \quad v \in K \otimes \bar{A}_1. \tag{4}$$

(We remind the reader that ψ_n is ψ acting in the n-th variable; before the boundedness of D^η_f is proved, in the first definition we also should restrict ourselves to $u \in K \otimes A_1$.)

We claim that nothing changes in (4) if on the right we replace u or v (or both) by $\mathbb{E}_{n-1}u$ and $\mathbb{E}_{n-1}v$ (respectively). Indeed, it suffices to show this for $u = x \otimes \alpha, v = y \otimes \bar{\beta}$, where $x, y \in K$, $\alpha, \beta \in A_1$. But since $\eta_{n-1}\psi_n(\mathbb{E}_n f)$ depends only on the first $n-1$ coordinates, we have

$$\langle D^\eta_f(u), v\rangle = \int \sum_{n\geq 1} \eta_{n-1}\psi_n(\mathbb{E}_n f)\mathbb{E}_{n-1}\langle V_n u, v\rangle_K d\nu^\infty$$

$$= \int \sum_{n\geq 1} \eta_{n-1}\psi_n(\mathbb{E}_n f)\mathbb{E}_{n-1}(\alpha\beta)\langle V_n x, y\rangle_K d\nu^\infty.$$

Now, $\mathbb{E}_{n-1}$ is multiplicative on A_1; consequently,

$$\mathbb{E}_{n-1}(\alpha\beta) = (\mathbb{E}_{n-1}\alpha)(\mathbb{E}_{n-1}\beta) = \mathbb{E}_{n-1}(\alpha\mathbb{E}_{n-1}\beta) = \mathbb{E}_{n-1}(\beta\mathbb{E}_{n-1}\alpha).$$

This proves the claim.

We define a mapping

$$\pi^\eta : H^\infty(\nu^\infty) \to B(\bar{H}^2(K, \nu^\infty) \oplus H^2(K, \nu^\infty))$$

by the formula

$$\pi^\eta(f) = \begin{pmatrix} \sigma_f, & D^\eta_f \\ 0, & M_f \end{pmatrix},$$

and prove that π^η is a bounded homomorphism. After this we shall verify that for some choices of the sequence η (in particular, for $\eta_j \equiv 1$) π^η is not completely bounded. (Observe also that $\pi^\eta(1) = id$.)

To show that η is a homomorphism, it suffices to prove that

$$D^\eta_{f_1 f_2} = \sigma_{f_1} D^\eta_{f_2} + D^\eta_{f_1} M_{f_2}.$$

For this, we use the above observation concerning formula (4), and also the fact that ψ is a φ-derivation. Specifically, $\psi_n(\mathbb{E}_n(f_1 f_2)) = (\mathbb{E}_{n-1}f)\psi_n(\mathbb{E}_n f_2) + \psi_n(\mathbb{E}_n f_1)\mathbb{E}_{n-1}f_2$, whence

$$\langle D^\eta_{f_1 f_2}(u), v\rangle$$

$$= \int \sum_{n\geq 1} \eta_{n-1}[\psi_n(\mathbb{E}_n f_2)\langle V_n u, \mathbb{E}_{n-1}(\bar{f}_1 v)\rangle_K$$

$$+ \psi_n(\mathbb{E}_n f_1)\langle V_n(\mathbb{E}_{n-1}(f_2 u)), v\rangle_K] d\nu^\infty$$

$$= \int \sum_{n\geq 1} \eta_{n-1}\psi_n(\mathbb{E}_n f_2)\langle V_n u, \bar{f}_1 v\rangle_K d\nu^\infty + \langle D^\eta_{f_1} M_{f_2} u, v\rangle$$

$$= \langle Q(f_1 D^\eta_{f_2}(u)), v\rangle + \langle D^\eta_{f_1} M_{f_2} u, v\rangle,$$

which is the required relation (in the last line we have used (3)).

Now, we check the continuity of π^η. For this, it suffices to show that the mapping $f \mapsto D_f^\eta$ is bounded.

Recalling the formula $\|\Sigma\alpha_n V_n\| = (\Sigma|\alpha_n|^2)^{1/2}$ for every complex scalars α_n, we see that $(\sum |\Phi(V_n)|^2)^{1/2} \le \|\Phi\|$ for every $\Phi \in B(K)^*$. In particular,

$$(\sum |\langle V_n x, y\rangle_K|^2)^{1/2} \le |x||y|, \quad x, y \in K, \tag{5}$$

where $|\cdot|$ stands for the norm in K.

We denote by $H^1_M(\ell^2, \nu^\infty)$ the space of all ℓ^2-valued $\{\mathcal{A}_n\}$-adapted martingales x for which

$$\|x\|_{1,M} \stackrel{\text{def}}{=} \int_{X^\infty} \sup_j \|\mathbb{E}_j x(\cdot)\|_{\ell^2} d\nu^\infty < \infty.$$

(We emphasize that, except for the measure ν, no structure related to the algebra A is involved in this definiton.) We refer the reader to the book [G] for the theory of this space and for the facts used below (the case of ℓ^2-valued martingales does not differ from the scalar case treated in [G]).

□

Lemma 2. *If* $u \in H^2(K, \nu^\infty)$, $v \in \bar{H}^2(K, \nu^\infty)$, *then* $x = \{\langle V_n u, v\rangle_K\}_{n\ge1} \in H^1_M(\ell^2, \nu^\infty)$ *and*

$$\|x\|_{1,M} \le C\|u\|_{L^2(K,\nu^\infty)}\|v\|_{L^2(K,\nu^\infty)}.$$

Proof. It is easily seen that $\mathbb{E}_j\langle V_n u, v\rangle_K = \langle V_n \mathbb{E}_j u, \mathbb{E}_j v\rangle_K$. (First, consider the case where $u \in K \otimes A_1, v \in K \otimes \bar{A}_1$, and then pass to the limit.) Consequently, by (5),

$$\begin{aligned}\sup_j \|\mathbb{E}_j x(\cdot)\|_{\ell^2} &\le \sup_j |\mathbb{E}_j u(\cdot)||\mathbb{E}_j v(\cdot)| \\ &\le \sup_j |\mathbb{E}_j u(\cdot)| \sup_j |\mathbb{E}_j v(\cdot)|.\end{aligned}$$

Integrating, we see that

$$\|x\|_{1,M} \le \Big(\int [\sup_j |\mathbb{E}_j u|]^2 d\nu^\infty\Big)^{1/2} \Big(\int [\sup_j |\mathbb{E}_j v|]^2 d\nu^\infty\Big)^{1/2},$$

and the lemma follows from the standard L^2-estimate for the martingale maximal function.

□

Recalling (4) and the duality between the martingale spaces $H^1_M(\ell^2, \nu^\infty)$ and $\mathrm{BMO}_M(\ell^2, \nu^\infty)$, we see that the continuity of the mapping $f \mapsto D_f^\eta$ will follow from the estimate

$$\|\sum_{n\ge1} \eta_{n-1}\psi_n(\mathbb{E}_n f)e_n\|_{\mathrm{BMO}_M(\ell^2,\nu^\infty)} \le C\|f\|_\infty, \tag{6}$$

where $f \in H^\infty(\nu^\infty)$ and the e_n denote the basic unit vectors of ℓ^2. To take the constants into account, we define

$$\|y\|_{\mathrm{BMO}_M}(\ell^2, \nu^\infty) = |||y||| + (\int \|y(.)\|_{\ell^2}^2 d\nu^\infty)^{1/2},$$

where

$$|||y||| = \Big(\sup_{n\geq 1} \|E_n(\|y - \mathbb{E}_{n-1} y\|_{\ell^2})^2\|_\infty\Big)^{1/2}.$$

Now, for $y = \sum_{n\geq 1} \eta_{n-1}\psi_n(\mathbb{E}_n f)e_n$ the estimate $(\int \|y(\cdot)\|_{\ell^2}^2 d\nu^\infty)^{1/2} \leq C\|f\|_\infty$ is quite clear:

$$\begin{aligned}&\int \sum |\psi_n(\mathbb{E}_n f)|^2 d\nu^\infty = \int \sum |\psi_n(d_n f)|^2 d\nu^\infty \\ &\leq \sum \int |d_n f|^2 d\nu^\infty = \|f\|_{L^2(\nu^\infty)}^2 \leq \|f\|_\infty^2.\end{aligned}$$

We show that $|||f||| \leq C\|f\|_\infty$, again using the fact that $\psi_n(\mathbb{E}_n f) = \psi_n(d_n f)$, and that the latter function depends only on the first $n-1$ coordinates:

$$\begin{aligned}&\mathbb{E}_j \|y - \mathbb{E}_{j-1} y\|^2 \\ &= \mathbb{E}_j \Big\| \sum_{n\geq j} e_n[\eta_{n-1}\psi_n(d_n f) - \mathbb{E}_{j-1}(\eta_{n-1}\psi_n(d_n d))]\Big\|_{\ell^2}^2 \\ &\leq C\mathbb{E}_j \Big(\sum_{n\geq j} |\psi_n(d_n f)|^2 + \sum_{n\geq j} (\mathbb{E}_{j-1}|\eta_{j-1}\psi_n(d_n f)|)^2\Big) \\ &\leq \Big(\mathbb{E}_j(\sum_{n\geq j} |d_n f|^2) + \mathbb{E}_j\mathbb{E}_{j-1}(\sum_{n\geq j} |d_n f|^2)\Big) \\ &\leq C(\mathbb{E}_j|f - \mathbb{E}_{j-1} f|^2 + \mathbb{E}_{j-1}\mathbb{E}_j|f - \mathbb{E}_{j-1} f|^2) \\ &\leq c' |||f|||^2 \leq C''\|f\|_\infty^2.\end{aligned}$$

It only remains to prove that for certain choices of the η_n (in particular, for $\eta_n \equiv 1$) the operator $\pi^\eta|_{A_1}$ is not completely bounded. Fixing $x, y \in K$ and putting $u = 1 \cdot x, v = 1 \cdot y$ in (4), we see that

$$\langle \pi^\eta(f)\begin{pmatrix} 0 \\ 1 \cdot x\end{pmatrix}, (1 \cdot y, 0)\rangle = \langle T_\eta(f)x, y\rangle_K, \tag{7}$$

where the operator $T_\eta : H^\infty(\nu^\infty) \to B(K)$ is given by the formula

$$T_\eta(f) = \sum_{s\geq 1}(\int_{X^\infty} \eta_{s-1}\psi_s(\mathbb{E}_s f)d\nu^\infty)V_s \tag{8}$$

Formula (7) implies that it suffices to show that $T_\eta|_{A_1}$ is not completely bounded.

We observe that if $\tau : A_1 \to B(K)$ is completely bounded, then for every n, ℓ, every $(n\times n)$-matrices $a_1, \dots, a_\ell$, and every $\alpha_1, \dots, \alpha_\ell \in A_1$ we have

$$\|\sum_k a_k \otimes \tau(\alpha_k)\|_{B(\ell^2_n(K))} \leq \|\tau\|_{cb} \sup_{t\in X^\infty} \|\sum a_k\alpha_k(t)\|_{M_n}. \tag{9}$$

Indeed, if $y = (y_1, \dots, y_n) \in \ell^2_n(K)$ and if $a_K\{a^k_{ij}\}_{1\leq i,j\leq n}$, then

$$\begin{aligned}
&\|(\sum_k a_k \otimes \tau(\alpha_k))y\|^2_{\ell^2_n(K)} = \sum_i \|\sum_k\sum_j a^{(k)}_{ij}\tau(\alpha_k)y_j\|^2 \\
&= \sum_i \|\sum_j \tau(\sum_k a^{(k)}_{ij}\alpha_k)y_j\|^2 \\
&\leq (\sum_j \|y_j\|^2)\|\tau\|_{cb} \sup_{t\in X^\infty} \|\sum_k a_k\alpha_k(t)\|^2_{M_n},
\end{aligned}$$

and (9) follows.

We want to disprove (9) for the operator (8). This is quite similar to a part of the arguments in [Pi1], [D-P], or [Ki2], so that we skip some calculations.

Consider the functions f_n occurring in Lemma 1. Fixing n, we put $\alpha_k(t) = f_n(t_k)$ $(t \in X^\infty; k = 1, \dots, \ell)$. Then

$$T_\eta(\alpha_k) = \int \eta_{k-1}d\nu^\infty \psi(f_n)C_k,$$

and (9) becomes

$$\begin{aligned}
&|\psi(f_n)|\Big\|\sum_{k=1}^{\ell}(a_k \otimes C_k)\int \eta_{k-1}d\nu^\infty\Big\|_{B(\ell^2_n(K))} \\
&\leq \|T_\eta\|_{cb} \sup_{t\in X^\infty} \Big\|\sum_{k=1}^{\ell} a_k f_n(t_k)\Big\|_{M_n}.
\end{aligned} \tag{10}$$

As in [Pi1] (and [D-P], [Ki2]), we realize $C_1, \dots, C_\ell$ by matrices of size $2^\ell \times 2^\ell$ (so that K in (10) must be replaced by $\ell^2_{2\ell}$), and put $a_k = \bar{C}_k$, $1 \leq k \leq \ell$. then the supremum on the right in (10) does not exceed $\sqrt{\ell}$. Assuming, e.g., that $\int \eta_{k-1}d\nu^\infty$ is nonzero and independent of k, and passing to the limit as $n \to \infty$, we obtain

$$\Big\|\sum_{k=1}^{\ell} \bar{C}_k \otimes C_k\Big\|_{B(\ell^2_{2\ell}(\ell^2_{2\ell}))} \leq C\|T_\eta\|_{cb}\sqrt{\ell}.$$

We refer the reader to [Pi1], [D-P], [Ki2] for a standard calculation showing that the right-hand side in the above inequality is at least $\ell/2$.

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