Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel

Niklas Lindholm, Göteborg University

March 30, 2000

Abstract

The kind of necessary density condition in $\mathbb{C}$ known for sampling and interpolation in the L^p space of entire functions with a subharmonic weight, is extended to the case of a 2-homogeneous, plurisubharmonic weight function in $\mathbb{C}^n$. The method is by estimating the eigenvalues of a certain Toeplitz concentration operator, using asymptotic estimates for the Bergman kernel of independent interest.

Keywords: Fock space, sampling, interpolation, Bargmann space, concentration operator, Bergman kernel, Toeplitz operator

2000 MSC: 46E20, 32A30, 30E05, 32A25

1 Introduction

The Shannon sampling theorem in Fourier analysis states that if $f \in L^2(\mathbb{R})$ is bandlimited to W Hz (i.e. $\hat{f}(x) = \int f(t)e^{-2\pi i x t}dt = 0$ for $|x| > W$) then

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2W}\right) \frac{\sin 2\pi W(t - k/2W)}{2\pi W(t - k/2W)}$$

and

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2W} \sum |f(k/2W)|^2.$$

This not only says that the sample values $f(k/2W)$ determine f, but also that the reconstruction of f from the values is stable. The proof follows from the fact that the functions $e^{2\pi i x(k/2W)}$ form an orthogonal basis for $L^2([-W, W])$.

If f is bandlimited to W then it is an entire function of exponential type not exceeding $2\pi W$, which also belongs to $L^2(\mathbb{R})$. The sampling theorem says that all such functions can be reconstructed from the values in the regularly spaced points $k/2W$. If we want to consider irregular sampling at a sequence of points $\{\lambda_k\}$ instead, we want to have an inequality of the type

$$A \int_{-\infty}^{\infty} |f(t)|^2 dt \leq \sum |f(\lambda_k)|^2 \leq B \int_{-\infty}^{\infty} |f(t)|^2 dt,$$

to be able to reconstruct the function in a stable way. Now look at the Fourier transform $g(x) = \hat{f}(x)$, which is supported in $[-W, W]$. We have

$$f(\lambda_k) = \int_{-W}^{W} g(x) e^{2\pi i x \lambda_k} dx, \quad \text{and} \quad \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-W}^{W} |g(x)|^2 dx.$$

A *sampling inequality* of the above kind is therefore equivalent to that

$$A \int_{-W}^{W} |g(x)|^2 dx \leq \sum \left| \int_{-W}^{W} g(x) e^{2\pi i x \lambda_k} dx \right|^2 \leq B \int_{-W}^{W} |g(x)|^2 dx,$$

for every $g \in L^2([-W, W])$. The system of functions $e^{2\pi i x \lambda_k}$ satisfying such an inequality will not be an orthogonal basis in general, but the functions will form what is known as a *frame* in $L^2([-W, W])$. Questions like this were considered e.g. by Duffin and Schaeffer in [DS52], where they proved a sufficient condition for some quite regularly spaced set of points $\{\lambda_k\}$ to be sampling.

The sampling theorem is of great practical importance for e.g. data transmission. It is a fair question to ask if the sampling rate (known as the Nyquist rate) cannot be improved in some way, for instance by very irregular sampling or by considering multiband functions instead. Landau studied this problem in [Lan67b] and [Lan67a], as well the more general problem in $\mathbb{R}^n$. He used density conditions introduced by Beurling ([CMNW89]) for the similar sampling problem for bounded functions (also known as balayage), and showed that in fact the Nyquist rate is the best possible.

For another example of a frame problem, consider the functions

$$\varphi_{a,b}(x) = \pi^{-1/4} e^{-iab/2} e^{iax} e^{-(x-b)^2/2}$$

in $L^2(\mathbb{R})$. They are known as Gabor wavelets, or in quantum mechanics as canonical coherent states. These functions are used a sort of "localised" Fourier transforms to reproduce functions $f \in L^2(\mathbb{R})$.

The space $L^2(\mathbb{R})$ is actually canonically isometric to the Bargmann-Fock space F of entire functions with norm

$$\|f\|^2 = \frac{1}{2\pi} \int |f(z)|^2 e^{-|z|^2/2} < \infty,$$

where the isometry is given by the so called Bargmann transform (see [DG88] for all the details). In the space F we have the reproducing kernel $B_w(z) = e^{z \cdot \overline{w}/2}$ so that

$$f(w) = (f, B_w) = \frac{1}{2\pi} \int f(z) e^{w \cdot \bar{z}/2} e^{-|z|^2/2}$$

for every $f \in F$. If we have a set $\{\varphi_{a_k,b_k}\}$ of Gabor wavelets, it turns out that in order to be able to represent any $f \in L^2(\mathbb{R})$ as $\sum c_k \varphi_{a_k,b_k}$ with square summable coefficients c_k, the functions φ_{a_k,b_k} should constitute a frame, i.e.

$$A\|f\|^2 \leq \sum |(f, \varphi_{a_k,b_k})|^2 \leq B\|f\|^2, \quad f \in L^2(\mathbb{R}).$$

If this is the case, the procedure of finding c_k and reconstructing f is stable. Now, the Bargmann transform U_B is unitary and it turns out that it maps the function φ_{a_k,b_k} to $e^{-|w_k|^2/4}B_{w_k}$, where $w_k = a_k + ib_k$. Hence $\|f\|^2_{L^2(\mathbb{R})} = \|U_B f\|^2_F$ and

$$(f, \varphi_{a_k,b_k}) = (U_B f, U_B \varphi_{a_k,b_k}) = e^{-|w_k|^2/4}(U_B f, B_{w_k}) = e^{-|w_k|^2/4} U_B f(w_k),$$

so that the frame inequality for φ_{a_k,b_k} in $L^2(\mathbb{R})$ is equivalent to the sampling inequality

$$A\|f\|^2 \leq \sum |f(w_k)|^2 e^{-|w_k|^2/2} \leq B\|f\|^2, \quad f \in F.$$

In[DG88], Daubechies and Grossmann studied this frame and sampling problem for some lattices of points $\{ma + inb : m, n \in \mathbb{Z}\}$. It was known that $ab \leq 2\pi$ was a necessary condition for the sampling inequality to hold for a lattice. They conjectured that $ab < 2\pi$ was a sufficient condition for any lattice, and obtained numerical estimated of the sampling constants in some concrete cases.

Just as in the case of bandlimited functions, it is natural to ask if the sampling can be made more efficient by sampling in an irregular set of points. The problem was considered by Seip and others in a series of articles [Sei91], [Sei92], [SW92], [BO95], [OS98]. The conclusion was that the criterion suggested by Daubechies and Grossmann was in fact both necessary and sufficient not only for lattices, but for *arbitrary* discrete sets. In fact, it is even valid in the generalised Fock spaces where we have the weight function $e^{-\varphi}$, with φ subharmonic, instead of $e^{-|z|^2/2}$. The precise formulations will be given in the next section.

In this paper we will extend this to 2-homogeneous, plurisubharmonic weight functions φ in $\mathbb{C}^n$, and show that a natural generalisation of the density condition is necessary for sampling. We will formulate the theorems in the next section. The case $\varphi = \alpha|z|^2$ is also covered by the calculations in [Sei91], but not stated therein.

The method we will use is the one employed by Landau in [Lan67a]. It consists in studying functions "concentrated" on compact sets, and it will lead us to a study of the Bergman kernel in the Fock space (where the weight function φ is not necessarily 2-homogeneous). We will prove a number of results of independent interest, concerning asymptotic behaviour of this kernel.

Estimates of the Bergman kernel in Fock spaces in $\mathbb{C}^n$ have also been considered by Delin [Del98]. The kind of asymptotic behaviour we are interested in has also been studied by e.g. Bouche [Bou90] and Tian [Tia90] in the different context of metrics on line bundles on compact complex manifolds.

The paper is organised as follows. In the next section we will formulate the problem we are working on more precisely, and state the main theorems.

In Section 3 we will discuss the method, introduce a concentration operator and demonstrate its connection to the sampling problem. In Section 4 we will investigate the asymptotic behaviour of the Bergman kernel, and in Section 5 we will treat the more general sampling problem in L^p spaces.

2 Sampling and interpolation

Let φ be a plurisubharmonic function in $\mathbb{C}^n$, and define the weighted L^p-norm of f by

$$\|f\|_{p,\varphi}^p = \int |f|^p e^{-p\varphi} dm$$

for $p < \infty$, and $\|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{C}^n} |f(z)|e^{-\varphi(z)}$. We let F_φ^p be the space of entire functions with $\|f\|_{p,\varphi} < \infty$.

A sequence of distinct points $\Gamma = \{\gamma_j\} \subset \mathbb{C}^n$ is said to be *sampling* for the space F_φ^p, $1 \leq p < \infty$, if there are positive constants A and B such that for any $f \in F_\varphi^p$

$$A\|f\|_{p,\varphi}^p \leq \sum_{\gamma_j\in\Gamma} |f(\gamma_j)|^p e^{-p\varphi(\gamma_j)} \leq B\|f\|_{p,\varphi}^p. \tag{1}$$

Γ is said to be sampling for F_φ^∞ if $A \sup |f(z)|e^{-\varphi(z)} \leq \sup_{\gamma_j\in\Gamma} |f(\gamma_j)|e^{-\varphi(\gamma_j)}$. The middle term in (1) is by definition the norm of $\{f(\gamma_j)\}$ in l_φ^p. The sequence Γ is called *interpolating* for F_φ^p, $1 \leq p \leq \infty$, if for any sequence $\{c_j\} \in l_\varphi^p$ there is a function $f \in F_\varphi^p$ such that $f(\gamma_j) = c_j$. Finally, a sequence of points is *uniformly separated* if the infimum of distances between distinct points is strictly positive. The infimum is called the *separation constant*, which we will always denote by δ_0. Hence we can refer to a uniformly separated sequence as δ_0-separated.

The following two theorems, due to Berndtsson, Ortega-Cerdà and Seip, characterise the sampling and interpolating sequences in $\mathbb{C}$ (the Laplacian is here defined as $\Delta = \partial^2/\partial z\partial\bar{z}$). The proofs are found in [BO95] (the sufficiency of the density condition) and [OS98] (the necessity).

Theorem (A). *Let ϕ be a subharmonic function satisfying*

$$0 < m \leq \Delta\phi(z) \leq M \tag{2}$$

for all $z \in \mathbb{C}$, for some positive constants m and M. A sequence $\Gamma \subset \mathbb{C}$ is sampling for F_ϕ^p if and only if it contains a uniformly separated sequence Γ' which is also sampling and satisfies

$$\liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#\Gamma' \cap D(z;r)}{\int_{D(z;r)} \Delta\phi} > \frac{2}{\pi}.$$

If $1 \leq p < \infty$ then Γ is in addition a finite union of uniformly separated sequences.

Theorem (B). *Let ϕ be a subharmonic function satisfying (2). A sequence $\Gamma \subset \mathbb{C}$ is interpolating for F^p_ϕ if and only if it is uniformly separated and satisfies*

$$\limsup_{r\to\infty}\sup_{z\in\mathbb{C}}\frac{\#\Gamma\cap D(z;r)}{\int_{D(z;r)}\Delta\phi}<\frac{2}{\pi}.$$

The type of density conditions considered in the two previous theorems were, as we have mentioned, introduced by Beurling.

We now want to study the corresponding problem in $\mathbb{C}^n$. Consider first the simple example with $\varphi(z)=\alpha_1|z_1|^2+\alpha_2|z_2|^2$ in $\mathbb{C}^2$. If we want a lattice $\{(a_1w_1,a_2w_2): w_1,w_2\in\mathbb{Z}\times i\mathbb{Z}\}$ in this space to be sampling, we can start by considering functions $f\in F^2_\varphi$ of the special type $f_1(z_1)f_2(z_2)$. We see that the lattice must be sampling in one variable in both directions independently, and the condition for this is by the theorems above that $1/a_1^2>2\alpha_1/\pi$ and $1/a_2^2>2\alpha_2/\pi$. But the asymptotic number of points from the lattice in a big ball $B(z;R)$ in $\mathbb{C}^2$ is $|B(z;R)|/a_1^2a_2^2$, which therefore should exceed $|B(z;R)|\cdot 4\alpha_1\alpha_2/\pi^2$. The latter expressions involves not $\Delta\varphi$ but $(i\partial\bar\partial\varphi)^n$. In our context, we therefore define the lower and upper densities of a sequence Γ with respect to φ as

$$D^-_\varphi(\Gamma)=\liminf_{r\to\infty}\inf_{z\in\mathbb{C}^n}\frac{\#\Gamma\cap B(z;r)}{\int_{B(z;r)}(i\partial\bar\partial\varphi)^n}$$

and

$$D^+_\varphi(\Gamma)=\limsup_{r\to\infty}\sup_{z\in\mathbb{C}^n}\frac{\#\Gamma\cap B(z;r)}{\int_{B(z;r)}(i\partial\bar\partial\varphi)^n}.$$

We see that $D^-_\varphi(\Gamma)\geq\alpha$ if and only if for all $\epsilon>0$

$$\#\Gamma\cap B(z;r)\geq(\alpha-\epsilon)\int_{B(z;r)}(i\partial\bar\partial\varphi)^n$$

for all sufficiently large r and all z.

It is clear that a density condition of this kind can never be sufficient in $\mathbb{C}^n$ when $n>1$. To see this, consider again $\varphi(z)=\alpha_1|z_1|^2+\alpha_2|z_2|^2$ in $\mathbb{C}^2$ and Γ a lattice. This lattice can be arbitrarily sparse in one direction, but still fulfil a density condition of the type $D^-_\varphi(\Gamma)\geq\alpha$ by being dense in the other direction. If the lattice is sparse enough in the z_1 direction, then it will be interpolating in that direction. In this case we can actually find a nonzero function $f_1(z_1)\in F^2_{\alpha_1|z_1|^2}(\mathbb{C})$ which is zero in all the points $\{ma_1+ina_1: m,n\in\mathbb{Z}\}$. Then $f_1(z_1)f_2(z_2)\in F^2_\varphi$ for any $f_2\in F^2_{\alpha_2|z_2|^2}(\mathbb{C})$, but it is zero in all the lattice points, which contradicts the sampling inequality.

The theorems we will prove are the following. The proofs will follow at the end of Section 5.

Theorem 1. *Let φ be a 2-homogeneous, plurisubharmonic function which is C^2 outside the origin. If a sequence Γ is sampling for F^p_φ, $1 \le p \le \infty$, then it contains a uniformly separated subsequence Γ' which is also sampling and satisfies*

$$D^-(\Gamma') \ge \frac{2^n}{\pi^n n!}. \tag{3}$$

If $1 \le p < \infty$ then Γ is in addition a finite union of uniformly separated sequences.

Theorem 2. *Let φ be as in the previous theorem. If a sequence Γ is interpolating for F^p_φ, $1 \le p \le \infty$, then it is uniformly separated and satisfies*

$$D^+(\Gamma) \le \frac{2^n}{\pi^n n!}. \tag{4}$$

It seems very likely that the strict inequalities should hold in Theorem 1 and Theorem 2, but it is still an open question in $\mathbb{C}^n$.

It may look like we demand more smoothness from φ than we do from ϕ in Theorem (A) and Theorem (B). If ϕ satisfies the properties in Theorem (A) we can however simply smooth it, without changing the density condition, and even assume that $\Delta\phi$ is uniformly Lipschitz. If we try to do that in our setting, φ will lose the homogeneity property.

We can remark that it actually follows from our theorems below that the choice of the balls $B(z;r)$ to measure density is not essential. We get the same result with any smooth set, or more generally a set where the boundary has measure zero.

To study the left inequality in (1) we will start with the case $p = 2$. Since the point evaluations in the space F^2_φ are bounded (by Lemma 7 below) we see that if the mass of a function f is very concentrated to a compact set Ω, then the contribution to the sum in (1) from those values of Γ outside of Ω is correspondingly small. If there were very many such functions f we could expect to find one which was zero in all the points of $\Gamma \cap \Omega$. If this were true for arbitrarily large sets Ω, the left inequality in (1) would be difficult to satisfy. This is the technique use by Landau in [Lan67a] and Seip in [Sei91]. The way to study functions concentrated on a certain set is by means of the concentration operator which we will now introduce. But before that, one last word on notation. We will write $f \lesssim g$ if there is a constant C such that $f \le Cg$.

3 A concentration operator

In this and the next section we will assume that we have positive constants so that

$$mi\partial\bar\partial|z|^2 \le i\partial\bar\partial\varphi \le Mi\partial\bar\partial|z|^2, \tag{5}$$

(as positive currents) and that φ is C^2 except at a finite number of points.

Let B_φ be the Bergman kernel for the space F^2_φ. Then $B_\varphi(z,\zeta)$ is holomorphic in z,

$$\overline{B_\varphi(z,\zeta)} = B_\varphi(\zeta, z)$$

and

$$P_\varphi f(z) = \int B_\varphi(z,\zeta) f(\zeta) e^{-2\varphi(\zeta)}$$

is the orthogonal projection from L^2_φ onto F^2_φ. We define the Toeplitz *concentration operator* $T_{\chi,\varphi}$ with symbol χ by

$$T_{\chi,\varphi} f(z) = P_\varphi(\chi f)(z) = \int f(\zeta)\chi(\zeta) B_\varphi(z,\zeta) e^{-2\varphi(\zeta)}.$$

If χ is a bounded function with compact support, the operator $T_{\chi,\varphi}$ is compact. This follows from the fact that its kernel belongs to L^2, since the reproducing property of the Bergman kernel implies that

$$\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |\chi(\zeta) B_\varphi(z,\zeta)|^2 e^{-2\varphi(z)} e^{-2\varphi(\zeta)} = \int_{\mathbb{C}^n} |\chi(\zeta)|^2 B_\varphi(\zeta,\zeta) e^{-2\varphi(\zeta)},$$

where the last integral is bounded by Theorem 10 below. We shall see that $T_{\chi,\varphi}$ is also of trace class.

If f is an eigenfunction of $T_{\chi,\varphi}$ with eigenvalue λ, then

$$\lambda \|f\|^2_{2,\varphi} = (T_{\chi,\varphi} f, f) = (\chi f, P_\varphi f) = (\chi f, f) = \int \chi(z) |f(z)|^2 e^{-2\varphi(z)}, \tag{6}$$

so if χ is the characteristic function of a set Ω, the eigenvalue measures the concentration of the mass of f to Ω.

We will denote the eigenvalues of $T_{\chi,\varphi}$ by $\lambda_m(\chi,\varphi)$, ordered in a non-increasing sequence (counted with multiplicity)

$$\lambda_0(\chi,\varphi) \ge \lambda_1(\chi,\varphi) \ge \lambda_2(\chi,\varphi) \ge \cdots$$

If χ is the characteristic function of the set Ω, we will also write $\lambda_m(\Omega,\varphi)$.

The following two lemmas is the connection between sampling or interpolation and the concentration operator.

Lemma 3. *Assume that the δ_0-separated sequence Γ is sampling for the space F^2_φ, and let $N = \#\Gamma \cap B(z; r+\delta_0/2)$. Then $\lambda_N(B(z;r),\varphi) \le \gamma$, for some $\gamma < 1$ independent of z and r.*

Proof. Let f_m be the orthonormal eigenfunctions connected with the eigenvalues $\lambda_m(B(z;r),\varphi)$, and let $f = \sum_{m=0}^N c_m f_m$ be a linear combination of the $N+1$ first eigenfunctions. Since we have N points in $\Gamma \cap B(z; r+\delta_0/2)$ we can choose c_m not all zero such that

$$f(\zeta) = \sum_{m=0}^{N} c_m f_m(\zeta) = 0, \ \zeta \in \Gamma \cap B(z; r+\delta_0/2).$$

Since Γ is sampling we then see that

$$A\|f\|_{2,\varphi}^2 \leq \sum_{\gamma_j \in \Gamma} |f(\gamma_j)|^2 e^{-2\varphi(\gamma_j)} = \sum_{\gamma_j \notin B(z;r+\delta_0/2)} |f(\gamma_j)|^2 e^{-2\varphi(\gamma_j)},$$

and as in Lemma 7 below we have

$$|f(\gamma_j)|^2 e^{-2\varphi(\gamma_j)} \leq C \int_{B(\gamma_j;\delta_0/2)} |f(w)|^2 e^{-2\varphi(w)} \tag{7}$$

(where C depends on δ_0). Hence we get

$$\begin{aligned} \frac{A}{C}\|f\|_{2,\varphi}^2 &\leq \int_{\mathbb{C}^n \setminus B(z;r)} |f(w)|^2 e^{-2\varphi(w)} \\ &= \|f\|_{2,\varphi}^2 - \int_{B(z;r)} |f(w)|^2 e^{-2\varphi(w)} = \|f\|_{2,\varphi}^2 - \sum_{m=0}^{N} |c_m|^2 \lambda_m, \end{aligned}$$

where the last equality follows as in (6). But $\|f\|_{2,\varphi}^2 = \sum_{m=0}^N |c_m|^2$ so we get

$$\lambda_N(B(z;r),\varphi) \leq \frac{\sum_{m=0}^N |c_m|^2 \lambda_m}{\sum_{m=0}^N |c_m|^2} \leq 1 - \frac{A}{C}.$$

Let $\gamma = 1 - A/C$. □

In the next proof we will need the following formulation of the Weyl-Courant Lemma:

$$\lambda_m(\chi,\varphi) = \min_{\dim E = m-1} \max_{\substack{f \perp E \\ \|f\|_{2,\varphi}=1}} (T_{\chi,\varphi} f, f) = \max_{\dim F = m} \min_{\substack{f \in F \\ \|f\|_{2,\varphi}=1}} (T_{\chi,\varphi} f, f) \tag{8}$$

where E and F are subspaces of F_φ^2.

Lemma 4. *Assume that the δ_0-separated sequence Γ is interpolating for the space F_φ^2, and let $N = \#\Gamma \cap B(z; r - \delta_0/2)$. Then $\lambda_N(B(z;r),\varphi) > \gamma$, for some $\gamma > 0$ independent of z and r.*

Proof. If Γ is interpolating for F_φ^p, a standard application of the closed graph theorem (see e.g. [Hof62, p. 196]) shows that interpolation can be performed in a stable way. If we write $\Gamma \cap B(z; r - \delta_0/2) = \{\zeta_1, \ldots, \zeta_N\}$ we can therefore find functions f_k with $f_k(\zeta_k) = 1$ and $f_k(\zeta) = 0$ for all other $\zeta \in \Gamma \setminus \{\zeta_k\}$, and such that $\|f_k\|_{2,\varphi}^2 \leq Ce^{-2\varphi(\zeta_k)}$, $k = 1, \ldots, N$.

If we let $F = \operatorname{span}\{f_k\}_{k=1}^N$ we see that any $f \in F$ can be written $f = \sum_{k=1}^N c_k f_k = \sum_{k=1}^N f(\zeta_k) f_k$, so that

$$
\begin{aligned}
\|f\|_{2,\varphi} &\le \sum_{k=1}^N |f(\zeta_k)| \cdot \|f_k\|_{2,\varphi} \le C \sum_{k=1}^N |f(\zeta_k)| e^{-\varphi(\zeta_k)} \\
&\le C \sum_{k=1}^N \left(\int_{B(\zeta_k;\delta_0/2)} |f(w)|^2 e^{-2\varphi(w)} \right)^{1/2} \\
&\le C \left(\int_{B(z;r)} |f(w)|^2 e^{-2\varphi(w)} \right)^{1/2}
\end{aligned}
$$

by (7). But since

$$
(T_{B(z;r),\varphi} f, f) = \int_{B(z;r)} |f(w)|^2 e^{-2\varphi(w)}
$$

as in (6), we then get that $(T_{B(z;r),\varphi} f, f)/\|f\|_{2,\varphi}^2 \ge 1/C$ for any $f \in F$. We finish the proof by appealing to (8). $\square$

From Lemma 3 it follows that if Γ is sampling for F_φ^2, then

$$
\#\Gamma \cap B(z; r + \delta_0/2) \ge \#\{\lambda_m(B(z;r), \varphi) : \lambda_m > \gamma\},
$$

and from Lemma 4 that

$$
\#\Gamma \cap B(z; r - \delta_0/2) \le \#\{\lambda_m(B(z;r), \varphi) : \lambda_m > \gamma\},
$$

if Γ is interpolating for F_φ^2. Since $\#\Gamma \cap B(z; r \pm \delta_0/2) = \#\Gamma \cap B(z;r) + O(r^{2n-1})$ when Γ is uniformly separated, we can prove the necessity of the density conditions (3) and (4) by demonstrating uniform estimates in z for $\#\{\lambda_m(B(z;r), \varphi) : \lambda_m > \gamma\}$ when $r \to \infty$.

If F is a biholomorphism, we have the formula

$$
B_{\varphi\circ F}(z,\zeta) = \det F'(z) B_\varphi(F(z), F(\zeta)) \overline{\det F'(\zeta)}, \tag{9}
$$

where $F'(z)$ is the complex Jacobian matrix. From this it is not difficult to see that in particular, the operators $T_{\chi(\frac{1}{k}z),\varphi}$ and $T_{\chi,\varphi(kz)}$ have the same eigenvalues. We therefore want to have uniform estimates in z for

$$
\#\{\lambda_m(B(z;1), \varphi(k\,\cdot)) : \lambda_m > \gamma\}, \;\; k \to \infty.
$$

In particular if φ is 2-homogeneous, we want to have uniform estimates for

$$
\#\{\lambda_m(B(z;1), k^2\varphi) : \lambda_m > \gamma\}, \;\; k \to \infty.
$$

In Theorem 13 and Theorem 15 below we will show the latter kind of estimates for plurisubharmonic functions φ (not necessarily 2-homogeneous).

The estimates will follow from comparisons between the traces of the operators $T_{\chi,k^2\varphi}$ and $T^2_{\chi,k^2\varphi} = T_{\chi,k^2\varphi} \circ T_{\chi,k^2\varphi}$.

The operator $T_{\chi,k^2\varphi}$ is compact if χ is bounded with compact support, but is not self-adjoint on $L^2_{k^2\varphi}$. It is self-adjoint when restricted to $F^2_{k^2\varphi}$, but the kernel $\chi(\zeta)B_{k^2\varphi}(z,\zeta)$ is not the canonical one, which is holomorphic in the first variable and anti-holomorphic in the second. All the same, we have the trace formula

$$\sum \lambda_m(\chi, k^2\varphi) = \int \chi(z) B_{k^2\varphi}(z,z) e^{-2k^2\varphi(z)}. \tag{10}$$

For this formula to be true, it is actually sufficient that $T_{\chi,k^2\varphi}$ is self-adjoint on its image $F^2_{k^2\varphi}$, but in this case it also follows from the reproducing properties of the Bergman kernel. To see this, note that if we restrict $T_{\chi,k^2\varphi}$ to $F^2_{k^2\varphi}$ it is given by the kernel $K(z,\zeta) = \sum \lambda_m f_m(z)\overline{f_m(\zeta)}$, where

$$\begin{aligned}
\overline{K(z,\zeta)} &= P_{k^2\varphi}\left(\overline{\chi(\cdot)B_{k^2\varphi}(z,\cdot)}\right)(\zeta) \\
&= \int B_{k^2\varphi}(\zeta,w)\overline{\chi(w)B_{k^2\varphi}(z,w)}e^{-2k^2\varphi(w)}dm(w).
\end{aligned}$$

Hence we can use the reproducing properties of the Bergman kernel to see that

$$\begin{aligned}
\int K(z,z)e^{-2k^2\varphi(z)} &= \iint \chi(w)B_{k^2\varphi}(z,w)B_{k^2\varphi}(w,z)e^{-2k^2\varphi(w)}e^{-2k^2\varphi(z)} \\
&= \int \chi(w)B_{k^2\varphi}(w,w)e^{-2k^2\varphi(w)},
\end{aligned}$$

and since the last integral is bounded by Theorem 10 below this implies that $T_{\chi,k^2\varphi}$ is of trace class and that (10) holds.

The operator $T^2_{\chi,k^2\varphi}$ is given by the kernel

$$\int \chi(\zeta)B_{k^2\varphi}(z,w)B_{k^2\varphi}(w,\zeta)\chi(w)e^{-2k^2\varphi(w)}dm(w),$$

and we have the corresponding trace formula

$$\begin{aligned}
\sum \lambda^2_m(\chi,k^2\varphi) &= \iint \chi(z)B_{k^2\varphi}(z,w)B_{k^2\varphi}(w,z)\chi(w)e^{-2k^2\varphi(w)}e^{-2k^2\varphi(z)} \\
&= \iint \left|B_{k^2\varphi}(z,\zeta)\right|^2 \chi(z)\chi(\zeta)e^{-2k^2\varphi(z)-2k^2\varphi(\zeta)}.
\end{aligned} \tag{11}$$

If χ is the characteristic function for an open and bounded set in $\mathbb{C}^n$, we notice by (6) that $0 \le \lambda_m \le 1$ and we will see that asymptotically the eigenvalues will be either close to 0 or 1. What we will show is actually that for a given $\delta > 0$ we have

$$(1-\delta)\sum \lambda_m(\chi,k^2\varphi) \le \sum \lambda^2_m(\chi,k^2\varphi) \tag{12}$$

for k large enough. If we define the number S_γ by $\sum_{\lambda_m \le \gamma} \lambda_m = S_\gamma \sum \lambda_m$ for $0 < \gamma < 1$ it follows from this that

$$\begin{aligned}(1-\delta)\sum \lambda_m &\le \sum_{\lambda_m > \gamma} \lambda_m^2 + \sum_{\lambda_m \le \gamma} \lambda_m^2 \\ &\le \sum_{\lambda_m > \gamma} \lambda_m + \gamma \sum_{\lambda_m \le \gamma} \lambda_m \\ &= (1-S_\gamma)\sum \lambda_m + \gamma S_\gamma \sum \lambda_m,\end{aligned}$$

so that $S_\gamma \le \delta/(1-\gamma)$. Hence

$$\begin{aligned}\#\{\lambda_m(\chi, k^2\varphi) : \lambda_m > \gamma\} &\ge \sum_{\lambda_m > \gamma} \lambda_m(\chi, k^2\varphi) \\ &\ge \left(1 - \frac{\delta}{1-\gamma}\right) \sum \lambda_m(\chi, k^2\varphi),\end{aligned} \tag{13}$$

and with $\gamma' > \gamma$

$$\begin{aligned}&\#\{\lambda_m(\chi, k^2\varphi) : \lambda_m > \gamma\} = \\ &= \#\{\lambda_m(\chi, k^2\varphi) : \lambda_m > \gamma'\} + \#\{\lambda_m(\chi, k^2\varphi) : \gamma' \ge \lambda_m > \gamma\} \\ &\le \frac{1}{\gamma'} \sum_{\lambda_m > \gamma'} \lambda_m(\chi, k^2\varphi) + \frac{1}{\gamma} \sum_{\gamma' \ge \lambda_m > \gamma} \lambda_m(\chi, k^2\varphi) \\ &\le \frac{1}{\gamma'} \sum \lambda_m(\chi, k^2\varphi) + \frac{1}{\gamma} S_{\gamma'} \sum \lambda_m(\chi, k^2\varphi) \\ &\le \left(\frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')}\right) \sum \lambda_m(\chi, k^2\varphi).\end{aligned} \tag{14}$$

We will use this inequality to get a good estimate from above by first choosing γ' close to 1 and then δ very small by letting k be large.

To make use of (13) and (14) we need a good estimate of the trace of $T_{\chi,k^2\varphi}$ in (10). We also need a good estimate of the trace of $T^2_{\chi,k^2\varphi}$ in (11) to obtain (12). To this end we will estimate the Bergman kernel.

4 Estimates of the Bergman kernel

We make the same assumptions on φ as in (5) in the previous section. To estimate the Bergman kernel, we will need a few lemmas. The following can be found e.g. in [Ber97].

Lemma 5. *Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$, and let ϕ be plurisubharmonic in B. Let*

$$M_\phi = \{v \le 0 : \partial\bar\partial v = \partial\bar\partial \phi\}$$

and put $a_\phi = \sup_{M_\phi} v(0)$. *Assume that* $u \in L^p_{loc}(B)$ *satisfies*

$$\int_B |u|^p e^{-p\phi} \leq 1 \quad \textit{and} \quad \sup_B |\bar\partial u|^p e^{-p\phi} \leq 1.$$

Then

$$|u(0)|^p e^{-p\phi(0)} \leq C e^{-pa_\phi}$$

where C *is a universal constant.*

The next lemma is a minor elaboration of Lemma 3 in [Del98].

Lemma 6. *Assume that* ω *is a positive, d-closed* $(1,1)$*-current satisfying* $\omega \leq Mi\partial\bar\partial|z|^2$ *on a neighbourhood of a smooth, strictly pseudoconvex star shaped domain for some* $M > 0$. *Then there exists a plurisubharmonic function* ψ *on the domain such that* $i\partial\bar\partial\psi = \omega$, *and* $\|\psi\|_{L^\infty} \leq C \cdot M$, *where the constant* C *only depends on the dimension and the domain.*

Proof. Let ρ_m be an approximative identity, which only depends on $|z|$. Define

$$\omega^m = \sum_{j,k} w^m_{j,k} dz_j \wedge d\bar z_k, \quad w^m_{j,k} = w_{j,k} * \rho_m.$$

Then ω^m is a smooth, positive form. Let β be the standard Kähler form $\beta = i\partial\bar\partial|z|^2$, and τ_z translation by $-z$. On our domain D we have

$$\begin{aligned}
\|\omega^m\|_{L^\infty(D)} = \sup_{z\in D}\left(\sum_{j,k}|w^m_{j,k}(z)|^2\right)^{1/2} &\leq \sup_{z\in D}\sum_{j,k}|w^m_{j,k}(z)| \\
&\leq C \sup_{z\in D}\sum_j w^m_{j,j}(z) \\
&= C \sup_{z\in D}\langle \omega\wedge\beta^{n-1}, \tau_z\rho_m\rangle \\
&\leq C \sup_{z\in D}\langle Mi\partial\bar\partial|z|^2\wedge\beta^{n-1}, \tau_z\rho_m\rangle \\
&= C \sup_{z\in D} M\int \tau_z\rho_m = CM,
\end{aligned}$$

where the inequality in the second line follows by positivity of ω and in the fourth line by the assumption that $Mi\partial\bar\partial|z|^2 - \omega$ is positive.

The Poincaré lemma implies that we can find u^m such that $du^m = \omega^m$. To be precise, we can decompose u^m and write $d(u^m_{1,0} + u^m_{0,1}) = \omega^m$, where

$$u^m_{0,1} = \sum_{j,k}\left(\int_0^1 t\omega^m_{j,k}(tz)z_j dt\right)d\bar z_k, \quad u^m_{1,0} = \overline{u^m_{0,1}},$$

and $\partial u^m_{1,0} = 0 = \bar\partial u^m_{0,1}$ by bidegree reasons. Hence both $\|u^m_{0,1}\|_{L^\infty(D)}$ and $\|u^m_{1,0}\|_{L^\infty(D)}$ are bounded by $\|\omega^m\|_{L^\infty(D)} \leq CM$.

By Theorem 2.6.1 in the book [HL84] by Henkin and Leiterer we can solve $i\bar{\partial}v^m = u^m_{0,1}$ with $\|v^m\|_{C^{1/2}(D)} \leq C\|u^m_{0,1}\|_{L^\infty(D)} \leq CM$. Now let $\psi^m = 2\mathrm{Re}\, v^m$. Then $i\partial\bar{\partial}\psi^m = \partial(i\bar{\partial}v^m) - \bar{\partial}(i\partial\overline{v^m}) = \partial u^m_{0,1} + \bar{\partial}\,\overline{u^m_{0,1}} = \omega^m$. Since $\|\psi^m\|_{L^\infty(D)}$ is bounded by CM for every m and ω^m converges to ω, we can find a function ψ in D which satisfies the same bound and solves $i\partial\bar{\partial}\psi = \omega$. $\square$

By the assumptions on φ we get an estimate on the point evaluations in the space F^p_φ from the two previous lemmas.

Lemma 7. *If $f \in F^p_\varphi$ where $i\partial\bar{\partial}\varphi \leq Mi\partial\bar{\partial}|z|^2$ (as positive currents) then*

$$|f(z)|^p e^{-p\varphi(z)} \leq C \int_{B(z;1)} |f(w)|^p e^{-p\varphi(w)},$$

where C only depends on M, p and the dimension.

Proof. We may assume that $z = 0$. By Lemma 6 we can find a function ψ which solves $i\partial\bar{\partial}\psi = i\partial\bar{\partial}\varphi$ in the unit ball B and satisfies $\|\psi\|_{L^\infty(B)} \leq CM$. We therefore find that the constant a_ϕ in Lemma 5 is greater than or equal to $-CM$, and conclude that

$$|f(0)|^p e^{-p\varphi(0)} \leq Ce^{p\cdot CM} \int_{B(0;1)} |f(w)|^p e^{-p\varphi(w)}. \qquad \square$$

We will now prove a uniform bound for the Bergman kernel. To do this, we will use the following estimate for solutions to the $\bar{\partial}$-equation from [Ber97].

Theorem 8. *Let D be a pseudoconvex domain in $\mathbb{C}^n$ and let $\phi \in PSH(D)$ be smooth. Let f be a $\bar{\partial}$-closed $(0,1)$-form in D and let u be the solution to $\bar{\partial}u = f$ of minimal norm in $L^2_\phi(D)$. Let Ω be a positive $(1,1)$-form. Then*

$$\int_D |u|^2 e^{-\phi}\omega \leq \int_D |f|^2_\Omega e^{-\phi}\omega$$

for all positive and smooth functions ω such that

$$i\partial\bar{\partial}\omega \leq \omega(i\partial\bar{\partial}\phi - \Omega).$$

Here $|f|_\Omega$ denotes the norm of the form f with respect to the metric defined by Ω. With the help of this we can show that the Bergman kernel decays rapidly off the diagonal.

Proposition 9. *We have the estimate*

$$|B_{k^2\varphi}(z,\zeta)| \leq Ck^{2n}e^{k^2\varphi(z)+k^2\varphi(\zeta)-kT|z-\zeta|}$$

for the Bergman kernel, where T is proportional to the lower bound of $i\partial\bar{\partial}\varphi$ and C depends on the upper bound.

Proof. Fix $z \in \mathbb{C}^n$, and let $f(w) = \overline{B_{k^2\varphi}(z,w)} \in F^2_{k^2\varphi}$. We will use Lemma 7 for f in a small ball centred at ζ. By a translation we may assume that $\zeta = 0$, so let us define

$$\tilde{f}(w) = k^{-n} f(\frac{1}{k}w),\ \tilde{\varphi}(w) = k^2\varphi(\frac{1}{k}w).$$

Then we still have $i\partial\bar{\partial}\tilde{\varphi} \leq Mi\partial\bar{\partial}|z|^2$ and we get

$$k^{-2n}|f(0)|^2 e^{-2k^2\varphi(0)} = |\tilde{f}(0)|^2 e^{-2\tilde{\varphi}(0)}.$$

By Lemma 7 we have that

$$|\tilde{f}(0)|^2 e^{-2\tilde{\varphi}(0)} \lesssim \int_{B(0;1)} |\tilde{f}(w)|^2 e^{-2\tilde{\varphi}(w)} = \int_{B(0;1/k)} |f(w)|^2 e^{-2k^2\varphi(w)},$$

so that in general

$$k^{-2n}|B_{k^2\varphi}(z,\zeta)|^2 e^{-2k^2\varphi(\zeta)} \lesssim \int_{B(\zeta;1/k)} |B_{k^2\varphi}(z,w)|^2 e^{-2k^2\varphi(w)}. \tag{15}$$

When $|z-\zeta|$ is small, say $|z-\zeta| \leq 8/k$, we can estimate the right hand side of (15) by

$$\begin{aligned}\int_{\mathbb{C}^n} |B_{k^2\varphi}(z,w)|^2 e^{-2k^2\varphi(w)} dm(w) &= B_{k^2\varphi}(z,z) \\ &= \sup_{\|f\|_{2,k^2\varphi}=1} |f(z)|^2 \lesssim k^{2n} e^{2\varphi(z)},\end{aligned}$$

where the last inequality follows by Lemma 7 again. Hence we get that

$$|B_{k^2\varphi}(z,\zeta)|^2 \leq Ck^{4n} e^{2k^2\varphi(\zeta)+2k^2\varphi(z)}$$

when $|z-\zeta| \leq 8/k$.

When $|z-\zeta| > 8/k$ let $\delta = |z-\zeta|/2$ and estimate the right hand side of (15) as

$$k^{-2n}|B_{k^2\varphi}(z,\zeta)|^2 e^{-2k^2\varphi(\zeta)} \lesssim \int_{|w-z|>\delta} |B_{k^2\varphi}(z,w)|^2 e^{-2k^2\varphi(w)} dm(w). \tag{16}$$

Choose $\chi \in C^\infty$ such that $\chi = 1$ outside $B(z;\delta)$, $\chi = 0$ in $B(z;\delta/2)$ and $|\bar{\partial}\chi(w)|^2 \leq C\chi(w)/\delta^2$. Then

$$\begin{aligned}\int_{|z-w|>\delta} \left|B_{k^2\varphi}(z,w)\right|^2 e^{-2k^2\varphi(w)} &\leq \int \left|B_{k^2\varphi}(z,w)\right|^2 \chi(w) e^{-2k^2\varphi(w)} \\ &= \sup_{\substack{f\in\mathcal{O}(\mathbb{C}^n\setminus \overline{B}(z;\delta/2)) \\ \int |f|^2\chi e^{-2k^2\varphi}=1}} \left|\int B_{k^2\varphi}(z,w) f(w)\chi(w) e^{-2k^2\varphi(w)}\right|^2.\end{aligned} \tag{17}$$

Now, the last integral is the Bergman projection of $f\chi$, i.e.

$$\left|\int B_{k^2\varphi}(z,w)f(w)\chi(w)e^{-2k^2\varphi(w)}\right|^2 = \left|P_{k^2\varphi}(f\chi)(z)\right|^2.$$

We have $P_{k^2\varphi}(f\chi)(\lambda) = f(\lambda)\chi(\lambda) - u(\lambda)$, where $u(\lambda)$ is the solution of minimal norm in $L^2_{k^2\varphi}$ to $\bar\partial u = \bar\partial(f\chi)$. In particular for $\lambda = z$ we have $\chi(\lambda) = 0$ so

$$|P_{k^2\varphi}(f\chi)(z)|^2 = |u(z)|^2.$$

In $B(z;\delta/2)$ the function u is holomorphic. Since $1/k < \delta/4$ we have $B(z;1/k) \subset B(z;\delta/2)$ and as above Lemma 7 implies that

$$|u(z)|^2 e^{-2k^2\varphi(z)} \le Ck^{2n}\int_{B(z,1/k)} |u(\xi)|^2 e^{-2k^2\varphi(\xi)}. \tag{18}$$

We will use Theorem 8 to estimate this integral. Let $\Omega = \frac{i}{2}\partial\bar\partial k^2\varphi$,

$$\rho(\xi) = kT\cdot \operatorname{dist}(\xi, B(z;\delta/3)),$$

for some constant T to be chosen below, and $\omega(\xi) = e^{-\rho(\xi)}$. We then have

$$i\partial\bar\partial\omega \le k^2T^2\omega i\partial\rho\wedge\bar\partial\rho \le \frac{k^2T^2}{4}\,\omega i\partial\bar\partial|z|^2$$

and since $mi\partial\bar\partial|z|^2 \le i\partial\bar\partial\varphi$ we get

$$i\partial\bar\partial\omega \le \omega(i\partial\bar\partial k^2\varphi - \Omega),$$

as needed in Theorem 8, if $T^2/4 \le m/2$. On the set $B(z;1/k) \subset B(z,\delta/3)$ we have $\omega(\xi) = 1$, so by Theorem 8 we have that

$$\begin{aligned}
k^{2n}\int_{B(z,1/k)} |u(\xi)|^2 e^{-2k^2\varphi(\xi)} &= k^{2n}\int_{B(z,1/k)} |u(\xi)|^2 e^{-2k^2\varphi(\xi)}\omega(\xi) \\
&\le k^{2n}\int_{\mathbb{C}^n\setminus B(z,\delta/2)} |f(\xi)\bar\partial\chi(\xi)|^2_\Omega e^{-2k^2\varphi(\xi)}\omega(\xi) \\
&\le Ck^{2n}\int_{\mathbb{C}^n\setminus B(z,\delta/2)} |f(\xi)|^2\frac{C}{\delta^2}\chi(\xi)\frac{1}{k^2}e^{-2k^2\varphi(\xi)}e^{-kT\delta/6} \\
&= \frac{C}{\delta^2}k^{2n-2}e^{-kT\delta/6}\int |f(\xi)|^2\chi(\xi)e^{-2k^2\varphi(\xi)} = \frac{C}{\delta^2}k^{2n-2}e^{-kT\delta/6}
\end{aligned} \tag{19}$$

by the properties of χ and the definitions of f and ρ. Tracing back through equations (16), (17), (18) and (19) we see that

$$k^{-2n}|B_{k^2\varphi}(z,\zeta)|^2 e^{-2k^2\varphi(\zeta)} \lesssim k^{2n}e^{-kT\delta/6}e^{2k^2\varphi(z)} = k^{2n}e^{-kT|z-\zeta|/12}e^{2k^2\varphi(z)},$$

when $|z-\zeta| > 8/k$. $\square$

We are now ready to prove the first convergence property of the Bergman kernel which will help us in studying the trace formulas in the previous section. The example to keep in mind is $\varphi(z) = k^2|z|^2$. Then the Bergman kernel is $B(z, \zeta) = 2^n \pi^{-n} k^{2n}\ e^{2k^2 z \cdot \bar{\zeta}}$, and we actually have

$$k^{-2n} B(z,z) e^{-2k^2|z|^2} dm(z) = 2^n \pi^{-n} (i\partial\bar{\partial}|z|^2)^n / n!$$

in this case.

Theorem 10. *We have that*

$$k^{-2n} B_{k^2\varphi}(z,z) e^{-2k^2\varphi(z)} dm(z) \to \frac{2^n}{\pi^n n!} (i\partial\bar{\partial}\varphi)^n(z)$$

pointwise wherever φ is C^2, and the left hand side is moreover uniformly bounded for every k and z.

Proof. For the convergence, we will use the fact that

$$B_\varphi(z,z) = \sup_{\|f\|_{2,\varphi}=1} |f(z)|^2, \tag{20}$$

and by a translation we may assume that $z = 0$. The boundedness is the first, easier part of Proposition 9 when z and ζ are close together. Here it can of course be proved in only on step using (20) and Lemma 7.

Assume that φ is C^2 in a neighbourhood of the origin. By the positivity of $i\partial\bar{\partial}\varphi$ and the change of variables formula (9) for the Bergman kernel we can then make a linear change of variables so that $i\partial\bar{\partial}\varphi(0) = i\partial\bar{\partial}|z|^2$. What we want to prove in this case is that

$$k^{-2n} B_{k^2\varphi}(0,0) e^{-2k^2\varphi(0)} \to 2^n \pi^{-n}.$$

In the ball $B(0;\tau)$ we can write

$$\varphi(z) = h(z) + |z|^2 + q(z),$$

where h is a pluriharmonic polynomial of degree two, $h(0) = \varphi(0)$ and $q(z) = o(|z|^2)$. We thus have

$$|2\varphi(z) - 2|z|^2 - 2h(z)| \leq c^2(\tau)\tau^2$$

in $B(0;\tau)$, for some function $c(\tau)$ such that $c(0) = 0$ and $c(\tau)$ is continuous and non decreasing. The function $c(\tau)\tau^2$ is then increasing, so given k we can choose the unique $\tau = \tau(k)$ so that $c(\tau)\tau^2 k^2 = 1$. Then τ is a strictly decreasing function of k, $\tau \to 0$ when $k \to \infty$, and in $B(0;\tau)$ we get the estimate

$$|2k^2\varphi(z) - 2k^2|z|^2 - 2k^2 h(z)| \leq k^2 c^2(\tau)\tau^2 = c(\tau).$$

Let us write $h(z) = \operatorname{Re} H(z)$ where H is holomorphic and let $F(z) = f(z)e^{-k^2H(z)}$ for a given $f \in F^2_{k^2\varphi}$. Then $|F(0)|^2 = |f(0)|^2 e^{-2k^2\varphi(0)}$, and by subharmonicity

$$k^{-2n}|F(0)|^2 \leq \frac{1}{C_{k,\tau}} \int_{B(0;\tau)} |F(z)|^2 e^{-2k^2|z|^2},$$

where

$$C_{k,\tau} = k^{2n} \int_{B(0;\tau)} e^{-2k^2|z|^2} = 2^{-n} \int_{B(0;k\tau)} e^{-|z|^2}.$$

Since $k^2\tau^2 = 1/c(\tau) \to \infty$ as $k \to \infty$, this tends to $2^{-n}\pi^n$. Meanwhile,

$$\begin{aligned} \int_{B(0;\tau)} |F(z)|^2 e^{-2k^2|z|^2} &= \int_{B(0;\tau)} |f(z)|^2 e^{-2k^2h(z) - 2k^2|z|^2} \\ &\leq e^{c(\tau)} \int_{\mathbb{C}^n} |f(z)|^2 e^{-2k^2\varphi(z)}, \end{aligned}$$

where $c(\tau) \to 0$ as $k \to \infty$. In view of (20) this proves that

$$\limsup_{k\to\infty} k^{-2n} B_{k^2\varphi}(0,0) e^{-2k^2\varphi(0)} \leq 2^n \pi^{-n}.$$

For the reverse estimate, choose $\chi \in C_c^\infty$ which is compactly supported in $B(0;1)$ and satisfies $\chi = 1$ in the smaller ball $B(0;1/2)$. With τ as above, let $g(z) = k^n \chi(\frac{1}{\tau}z) e^{k^2H(z)}$. Then

$$\begin{aligned} \|g\|^2_{2,k^2\varphi} &\leq \int_{B(0;\tau)} k^{2n} e^{2k^2h(z)} e^{-2k^2\varphi(z)} \\ &\leq \int_{B(0;\tau)} k^{2n} e^{-2k^2|z|^2 - c(\tau)} \\ &< e^{-c(\tau)} \int_{B(0;k\tau)} e^{-|z|^2}, \end{aligned}$$

which converges to $2^{-n}\pi^n$. We also have $|\bar\partial g(z)|^2 \leq C\frac{k^{2n}}{\tau^2} e^{2k^2h(z)}$ and $\bar\partial g(z) = 0$ except when $\tau/2 < |z| < \tau$, so by Hörmander's L^2-method [Hör90, Section 4.4] we can solve $\bar\partial u = \bar\partial g$ with

$$\begin{aligned} \int_{\mathbb{C}^n} |u(z)|^2 e^{-2k^2\varphi(z)} &\leq C \int_{\tau/2<|z|<\tau} \frac{k^{2n}}{\tau^2} e^{2k^2h(z)} \frac{1}{k^2} e^{-2k^2\varphi(z)} \\ &\leq C \frac{k^{2n}}{k^2\tau^2} \int_{\tau/2<|z|<\tau} e^{-2k^2|z|^2 + c(\tau)} \\ &\leq C\pi^n \frac{e^{-k^2\tau^2/4}}{k^2\tau^2} e^{c(\tau)}, \end{aligned}$$

which tends to 0 since $k^2\tau^2 = 1/c(\tau) \to \infty$ as $k \to \infty$. By the estimates for f above applied to u we also have

$$k^{-2n}|u(0)|^2 e^{-2k^2\varphi(0)} \lesssim e^{c(\tau)} \int_{\mathbb{C}^n} |u(z)|^2 e^{-2k^2\varphi(z)} \to 0.$$

Hence $G(z) = g(z) - u(z)$ is holomorphic and

$$\frac{|G(0)|}{\|G\|_{2,k^2\varphi}} \geq \frac{|g(0)| - |u(0)|}{\|g\|_{2,k^2\varphi} + \|u\|_{2,k^2\varphi}}.$$

By the estimates for g and u we conclude that

$$\liminf_{k\to\infty} k^{-2n} B_{k^2\varphi}(0,0) e^{-2k^2\varphi(0)} \geq 2^n\pi^{-n},$$

and the proof is complete. ☐

When we are not on the diagonal, the decay of the Bergman kernel obtained in Proposition 9 implies the following.

Theorem 11. *We have that*

$$k^{-2n} \left|B_{k^2\varphi}(z,\zeta)\right|^2 e^{-2k^2\varphi(z)-2k^2\varphi(\zeta)} dm(z,\zeta) \to \frac{2^n}{\pi^n n!}(i\partial\bar\partial\varphi)^n\Big|_{\{z=\zeta\}}$$

weakly as positive measures on $\mathbb{C}^n \times \mathbb{C}^n$*, i.e.*

$$k^{-2n} \iint g(z,\zeta) \left|B_{k^2\varphi}(z,\zeta)\right|^2 e^{-2k^2\varphi(z)-2k^2\varphi(\zeta)} \to \frac{2^n}{\pi^n n!} \int g(z,z)(i\partial\bar\partial\varphi)^n(z)$$

for every $g \in C_c(\mathbb{C}^n \times \mathbb{C}^n)$.

Proof. By the reproducing properties of the Bergman kernel we have

$$\begin{aligned}\int \left|B_{k^2\varphi}(z,\zeta)\right|^2 e^{-2k^2\varphi(\zeta)} dm(\zeta) &= \int B_{k^2\varphi}(z,\zeta) B_{k^2\varphi}(\zeta,z) e^{-2k^2\varphi(\zeta)} dm(\zeta) \\ &= B_{k^2\varphi}(z,z).\end{aligned}$$

Using this we can write

$$\begin{aligned}k^{-2n} \iint g(z,\zeta) \left|B_{k^2\varphi}(z,\zeta)\right|^2 e^{-2k^2\varphi(z)-2k^2\varphi(\zeta)} = \\ k^{-2n} \int g(z,z) B_{k^2\varphi}(z,z) e^{-2k^2\varphi(z)} dm(z) \\ - k^{-2n} \iint (g(z,z) - g(z,\zeta)) \left|B_{k^2\varphi}(z,\zeta)\right|^2 e^{-2k^2\varphi(z)-2k^2\varphi(\zeta)}.\end{aligned} \tag{21}$$

By Theorem 10 and dominated convergence we know that

$$k^{-2n} \int g(z,z) B_{k^2\varphi}(z,z) e^{-2k^2\varphi(z)} dm(z) \to \frac{2^n}{\pi^n n!} \int g(z,z)(i\partial\bar\partial\varphi)^n(z),$$

so we need to estimate the last integral in (21).

Fix $\epsilon > 0$ and choose $\delta > 0$ such that $|g(z,z) - g(z,\zeta)| < \epsilon$ when $|z-\zeta| \leq \delta$. We split the last integral in (21) into the integrals over the domains where $|z-\zeta| \leq \delta$ and $|z-\zeta| > \delta$ respectively. The function g has compact support, so assume that $g = 0$ for $|z| > R$. The first of these two integrals can then be estimated by

$$\epsilon \cdot k^{-2n} \int_{|z|<R} \left(\int_{\mathbb{C}^n} \left| B_{k^2\varphi}(z,\zeta) \right|^2 e^{-2k^2\varphi(\zeta)} dm(\zeta) \right) e^{-2k^2\varphi(z)} dm(z) = \\ \epsilon \cdot k^{-2n} \int_{|z|<R} B_{k^2\varphi}(z,z) e^{-2k^2\varphi(z)} dm(z),$$

which is bounded by a constant times ϵ by Theorem 10.

The other integral we estimate with the help of Proposition 9 as

$$\begin{aligned} 2 \sup |g| \cdot k^{-2n} \iint_{|z|<R,\ |z-\zeta|>\delta} &\left| B_{k^2\varphi}(z,\zeta) \right|^2 e^{-2k^2\varphi(\zeta) - 2k^2\varphi(z)} \\ &\leq C \sup |g| \cdot k^{-2n} \iint_{|z|<R,\ |z-\zeta|>\delta} k^{4n} e^{-2kT|z-\zeta|} \\ &= C \sup |g| \int_{|z|<R} \int_{|\zeta|>\delta} k^{2n} e^{-2kT|\zeta|}, \end{aligned}$$

which tends to 0 as $k \to \infty$. Hence the remainder term in (21) is bounded by a constant times ϵ for large k, with $\epsilon > 0$ arbitrary. □

With the help of the decay of the Bergman kernel we obtained in Proposition 9 we can also calculate the dual space of F^p_φ. If we let $F^{\infty,0}_\varphi$ be the subspace of F^∞_φ consisting of functions f such that $|f(z)|e^{-\varphi(z)} \to 0$ as $z \to \infty$, we get the following theorem which we will use in the next section.

Theorem 12. *The Bergman projection P_φ projects L^p_φ boundedly onto F^p_φ for $1 \leq p \leq \infty$. For $1 \leq p < \infty$ we have $(F^p_\varphi)^* = F^q_\varphi$, with $1/p + 1/q = 1$, and furthermore $(F^{\infty,0}_\varphi)^* = F^1_\varphi$.*

Proof. Let us first see that the Bergman projection is well defined on L^p_φ. With $1 \leq p \leq \infty$ and $f \in L^p_\varphi$ we get by Proposition 9 and Hölder's inequality that

$$\begin{aligned} |P_\varphi f(z)| &= \left| \int B_\varphi(z,\zeta) f(\zeta) e^{-2\varphi(\zeta)} dm(\zeta) \right| \\ &\lesssim e^{\varphi(z)} \int |f(\zeta)| e^{-\varphi(\zeta)} e^{-2T|z-\zeta|} dm(\zeta) \\ &\leq e^{\varphi(z)} \left\| f e^{-T|z-\cdot|} \right\|_{L^p_\varphi} \cdot \left\| e^{-T|z-\cdot|} \right\|_{L^q}, \end{aligned}$$

which shows in particular that $P_\varphi f \in F^\infty_\varphi$. For $1 \le p < \infty$ if follows that

$$\begin{aligned}\int |P_\varphi f(z)|^p e^{-p\varphi(z)} dm(z) &\lesssim \iint |f(\zeta)|^p e^{-pT|z-\zeta|} e^{-p\varphi(\zeta)} dm(\zeta) dm(z)\\ &= \int |f(\zeta)|^p e^{-p\varphi(\zeta)} \left(\int e^{-pT|z-\zeta|} dm(z) \right) dm(\zeta),\end{aligned}$$

so that P_φ is bounded into F^p_φ for $1 \le p \le \infty$.

To show that P_φ is surjective, we want to show that $P_\varphi f = f$ for $f \in F^p_\varphi$. By Lemma 7 we have that $F^p_\varphi \subseteq F^{\infty,0}_\varphi \subseteq F^\infty_\varphi$ for $1 \le p < \infty$, so it is enough to show that P_φ acts reproducingly on F^∞_φ. To see this we will approximate $f \in F^\infty_\varphi$ by functions in F^2_φ.

Take a radial cutoff function $\chi \in C^\infty_c$ with $\chi(z) = 1$ when $|z| \le 1$, and $\chi(z) = 0$ when $|z| > 2$. Let $\chi_n(z) = \chi(z/n)$ and write

$$\chi_n f = P_\varphi(\chi_n f) + K(\bar\partial \chi_n \wedge f),$$

where $K(\bar\partial\chi_n \wedge f)$ as in the proof of Theorem 11 is the L^2_φ-minimal solution to $\bar\partial u = \bar\partial \chi_n \wedge f$. Since we have the uniform bound (5) from below on $i\partial\bar\partial\varphi$, a theorem in [Ber97] implies that

$$|K(\bar\partial\chi_n \wedge f)(z)| e^{-\varphi(z)} \lesssim \sup |\bar\partial \chi_n \wedge f(z)| e^{-\varphi(z)} \lesssim \frac{1}{n} \|f\|_{\infty,\varphi}.$$

If we let $f_n = P_\varphi(\chi_n f)$ we thus have that $f_n \in F^2_\varphi$, and

$$|f(z) - f_n(z)| \cdot e^{-\varphi(z)} \le |f(z) - \chi_n f(z)| \cdot e^{-\varphi(z)} + |\chi_n f(z) - P_\varphi(\chi_n f)(z)| \cdot e^{-\varphi(z)}.$$

Hence

$$\sup |f(z) - f_n(z)| e^{-\varphi(z)} \lesssim \|f\|_{\infty,\varphi} \tag{22}$$

uniformly in n, and $|f(z) - f_n(z)| e^{-\varphi(z)} \to 0$ uniformly on compacts.

Let us fix z. By Proposition 9 and (22) we have that

$$\begin{aligned}&|P_\varphi(f - f_n)(z)| \le\\ &\int_{|\zeta|\le R} |f(\zeta) - f_n(\zeta)||B_\varphi(z,\zeta)| e^{-2\varphi(\zeta)} + \int_{|\zeta|>R} |f(\zeta) - f_n(\zeta)||B_\varphi(z,\zeta)| e^{-2\varphi(\zeta)}\\ &\qquad \lesssim e^{\varphi(z)} \int_{|\zeta|\le R} |f(\zeta) - f_n(\zeta)| e^{-\varphi(\zeta)} e^{-T|z-\zeta|} + e^{\varphi(z)} \int_{|\zeta|>R} e^{-T|z-\zeta|}.\end{aligned}$$

If we first choose R big enough to make the second term small, and then n to take care of the first term, this will be small. Since $f_n \in F^2_\varphi$ we have $P_\varphi f_n = f_n$. Hence

$$|P_\varphi f(z) - f(z)| \le |P_\varphi(f - f_n)(z)| + |f_n(z) - f(z)|,$$

which tends to 0, and we have indeed that $P_\varphi f(z) = f(z)$ for every z.

Now, let $1 \le p < \infty$ and take $L \in (F^p_\varphi)^*$. Then L can be represented by $g \in L^q_\varphi$ so that

$$L(f) = \langle f, g\rangle = \langle P_\varphi f, g\rangle = \langle f, P_\varphi g\rangle$$

for every $f \in F^p_\varphi$. Hence L can be represented by $P_\varphi g \in F^q_\varphi$. Furthermore, if we have $h \perp F^p_\varphi$ for $h \in F^q_\varphi$, then $h = P_\varphi h \perp L^p_\varphi$ so that $h = 0$. We get that $(F^p_\varphi)^* = F^q_\varphi$.

It remains to show that $(F^{\infty,0}_\varphi)^* = F^1_\varphi$, so take $L \in (F^{\infty,0}_\varphi)^*$. Notice first that with the same approximation functions f_n as above, we see that F^2_φ is dense in $F^{\infty,0}_\varphi$. For $f \in F^2_\varphi$ we have $|Lf| \le C\|f\|_{F^{\infty,0}_\varphi} \le C'\|f\|_{2,\varphi}$ by Lemma 7. Hence there is a $g \in (F^2_\varphi)^* = F^2_\varphi$ so that $Lf = \langle f, g\rangle$ for every $f \in F^2_\varphi$. We want to show that in fact $g \in F^1_\varphi$, and for that it is enough to check that

$$\left|\int \overline{h} g e^{-2\varphi}\right| \le C\|h\|_{L^\infty_\varphi}$$

for every compactly supported $h \in L^\infty_\varphi$. But if h is compactly supported we have $h \in L^2_\varphi$ so that

$$\begin{aligned}\left|\int \overline{h} g e^{-2\varphi}\right| &= |\langle h, g\rangle| = |\langle h, P_\varphi g\rangle| = |\langle P_\varphi h, g\rangle| \\ &= |L(P_\varphi h)| \le C\|P_\varphi h\|_{F^\infty_\varphi} \le C'\|h\|_{L^\infty_\varphi}.\end{aligned}$$

For $f \in F^{\infty,0}_\varphi$ we now have $\|f - f_n\|_{F^{\infty,0}_\varphi} \to 0$, with $f_n \in F^2_\varphi$ as above. Hence

$$|Lf - \langle f, g\rangle| \le |L(f - f_n)| + |\langle f_n - f, g\rangle|,$$

which tends to 0 since $g \in F^1_\varphi$. We see that $L(f) = \langle f, g\rangle$ on $F^{\infty,0}_\varphi$. To show that g is unique, we must check that $g \perp F^{\infty,0}_\varphi$ implies that $g = 0$. But it is a simple calculation to check that P_φ maps compactly supported functions in L^∞ into $F^{\infty,0}_\varphi$. We then have that $(g, h) = (P_\varphi g, h) = (g, P_\varphi h) = 0$ for every compactly supported $h \in L^\infty$, which implies that $g = 0$. $\square$

We are now ready to address the question of the asymptotic behaviour of the number of eigenvalues.

Theorem 13. *Let χ be the characteristic function for an open, smooth and bounded set in $\mathbb{C}^n$, and let $0 < \gamma < 1$. Then we have*

$$\lim_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) > \gamma\} \cdot k^{-2n} = \frac{2^n}{\pi^n n!}\int \chi(i\partial\bar\partial\varphi)^n$$

for the number of eigenvalues of the concentration operator $T_{\chi,k^2\varphi}$.

Proof. The proof will follow the discussion at the end of Section 3, and use the estimates (13) and (14). Recall that these estimates followed from equation (12) for the traces, so we must start there. Fix $\delta > 0$ and $\epsilon > 0$ so small that at least $\gamma < 1/(1+\epsilon)$.

We can approximate the function χ with $0 \le \chi_0 \le \chi$ so that $\chi_0 \in C_c(\mathbb{C}^n)$. By the trace formula (11) we then have

$$k^{-2n} \sum \lambda_m^2(\chi, k^2\varphi) \ge k^{-2n} \iint \left|B_{k^2\varphi}(z,\zeta)\right|^2 \chi_0(z)\chi_0(\zeta) e^{-2k^2\varphi(z)-2k^2\varphi(\zeta)},$$

and by Theorem 11 this tends to $\frac{2^n}{\pi^n n!}\int \chi_0^2(z)(i\partial\bar\partial\varphi)^n(z)$. In the same way we have by (10) that

$$k^{-2n} \sum \lambda_m(\chi, k^2\varphi) = k^{-2n} \int \chi(z) B_{k^2\varphi}(z,z) e^{-2k^2\varphi(z)},$$

which tends to $\frac{2^n}{\pi^n n!}\int \chi(z)(i\partial\bar\partial\varphi)^n(z)$ by Theorem 10. Since our set is smoothly bounded, we can choose χ_0 so close to χ that

$$(1-\delta/2)\int \chi(z)(i\partial\bar\partial\varphi)^n(z) \le \int \chi_0^2(z)(i\partial\bar\partial\varphi)^n(z).$$

We then have that

$$(1-\delta)\sum \lambda_m(\chi, k^2\varphi) \le \sum \lambda_m^2(\chi, k^2\varphi) \tag{23}$$

for large k, which is (12).

We will now use (14) to estimate the number of eigenvalues from above. Let $\gamma' = 1/(1+\epsilon)$, and now choose $\delta = \gamma\epsilon^2/(1+\epsilon)$. Since $\gamma' > \gamma$ by the assumption on ϵ we have from (14) and (10) that

$$\begin{aligned}\#\{\lambda_m(\chi, k^2\varphi) : \lambda_m > \gamma\} \cdot k^{-2n} &\le k^{-2n}\left(\frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')}\right)\sum \lambda_m(\chi, k^2\varphi) \\ &= (1+2\epsilon)\, k^{-2n}\int \chi(z) B_{k^2\varphi}(z,z)e^{-2k^2\varphi(z)}\end{aligned}$$

for large k. By Theorem 10 it follows that

$$\limsup_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) : \lambda_m > \gamma\} \cdot k^{-2n} \le (1+2\epsilon)\frac{2^n}{\pi^n n!}\int \chi(z)(i\partial\bar\partial\varphi)^n(z).$$

With the same δ, and k large enough, we get from (13) and (10) that

$$\begin{aligned}\#\{\lambda_m(\chi, k^2\varphi) : \lambda_m > \gamma\} \cdot k^{-2n} &\ge k^{-2n}(1-\epsilon)\sum \lambda_m(\chi, k^2\varphi) \\ &= (1-\epsilon)\, k^{-2n}\int \chi(z) B_{k^2\varphi}(z,z)e^{-2k^2\varphi(z)},\end{aligned}$$

and hence by Theorem 10 that

$$\liminf_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) : \lambda_m > \gamma\} \cdot k^{-2n} \ge (1-\epsilon)\frac{2^n}{\pi^n n!}\int \chi(z)(i\partial\bar\partial\varphi)^n(z).$$

Since ϵ was arbitrary, the theorem follows. ∎

With the help of the last theorem, we are finally able to prove the density conditions for sampling and interpolation sequences. We will discuss uniformly separated sequences in F^2_φ here, and defer the general discussion to the next section.

Corollary 14. *Let $\varphi \in C^2(\mathbb{C}^n \setminus \{0\})$ be a 2-homogeneous plurisubharmonic function. If the uniformly separated sequence Γ is sampling for the space F^2_φ, then*

$$D^-(\Gamma) \geq \frac{2^n}{\pi^n n!}.$$

If the uniformly separated sequence Γ is interpolating for the space F^2_φ, then

$$D^+(\Gamma) \leq \frac{2^n}{\pi^n n!}.$$

Proof. Remember that $D^-(\Gamma) \geq \frac{2^n}{\pi^n n!}$ if and only if for all $\epsilon > 0$

$$\#\Gamma \cap B(z;r) \geq \left(\frac{2^n}{\pi^n n!} - \epsilon\right) \int_{B(z;r)} (i\partial\bar{\partial}\varphi)^n$$

for all sufficiently large r and all z. We saw in Section 3 that for a 2-homogeneous function φ it is enough to show that

$$\#\{\lambda_m(B(z;1), k^2\varphi) : \lambda_m > \gamma\} \cdot k^{-2n} \geq \left(\frac{2^n}{\pi^n n!} - \epsilon\right) \int_{B(z;1)} (i\partial\bar{\partial}\varphi)^n$$

for all large k and all z. The estimate $D^+(\Gamma) \leq \frac{2^n}{\pi^n n!}$ for interpolation sequences is the opposite one.

If φ is 2-homogeneous and C^2 outside of the origin, it automatically satisfies (5), so that we can use Theorem 13. Since we however want our estimates to be uniform for all balls $B(z;1)$, we need to be a bit careful.

In Theorem 11 the convergence depends on the diameter of the support and the supremum norm of the function g in the statement of the theorem. Since we only consider balls $B(z;1)$ this poses no problem.

The constants and the rate of convergence in Theorem 10 depend on the modulus of continuity of the second derivatives of φ. In this case all the second derivates will be uniformly continuous, and the convergence uniform, outside any ball $B(0;\tau)$. Hence we get by Theorem 13 that

$$\#\{\lambda_m(B(z;1) \setminus B(0;\tau), k^2\varphi) : \lambda_m > \gamma\} \cdot k^{-2n} \to \frac{2^n}{\pi^n n!} \int_{B(z;1)\setminus B(0;\tau)} (i\partial\bar{\partial}\varphi)^n$$

uniformly for all z.

We want to see that we can subtract the ball $B(0;\tau)$ without changing the estimates too much. We can choose τ such that

$$\int_{B(0;\tau)} (i\partial\bar{\partial}\varphi)^n \leq \epsilon \int_{B(z;1)} (i\partial\bar{\partial}\varphi)^n$$

for all z. With χ as the characteristic function for $B(z;1) \setminus B(0;\tau)$ we then see by the formulas in the proof of Theorem 13 that the estimates for $k^{-2n} \sum \lambda_m(B(z;1), k^2\varphi)$ differ very little from $k^{-2n} \sum \lambda_m(\chi, k^2\varphi)$, and the same for the squares of the eigenvalues. Most importantly, this difference is uniform in z, and can be made arbitrarily small by choosing τ small enough. If we continue in the proof of Theorem 13 this will imply that the number of eigenvalues greater then γ will differ very little irrespective if we consider the eigenvalues $\lambda_m(B(z;1), k^2\varphi)$ or $\lambda_m(\chi, k^2\varphi)$. Since we have a uniform estimate of the latter number, we have it also for the former. Alternatively, we could go through the proof of Theorem 13 and see that by the way we choose the radius τ here we will have the analogue to formula (23) for $\lambda_m(B(z;1), k^2\varphi)$, uniformly in z, which means that we will have the wanted uniform convergence. □

For a general χ we will of course not have that the eigenvalues will accumulate to 0 and 1 only. Instead we have the following statement.

Theorem 15. *Let $\chi \in C_c(\mathbb{C}^n)$ be a nonnegative function. Then for every $\gamma > 0$, except possibly for a countable sequence, we have*

$$\lim_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) > \gamma\} \cdot k^{-2n} = \frac{2^n}{\pi^n n!} \int_{\chi>\gamma} (i\partial\bar{\partial}\varphi)^n$$

for the eigenvalues of the operator $T_{\chi,k^2\varphi}$.

Proof. First of all we see that if $\chi_1 \geq \chi_2 \geq 0$ are two compactly supported and bounded functions, then for any $f \in F^2_\varphi$

$$(T_{\chi_1,\varphi}f, f) = (\chi_1 f, f) = \int \chi_1 |f|^2 e^{-2\varphi} \geq \int \chi_2 |f|^2 e^{-2\varphi} = (T_{\chi_2,\varphi}f, f).$$

From equation (8) above it follows that $\lambda_m(\chi_1, \varphi) \geq \lambda_m(\chi_2, \varphi)$ for every m. We want to use this to compare the eigenvalues of $T_{\chi,k^2\varphi}$ to eigenvalues corresponding to characteristic functions, so fix $\epsilon > 0$ and let η be the characteristic function of the set $\{z : \chi(z) > \gamma + \epsilon\}$. Then $\chi \geq (\gamma + \epsilon)\eta$ so that

$$\begin{aligned} \#\{\lambda_m(\chi, k^2\varphi) > \gamma\} &\geq \#\{\lambda_m((\gamma + \epsilon)\eta, k^2\varphi) > \gamma\} \\ &= \#\{\lambda_m(\eta, k^2\varphi) > \gamma/(\gamma + \epsilon)\} \end{aligned}$$

and by Theorem 13 we know that

$$\#\{\lambda_m(\eta, k^2\varphi) > \gamma/(\gamma + \epsilon)\} \cdot k^{-2n} \to \frac{2^n}{\pi^n n!} \int \eta (i\partial\bar{\partial}\varphi)^n.$$

Hence

$$\liminf_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) > \gamma\} \cdot k^{-2n} \geq \frac{2^n}{\pi^n n!} \int_{\chi>\gamma+\epsilon} (i\partial\bar{\partial}\varphi)^n. \tag{24}$$

For the reverse estimate, let instead η be the characteristic function of the set $\{z : \chi(z) > \gamma - \epsilon\}$ and $M = \sup \chi$. If $T_{\chi, k^2\varphi} f = \lambda f$ with $\lambda > \gamma$ and $\|f\|_{2,k^2\varphi} = 1$ then

$$\begin{aligned}\lambda = (T_{\chi,k^2\varphi} f, f) &= \int \chi \cdot (1-\eta)|f|^2 e^{-2k^2\varphi} + \int \chi \cdot \eta |f|^2 e^{-2k^2\varphi} \\ &< (\gamma - \epsilon)\|f\|_{2,k^2\varphi}^2 + \int M\eta |f|^2 e^{-2k^2\varphi},\end{aligned}$$

so that

$$(T_{M\eta,k^2\varphi} f, f) = \int M\eta |f|^2 e^{-2k^2\varphi} > \epsilon. \tag{25}$$

If we let $N = \#\{\lambda_m(\chi, k^2\varphi) > \gamma\}$ we see that there are N linearly independent functions which satisfy (25). The Weyl-Courant Lemma in (8) then implies that $\lambda_N(M\eta, k^2\varphi) > \epsilon$, so that

$$\#\{\lambda_m(\chi, k^2\varphi) > \gamma\} = N \le \#\{\lambda_m(M\eta, k^2\varphi) > \epsilon\}.$$

Since

$$\#\{\lambda_m(M\eta, k^2\varphi) > \epsilon\} = \#\{\lambda_m(\eta, k^2\varphi) > \epsilon/M\} \to \frac{2^n}{\pi^n n!} \int \eta (i\partial\bar\partial\varphi)^n$$

by Theorem 13, we get that

$$\limsup_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) > \gamma\} \cdot k^{-2n} \le \frac{2^n}{\pi^n n!} \int_{\chi > \gamma - \epsilon} (i\partial\bar\partial\varphi)^n. \tag{26}$$

By combining (24) and (26) and letting $\epsilon \to 0$ we find that

$$\lim_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) > \gamma\} \cdot k^{-2n} = \frac{2^n}{\pi^n n!} \int_{\chi > \gamma} (i\partial\bar\partial\varphi)^n$$

for every γ such that $\int_{\chi=\gamma} (i\partial\bar\partial\varphi)^n = 0$. $\square$

In the previous theorem we can interpret the left hand side as the integral over the interval (γ, ∞) of the measure with a point mass of weight k^{-2n} in all the points $\lambda_m(\chi, k^2\varphi)$ on the real axis. A more general statement would then be the following.

Theorem 16. *Let $\chi \in C_c(\mathbb{C}^n)$ be a nonnegative function. Then*

$$\nu_k = k^{-2n} \cdot \sum \delta_{\lambda_m(\chi, k^2\varphi)} \to \chi_*(i\partial\bar\partial\varphi)^n$$

as positive measures on the open positive half axis, i.e.

$$\int f d\nu_k \to \int f \circ \chi \, (i\partial\bar\partial\varphi)^n$$

for every continuous function f with compact support in $(0, \infty)$.

Proof. If $t > t_0$ then

$$\#\{\lambda_m(\chi, k^2\varphi) > t\} \cdot k^{-2n} \leq \#\{\lambda_m(\chi, k^2\varphi) > t_0\} \cdot k^{-2n}$$

and by (the proof of) the previous theorem

$$\limsup_{k\to\infty} \#\{\lambda_m(\chi, k^2\varphi) > t_0\} \cdot k^{-2n} \leq \frac{2^n}{\pi^n n!} \int_{\chi \geq t_0} (i\partial\bar{\partial}\varphi)^n.$$

Hence the measures ν_k have uniformly bounded mass on any interval (t_0, ∞). It is therefore safe to approximate f with a smooth function so we may assume that $f \in C_c^\infty$ and is compactly supported in $(0, \infty)$.

If we let $N_{\chi,k}(t) = \#\{\lambda_m(\chi, k^2\varphi) > t\}$ then $-k^{-2n}N_{\chi,k}(t)$ is a primitive distribution to ν_k, i.e.

$$\langle \nu_k, f\rangle = \langle -k^{-2n}N'_{\chi,k}(t), f\rangle = \langle k^{-2n}N_{\chi,k}(t), f'\rangle = \int_0^\infty f'(t)k^{-2n}N_{\chi,k}(t).$$

By Theorem 15 and dominated convergence we get

$$\int_0^\infty f d\nu_k \to \int_0^\infty f'(t)\left(\frac{2^n}{\pi^n n!}\int_{\chi > t} (i\partial\bar{\partial}\varphi)^n(z)\right) = \int f\circ\chi\,(i\partial\bar{\partial}\varphi)^n(z),$$

which is what we wanted to prove. ☐

5 Sampling and interpolation in F^p_φ

In Corollary 14 we have proved the density criterion for uniformly separated sequences in F^2_φ. In this section we will prove the remaining statements in Theorem 1 and Theorem 2 and show that the density criterion in F^p_φ follows from the one in F^2_φ.

The following lemma should be compared to Lemma 7. It is a small modification of Lemma 1 in [OS98].

Lemma 17. *If $f \in F^p_\varphi$ where $i\partial\bar{\partial}\varphi \leq Mi\partial\bar{\partial}|z|^2$ (as positive currents), φ is C^2 except at a finite number of points and $r > 0$, then*

$$|\nabla(|f|^r e^{-r\varphi})(z)| \leq C\left(\int_{B(z;2)} |f(w)|^p e^{-p\varphi(w)}\right)^{r/p},$$

in every point z where φ is smooth and $f(z) \neq 0$.

Proof. We can assume that $z = 0$. Since φ is smooth except at a finite number of points, we can rotate the axes in $\mathbb{C}^n$ so that the vectors $e_1, \ldots, e_n$ are the coordinate directions and $\varphi(\lambda e_j)$ is smooth when $|\lambda| < 1$.

Let $f_j(\lambda) = f(\lambda e_j)$ and $\varphi_j(\lambda) = \varphi(\lambda e_j)$. In Lemma 1 in [OS98] the result in one variable is proved assuming that φ_j is smooth and that $\Delta\varphi_j$ is uniformly bounded. Hence we get that

$$\begin{aligned} \left|\frac{\partial}{\partial z_j}(|f|^r e^{-r\varphi})(0)\right| &= \left|\frac{\partial}{\partial \lambda}(|f_j|^r e^{-r\varphi_j})(0)\right| \\ &\lesssim \left(\int_{|\lambda|<1} |f_j|^p e^{-p\varphi_j}(\lambda)\right)^{r/p} \\ &= \left(\int_{|\lambda|<1} |f|^p e^{-p\varphi}(\lambda e_j)\right)^{r/p}. \end{aligned}$$

If we now use Lemma 7 on the integrand in the last line, we see that

$$\left|\frac{\partial}{\partial z_j}(|f|^r e^{-r\varphi})(0)\right| \lesssim \left(\int_{B(0;2)} |f|^p e^{-p\varphi}(w)\right)^{r/p},$$

whence

$$|\nabla(|f|^r e^{-r\varphi})(0)| = 2\,|\partial(|f|^r e^{-r\varphi})(0)| \lesssim \left(\int_{B(0;2)} |f|^p e^{-p\varphi}(w)\right)^{r/p}. \qquad \square$$

The right inequality in the sampling inequality (1) is a consequence of the separation property of the sequence. The way to see this is first to notice that if Γ is a finite union of uniformly separated sequences, then

$$\sum_{\gamma_j\in\Gamma} |f(\gamma_j)|^p e^{-p\varphi(\gamma_j)} \lesssim \sum_{\gamma_j\in\Gamma} \int_{B(\gamma_j;1)} |f(w)|^p e^{-p\varphi(w)} \lesssim \|f\|_{p,\varphi}^p,$$

by Lemma 7. The following proposition is then proved exactly as in [OS98].

Proposition 18. *If $1 \le p < \infty$ we have*

$$\sum_{\gamma_j\in\Gamma} |f(\gamma_j)|^p e^{-p\varphi(\gamma_j)} \lesssim \|f\|_{p,\varphi}^p$$

for all $f \in F_\varphi^p$ if and only if Γ can be expressed as a finite union of uniformly separated sequences.

As a consequence of the bound on the gradient, we just as in [OS98] get the two following propositions.

Proposition 19. *If Γ is a sampling sequence for F_φ^p, then there exists a uniformly separated subsequence $\Gamma' \subset \Gamma$ which is also sampling for F_φ^p.*

Proposition 20. *If Γ is an interpolation sequence for F_φ^p, then it is uniformly separated.*

By Proposition 18, Proposition 19 and Proposition 20 we see that it is sufficient to consider uniformly separated sequences Γ. In the case $p = 2$, Theorem 1 and Theorem 2 are now completely proved by Corollary 14.

To treat sampling sequences for F^∞_φ it is enough to look at $F^{\infty,0}_\varphi$, since Γ actually is sampling for F^∞_φ if and only if it is sampling for $F^{\infty,0}_\varphi$. In one direction it is indeed clear. Assume on the other hand that Γ is sampling for $F^{\infty,0}_\varphi$. Having this sampling inequality is equivalent to that the restriction operator $R : F^{\infty,0}_\varphi \to l^{\infty,0}_\varphi$ defined by

$$Rf = \{f(\gamma_k)\} = \{\langle f, B_\varphi(\cdot, \gamma_k)\rangle\}$$

is injective and bounded, with closed range. This is in turn the same as saying that the adjoint operator $R^* : l^1_\varphi \to F^1_\varphi$ defined by $R^*(\{c_k\}) = \sum c_k B_\varphi(z, \gamma_k) e^{-2\varphi(\gamma_k)}$ is bounded and surjective. By duality again, this means that $R^{**} : F^\infty_\varphi \to l^\infty_\varphi$ is injective and bounded, with closed range, which means that Γ is sampling for F^∞_φ.

There is an alternative characterisation of sampling sequences, close in spirit to Beurling's original dual formulation of balayage in [CMNW89]. We formulate it in the following lemma.

Lemma 21. *The uniformly separated sequence Γ is sampling for F^p_φ, $1 \leq p < \infty$, if and only if there is a $K > 0$ such that every $g \in (F^p_\varphi)^* = F^q_\varphi$ can be represented as*

$$g.f = \sum_{\gamma_k \in \Gamma} c_k f(\gamma_k) e^{-2\varphi(\gamma_k)}, \quad \|\{c_k\}\|_{l^q_\varphi} \leq K \|g\|_{q,\varphi},$$

where q is the dual index to p. Γ is sampling for $F^{\infty,0}_\varphi$ if and only if the above holds with $g \in F^1_\varphi$.

Proof. We will only consider $1 < p < \infty$. The same arguments work for $F^{\infty,0}_\varphi$ and F^1_φ.

To show that the condition is sufficient, consider

$$\begin{aligned} \|f\|_{p,\varphi} &= \sup_{\|g\|_{q,\varphi}=1} |g.f| \\ &= \sup_{\|g\|_{q,\varphi}=1} \left|\sum c_k f(z_k) e^{-2\varphi(z_k)}\right| \\ &\leq \left(\sum |f(z_k)|^p e^{-p\varphi(z_k)}\right)^{1/p} \left(\sum |c_k|^q e^{-q\varphi(z_k)}\right)^{1/q} \\ &\leq K \left(\sum |f(z_k)|^p e^{-p\varphi(z_k)}\right)^{1/p}, \end{aligned}$$

which is the left sampling inequality. The right inequality is, as above, a consequence of the separation property of Γ.

To show the necessity, we remember that Γ being sampling is equivalent to that the restriction operator $R : F^p_\varphi \to l^p_\varphi$ is bounded and injective, with closed range. Therefore, R has a bounded inverse with $\|R^{-1}\| \le K < \infty$, and every $L \in (F^p_\varphi)^*$ can be written $L = (L \circ R^{-1}) \circ R$, with $\|L \circ R^{-1}\| \le K\|L\|$. Since $L \circ R^{-1}$ is an functional on a closed subspace of l^p_φ, it can be represented by $\{c_k\} \in l^q_\varphi$. □

The following proposition is a modification of a proposition in [OS98].

Proposition 22. *If the uniformly separated sequence* Γ *is sampling for* $F^p_\varphi, 1 \le p \le \infty$, *then* Γ *is sampling for* $F^2_{\varphi-\epsilon|z|^2}$, *for all small* $\epsilon > 0$.

Proof. We will only consider $1 < p < \infty$. The same arguments work for the spaces $F^{\infty,0}_\varphi$ and F^1_φ.

Assume that Γ is sampling for F^p_φ. Take $f \in F^2_{\varphi-\epsilon|z|^2}$, and let $f_z(w) = f(w)e^{2\epsilon w\cdot\bar{z}-2\epsilon|z|^2}$. By Lemma 7

$$\begin{aligned}
|f_z(w)|^p &= |f(w)|^p \cdot e^{2p\epsilon \operatorname{Re} w\cdot\bar{z}-2p\epsilon|z|^2} \\
&\lesssim \|f\|^p_{2,\varphi-\epsilon|z|^2} e^{p\varphi(w)-p\epsilon|w|^2} \cdot e^{2p\epsilon \operatorname{Re} w\cdot\bar{z}-2p\epsilon|z|^2} \\
&= \|f\|^p_{2,\varphi-\epsilon|z|^2} e^{p\varphi(w)} e^{-p\epsilon|w-z|^2-p\epsilon|z|^2}.
\end{aligned}$$

Hence

$$\|f_z\|^p_{p,\varphi} \lesssim \|f\|^p_{2,\varphi-\epsilon|z|^2} e^{-p\epsilon|z|^2} \int e^{-p\epsilon|w-z|^2} dm(w),$$

so that $f_z \in F^p_\varphi$.

By Lemma 7 all point evaluations in F^p_φ are uniformly bounded. By Lemma 21 we can therefore write

$$f_z(w)e^{-\varphi(w)} = \sum c_k(w) f_z(\gamma_k) e^{-2\varphi(\gamma_k)},$$

where $\sum |c_k(w)|^q e^{-q\varphi(\gamma_k)} \le K$ for every w. In particular with $w = z$, we have

$$\begin{aligned}
|f(z)|e^{-\varphi(z)+\epsilon|z|^2} &= e^{\epsilon|z|^2} \left|\sum c_k(z) f(\gamma_k) e^{2\epsilon\gamma_k\cdot\bar{z}-2\epsilon|z|^2} e^{-2\varphi(\gamma_k)}\right| \\
&\le \sum |c_k(z)||f(\gamma_k)| e^{-\epsilon|z-\gamma_k|^2+\epsilon|\gamma_k|^2} e^{-2\varphi(\gamma_k)},
\end{aligned}$$

so that

$$\begin{aligned}
|f(z)|^2 e^{-2\varphi(z)+2\epsilon|z|^2} \le & \sum |f(\gamma_k)|^2 e^{-2\varphi(\gamma_k)+2\epsilon|\gamma_k|^2} e^{-\epsilon|z-\gamma_k|^2} \\
& \cdot \sum |c_k(z)|^2 e^{-2\varphi(\gamma_k)} e^{-\epsilon|z-\gamma_k|^2}.
\end{aligned}$$

If $q \leq 2$ then $l^q_\varphi \subseteq l^2_\varphi$ and

$$\begin{aligned}\sum |c_k(z)|^2 e^{-2\varphi(\gamma_k)} e^{-\epsilon|z-\gamma_k|^2} &\leq \sum |c_k(z)|^2 e^{-2\varphi(\gamma_k)} \\ &\leq \left(\sum |c_k(z)|^q e^{-q\varphi(\gamma_k)}\right)^{2/q}.\end{aligned}$$

If $q > 2$ we use Hölder's inequality with the dual indices $r_1 = q/2$ and $1/r_2 = 1 - 1/r_1 = 1 - 2/q$. We get

$$\begin{aligned}\sum |c_k(z)|^2 e^{-2\varphi(\gamma_k)} e^{-\epsilon|z-\gamma_k|^2} \leq &\left(\sum |c_k(z)|^q e^{-q\varphi(\gamma_k)}\right)^{2/q} \\ &\cdot \left(\sum e^{-\frac{q}{q-2}\epsilon|z-\gamma_k|^2}\right)^{1-2/q}.\end{aligned}$$

In either case this is bounded, and we find that

$$\int |f(z)|^2 e^{-2\varphi(z)+2\epsilon|z|^2} \lesssim \sum |f(\gamma_k)|^2 e^{-2\varphi(\gamma_k)+2\epsilon|\gamma_k|^2} \int e^{-\epsilon|z-\gamma_k|^2},$$

which is the left sampling inequality. The right inequality is, as above, a consequence of the separation property of Γ. □

For interpolation we have instead the following, as in Theorem 3.3 in the article [MT99].

Proposition 23. *If the uniformly separated sequence Γ is interpolating for $F^p_\varphi, 1 \leq p \leq \infty$, then Γ is interpolating for $F^2_{\varphi+\epsilon|z|^2}$, for all small $\epsilon > 0$.*

Proof. As we remarked in the proof of Lemma 4 we can perform the interpolation in a stable way. We can therefore find $f_k \in F^p_\varphi$ such that $f_k(\gamma_j) = \delta_{jk} e^{\varphi(\gamma_k)}$ and $\|f_k\|_{p,\varphi} \leq C$ for every k. Let

$$G_k(z) = f_k(z) e^{2\epsilon z \cdot \overline{\gamma_k} - 2\epsilon|\gamma_k|^2} e^{-\varphi(\gamma_k)},$$

and take $\{v_k\} \in l^2_{\varphi+\epsilon|z|^2}$. We now claim that $G(z) = \sum v_k G_k(z)$ is an interpolating function in $F^2_{\varphi+\epsilon|z|^2}$.

It is clear that $G(\gamma_j) = \sum v_k G_k(\gamma_j) = v_j f_j(\gamma_j) e^{-\varphi(\gamma_j)} = v_j$, so we need to estimate the norm of G. Lemma 7 implies that $|f_k(z)| e^{-\varphi(z)} \lesssim \|f_k\|_{p,\varphi} \leq C$, and hence

$$\begin{aligned}|G(z)|^2 e^{-2\varphi(z)-2\epsilon|z|^2} &= e^{-2\varphi(z)-2\epsilon|z|^2} \left|\sum v_k f_k(z) e^{2\epsilon z\cdot\overline{\gamma_k} - 2\epsilon|\gamma_k|^2} e^{-\varphi(\gamma_k)}\right|^2 \\ &\leq \left(\sum |v_k| e^{-\varphi(\gamma_k)-\epsilon|\gamma_k|^2} |f_k(z)| e^{-\varphi(z)} e^{-\epsilon|z-\gamma_k|^2}\right)^2 \\ &\leq C \cdot \sum |v_k|^2 e^{-2\varphi(\gamma_k)-2\epsilon|\gamma_k|^2} e^{-\epsilon|z-\gamma_k|^2} \cdot \sum e^{-\epsilon|z-\gamma_k|^2}.\end{aligned}$$

Since Γ is uniformly separated the last sum is bounded, and we get the estimate

$$\|G\|^2_{2,\varphi+\epsilon|z|^2} \lesssim \sum |v_k|^2 e^{-2\varphi(\gamma_k)-2\epsilon|\gamma_k|^2} \int e^{-\epsilon|z-\gamma_k|^2},$$

which is bounded since $\{v_k\} \in l^2_{\varphi+\epsilon|z|^2}$. $\square$

In Corollary 14 we have proved the density conditions in Theorem 1 and Theorem 2 for sequences in F^2_φ. The density conditions for the spaces F^p_φ now follow from Proposition 22 and Proposition 23.

References

[Ber97] Bo Berndtsson. Uniform estimates with weights for the $\bar{\partial}$-equation. *The Journal of Geometric Analysis*, Vol. 7, No 2:195–215, 1997.

[BO95] Bo Berndtsson and Joaquim Ortega-Cerdà. On interpolation and sampling in Hilbert spaces of analytic functions. *J. reine angew. Math*, 464:109–128, 1995.

[Bou90] Thierry Bouche. Convergence de la métrique de Fubini-Study d'un fibré linéaire positif. *Ann. Inst. Fourier (Grenoble)*, 40(no. 1):117–130, 1990.

[CMNW89] L. Carleson, P. Malliavin, J. Neuberger, and J. Wermer, editors. *The Collected Works of Arne Beurling*, volume 2, pages 341–365. Birkhäuser, 1989.

[Del98] Henrik Delin. Pointwise estimates for the weighted Bergman projection kernel in $\mathbb{C}^n$, using a weighted L^2 estimate for the $\overline{\partial}$ equation. *Ann. Inst. Fourier (Grenoble)*, 48(no. 4):967–997, 1998.

[DG88] Ingrid Daubechies and A. Grossmann. Frames in the Bargmann space of entire functions. *Comm. Pure Appl. Math.*, 41, No. 2:151–164, 1988.

[DS52] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[HL84] Gennadi M. Henkin and Jürgen Leiterer. *Theory of Functions on Complex Manifolds*, volume 79 of *Monographs in Mathematics*. Birkhäuser, 1984.

[Hof62] Kenneth Hoffman. *Banach Spaces of Analytic Functions.* Prentice-Hall, Inc., 1962.

[Hör90] Lars Hörmander. *An Introduction to Complex Analysis in Several Variables.* North-Holland, Amsterdam, third edition, 1990.

[Lan67a] H. J. Landau. Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Mathematica*, 117:37–52, 1967.

[Lan67b] H. J. Landau. Sampling, data transmission, and the Nyquist rate. *Proceedings of the IEEE*, Vol. 55(no. 10):1701–1706, 1967.

[MT99] Xavier Massaneda and Pascal J. Thomas. Interpolating sequences for Bargmann-Fock spaces in $\mathbb{C}^n$. Preprint, 1999.

[OS98] Joaquim Ortega-Cerdà and Kristian Seip. Beurling-type density theorems for weighted L^p spaces of entire functions. *J. Analyse Math.*, 75:247–266, 1998.

[Sei91] Kristian Seip. Reproducing formulas and double orthogonality in Bargmann and Bergman spaces. *SIAM J. Math. Anal.*, 22(3):856–876, 1991.

[Sei92] Kristian Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. I. *J. Reine Angew. Math.*, 429:91–106, 1992.

[SW92] Kristian Seip and Robert Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

[Tia90] Gang Tian. On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geometry*, 32:99–130, 1990.

Niklas Lindholm, Matematik, Chalmers University of Technology and
Göteborg University, S-412 96 Göteborg, Sweden
niklin@math.chalmers.se