CHARACTERIZATION OF THE ATOMIC SPACE H^1 FOR NON DOUBLING MEASURES IN TERMS OF A GRAND MAXIMAL OPERATOR

XAVIER TOLSA

ABSTRACT. Let μ be a Radon measure on $\mathbb{R}^d$, which may be non doubling. The only condition that μ must satisfy is the size condition $\mu(B(x,r)) \leq C\,r^n$, for some fixed $0 < n \leq d$. Recently, the author introduced spaces of type $BMO(\mu)$ and $H^1(\mu)$ with properties similar to ones of the classical spaces BMO and H^1 defined for doubling measures. These new spaces proved to be useful to study the $L^p(\mu)$ boundedness of Calderón-Zygmund operators without assuming doubling conditions. In this paper a characterization of this new atomic Hardy space $H^1(\mu)$ in terms of a maximal operator M_Φ is given. It is shown that f belongs to $H^1(\mu)$ if and only if $f \in L^1(\mu)$, $\int f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$, as in the usual doubling situation.

1. INTRODUCTION

The aim of this paper is to characterize the atomic Hardy space $H^{1,\infty}_{atb}(\mu)$ introduced in [To3] in terms of a maximal operator. Throughout all the paper μ will be a (positive) Radon measure on $\mathbb{R}^d$ satisfying the growth condition

$$\mu(B(x,r)) \leq C_0\,r^n \qquad \text{for all } x \in \operatorname{supp}(\mu),\ r > 0, \tag{1.1}$$

where n is some fixed number with $0 < n \leq d$. However we do *not* assume that μ is doubling (μ is said to be doubling if there exists some constant C such that $\mu(B(x,2r)) \leq C\,\mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)$, $r > 0$).

The doubling condition on μ is an essential assumption in most results of classical Calderón-Zygmund theory. Nevertheless, recently it has been shown that many results in this theory also hold without the doubling assumption. For example, in [To1] a $T(1)$ theorem and weak $(1,1)$ estimates for the Cauchy tranforms are obtained. For general Calderón-Zygmund operators (CZO's) a $T(1)$ theorem in [NTV1], and weak $(1,1)$ estimates and Cotlar's inequality in [NTV2] are proved. A $T(b)$ is also given in [NTV3].

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For more results, see [MMNO], [NTV4], [OP], [To2], [To3], [To4] and [Ve], for example.

In [To3] some variants of the classical spaces $BMO(mu)$ and $H^1(mu)$ are introduced. These variants are denoted by $RBMO(\mu)$ and $H^{1,\infty}_{atb}(\mu)$ respectively. There, it is shown that many of the properties fulfiled by $BMO(\mu)$ and $H^1(\mu)$ when μ is doubling are also satisfied by $RBMO(\mu)$ and $H^{1,\infty}_{atb}(\mu)$ without assuming μ doubling. For example, the functions from $RBMO(\mu)$ fulfil a John-Nirenberg type inequality (see Section 5 for the precise statement of this inequality), $RBMO(\mu)$ is the dual of $H^{1,\infty}_{atb}(\mu)$, CZO's which are bounded in $L^2(\mu)$ are also bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$ and from $L^\infty(\mu)$ into $RBMO(\mu)$ and, on the other hand, any operator which is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$ and from $L^\infty(\mu)$ into $RBMO(\mu)$ is bounded in $L^p(\mu)$, $1 < p < \infty$.

Let us remark that if μ is non doubling and one defines $BMO(\mu)$ and the atomic space $H^{1,\infty}_{at}(\mu) \equiv H^1(\mu)$ exactly as in the classical doubling situation (see [GR], [Jo] or [St], for instance), then these spaces still fulfil some of the properties stated above [MMNO]. However a basic one fails: CZO's may be bounded in $L^2(\mu)$ but not from $H^{1,\infty}_{at}(\mu)$ into $L^1(\mu)$ or from $L^\infty(\mu)$ into $BMO(\mu)$ (see [Ve] and [MMNO]). For this reason, if one wants to study the L^p-boundedness of CZO's, the spaces $BMO(\mu)$ and $H^{1,\infty}_{at}(\mu)$ are not appropiate. This is the main reason for the introduction of $RBMO(\mu)$ and $H^{1,\infty}_{atb}(\mu)$ in [To3].

Before stating our main result, we need some notation and terminology. By a cube $Q \subset \mathbb{R}^d$ we mean a closed cube centered at some point in $\mathrm{supp}(\mu)$ with sides parallel to the axes. Its side length is denoted by $\ell(Q)$ and its center by z_Q. Given $\rho > 0$, we denote by ρQ the cube concentric with Q with side length $\rho\,\ell(Q)$. Recall that a function $f \in L^1_{loc}(\mu)$ belongs to the classical space $H^{1,\infty}_{at}(\mu)$ if it can be written as $f = \sum_i \lambda_i\, a_i$, where $\lambda_i \in \mathbb{R}$ are numbers such that $\sum_i |\lambda_i| < \infty$ and a_i are functions called *atoms* such that

1. there exists some cube Q_i such that $\mathrm{supp}(a_i) \subset Q_i$,
2. $\displaystyle\int a_i\, d\mu = 0$,
3. $\|a_i\|_{L^\infty(\mu)} \leq \mu(Q_i)^{-1}$.

In order to recall the precise definition of $H^{1,\infty}_{atb}(\mu)$ we have to introduce the coefficients $K_{Q,R}$. Given two cubes $Q \subset R$, we set

$$K_{Q,R} = 1 + \int_{Q_R \setminus Q} \frac{1}{|x - z_Q|^n}\, d\mu(x),$$

where Q_R is the smallest cube concentric with Q containing R.

For a fixed $\rho > 1$, a function $b \in L^1_{loc}(\mu)$ is called an *atomic block* if

1. there exists some cube R such that $\mathrm{supp}(b) \subset R$,
2. $\displaystyle\int b\, d\mu = 0$,

3. there are functions a_j supported on cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \sum_{j=1}^{\infty} \lambda_j a_j$, and
$$\|a_j\|_{L^\infty(\mu)} \leq \big(\mu(\rho Q_j)\, K_{Q_j,R}\big)^{-1}.$$

We denote
$$|b|_{H^{1,\infty}_{atb}(\mu)} = \sum_j |\lambda_j|$$
(to be rigorous, we should think that b is not only a function, but a 'structure' formed by the function b, the cubes R and Q_j, the functions a_j, etc.). Then, we say that $f \in H^{1,\infty}_{atb}(\mu)$ if there are atomic blocks b_i such that
$$f = \sum_{i=1}^{\infty} b_i, \tag{1.2}$$
with $\sum_i |b_i|_{H^{1,\infty}_{atb}(\mu)} < \infty$ (notice that this implies that the sum in (1.2) converges in $L^1(\mu)$). The $H^{1,\infty}_{atb}(\mu)$ norm of f is
$$\|f\|_{H^{1,\infty}_{atb}(\mu)} = \inf \sum_i |b_i|_{H^{1,\infty}_{atb}(\mu)},$$
where the infimum is taken over all the possible decompositions of f in atomic blocks.

The definition of $H^{1,\infty}_{atb}(\mu)$ does not depend on the constant $\rho > 1$. The $H^{1,\infty}_{atb}(\mu)$ norms for different choices of $\rho > 1$ are equivalent. Nevertheless, for definiteness, we will assume $\rho = 2$ in the definition.

Compare the definitions of the spaces $H^{1,\infty}_{at}(\mu)$ and $H^{1,\infty}_{atb}(\mu)$: In $H^{1,\infty}_{at}(\mu)$ the cancellation condition 2 and the size condition 3 are imposed over the atoms a_j. On the other hand, in $H^{1,\infty}_{atb}(\mu)$ the cancellation condition 2 is imposed over the atomic blocks b_i, and the size condition 3 is satisfied by the "components" $a_{i,j}$ of b_i separately for each j. It is not difficult to check that $H^{1,\infty}_{at}(\mu) \equiv H^{1,\infty}_{atb}(\mu)$ if $\mu(B(x,r)) \approx r$ for all $x \in \operatorname{supp}(\mu)$, $r > 0$ (the notation $A \approx B$ means that there exists some constant $C > 0$ such that $C^{-1} A \leq B \leq C\, A$, that is $A \lesssim B \lesssim A$). If the latter condition does not hold, then $H^{1,\infty}_{at}(\mu)$ may be different from $H^{1,\infty}_{atb}(\mu)$, even when μ is doubling (see [To3]).

Now we are going to introduce the "grand" maximal operator M_Φ, which is the main tool in our characterization of $H^{1,\infty}_{atb}(\mu)$.

Definition 1.1. Given $f \in L^1_{loc}(\mu)$, we set
$$M_\Phi f(x) = \sup_{\varphi \sim x} \left| \int f\, \varphi\, d\mu \right|,$$
where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap \mathcal{C}^1(\mathbb{R}^d)$ and satisfies

1. $\|\varphi\|_{L^1(\mu)} \leq 1$,
2. $0 \leq \varphi(y) \leq \dfrac{1}{|y-x|^n}$ for all $y \in \mathbb{R}^d$, and

3. $|\varphi'(y)| \leq \dfrac{1}{|y-x|^{n+1}}$ for all $y \in \mathbb{R}^d$.

In this paper we will prove the following result.

Theorem 1.2. *A function f belongs to $H^{1,\infty}_{atb}(\mu)$ if and only if $f \in L^1(\mu)$, $\int f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, in this case*

$$\|f\|_{H^{1,\infty}_{atb}(\mu)} \approx \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

Theorem 1.2 can be considered as a version for non doubling measures of some results that are already known in more classical situations. When μ is the Lebesgue in the real line, a characterization of $H^{1,\infty}_{at}(\mu)$ such as the one of Theorem 1.2 was proved by Coifman [Co]. This result was extended to the Lebesgue measure in $\mathbb{R}^d$ by Latter [La]. Let us remark that in these cases, in the definition of M_Φ, for each x it is enough to take the supremum over functions $\varphi_{x,r}$, $r > 0$, of the form

$$\varphi_{x,r}(y) = \frac{1}{r^d}\,\psi\left(\frac{y-x}{r}\right),$$

where $0 \not\equiv \psi \in \mathcal{S}$ is some fixed function.

If

$$\mu(B(x,r)) \approx r^n \qquad \text{for all } x \in \operatorname{supp}(\mu),\ r > 0, \tag{1.3}$$

then $\operatorname{supp}(\mu)$ is a homogeneous space in the sense of [CW]. For some homogeneous spaces satisfying (1.3), Coifman, Meyer and Weiss [CW] showed that there exists a description of $H^{1,\infty}_{at}(\mu)$ in terms of a maximal operator similar to the one in Theorem 1.2. They observed that a proof of this description by Carleson [Ca] using the duality $H^{1,\infty}(\mu)$–$BMO(\mu)$ in the case where μ is the Lebesgue measure on $\mathbb{R}^n$ can be easily extended to the more general situation of homogeneous spaces.

For a measure μ on $\mathbb{R}^d$ which is doubling but which may not satisfy (1.3), Macías and Segovia ([MS1], [MS2]) obtained a characterization of $H^{1,\infty}_{at}(\mu)$ by means of a maximal operator too (see also [Uc]). They showed that if μ is doubling, then taking a suitable quasimetric one can assume that (1.3) holds. Their result applies not only to doubling measures on $\mathbb{R}^d$, but to more general homogeneous spaces. Since $H^{1,\infty}_{at}(\mu)$ may be different from $H^{1,\infty}_{atb}(\mu)$ if μ is a doubling measure on $\mathbb{R}^d$ which does not satisfy (1.3), their result (in the particular case we are dealing with) cannot be derived as a particular instance of Theorem 1.2.

In Theorem 1.2 we only assume the size condition (1.1). The absence of any other regularity condition on μ makes impossible to extend classical arguments to the present situation without major changes. We will not consider any quasimetric on $\mathbb{R}^d$ different from the Euclidean distance and we are not able to reduce our case to a situation where (1.3) holds.

Let us remark that the results of [Co], [La], [MS1] and [MS2] concern not only the Hardy space H^1 but also the Hardy spaces H^p, with $0 < p < 1$.

However, it is not possible to extend our proof of Theorem 1.2 to $p < 1$ because we have obtained it by duality (following the same approach as Carleson [Ca]).

2. Preliminaries

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, do not change in different occurrences.

We will assume that the constant C_0 in (1.1) has been chosen big enough so that for all the cubes $Q \subset \mathbb{R}^d$ we have

$$\mu(Q) \leq C_0\, \ell(Q)^n. \tag{2.1}$$

Given a function $f \in L^1_{loc}(\mu)$, we denote by $m_Q f$ the mean of f over Q with respect to μ, i.e. $m_Q f = \frac{1}{\mu(Q)} \int_Q f\, d\mu$.

Definition 2.1. Given $\alpha > 1$ and $\beta > \alpha^n$, we say that the cube $Q \subset \mathbb{R}^d$ is (α, β)-doubling if $\mu(\alpha Q) \leq \beta\, \mu(Q)$.

Remark 2.2. As shown in [To3], due to the fact that μ satisfies the growth condition (1.1), there are a lot "big" doubling squares. To be precise, given any point $x \in \operatorname{supp}(\mu)$ and $c > 0$, there exists some (α, β)-doubling cube Q centered at x with $l(Q) \geq c$. This follows easily from (1.1) and the fact that $\beta > \alpha^n$.

On the other hand, if $\beta > \alpha^d$, then for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence of (α, β)-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. So there are a lot of "small" doubling squares too.

For definiteness, if α and β are not specified, by a doubling square we mean a $(2, 2^{d+1})$-doubling square.

Now we are going to recall the definition of *RBMO*(μ). In fact, in Section 2 of [To3] several equivalent definitions are given. Maybe the easiest one is the following. Let $f \in L^1_{loc}(\mu)$. We say that $f \in$ *RBMO*(μ) if there exists some constant C_1 such that for any *doubling* cube Q

$$\int_Q |f - m_Q f|\, d\mu \leq C_1\, \mu(Q) \tag{2.2}$$

and

$$|m_Q f - m_R f| \leq C_1\, K_{Q,R} \quad \text{for any two } \textit{doubling} \text{ cubes } Q \subset R. \tag{2.3}$$

The best constant C_1 is the *RBMO*(μ) norm of f, that we denote as $\|f\|_*$.

Given any pair of constants $0 < \alpha, \beta$, with $\beta > \alpha^n$, if in the definition of *RBMO*(μ) we ask (2.2) and (2.3) to hold for (α, β)-doubling cubes (instead of doubling cubes), we will get the same space *RBMO*(μ), with an equivalent norm [To3]. In fact, *RBMO*(μ) can be defined also without talking about doubling cubes: Given some fixed constant $\rho > 1$, $f \in$ *RBMO*(μ) if and only if there exists a collection of numbers $\{f_Q\}_Q$ (i.e. for each cube Q some number f_Q) and some constant C_2 such that

$$\int_Q |f(x) - f_Q|\, d\mu(x) \leq C_2\, \mu(\rho Q) \quad \text{for any cube } Q \subset \mathbb{R}^d$$

and,

$$|f_Q - f_R| \leq C_2\, K_{Q,R} \quad \text{for any two cubes } Q \subset R.$$

The best constant C_2 is comparable to the *RBMO*(μ) norm of f given by (2.2) and (2.3).

Definition 2.3. Consider two cubes $Q, R \subset \mathbb{R}^d$ (we do *not* assume $Q \subset R$). We denote

$$\delta(Q,R) = \max\left(\int_{Q_R \setminus Q} \frac{1}{|x - z_Q|^n}\, d\mu(x),\ \int_{R_Q \setminus R} \frac{1}{|x - z_R|^n}\, d\mu(x) \right).$$

(see the definition of the coefficients $K_{Q,R}$ for notations).

Notice that $\ell(Q_R) \approx \ell(R_Q) \approx \ell(Q) + \ell(R) + \operatorname{dist}(Q,R)$, and if $Q \subset R$, then $R_Q = R$ and $\ell(R) \leq \ell(Q_R) \leq 2\ell(R)$.

It is clear that if $Q \subset R$, then $K_{Q,R} = 1 + \delta(Q,R)$. Quite often we will treat points $x \in \operatorname{supp}(\mu)$ as if they were cubes (with $\ell(x) = 0$). So for $x, y \in \operatorname{supp}(\mu)$ and some cube Q, the notations $\delta(x,Q)$ and $\delta(x,y)$ make sense. In some way, they are particular cases of Definition 2.3. Of course, it may happen $\delta(x,Q) = \infty$ or $\delta(x,y) = \infty$.

In the following lemma we show that $\delta(\cdot,\cdot)$ satisfies some very useful properties.

Lemma 2.4. *The following properties hold:*

(a) *If* $\ell(Q) \approx \ell(R)$ *and* $\operatorname{dist}(Q,R) \lesssim \ell(Q)$, *then* $\delta(Q,R) \leq C$. *In particular,* $\delta(Q, \rho Q) \leq C_0\, 2^n \rho^n$ *for* $\rho > 1$.

(b) *Let* $Q \subset R$ *be concentric cubes such that there are no doubling cubes of the form* $2^k Q$, $k \geq 0$, *with* $Q \subset 2^k Q \subset R$. *Then,* $\delta(Q,R) \leq C_3$.

(c) *If* $Q \subset R$, *then*

$$\delta(Q,R) \leq C\left(1 + \log \frac{\ell(R)}{\ell(Q)}\right).$$

(d) *If* $P \subset Q \subset R$, *then*

$$\big|\delta(P,R) - [\delta(P,Q) + \delta(Q,R)]\big| \leq \varepsilon_0.$$

That is, with a different notation, $\delta(P,R) = \delta(P,Q) + \delta(Q,R) \pm \varepsilon_0$. *If* P *and* Q *are concentric, then* $\varepsilon_0 = 0$: $\delta(P,R) = \delta(P,Q) + \delta(Q,R)$.

(e) *For* $P, Q, R \subset \mathbb{R}^d$,

$$\delta(P,R) \leq C_4 + \delta(P,Q) + \delta(Q,R).$$

The constants that appear in (b), (c), (d) and (e) depend on C_0, n, d. The constant C in (a) depends, further, on the constants that are implicit in the relations $\approx$, $\lesssim$.

Let us insist on the fact that a notation such as $a = b \pm \varepsilon$ does not mean any precise equality but the estimate $|a - b| \leq \varepsilon$.

Proof. The estimates in (a) are immediate. The proof of (b) is also an easy estimate, which can be found in [To3, Lemma 2.1], for example. The arguments for (c) are also quite standard. We leave the proof for the reader.

Let us see that (d) holds. If P and Q are concentric, the identity $\delta(P, R) = \delta(P, Q) + \delta(Q, R)$ is a direct consequence of the definition. In case P and Q are not concentric we have to make some calculations:

$$\begin{aligned} \delta(P,R) &= \delta(P,P_Q) + \int_{P_R\setminus P_Q} \frac{1}{|y-z_P|^n}\,d\mu(y) \\ &= \delta(P,Q) + \int_{P_R\setminus P_Q} \frac{1}{|y-z_P|^n}\,d\mu(y). \end{aligned}$$

So we must show that

$$S := \left| \int_{P_R\setminus P_Q} \frac{1}{|y-z_P|^n}\,d\mu(y) - \delta(Q,R) \right| \leq C.$$

We set

$$\begin{aligned} S &\leq \int_{P_Q\setminus Q} \frac{1}{|y-z_Q|^n}\,d\mu(y) + \int_{P_R\Delta Q_R} \left(\frac{1}{|y-z_P|^n} + \frac{1}{|y-z_Q|^n} \right) d\mu(y) \\ &\quad + \int_{\mathbb{R}^d\setminus P_Q} \left| \frac{1}{|y-z_P|^n} - \frac{1}{|y-z_Q|^n} \right| d\mu(y) \\ &= S_1 + S_2 + S_3. \end{aligned}$$

The integral S_2 is easily estimated above by some constant C, since $|y - z_P|, |y - z_Q| \leq C\,\ell(R)$ for $y \in P_R \Delta Q_R$. An analogous calculation yields $S_1 \leq C$. For S_3 we have

$$S_3 \leq C \int_{|y-z_Q|\geq \ell(Q)/2} \frac{|z_P - z_Q|}{|y-z_Q|^{n+1}}\,d\mu(y) \leq C\,\frac{|z_P - z_Q|}{\ell(Q)} \leq C,$$

and we are done with (d).

We leave the proof of (e) for the reader too. □

Notice that if we set $D(Q, R) = 1 + \delta(Q, R)$ for $Q \neq R$ and $D(Q, Q) = 0$, then $D(\cdot, \cdot)$ is a quasidistance on the set of cubes, by (e) in the preceeding lemma.

From (a) and the fact that Q_R and R_Q have comparable sizes and $Q_R \cap R_Q \neq \varnothing$, we get that Q_R and R_Q are close in the quasimetric $D(\cdot, \cdot)$. Also, if we denote by $\widetilde{Q}$ the smallest doubling cube of the form $2^k Q$, $k \geq 0$, by (b) we know that $\widetilde{Q}$ is not far from Q (using again the quasidistance D). So Q and $\widetilde{Q}$ may have very different sizes, but we still have $D(Q, \widetilde{Q}) \leq C$.

In Remark 2.2 we have explained that there a lot of big and small doubling cubes. In the following lemma we state a more precise result about the existence of small doubling cubes in terms of $\delta(\cdot, \cdot)$.

Lemma 2.5. *There exists some (big) constant $\eta > 0$ depending only on C_0, n and d such that if R_0 is some cube centered at some point of* $\operatorname{supp}(\mu)$ *and $\alpha > \eta$, then for each $x \in R_0 \cap \operatorname{supp}(\mu)$ such that $\delta(x, 2R_0) > \alpha$ there exists some doubling cube $Q \subset 2R_0$ centered at x satisfying*

$$|\delta(Q, 2R_0) - \alpha| \leq \varepsilon_1, \tag{2.4}$$

where ε_1 depends only on C_0, n and d (but not on α).

Proof. Let Q_1 be the biggest cube centered at x with side length $2^{-k}\,\ell(R_0)$, $k \geq 1$, such that $\delta(Q_1, 2R_0) \geq \alpha$. Then, $\delta(2Q_1, 2R_0) < \alpha$. Otherwise, $k = 1$ and since $\ell(Q_1) = \ell(R_0)/2$ and $\ell(Q_{1,R_0}) \leq 4\,\ell(R_0)$ we get

$$\delta(Q_1, 2R_0) \leq \int_{\ell(Q_1)/2<|y-x|,\, y\in Q_{1,R_0}} \frac{1}{|y-x|^n}\, d\mu(y) \leq \frac{C_0\, 8^n\, \ell(R_0)^n}{\ell(Q_1)^n} = C_0\, 16^n,$$

which contradicts the choice of Q_1, assuming $\eta > C_0\, 16^n$.

Now we have $\delta(Q_1, 2R_0) \leq \alpha + \delta(Q_1, 2Q_1) \leq \alpha + C_0\, 16^n$. Thus

$$|\delta(Q_1, 2R_0) - \alpha| \leq C_0\, 16^n.$$

Let Q be the smaller doubling cube of the form $2^k\, Q_1$, $k \geq 0$. Then $\delta(Q_1, Q) \leq C_3$. Also, $\ell(Q) \leq \ell(R_0)$. Otherwise, $R_0 \subset 3Q$ and

$$\delta(Q_1, 2R_0) \leq \delta(Q_1, 3Q) = \delta(Q_1, Q) + \delta(Q, 3Q) \leq C_3 + 6^n\, C_0.$$

This is not possible if we assume $\eta > C_3 + 6^n\, C_0$.

Now Q satisfies the required properties, since it is doubling, it is contained in $2R_0$, and

$$\begin{aligned} |\delta(Q, 2R_0) - \alpha| &\leq |\delta(Q, 2R_0) - \delta(Q_1, 2R_0)| + |\delta(Q_1, 2R_0) - \alpha| \\ &\leq \delta(Q, Q_1) + C_0\, 16^n \leq C_3 + C_0\, 16^n =: \varepsilon_1. \end{aligned}$$

□

As in (d) of Lemma 2.4, instead of (2.4), often we will write $\delta(Q, 2R_0) = \alpha \pm \varepsilon_1$.

Notice that by (e) and (a) of Lemma 2.4, we get

$$\begin{aligned} |\delta(Q, R_0) - \alpha| &\leq |\delta(Q, 2R_0) - \alpha| + |\delta(Q, 2R_0) - \delta(Q, R_0)| \\ &\leq \varepsilon_1 + \delta(R_0, 2R_0) + C_4 \\ &\leq \varepsilon_1 + C + C_4 := \varepsilon_1'. \end{aligned}$$

However we prefer the estimate (2.4), because we have $Q \subset 2R_0$ but $Q \not\subset R_0$, in general. So the cube $2R_0$, in some sense, is a more appropriate reference.

Results analogous to the ones in Lemma 2.5 can be stated about the existence of cubes Q centered at some point $x \in R_0$ with $Q \supset R_0$, but since we will not need this fact below, we will not show any precise result of this kind.

If $Q \subset R$ are doubling cubes and $f \in \mathit{RBMO}(\mu)$, then $|m_Q f - m_R f| \leq (1 + \delta(Q, R))\, \|f\|_*$. Without assuming $Q \subset R$, we have a similar result:

Proposition 2.6. *Let $Q, R \subset \mathbb{R}^d$ be doubling cubes. If $f \in RBMO(\mu)$, then*

$$|m_Q f - m_R f| \leq (C + 2\,\delta(Q,R))\, \|f\|_*.$$

Proof. Suppose, for example, $\ell(R_Q) \geq \ell(Q_R)$. Then, $Q_R \subset 3R_Q$.

Let $\widetilde{3R_Q}$ be the smallest doubling cube of the form $2^k\, 3R_Q$, $k \geq 0$. We have

$$\delta(R, \widetilde{3R_Q}) = \delta(R, R_Q) + \delta(R_Q, \widetilde{3R_Q}) \leq \delta(R,Q) + C.$$

Thus

$$|m_R f - m_{\widetilde{3R_Q}} f| \leq (1 + C + \delta(R,Q))\, \|f\|_*. \tag{2.5}$$

We also have

$$\delta(Q, \widetilde{3R_Q}) \leq C + \delta(Q, 3R_Q) + \delta(3R_Q, \widetilde{3R_Q}) \leq C + \delta(Q, Q_R) + \delta(Q_R, 3R_Q).$$

Since Q_R and R_Q have comparable sizes, $\delta(Q_R, 3R_Q) \leq C$, and so

$$\delta(Q, \widetilde{3R_Q}) \leq C + \delta(Q,R).$$

Therefore,

$$|m_Q f - m_{\widetilde{3R_Q}} f| \leq (1 + C + \delta(Q,R))\, \|f\|_*. \tag{2.6}$$

By (2.5) and (2.6), the proposition follows. □

3. THE EASY IMPLICATION OF THEOREM 1.2

In this section we will prove the "only if" part of Theorem 1.2.

Lemma 3.1. *The operator M_Φ is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$.*

Proof. Let $b = \sum_i \lambda_i\, a_i$ be an atomic block supported on some cube R, with $\lambda_i \in \mathbb{R}$, where a_i are functions supported on cubes $Q_i \subset R$ such that $\|a_i\|_\infty \leq ((1 + \delta(Q_i, R))\, \mu(2Q_i))^{-1}$. We will show that $\|M_\Phi b\|_{L^1(\mu)} \leq C \sum_i |\lambda_i|$.

First we will estimate the integral $\int_{\mathbb{R}^d \setminus 2R} M_\Phi b\, d\mu$. For $x \in \mathbb{R}^d \setminus 2R$ and $\varphi \sim x$, since $\int b\, d\mu = 0$, we have

$$\begin{aligned} \left| \int b\, \varphi\, d\mu \right| &= \left| \int b(y)\, (\varphi(y) - \varphi(z_R))\, d\mu(y) \right| \\ &\leq C \int |b(y)|\, \frac{\ell(R)}{|x - z_R|^{n+1}}\, d\mu(y). \end{aligned} \tag{3.1}$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^d \setminus 2R} M_\Phi b\, d\mu &\leq C\, \|b\|_{L^1(\mu)} \int_{\mathbb{R}^d \setminus 2R} \frac{\ell(R)}{|x - z_R|^{n+1}}\, d\mu(x) \\ &\leq C\, \|b\|_{L^1(\mu)} \leq C \sum_i |\lambda_i|. \end{aligned} \tag{3.2}$$

Now we will show that

$$\int_{2R} M_\Phi a_i\, d\mu \leq C, \tag{3.3}$$

and we will be done. If $x \in 2Q_i$ and $\varphi \sim x$, then

$$\left|\int a_i\,\varphi\,d\mu\right| \leq C\,\|a_i\|_{L^\infty(\mu)}\,\|\varphi\|_{L^1(\mu)} \leq C\,\|a_i\|_{L^\infty(\mu)}.$$

So

$$\int_{2Q_i} M_\Phi a_i\,d\mu \leq C\,\|a_i\|_{L^\infty(\mu)}\,\mu(2Q_i) \leq C.$$

For $x \in 2R \setminus 2Q_i$ and $\varphi \sim x$, we have

$$\left|\int a_i\,\varphi\,d\mu\right| \leq C\,\|a_i\|_{L^1(\mu)}\,\frac{1}{|x - z_{Q_i}|^n}.$$

Therefore,

$$\begin{aligned} \int_{2R\setminus 2Q_i} M_\Phi a_i\,d\mu &\leq C\,\|a_i\|_{L^1(\mu)} \int_{2R\setminus 2Q_i} \frac{1}{|x - z_{Q_i}|^n}\,d\mu(x) \\ &\leq C\,\|a_i\|_{L^1(\mu)}\,(1 + \delta(Q_i, R)) \leq C, \end{aligned} \tag{3.4}$$

and (3.3) follows. □

4. An approach by duality for the other implication

We have to show that if $f \in L^1(\mu)$, $\int f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$, then $f \in H^{1,\infty}_{atb}(\mu)$. We will obtain this result by duality, following the ideas of Carleson [Ca]. So we will prove

Lemma 4.1 (Main Lemma). *Let $f \in RBMO(\mu)$ with compact support and $\int f\,d\mu = 0$. There exist functions $h_m \in L^\infty(\mu)$, $m \geq 0$, such that*

$$f(x) = h_0(x) + \sum_{m=1}^{\infty} \int \varphi_{y,m}(x)\,h_m(y)\,d\mu(y), \tag{4.1}$$

with convergence in $L^1(\mu)$ where, for each $m \geq 1$, $\varphi_{y,m} \sim y$, and

$$\sum_{m=0}^{\infty} |h_m| \leq C\,\|f\|_*. \tag{4.2}$$

Let us see that from this lemma the "if" part of Theorem (1.2) follows. Consider $f \in L^1(\mu)$ such that $\int f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Assume first that $f \in L^\infty(\mu)$ and has compact support. In this case, $f \in H^{1,\infty}_{atb}(\mu)$ and so we only have to estimate the norm of f.

Since $RBMO(\mu)$ is the dual of $H^{1,\infty}_{atb}(\mu)$ [To3], given $f \in L^1(\mu)$, by the Hahn-Banach theorem we have

$$\|f\|_{H^{1,\infty}_{atb}(\mu)} = \sup_{\|g\|_* \leq 1} |\langle f,\, g\rangle|.$$

Since $\int f\,d\mu = 0$, we can assume that g has compact support and $\int g\,d\mu = 0$. Then, applying the Main Lemma to g we get

$$|\langle f,\, g\rangle| \leq \left|\int f\,h_0\,d\mu\right| + \left|\sum_{m=1}^{\infty} \iint \varphi_{y,m}(x)\,h_m(y)\,f(x)\,d\mu(x)\,d\mu(y)\right|.$$

Since $\int |\varphi_{y,m}(x)\, f(x)|\, d\mu(x) \leq M_\Phi f(y)$, we have

$$\begin{aligned} |\langle f,\, g\rangle| &\leq \|f\|_{L^1(\mu)}\, \|h_0\|_{L^\infty(\mu)} + \sum_{m=1}^{\infty} \int M_\Phi f(y)\, |h_m(y)|\, d\mu(y) \\ &\leq \|f\|_{L^1(\mu)}\, \|h_0\|_{L^\infty(\mu)} + \|M_\Phi f\|_{L^1(\mu)} \left\| \sum_{m=1}^{\infty} |h_m| \right\|_{L^\infty(\mu)} \\ &\leq C\, \left(\|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)} \right) \|g\|_*. \end{aligned}$$

That is, $\|f\|_{H^{1,\infty}_{atb}(\mu)} \leq C\, \left(\|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}\right)$.

In the general case where we don't know a priori that $f \in H^{1,\infty}_{atb}(\mu)$, we can consider a sequence of functions f_n bounded with compact support such that $\int f_n\, d\mu = 0$, $f_n \to f$ in $L^1(\mu)$ and $\|M_\Phi(f - f_n)\|_{L^1(\mu)} \to 0$, and then we apply the usual arguments. The existence of such a sequence is showed in Lemma 9.1, in the Appendix.

The rest of the paper, with the exception of the Appendix, is devoted to the proof of the Main Lemma.

5. The inequality of John-Nirenberg

In [To3] it is shown that the functions of the space *RBMO*(μ) satisfy a John-Nirenberg type inequality. Let us state the precise result.

Theorem 5.1. *Let $Q \subset \mathbb{R}^d$ be a doubling cube. If $f \in$ RBMO(μ), then*

$$\mu\{x \in Q : |f - m_Q f| > \lambda\} \leq C_5\, \mu(Q)\, \exp\left(\frac{-C_6\, \lambda}{\|f\|_*}\right), \qquad \lambda > 0,$$

where $C_5, C_6 > 0$ are constants that only depend on C_0, n, d.

In the proof of the Main Lemma we will need a version of the above inequality which appears to be stronger (although it is equivalent). In this section we will state and prove this new version of John-Nirenberg inequality.

Definition 5.2. Given a doubling cube Q, we denote by $Z(Q, \lambda)$ the set of points $x \in Q$ such that any doubling cube P with $x \in P$ and $\ell(P) \leq \ell(Q)/4$ satisfies $|m_P f - m_Q f| \leq \lambda$.

In other other words, $Q \setminus Z(Q, \lambda)$ is the subset of Q such that for some doubling cube P with $x \in P$ and $\ell(P) \leq \ell(Q)/4$ we have

$$|m_P f - m_Q f| > \lambda.$$

Proposition 5.3. *Let $Q \subset \mathbb{R}^d$ be a doubling cube. If $f \in$ RBMO(μ), then*

$$\mu(Q \setminus Z(Q, \lambda)) \leq C_5'\, \mu(Q)\, \exp\left(\frac{-C_6'\, \lambda}{\|f\|_*}\right), \qquad \lambda > 0.$$

where $C_5', C_6' > 0$ are constants that only depend on C_0, n, d.

Proof. The arguments are quite standard. For any $x \in Q \setminus Z(Q,\lambda)$ there exists some cube P_x which contains x, with $\ell(P_x) \leq \ell(Q)/4$ and such that $|m_{P_x} f - m_Q f| > \lambda$. Then by Besicovich's Covering Theorem, there are points $x_i \in Q \setminus Z(Q,\lambda)$ such that

$$Q \setminus Z(Q,\lambda) \subset \bigcup_i 2P_i,$$

and so that the cubes $2P_i$, $i = 1, 2, \ldots$, form an almost disjoint family. Observe that the Covering Theorem of Besicovich cannot be applied to the cubes P_x (they are non centered), however we have applied it to the cubes $2P_x$, which are non centered too, but fulfil the condition

$$x \in \tfrac{1}{2} 2P_x.$$

That is, the point x is "far" from the boundary of $2P_x$. Under this condition, Besicovich's Covering Theorem also holds.

Since, for each i, $\ell(P_i) \leq \ell(Q)/4$ and $P_i \cap Q \neq \varnothing$, it is easlily seen that $2P_i \subset \frac{7}{4}Q$. Then,

$$\begin{aligned} \mu(Q \setminus Z(Q,\lambda)) &\leq \sum_i \mu(2P_i) \\ &\leq \sum_i \int_{P_i} \exp\left(|f(x) - m_Q f|\, k\right) \exp(-\lambda\, k)\, d\mu(x) \\ &\leq C \int_{\frac{7}{4}Q} \exp\left(|f(x) - m_Q f|\, k\right) \exp(-\lambda\, k)\, d\mu(x), \end{aligned}$$

where k is some constant that will be fixed below. Now, we have

$$\begin{aligned} \exp\left(|f(x) - m_Q f|\, k\right) &\leq \exp\left(|f(x) - m_{\frac{7}{4}Q} f|\, k\right) \exp\left(|m_{\frac{7}{4}Q} f - m_Q f|\, k\right) \\ &\leq \exp\left(|f(x) - m_{\frac{7}{4}Q} f|\, k\right) \exp\left(C\, \|f\|_*\, k\right). \end{aligned}$$

The last inequality follows from $|m_{\frac{7}{4}Q} f - m_Q f| \leq C\, \|f\|_*$ (notice that the square $\frac{7}{4}Q$ is $(\frac{8}{7}, 2^{d+1})$-doubling).

Therefore, by Theorem 5.1 (which also holds for cubes that are $(\frac{8}{7}, 2^{d+1})$-doubling instead of $(2, 2^{d+1})$-doubling, with constants $\widetilde{C}_1$ and $\widetilde{C}_2$ instead of C_1 and C_2) we have

$$\begin{aligned} &\mu(Q \setminus Z(Q,\lambda)) \\ &\leq C \exp(-\lambda\, k) \exp\left(C\, \|f\|_*\, k\right) \int_{\frac{7}{4}Q} \exp\left(|f(x) - m_{\frac{7}{4}Q} f|\, k\right) d\mu(x) \\ &= C \exp(-\lambda\, k) \exp\left(C\, \|f\|_*\, k\right) \\ &\quad \times \int_0^\infty \mu\left\{x \in \tfrac{7}{4}Q : \exp\left(|f(x) - m_{\frac{7}{4}Q} f|\, k\right) > t\right\} dt \\ &\leq C\, \mu(\tfrac{7}{4}Q) \exp(-\lambda\, k) \exp\left(C\, \|f\|_*\, k\right) \int_0^\infty \widetilde{C}_1 \exp\left(\frac{-\widetilde{C}_2 \log t}{k\, \|f\|_*}\right) dt. \end{aligned}$$

So if we choose $k := \widetilde{C}_2/2\|f\|_*$, we get

$$\mu(Q \setminus Z(Q,\lambda)) \leq C\,\mu(\tfrac{7}{4}Q)\,\exp\left(\frac{-\widetilde{C}_2\,\lambda}{2\|f\|_*}\right) \leq C\,\mu(Q)\,\exp\left(\frac{-\widetilde{C}_2\,\lambda}{2\|f\|_*}\right).$$

□

6. The “dyadic” cubes

In [Ca], Carleson proves a result analogous to the one stated in the Main Lemma for μ being the Lebesgue measure on $\mathbb{R}^d$. He uses dyadic cubes of side length 2^{-mA}, where A is some big positive integer. In our proof, we will also consider some cubes which will play the role of the dyadic cubes with side length 2^{-mA} of Carleson. In this subsection we will introduce these new “dyadic” cubes and we will show some of the properties that they satisfy and that will be needed in the proof of the Main Lemma.

As in [Ca], we will take some big positive integer A whose precise value will be fixed after knowing or choosing several additional constants. In particular, we assume that A is much bigger than the constants ε_0, ε_1 and η of Section 2.

Definition 6.1. Suppose that the support of the function f of the Main Lemma is contained in a doubling cube R_0. Let $m \geq 1$ be some fixed integer and $x \in \operatorname{supp}(\mu) \cap R_0$. If $\delta(x, 2R_0) > m\,A$, we denote by $Q_{x,m}$ a doubling cube (with $Q_{x,m} > 0$) such that

$$|\delta(Q_{x,m}, 2R_0) - m\,A| \leq \varepsilon_1. \tag{6.1}$$

Also, $\mathcal{D}'_m = \{Q_{i,m}\}_{i \in I'_m}$, is a subfamily with finite overlap of the cubes $Q_{x,m}$, such that each cube $Q_{i,m} \equiv Q_{y_i,m}$ is centered at some point $y_i \in \operatorname{supp}(\mu) \cap R_0$ with $\delta(y_i, 2R_0) > m\,A$, and

$$\{x \in \operatorname{supp}(\mu) \cap R_0 \,:\, \delta(x, 2R_0) > m\,A\} \subset \bigcup_{i \in I'_m} Q_{i,m}$$

(this family exists because of Besicovich’s Covering Theorem).

If $\delta(x, 2R_0) \leq m\,A$, we set $Q_{x,m} = \{x\}$. We denote by $\mathcal{D}''_m$ the family of cubes $Q_{x,m} \equiv \{x\}$ such that $\delta(x, 2R_0) \leq m\,A$ and $x \notin \bigcup_{i \in I'_m} Q_{i,m}$. We set $\mathcal{D}_m = \mathcal{D}'_m \cup \mathcal{D}''_m$.

The cubes $Q_{x,m}$, $x \in \operatorname{supp}(\mu) \cap R_0$ (not necessarily from the family $\mathcal{D}_m$) are called cubes of the m-th generation.

Obviously, the whole family of cubes in $\mathcal{D}_m$ has also finite overlap. Notice that if x is a point in $\operatorname{supp}(\mu)$ such that $\delta(x, 2R_0) = \infty$, then $\ell(Q_{x,m}) > 0$ for all $m \geq 1$. Otherwise, there exists some m_0 such that $\ell(Q_{x,m}) = 0$ for all $m \geq m_0$.

It is easily seen that if A is big enough, then $\ell(Q_{x,m+1}) \leq \ell(Q_{x,m})/10$ (a more precise version of this result will be proved in Lemma 6.3 below). So $\ell(Q_{x,m}) \to 0$ as $m \to \infty$.

If A is much bigger than ε_1 and $Q_{x,m} \neq \{x\}$, then $\delta(Q_{x,m}, 2R_0) \approx mA$. However, the estimate (6.1) is much sharper. This will very useful in our construction.

Lemma 6.2. *Assume that P and Q are cubes contained in $2R_0$ whose centers are in R_0. Let S be a cube such that $P, Q \subset S \subset 2R_0$.*

(a) *If $|\delta(P, 2R_0) - \delta(Q, 2R_0)| \leq \beta$, then*

$$|\delta(P, S) - \delta(Q, S)| \leq \beta + 2\varepsilon_0.$$

(b) *If $|\delta(P, S) - \delta(Q, S)| \leq \beta$, then*

$$|\delta(P, 2R_0) - \delta(Q, 2R_0)| \leq \beta + 2\varepsilon_0.$$

In particular, this lemma can be applied to cubes P and Q belonging to the same generation m, with $\beta = 2\varepsilon_1$ (assuming $\ell(P), \ell(Q) \neq 0$).

Proof. Both statements are a straightforward consequence of (d) in Lemma 2.4, since

$$\delta(P, 2R_0) = \delta(P, S) + \delta(S, 2R_0) \pm \varepsilon_0$$

and

$$\delta(Q, 2R_0) = \delta(Q, S) + \delta(S, 2R_0) \pm \varepsilon_0.$$

□

The constants ε_0 and ε_1 should be understood as upper bounds for some "errors" and deviations of our construction from the classical dyadic lattice.

We will need the following result too.

Lemma 6.3. *Assume that A is big enough. There exists some $\gamma > 0$ such that if $Q_{x,m} \cap Q_{y,m+1} \neq \varnothing$, $x, y \in \operatorname{supp}(\mu)$, then $\ell(Q_{y,m+1}) \leq 2^{-\gamma A} \ell(Q_{x,m})$.*

Proof. We can assume $Q_{y,m+1} \neq \{y\}$. Let $B > 1$ be some fixed constant. If $\ell(Q_{y,m+1}) > B^{-1} \ell(Q_{x,m})$, then $Q_{x,m} \subset 3B\, Q_{y,m+1}$. So, if R_x is a cube centered at x with side length $6B\, \ell(Q_{y,m+1})$, we have $Q_{x,m}, Q_{y,m+1} \subset R_x$.

By (c) of Lemma 2.4 we get

$$\delta(Q_{y,m+1}, R_x) \leq C \left(1 + \log \left(\frac{\ell(R_x)}{\ell(Q_{y,m+1})}\right)\right) \leq C\,(1 + \log B).$$

Since

$$\delta(Q_{y,m+1}, 2R_0) = \delta(Q_{y,m+1}, R_x) + \delta(R_x, 2R_0) \pm \varepsilon_0,$$

if we set $B = 2^{\gamma A}$, we obtain

$$\delta(R_x, 2R_0) > (m+1)\,A - \varepsilon_1 - \varepsilon_0 - C(1 + \gamma\, A\, \log 2).$$

Then for γ small enough we have

$$\delta(R_x, 2R_0) > (m+1)\,A - \varepsilon_1 - \varepsilon_0 - C - \frac{1}{2}\,A > mA + \varepsilon_1.$$

This implies $\delta(Q_{x,m}, 2R_0) > mA + \varepsilon_1$, which is not possible. □

As a consequence, we obtain

Lemma 6.4. *Assume that A is big enough. If $x, y \in \operatorname{supp}(\mu)$ are such that $Q_{x,m} \cap Q_{y,m+k} \neq \varnothing$ (with $k \geq 1$), then $\ell(Q_{y,m+k}) \leq 2^{-\gamma A k}\, \ell(Q_{x,m})$.*

Proof. By the previous lemma, $\ell(Q_{y,j+1}) \leq 2^{-\gamma A}\, \ell(Q_{y,j})$ and $\ell(Q_{y,m+1}) \leq 2^{-\gamma A}\, \ell(Q_{x,m})$. This gives $\ell(Q_{y,m+1}) \leq 2^{-\gamma A k}\, \ell(Q_{x,m})$. ☐

7. The "convolution"

The proof of the Main Lemma will be constructive. At the level of cubes of generation m we will construct a function h_m yielding the "potential"

$$U_m(x) = \int \varphi_{y,m}(x)\, h_m(y)\, d\mu(y)$$

(to be precise, instead of one function h_m, for each m we will have N functions $h_m^1, \ldots, h_m^N$, but this is a rather technical detail that we can skip now). The potentials U_m will compensate the large values of f at the scale of cubes of the generation m. So the arguments will be similar to the ones of [Ca].

However, in our situation several problems arise, in general, because of the absence of any kind of regularity in the measure μ (except the growth condition (1.1)). For example, in [Ca] the potentials U_m are convolutions with approximations of the identity: $U_m = \varphi_m * h_m$. Using the previous notation, we have

$$\varphi_{y,m}(x) = \varphi_m(y-x) = 2^{mAn}\, \varphi(2^{mA}(y-x)).$$

This is not our case. The measure μ is not invariant by translations and we don't know how it behaves under dilations (notice that if μ were doubling, we would have some information, at least, about the behaviour under dilations). We need to use functions $\varphi_{y,m}$ such that $\|\varphi_{y,m}\|_{L^1(\mu)} = 1$ (or at least equal to some value close to 1). So $\varphi_{y',m}$ cannot be obtained as a translation of $\varphi_{y,m}$ for $y' \neq y$, neither as a dilation of $\varphi_{y',k}$, $k \neq m$. In this subsection we will show how these problems can be overcome.

We denote

$$\sigma := 10\varepsilon_0 + 10\varepsilon_1 + 12^{n+1} C_0.$$

We introduce two new constants $\alpha_1, \alpha_2 > 0$ whose precise value will be fixed below. For the moment, let us say that $\varepsilon_0, \varepsilon_1, C_0, \sigma \ll \alpha_1 \ll \alpha_2 \ll A$.

Definition 7.1. Let $y \in \operatorname{supp}(\mu)$. We denote by $Q_{y,m}^1$, $\widehat{Q}_{y,m}^1$, $Q_{y,m}^2$, $\widehat{Q}_{y,m}^2$, $Q_{y,m}^3$ some doubling cubes (with positive side length) centered at y such that

$$
\begin{aligned}
\delta(Q_{y,m}, 2R_0) &= m\,A \pm \varepsilon_1,\\
\delta(Q^1_{y,m}, 2R_0) &= m\,A - \alpha_1 \pm \varepsilon_1,\\
\delta(\widehat{Q}^1_{y,m}, 2R_0) &= m\,A - \alpha_1 - \sigma \pm \varepsilon_1,\\
\delta(Q^2_{y,m}, 2R_0) &= m\,A - \alpha_1 - \alpha_2 \pm \varepsilon_1,\\
\delta(\widehat{Q}^2_{y,m}, 2R_0) &= m\,A - \alpha_1 - \alpha_2 - \sigma \pm \varepsilon_1,\\
\delta(Q^3_{y,m}, 2R_0) &= m\,A - \alpha_1 - \alpha_2 - 2\,\sigma \pm \varepsilon_1
\end{aligned}
\tag{7.1}
$$

By Lemma 2.5 we know that if $\delta(y, 2R_0) > m\,A$, then all the cubes $Q^1_{y,m}$, $\widehat{Q}^1_{y,m}$, $Q^2_{y,m}$, $\widehat{Q}^2_{y,m}$, $Q^3_{y,m}$ exist. Otherwise only some (or none) of them may exist. If any of these cubes does not exists, we let this cube be the point $\{y\}$.

Notice that we can only assume that the estimates in (7.1) hold for the cubes Q wich are different from $\{y\}$ (i.e. with $\ell(Q) > 0$). So if $\widehat{Q}^1_{y,m} = \{y\}$, say, then, we only know that $\delta(\widehat{Q}^1_{y,m}, 2R_0) \leq m\,A - \alpha_1 - \sigma + \varepsilon_1$.

Lemma 7.2. *Let $y \in \operatorname{supp}(\mu)$. If we choose the constants α_1, α_2 and A big enough, we have*

$$
Q_{y,m} \subset Q^1_{y,m} \subset \widehat{Q}^1_{y,m} \subset Q^2_{y,m} \subset \widehat{Q}^2_{y,m} \subset Q^3_{y,m} \subset Q_{y,m-1}. \tag{7.2}
$$

Proof. Notice first that for α_1, α_2 and A big enough, then the numbers that appear in the right hand side of the estimates in (7.1) form an estrictly decreasing sequence. That is,

$$
\begin{aligned}
&m\,A - \varepsilon_1 \;>\; m\,A - \alpha_1 + \varepsilon_1,\\
&m\,A - \alpha_1 - \varepsilon_1 \;>\; m\,A - \alpha_1 - \sigma + \varepsilon_1,\\
&m\,A - \alpha_1 - \sigma - \varepsilon_1 \;>\; m\,A - \alpha_1 - \alpha_2 + \varepsilon_1\\
&m\,A - \alpha_1 - \alpha_2 - \varepsilon_1 \;>\; m\,A - \alpha_1 - \alpha_2 - \sigma + \varepsilon_1,\\
&m\,A - \alpha_1 - \alpha_2 - \sigma - \varepsilon_1 \;>\; m\,A - \alpha_1 - \alpha_2 - 2\sigma + \varepsilon_1,\\
&m\,A - \alpha_1 - \alpha_2 - 2\sigma - \varepsilon_1 \;>\; (m-1)\,A + \varepsilon_1.
\end{aligned}
$$

Let us check the inclusion $\widehat{Q}^1_{y,m} \subset Q^2_{y,m}$, for example. Suppose first that $Q^2_{y,m} \neq \{y\}$, then

$$\delta(Q^2_{y,m}, 2R_0) = m\,A - \alpha_1 - \alpha_2 \pm \varepsilon_1.$$

If $\widehat{Q}^1_{y,m} = \{y\}$, the inclusion is obvious. Otherwise,

$$\delta(\widehat{Q}^1_{y,m}, 2R_0) = m\,A - \alpha_1 - \sigma \pm \varepsilon_1.$$

Then $\delta(\widehat{Q}^1_{y,m}, 2R_0) > \delta(Q^2_{y,m}, 2R_0)$, and so $\widehat{Q}^1_{y,m} \subset Q^2_{y,m}$. Assume now $Q^2_{y,m} = \{y\}$. Then,

$$\delta(y, 2R_0) \leq m\,A - \alpha_1 - \alpha_2 + \varepsilon_1.$$

In this case there is not any cube $\widehat{Q}^1_{y,m}$ satisfying

$$\delta(\widehat{Q}^1_{y,m}, 2R_0) = m\,A - \alpha_1 - \sigma \pm \varepsilon_1,$$

and so, by our convention, $\widehat{Q}^1_{y,m} = \{y\}$. That is, the inclusion holds in any case.

The other inclusions are proved in a similar way. $\square$

For a fixed m, the cubes $Q^1_{y,m}$ may have very diferent sizes for different y's. The same happens for the cubes $Q^2_{y,m}$ Nevertheless, in the following lemma we show that we still have some kind of regularity. This regularity property will be essential for our purposes.

Lemma 7.3. *Let x, y be points in* $\operatorname{supp}(\mu)$. *Then,*

(a) *If $Q^1_{x,m} \cap Q^1_{y,m} \neq \varnothing$, then $Q^1_{x,m} \subset \widehat{Q}^1_{y,m}$, in particular $x \in \widehat{Q}^1_{y,m}$.*
(b) *If $Q^2_{x,m} \cap Q^2_{y,m} \neq \varnothing$,then $Q^2_{x,m} \subset \widehat{Q}^2_{y,m}$, in particular $x \in \widehat{Q}^2_{y,m}$.*

So, although we cannot expect to have the equivalence

$$y \in Q^1_{x,m} \Leftrightarrow x \in Q^1_{y,m},$$

we still have something quite close to it, because the cubes $Q^1_{x,m}$ and $\widehat{Q}^1_{x,m}$ are close one each other in the quasimetric $D(\cdot,\cdot)$, since $\delta(Q^1_{x,m}, \widehat{Q}^1_{x,m})$ is small (at least in front of A). Of course, the same idea applies if we change 1 by 2 in the superscripts of the cubes.

Proof of Lemma 7.3. Let us proof the statement (a). The second statement is proved in an analogous way. Let x, y be as in (a). If $\ell(Q^1_{y,m}) > \ell(Q^1_{x,m})$ (in particular, $Q^1_{y,m} \neq \{y\}$), then $Q^1_{x,m} \subset 3Q^1_{y,m} \subset \widehat{Q}^1_{y,m}$ (the latter inclusion holds provided $\delta(\widehat{Q}^1_{y,m}, 2R_0) < \delta(Q^1_{y,m}, 2R_0) - 6^n C_0$).

Assume now $\ell(Q^1_{y,m}) \leq \ell(Q^1_{x,m})$. If $Q^1_{x,m} = \{x\}$, then $x = y$ and the result is trivial. If $Q^1_{x,m} \neq \{x\}$, we denote by P_y a cube centered at y with side length $3\ell(Q^1_{x,m})$. Then, $Q^1_{x,m} \subset P_y \subset 6Q^1_{x,m}$ and so $\delta(Q^1_{x,m}, P_y) \leq 12^n\,C_0$. Thus

$$\begin{aligned}
\delta(P_y, 2R_0) &\geq \delta(Q^1_{x,m}, 2R_0) - \delta(Q^1_{x,m}, P_y) - \varepsilon_0 \\
&\geq \delta(Q^1_{x,m}, 2R_0) - 12^n\,C_0 - \varepsilon_0 \\
&\geq m\,A - \alpha_1 - \sigma + \varepsilon_1.
\end{aligned}$$

Therefore, $\widehat{Q}^1_{y,m} \neq \{y\}$ and $\widehat{Q}^1_{y,m} \supset P_y \supset Q^1_{x,m}$. $\square$

Now we are going to define the functions $\varphi_{y,m}$. First we introduce the auxiliary functions $\psi_{y,m}$.

Definition 7.4. For any $y \in \operatorname{supp}(\mu) \cap 2R_0$, the function $\psi_{y,m}$ is a function such that

1. $0 \leq \psi_{y,m}(x) \leq \min\left(\dfrac{4}{\ell(Q^1_{y,m})^n}, \dfrac{1}{|y-x|^n}\right)$,

2. $\psi_{y,m}(x) = \dfrac{1}{|x-y|^n}$ if $x \in \widehat{Q}^2_{y,m} \setminus Q^1_{y,m}$,
3. $\operatorname{supp}(\psi_{y,m}) \subset Q^3_{y,m}$,
4. $|\psi'_{y,m}(x)| \leq C_{12} \min\left(\dfrac{1}{\ell(Q^1_{y,m})^{n+1}}, \dfrac{1}{|y-x|^{n+1}}\right)$.

It is not difficult to check that such a function exists if we choose C_{12} big enough. We have to take into account that $2\widehat{Q}^2_{y,m} \subset Q^3_{y,m}$. This is due to the fact that $\delta(\widehat{Q}^2_{y,m}, 2\widehat{Q}^2_{y,m}) \leq 4^n C_0 < \delta(\widehat{Q}^2_{y,m}, Q^3_{y,m})$ if $\ell(\widehat{Q}^2_{y,m}) \neq 0$.

In the definition of $\psi_{y,m}$, if $Q^1_{y,m} = \{y\}$, then one must take $1/\ell(Q^1_{y,m}) = \infty$. If $\widehat{Q}^2_{y,m} = \{y\}$, then we set $\psi_{y,m} \equiv 0$. This choice satisfies the conditions for the definition of $\psi_{y,m}$ stated above.

Choosing α_2 big enough, the largest part of the $L^1(\mu)$ norm of $\psi_{y,m}$ will come from the integral over $Q^2_{y,m} \setminus \widehat{Q}^1_{y,m}$. We state this in a precise way in the following lemma.

Lemma 7.5. *There exists some constant ε_2 depending on n, d, C_0, ε_0, ε_1 and σ (but not on α_1, α_2 nor A) such that if $Q^1_{y,m} \neq \{y\}$, then*

$$\big| \|\psi_{y,m}\|_{L^1(\mu)} - \alpha_2 \big| \leq \varepsilon_2 \tag{7.3}$$

and

$$\left| \|\psi_{y,m}\|_{L^1(\mu)} - \int_{Q^2_{y,m} \setminus \widehat{Q}^1_{y,m}} \frac{1}{|y-x|^n}\, d\mu(x) \right| \leq \varepsilon_2. \tag{7.4}$$

The proof of this result is an easy calculation that we will skip. A direct consequence of it is

$$\lim_{\alpha_2 \to \infty} \frac{1}{\alpha_2} \int_{Q^2_{y,m} \setminus \widehat{Q}^1_{y,m}} \frac{1}{|y-x|^n}\, d\mu(x) = 1$$

for $y \in \operatorname{supp}(\mu)$ such that $\delta(y, 2R_0) > m\,A$.

Definition 7.6. Let $w_{i,m}$ be the weight function defined for $y \in \bigcup_{i \in I'_m} Q_{i,m}$ (these are the cubes of $\mathcal{D}_m$ with $\ell(Q_{i,m}) > 0$) by

$$w_{i,m}(y) = \frac{\chi_{Q_{i,m}}(y)}{\sum_{j \in I'_m} \chi_{Q_{j,m}(y)}}.$$

If $y \in \operatorname{supp}(\mu) \cap 2R_0$ belongs to some cube $Q_{i,m}$ centered at some point y_i, with $\ell(Q_{i,m}) > 0$, then we set

$$\varphi_{y,m}(x) = \alpha_2^{-1} \sum_i w_{i,m}(y)\, \psi_{y_i,m}(x).$$

If y does not belong to any cube $Q_{i,m}$ with $\ell(Q_{i,m}) > 0$ (this implies $\delta(y, 2R_0) \leq mA$ and $Q_{y,m} = \{y\}$), then we set

$$\varphi_{y,m}(x) = \alpha_2^{-1}\, \psi_{y,m}(x).$$

Setting $w_{i,m}(y) = \chi_{Q_{i,m}}(y)$ if $\ell(Q_{i,m}) = 0$, we can write

$$\varphi_{y,m}(x) = \alpha_2^{-1} \sum_i w_{i,m}(y)\, \psi_{y_i,m}(x),$$

for *any* y.

Let us remark that a more natural definition for $\varphi_{y,m}$ would have been the choice $\varphi_{y,m}(x) = \alpha_2^{-1}\, \psi_{y,m}(x)$ for all y. However, as we shall see, for some of the arguments in the proof of the Main Lemma below (in Subsection 8.2), the choice of Definition 7.6 is better.

In order to study some of the properties of the functions $\varphi_{y,m}$, we need to introduce some additional notation.

Definition 7.7. Given $x \in \operatorname{supp}(\mu)$, we denote by $\widehat{\widehat{Q}}{}^3_{x,m}$ a doubling cube centered at x such that $\delta(\widehat{\widehat{Q}}{}^3_{x,m}, 2R_0) = m\,A - \alpha_1 - \alpha_2 - 3\,\sigma \pm \varepsilon_1$. Also, we denote by $\check{Q}^1_{x,m}$ and $\check{\check{Q}}{}^1_{x,m}$ some doubling cubes centered at x such that

$$\delta(\check{Q}^1_{x,m}, 2R_0) = m\,A - \alpha_1 + \sigma \pm \varepsilon_1,$$
$$\delta(\check{\check{Q}}{}^1_{x,m}, 2R_0) = m\,A - \alpha_1 + 2\sigma \pm \varepsilon_1$$

(the idea is that the symbols ^ and ˇ are inverse operations, modulo some small errors). If any of the cubes $\check{Q}^1_{x,m}$, $\check{\check{Q}}{}^1_{x,m}$, $\widehat{\widehat{Q}}{}^3_{x,m}$ does not exist, then we let it be the point x.

So, when $\delta(x, 2R_0)$ is big enough, one should think that $\widehat{\widehat{Q}}{}^3_{x,m}$ is a cube a little bigger than $\widehat{Q}^3_{x,m}$, while $\check{Q}^1_{x,m}$ is a little smaller than $Q^1_{x,m}$. Also, $\check{\check{Q}}{}^1_{x,m}$ is a little smaller than $\check{Q}^1_{x,m}$, but still much bigger than $Q_{x,m}$.

Lemma 7.8. *Let $x, y \in \operatorname{supp}(\mu)$. For α_1 and α_2 big enough, we have:*

(a) *If $x \in Q_{x_0,m}$ and $y \notin \widehat{Q}^3_{x_0,m}$, then $\varphi_{y,m}(x) = 0$. In particular, $\varphi_{y,m}(x) = 0$ if $y \notin \widehat{Q}^3_{x,m}$.*

(b) *If $y \in \check{Q}^1_{x,m}$, then $\varphi_{y,m}(x) \le C \dfrac{\alpha_2^{-1}}{\ell(\check{Q}^1_{x,m})^n}$.*

(c) *Let $\varepsilon_3 > 0$ be an arbitrary constant. If α_1 is big enough (depending on ε_3, C_0, n, d but not on α_2), then*

$$\varphi_{y,m}(x) \le \frac{\alpha_2^{-1}\,(1 + \varepsilon_3/2)}{|y - x|^n} \qquad \text{if } y \notin \check{Q}^1_{x,m},$$

and

$$\varphi_{y,m}(x) \ge \frac{\alpha_2^{-1}\,(1 - \varepsilon_3/2)}{|y - x|^n} \qquad \text{if } y \in Q^2_{x,m} \setminus \widehat{Q}^1_{x,m}.$$

(d) *If $x \in Q_{x_0,m}$, then*

$$|\varphi'_{y,m}(x)| \le C\, \alpha_2^{-1} \min\left(\frac{1}{\ell(\check{Q}^1_{x_0,m})^{n+1}}, \frac{1}{|y - x|^{n+1}} \right).$$

Notice that, in Definition 7.4 of the functions $\psi_{y,m}$, the properties that define these functions are stated with respect to cubes centered at y ($Q^1_{y,m}$, $Q^2_{y,m}$, $Q^3_{y,m}$...). In this lemma some analogous properties are stated, but these properties have to do with cubes centered at x or containing x ($Q_{x_0,m}$, $\check{Q}^1_{x,m}$, $Q^2_{x,m}$, $\widehat{Q}^3_{x,m}$...).

Proof. (a) Let $x_0 \in \operatorname{supp}(\mu)$ and $x \in Q_{x_0,m}$. If $\varphi_{y,m}(x) \neq 0$, there exists some i with $y \in Q_{i,m} \equiv Q_{y_i,m}$ and $x \in Q^3_{y_i,m}$. Then $Q^3_{x_0,m} \cap Q^3_{y_i,m} \neq \varnothing$ and so $y \in Q^3_{y_i,m} \subset \widehat{Q}^3_{x_0,m}$ (as in Lemma 7.3).

(b) Let $y \in \check{Q}^1_{x,m}$ and let y_i be such that $y \in Q_{y_i,m}$. We know that

$$\varphi_{y_i,m}(x) \leq C\,\alpha_2^{-1}\,\frac{1}{\ell(Q^1_{y_i,m})^n}.$$

So we are done if we see that $\ell(Q^1_{y_i,m}) \geq \ell(\check{Q}^1_{x,m})$.

As in Lemma 7.3, we have

$$y \in \check{Q}^1_{x,m} \Rightarrow \check{Q}^1_{y_i,m} \cap \check{Q}^1_{x,m} \neq \varnothing \Rightarrow \check{Q}^1_{x,m} \subset Q^1_{y_i,m}.$$

Thus $\ell(\check{Q}^1_{x,m}) \leq \ell(Q^1_{y_i,m})$.

(c) Let us see the first inequality. If $y \notin \check{Q}^1_{x,m}$ and y belongs to some cube $Q_{y_i,m}$ with $\ell(Q_{y_i,m}) > 0$, then $x \notin \check{\check{Q}}^1_{y_i,m}$ because otherwise, as in Lemma 7.3, we would get $\check{\check{Q}}^1_{y_i,m} \subset \check{Q}^1_{x,m}$. However, since we assume $\alpha_1 \gg \sigma$, the cube $\check{\check{Q}}^1_{y_i,m}$ is bigger than $Q_{y_i,m}$ and contains y. So $y \in \check{Q}^1_{x,m}$, which is a contradiction.

Since $x \notin \check{\check{Q}}^1_{y_i,m}$ and this cube is much bigger than $Q_{y_i,m}$, if α_1 is big enough we get

$$\frac{\alpha_2^{-1}}{|y_i - x|^n} \leq \frac{\alpha_2^{-1}\,(1+\varepsilon_3)}{|y-x|^n}.$$

As this holds for all i with $w_{i,m}(y) \neq 0$, we obtain

$$\varphi_{y,m}(x) \leq \frac{\alpha_2^{-1}\,(1+\varepsilon_3)}{|y-x|^n}.$$

This inequality also holds if $\ell(Q_{y_i,m}) = 0$ with $\varepsilon_3 = 0$, since in this case $y_i = y$.

We consider now the second inequality in (c). Let $y \in \operatorname{supp}(\mu)$ be such that $y \in Q^2_{x,m} \setminus \widehat{Q}^1_{x,m}$. If $y \in Q_{y_i,m}$ with $\ell(Q_{y_i,m}) > 0$ for some i, by Lemma 7.3 we get $x \in \widehat{Q}^2_{y_i,m} \setminus Q^1_{y_i,m}$. Since this is satisfied for all i such that $w_{i,m}(y) \neq 0$,

$$\varphi_{y,m}(x) = \sum_i w_{i,m}(y)\,\frac{\alpha_2^{-1}}{|y_i - x|^n}.$$

If α_1 has been chosen big enough, then $\ell(Q^1_{y_i,m}) \gg \ell(Q_{y_i,m})$ and one has

$$\frac{\alpha_2^{-1}}{|y_i - x|^n} \geq \frac{\alpha_2^{-1}\,(1-\varepsilon_3/2)}{|y-x|^n}.$$

Thus

$$\varphi_{y,m}(x) \geq \frac{\alpha_2^{-1}\,(1-\varepsilon_3/2)}{|y-x|^n}. \tag{7.5}$$

If $y \in Q^2_{x,m} \setminus \widehat{Q}^1_{x,m}$ and $y \in Q_{i,m}$ with $\ell(Q_{i,m}) = 0$, then by Lemma 7.3 we also get $x \in \widehat{Q}^2_{y,m} \setminus Q^1_{y,m}$ (in particular $\widehat{Q}^2_{y,m} \neq \{y\}$). Then (7.5) holds in this case too (with $\varepsilon_3 = 0$).

(d) Suppose first that $y \in \check{Q}^1_{x_0,m}$. In this case we must show that

$$|\varphi'_{y,m}(x)| \leq C\,\frac{\alpha_2^{-1}}{\ell(\check{Q}^1_{x_0,m})^{n+1}}.$$

Let y_i be such that $y \in Q_{y_i,m}$. We know that

$$|\varphi'_{y_i,m}(x)| \leq C\,\frac{\alpha_2^{-1}}{\ell(Q^1_{y_i,m})^{n+1}}.$$

By the definition of $\varphi_y(x)$, it is enough to see that $\ell(Q^1_{y_i,m}) \geq \ell(\check{Q}^1_{x_0,m})$. This follows from the inclusion $Q^1_{y_i,m} \supset \check{Q}^1_{x_0,m}$, which holds because $y \in \check{Q}^1_{y_i,m} \cap \check{Q}^1_{x_0,m}$ and then we can apply Lemma 7.3 (in fact, a slight variant of Lemma 7.3).

Suppose now that $y \notin \check{Q}^1_{x_0,m}$. It is enough to show that

$$|\varphi'_{y,m}(x)| \leq C\,\frac{\alpha_2^{-1}}{|y-x|^{n+1}}.$$

Let y_i be such that $y \in Q_{y_i,m}$. By definition we have

$$|\varphi'_{y_i,m}(x)| \leq C\,\frac{\alpha_2^{-1}}{|y_i-x|^{n+1}}.$$

We are going to see that

$$|y - y_i| \leq |y-x|/2. \tag{7.6}$$

Assume $|y-y_i| > |y-x|/2$. Then, since $x \in \frac{1}{2}\check{Q}^1_{x_0,m}$ (for α_1 big enough),

$$\ell(Q_{y_i,m}) > C^{-1}\,|y-x| \geq C^{-1}\,\ell(\check{Q}^1_{x_0,m}). \tag{7.7}$$

Notice that from the first inequality in (7.7) we get $\operatorname{dist}(x, Q_{y_i,m}) \leq C\,\ell(Q_{y_i,m})$. In this situation we have $\check{Q}^1_{x_0,m} \subset C\,Q_{y_i,m} \subset \check{\check{Q}}^1_{y_i,m}$. This is not possible, since by Lemma 7.3 we would have $\check{Q}^1_{x_0,m} \supset \check{\check{Q}}^1_{y_i,m}$, and then we would get $\check{Q}^1_{x_0,m} = \check{\check{Q}}^1_{y_i,m}$. This would imply $x_0 = y_i$ and also

$x_0 = y_i = \check{Q}^1_{x_0,m} = \check{Q}^1_{y_i,m}$, and then $y = y_i$ which is a contradiction because we are assuming that (7.6) does not hold.

So (7.6) is true and $|y_i - x| \approx |y - x|$. Thus

$$|\varphi'_{y_i,m}(x)| \leq C\, \frac{\alpha_2^{-1}}{|y-x|^{n+1}}.$$

Since this holds for any i such that $y \in Q_{y_i,m}$, we get

$$|\varphi'_{y,m}(x)| \leq C\, \frac{\alpha_2^{-1}}{|y-x|^{n+1}}.$$

□

Some of the estimates in the preceeding lemma will be used to prove next result, which was one of our main goals in this section.

Lemma 7.9. *For any* $\varepsilon_3 > 0$, *if* α_1 *and* α_2 *are big enough, for all* $x \in \operatorname{supp}(\mu)$ *we have*

$$\int \varphi_{y,m}(x)\, d\mu(y) \leq 1 + \varepsilon_3. \tag{7.8}$$

If $x \in \operatorname{supp}(\mu)$ *is such that there exists some cube* $Q \in \mathcal{D}_m$ *with* $Q \ni x$ *and* $\ell(Q) > 0$ *(in particular if* $\delta(x, 2R_0) > m\,A$*), then*

$$1 - \varepsilon_3 \leq \int \varphi_{y,m}(x)\, d\mu(y) \tag{7.9}$$

Let us observe that if μ were invariant by translations and $\varphi_{y,m}(x) = \varphi_m(y-x)$, then (7.8) and (7.9) would hold with $\varepsilon_3 = 0$ (choosing $\|\varphi_{y,m}\|_{L^1(\mu)} = 1$).

Proof. Let us see (7.9) first. So we assume that there exist some cube $Q_{i,m} \in \mathcal{D}_m$ containing x with $\ell(Q_{i,m}) > 0$. Since $x \in Q_{i,m} \subset \check{Q}^1_{i,m}$, we have $\check{Q}^1_{i,m} \subset Q^1_{x,m}$. In particular, $\ell(Q^1_{x,m}) > 0$. By Lemma 7.5 and the second inequality of (c) in Lemma 7.8 we get

$$\begin{aligned}
\int \varphi_{y,m}(x)\, d\mu(y) &\geq \int_{Q^2_{x,m} \setminus \widehat{Q}^1_{x,m}} \varphi_{y,m}(x)\, d\mu(y) \\
&\geq \int_{Q^2_{x,m} \setminus \widehat{Q}^1_{x,m}} \frac{\alpha_2^{-1}\,(1-\varepsilon_3/2)}{|y-x|^n}\, d\mu(y) \\
&\geq \alpha_2^{-1}\,(\alpha_2 - 2\varepsilon_2)\,(1-\varepsilon_3/2).
\end{aligned}$$

So (7.9) holds if we take α_2 big enough.

Consider now (7.8). By (a) in Lemma 7.8 have

$$\int \varphi_{y,m}(x)\, d\mu(y) = \int_{\widehat{Q}^3_{x,m}} \varphi_{y,m}(x)\, d\mu(y).$$

Thus we can write

$$\int \varphi_{y,m}(x)\,d\mu(y) = \int_{\widehat{Q}^3_{x,m}\setminus \check{Q}^1_{x,m}} \varphi_{y,m}(x)\,d\mu(y) + \int_{\check{Q}^1_{x,m}} \varphi_{y,m}(x)\,d\mu(y). \tag{7.10}$$

Let us estimate the first integral on the right hand side of (7.10). Using the first inequality in (c) of Lemma 7.8 we obtain

$$\begin{aligned}\int_{\widehat{Q}^3_{x,m}\setminus \check{Q}^1_{x,m}} \varphi_{y,m}(x)\,d\mu(y) &\leq \int_{\widehat{Q}^3_{x,m}\setminus \check{Q}^1_{x,m}} \frac{\alpha_2^{-1}\,(1+\varepsilon_3/2)}{|y-x|^n}\,d\mu(y)\\ &= \delta(\check{Q}^1_{x,m}, \widehat{Q}^3_{x,m})\,\alpha_2^{-1}\,(1+\varepsilon_3/2)\\ &\leq \alpha_2^{-1}\,(\alpha_2+4\,\sigma+2\,\varepsilon_1)\,(1+\varepsilon_3/2). \qquad (7.11)\end{aligned}$$

Let us consider the last integral in (7.10) (only in the case $\check{Q}^1_{x,m} \neq \{x\}$). By (b) in Lemma 7.8 we have

$$\int_{\check{Q}^1_{x,m}} \varphi_{y,m}(x)\,d\mu(y) \leq \int_{\check{Q}^1_{x,m}} \frac{C\,\alpha_2^{-1}}{\ell(\check{Q}^1_{x,m})^n}\,d\mu(y) \leq C\,C_0\,\alpha_2^{-1}. \tag{7.12}$$

From (7.11) and (7.12) we get (7.8). □

8. Proof of the Main Lemma

8.1. **The argument.** As stated above, A is a large positive integer that will be fixed at the end of the proof. We assume that the support of f is contained in some doubling cube R_0, and for each integer $m \geq 1$ we consider the family $\mathcal{D}_m$ of "dyadic" cubes $Q_{i,m}$, $i \in I_m$, introduced in Definition 6.1, and we set $\mathcal{D} = \bigcup_{m\geq 1} \mathcal{D}_m$. Recall that the elements of $\mathcal{D}$ may be cubes with side length 0, i.e. points.

For each m we will construct functions g_m and b_m. The function g_m will be supported on a subfamily $\mathcal{D}^G_m$ of the cubes in $\mathcal{D}_m$. On the other hand, b_m will be supported on a subfamily $\mathcal{D}^B_m$ of the cubes in $\mathcal{D}_m$. We set $\mathcal{D}^G = \bigcup_{m\geq 1} \mathcal{D}^G_m$ and $\mathcal{D}^B = \bigcup_{m\geq 1} \mathcal{D}^B_m$. The cubes in $\mathcal{D}^G$ will be called good cubes and the ones in $\mathcal{D}^B$ bad cubes (let us remark that in the family $\mathcal{D}_m$, in general, there are also cubes which are neither good nor bad).

From g_m and b_m, we will obtain the following potentials:

$$\begin{aligned} U^G_m(x) &= \int \varphi_{y,m}(x)\,g_m(y)\,d\mu(y),\\ U^B_m(x) &= \int \varphi_{y,m}(x)\,b_m(y)\,d\mu(y),\\ U_m(x) &= U^G_m(x) + U^B_m(x).\end{aligned}$$

This potentials will be successively subtracted from f. We will set

$$f_{m+1}(x) = f(x) - \sum_{j=1}^m U_j(x) = f_m(x) - U_m(x)$$

and

$$h_0 = f - \sum_{m=1}^{\infty} U_m = \lim_{m\to\infty} f_m. \tag{8.1}$$

The support of the functions g_m, b_m, U_m^G, U_m^B will be contained in $2R_0$.

By induction we will show that the functions g_m, b_m, U_m and f_m fulfil the following properties:

(a) $|g_m|, |b_m| \leq C_8\, A\, \|f\|_*$.
(b) $|m_Q f_{m+1}| \leq A\, \|f\|_*$ if $Q \in \mathcal{D}_m$ and $\ell(Q) > 0$.
(c) If $g_m \not\equiv 0$ on Q, $Q \in \mathcal{D}_m$, with $\ell(Q) > 0$, then $|m_Q f_{m+1}| \leq \dfrac{7}{20}\, A\, \|f\|_*$.
(d) If $Q \in \mathcal{D}_m$ and $|m_Q f_m| \leq \dfrac{8}{20}\, A\, \|f\|_*$, then $U_m \equiv 0$ and $g_m \equiv b_m \equiv 0$ on Q.
(e) If $Q \in \mathcal{D}_m$ and $\delta(Q, 2R_0) \leq (m - \frac{1}{10})\, A$ (so $\ell(Q) = 0$), then $U_m \equiv 0$ and $g_m \equiv b_m \equiv 0$ on Q.

Finally, we will see that our construction satisfies the following properties too:

(f) If $\delta(x, 2R_0) < \infty$, then $|h_0(x)| \leq C_9\, A\, \|f\|_*$, and if $Q \in \mathcal{D}_m$ and $\ell(Q) = 0$, then $|m_Q f_{m+1}| \equiv |f_{m+1}(z_Q)| \leq C_9\, A\, \|f\|_*$.
(g) For each m, there are functions $g_m^1, \ldots, g_m^N$ such that
  (g.1) $U_m^G(x) = \displaystyle\sum_{p=1}^{N} \int \varphi_{y,m}^p(x)\, g_m^p(y)\, d\mu(y)$, where $\varphi_{y,m}^p$ is defined below.
  (g.2) $|g_m^p| \leq 2C_8\, A\, \|f\|_*$ for $p = 1, \ldots, N$,
  (g.3) The functions $\sum_{p=1}^N |g_m^p|$ have disjoint supports for different m's.
(h) The family of cubes $\mathcal{D}^B$ that support the functions b_m, $m \geq 1$, satisfies the following Carleson packing condition for each cube $R \in \mathcal{D}_m$ with $\ell(R) > 0$:

$$\sum_{\substack{Q:\, Q\cap R \neq \varnothing \\ Q \in D_k^B,\, k>m}} \mu(Q) \leq C\, \mu(R). \tag{8.2}$$

Let us remark that if some cube Q coincides with a point $\{x\}$, then we set $m_Q f_m \equiv f_m(x)$. Also, the notation for the sum in (h) is an abuse of notation. This sum has to be undestood as

$$\sum_{\substack{Q:\, Q\subset 2R \\ Q \in D_k^B,\, k>m}} \mu(Q) \equiv \sum_{\substack{Q:\, \ell(Q)>0,\, Q\subset 2R \\ Q \in D_k^B,\, k>m}} \mu(Q) + \sum_{k>m} \mu\left\{x \in 2R :\, \{x\} \in D_k^B\right\}.$$

On the other hand, the number N that appears in (g) is the number of disjoint families of cubes given in the Covering Theorem of Besicovich, which only depends only on d.

The functions $\varphi_{y,m}^p$ of (g) are defined as follows. We set $\mathcal{D}_m = \mathcal{D}_m^1 \cup \cdots \cup \mathcal{D}_m^N$, where each subfamily $\mathcal{D}_m^p$ is disjoint (recall that the cubes of $\mathcal{D}_m$

originated from Besicovich's Covering Theorem). Then we set

$$\varphi^p_{y,m}(x) = \varphi_{y_i,m}(x)$$

if $y \in Q_{i,m}$ with $Q_{i,m} \in \mathcal{D}^p_m$, and $\varphi^p_{y,m}(x) \equiv 0$ if there does not exist any cube of the subfamily $\mathcal{D}^p_m$ containing y.

First we will show that if there exist functions g_m and b_m satisfying (a)–(h) then the Main Lemma follows, and later we will show the existence of these functions.

It is not difficult to check that if (4.1) and (4.2) hold,mi then the sum of (8.1) converges in $L^1_{loc}(\mu)$ (this is left to the reader). Since the support of all the functions involved is contained in $2R_0$, the convergence is in $L^1(\mu)$.

Let us see now that if (b) and (f) hold, then $\|h_0\|_{L^\infty(\mu)} \leq C\,A\,\|f\|_*$. Taking into account (f), we only have to see that $|h_0(x)| \leq C\,A\,\|f\|_*$ for $x \in \operatorname{supp}(\mu)$ such that $\delta(x, 2R_0) = \infty$. In this case, if $Q \in \mathcal{D}_k$ is such that $x \in Q$, then $\ell(Q) > 0$. We are going to see that

$$|m_Q f_m| \leq C\,A\,\|f\|_* \qquad \text{for } Q \in \mathcal{D}_k,\ k \leq m-1 \tag{8.3}$$

(not only for $k = m-1$, which is a direct consequence of (b) and (f)). Take $Q \in D_k$, $k < m-1$. This cube is covered with finite overlap by the family of cubes $\mathcal{D}_{m-1}$. Moreover, if $P \in \mathcal{D}_{m-1}$ and $P \cap Q \neq \varnothing$, then $\ell(P) \leq \ell(Q)/10$ by Lemma 6.3, and so $P \subset 2Q$. Thus we get

$$\int_Q |f_m|\,d\mu \leq \sum_i \int_{Q\cap Q_{i,m-1}} |f_m|\,d\mu \leq C\,A\,\|f\|_*\,\mu(2Q) \leq C\,A\,\|f\|_*\,\mu(Q),$$

and (8.3) follows (notice that, as remarked above, we have abused notation for the cubes which are single points).

Then h_0 will satisfy $|m_Q h_0| \leq C\,A\,\|f\|_*$ for all $Q \in \mathcal{D}$ containing x, because the sequence $\{f_m\}_m$ converges to h_0 in $L^1(\mu)$. Then, by the Lebesgue differentiation theorem we will get that $|h_0(x)| \leq C\,A\,\|f\|_*$ (this theorem can be applied to the cubes $Q \in \mathcal{D}$ which are non centered because they are doubling) for μ-a.e. $x \in \operatorname{supp}(\mu)$ with $\delta(x, 2R_0) = \infty$. Therefore, $\|h_0\|_{L^\infty(\mu)} \leq C\,A\,\|f\|_*$.

Observe that the functions g^p_m in (g.1) originate the same potential as g_m. In fact, they will be constructed modifying slightly the function g_m in such a way that they are supported in disjoint sets for different m's. By (g.2) we have

$$\sum_m \sum_{p=1}^N |g^p_m| \leq 2N\,C_8\,A\,\|f\|_*.$$

The supports of the functions b_m may be not disjoint. To solve this problem, we will construct "corrected" versions (b^p_m, $p = 1, \ldots, N$) of $w_{i,m}\,b_m$. Moreover, as in the case of g_m, the modifications will be made in such a way that the potentials U^B_m will not change.

8.2. **The "correction" of b_m.** We assume that the functions b_m, $m \geq 1$, have been obtained and they satisfy (a)–(h). We will start the construction of some new functions (the corrected versions of $w_{i,m}\, b_m$) in the small cubes, and then we will go over the cubes from previous generations. However, since there is an infinite number of generations, we will need to use a limitting argument.

For each j we can write the potential originated by b_j as

$$U_j^B(x) = \sum_{i\in I_j} \varphi_{y_i,j}(x) \int w_{i,j}(y)\, b_j(y)\, d\mu(y).$$

For a fixed $m \geq 1$ we are going to define functions $v_{i,j}^m$, for $j = m, m-1, \ldots, 1$, and all $i \in I_j$. The functions $v_{i,j}^m$ will satisfy

$$\operatorname{supp}(v_{i,j}^m) \subset Q_{i,j}, \tag{8.4}$$

where $Q_{i,j} \in \mathcal{D}_J^B$, the sign of $v_{i,j}^m$ will be constant on $Q_{i,j}$, and

$$\int v_{i,j}^m(y)\, d\mu(y) = \int w_{i,j}(y)\, b_j(y)\, d\mu(y). \tag{8.5}$$

Moreover, we will also have

$$\sum_{j=1}^m \sum_{i\in I_j} |v_{i,j}^m| \leq C_{11}\, A\, \|f\|_*. \tag{8.6}$$

We set $v_{i,m}^m(y) = w_{i,m}(y)\, b_m(y)$ for all $i \in I_m$. Assume that we have obtained functions $v_{i,m}^m, v_{i,m-1}^m, \ldots v_{i,k+1}^m$ for all the i's, fulfiling (8.4), (8.5), and such that

$$\sum_{j=k+1}^m \sum_{i\in I_j} |v_{i,j}^m| \leq B\, A\, \|f\|_*,$$

where B is some constant that will be fixed below. We are going to construct $v_{i,k}^m$ now.

Let $Q_{i_0,k} \in \mathcal{D}_k$ be some fixed cube from the k-th generation. Assume first that $Q_{i_0,k}$ is not a single point. Since the cubes in the family $\mathcal{D}^B$ satisfy the packing condition (8.2), for any $t > 0$ we get

$$\begin{aligned}
&\mu\left\{y \in Q_{i_0,k} : \sum_{j=k+1}^{m} \sum_{i\in I_j} |v_{i,j}^m(y)| > t\right\} \\
&\quad\leq \frac{1}{t} \sum_{j=k+1}^{m} \sum_{i\in I_j} \int_{Q_{i_0,k}} |v_{i,j}^m(y)|\, d\mu(y) \\
&\quad\leq \frac{1}{t} \sum_{j=k+1}^{m} \sum_{i\in I_j} \int_{Q_{i_0,k}} |w_{i,j}(y)\, b_j(y)|\, d\mu(y) \\
&\quad\leq \frac{C_8\, A\, \|f\|_*}{t} \sum_{\substack{Q:\, Q\cap Q_{i_0,k}\neq\varnothing \\ Q\in D_j^B,\, j>k}} \mu(Q) \leq \frac{C_{12}\, A\, \|f\|_*}{t}\, \mu(Q_{i_0,k}).
\end{aligned}$$

Therefore, if we choose $t = 2C_{12}\, A\, \|f\|_*$ and we denote

$$V_{i_0,k}^m = \left\{y \in Q_{i_0,k} : \sum_{j=k+1}^{m} \sum_{i\in I_j} |v_{i,j}^m(y)| \leq t\right\},$$

we have $\mu(V_{i_0,k}^m) \geq \frac{1}{2}\mu(Q_{i_0,k})$. If we set $v_{i_0,k}^m = c_{i_0,k}^m\, \chi_{V_{i_0,k}}$, where $c_{i_0,k}^m \in \mathbb{R}$ is such that (8.5) holds for $i = i_0$, then

$$|c_{i_0,k}^m| \leq \frac{1}{\mu(V_{i_0,k})} \int |w_{i,k}(y)\, b_k(y)|\, d\mu(y) \leq 2\, C_8\, A\, \|f\|_*.$$

By the finite overlap of the cubes in $\mathcal{D}_k^B$, we get

$$\sum_{\substack{i_0:\, Q_{i_0,k}\in\mathcal{D}_k^B \\ \ell(Q_{i_0,k})\neq 0}} |v_{i_0,k}^m| \leq C_B\, 2\, C_8\, A\, \|f\|_*,$$

where C_B is the overlap constant in the Covering Theorem of Besicovich. Now if we take $B := 2\, C_B\, C_8 + 2C_{12}$, we will have

$$\sum_{\substack{i_0:\, Q_{i_0,k}\in\mathcal{D}_k^B \\ \ell(Q_{i_0,k})\neq 0}} |v_{i_0,k}^m| + \sum_{j=k+1}^{m} \sum_{i\in I_j} |v_{i,j}^m| \leq B\, A\, \|f\|_*. \tag{8.7}$$

In case $Q_{i_0,k}$ is a single point $\{y\}$, then we set $v_{i_0,k}^m(y) = w_{i_0,k}(y)\, b_k(y) = b_k(y)$. All the cubes of the generations $k+1, \ldots, m$ that intersect $Q_{i_0,k} \equiv \{y\}$ coincide with $\{y\}$ by Lemma 6.3. From (e) we get that $b_{k+1}(y) = b_{k+2}(y) = \cdots = 0$, which is the same as saying that $v_{i,k+1}^m(y) = v_{i,k+2}^m(y) = \cdots = 0$ for all i. So we have

$$\sum_{j=k}^{m} \sum_{i\in I_j} |v_{i,j}^m(y)| = |b_k(y)| \leq C_8\, A\, \|f\|_* \leq B\, A\, \|f\|_*. \tag{8.8}$$

From (8.7) and (8.8) we get

$$\sum_{j=k}^{m} \sum_{i\in I_j} |v_{i,j}^m| \leq B\, A\, \|f\|_*.$$

Operating in this way, the functions $v_{i,j}^m$, $j = m, m-1, \ldots, 1$, $i \in I_j$, will satisfy the conditions (8.4), (8.5) and (8.6) (with $C_{11} = B$).

Now we can take a subsequence $\{m_k\}_k$ such that for all $i \in I_1$ (i.e. for all the cubes of the first generation) the functions $\{v_{i,1}^{m_k}\}_k$ converge weakly in $L^\infty(\mu)$ to some function $v_{i,1} \in L^\infty(\mu)$. Let us remark that the sequence $\{m_k\}_k$ can be chosen independently of i since, by the Besicovich's Covering Theorem, there is a bounded number N of subfamilies $\mathcal{D}_1^1, \ldots, \mathcal{D}_1^N$ of $\mathcal{D}_1$ such that each subfamily $\mathcal{D}_1^p$ is disjoint. If we denote by $\mathcal{D}_1^{p,B}$ the subfamily of bad cubes of $\mathcal{D}_1^p$, we can write

$$\sum_{i\in I_1} v_{i,1}^m = \sum_{p=1}^{N} \sum_{i:\, Q_{i,1}\in \mathcal{D}_1^{p,B}} v_{i,1}^m,$$

and we can choose $\{m_k\}_k$ such that, for each p, $\sum_{i:\, Q_{i,1}\in\mathcal{D}_1^{p,B}} v_{i,1}^{m_k}$ converges weakly to $\sum_{i:\, Q_{i,1}\in\mathcal{D}_1^{p,B}} v_{i,1}$.

In a similar way, we can consider another subsequence of $\{m_{k_j}\}_j$ of $\{m_k\}_k$ such that for all $i \in I_2$ the functions $\{v_{i,2}^{m_{k_j}}\}_j$ converge weakly in $L^\infty(\mu)$ to some function $v_{i,2} \in L^\infty(\mu)$. Going on with this process, we will obtain functions $v_{i,j}$, $j \geq 1$, that satisfy (8.4), (8.5) (without the superscript m) and

$$\sum_{j=1}^{\infty} \sum_{i\in I_j} |v_{i,j}| \leq C_{11}\, A\, \|f\|_*. \tag{8.9}$$

Also, we have

$$U_j^B(x) = \sum_{i\in I_j} \varphi_{y_i,j}(x) \int v_{i,j}(y)\, d\mu(y).$$

We denote $\mathcal{D}_m^{p,B} = \mathcal{D}_m^p \cap \mathcal{D}-m$ (recall $\mathcal{D}_m 0 \bigcup_{p=1}^N \mathcal{D}_m^p$, where each subfamily $\mathcal{D}_m^p$ is disjoint) and

$$b_m^p(y) = \sum_{i:\, Q_{i,m}\in\mathcal{D}_m^{p,B}} v_{i,m}(y).$$

Recall also that $\varphi_{y,m}^p(x) = \varphi_{y_i,m}(x)$ if $y \in Q_{i,m}$ with $Q_{i,m} \in \mathcal{D}_m^p$, and $\varphi_{y,m}^p(x) = 0$ if there does not exist any cube of the subfamily $\mathcal{D}_m^p$ containing y. Then we have

$$U_m^B(x) = \sum_{p=1}^{N} \int \varphi_{y,m}^p(x)\, b_m^p(y)\, d\mu(y).$$

Now we set $h_m^p = g_m^p + b_m^p$, and we get

$$f(x) = h_0(x) + \sum_{p=1}^{N} \sum_{m=1}^{\infty} \int \varphi_{y,m}^p(x)\, h_m^p(y)\, d\mu(y),$$

with $C\,\varphi_{y,m} \sim y$ for some constant $C > 0$, and

$$|h_0| + \sum_{p=1}^{N} \sum_{m=1}^{\infty} |h_m^p| \leq C\,A\,\|f\|_*,$$

and the Main Lemma follows, by (g) and (8.9).

8.3. **The construction of g_m and b_m.** In this subsection we will construct inductively functions g_m and b_m satisfying the properties (a)–(e). We will check in Subsection 8.4 that these functions fulfil (f)–(h) too.

Assume that $g_1, \ldots, g_{m-1}$ and $b_1, \ldots, b_{m-1}$ have been constructed and they satisfy (a)–(e). Let Ω_m be the set of points $x \in \operatorname{supp}(\mu)$ with $\delta(x, 2R_0) > m\,A$ such that that there exists some $Q \in \mathcal{D}_m$, $\ell(Q) > 0$, with $Q \ni x$ and $|m_Q f_m| \geq \frac{3}{4}A$. For each $x \in \Omega_m$, we consider a doubling cube $S_{x,m}$ centered at x such that $\delta(S_{x,m}, 2R_0) = mA - \alpha_1 - \alpha_2 - \alpha_3 \pm \varepsilon_1$, where α_3 is some big constant with $10\alpha_2 < \alpha_3 \ll A$, whose precise value will be fixed below. One has to think that $S_{x,m}$ is much bigger than $Q_{x,m}^3$ but much smaller than $Q_{x,m-1}$ (observe that all these cubes have positive side length).

Now we take a Besicovich covering of Ω_m with cubes of type $S_{x,m}$, $x \in \Omega_m$:

$$\Omega_m \subset \bigcup_j S_{j,m},$$

where $S_{j,m}$ stands for $S_{x_j,m}$, with $x_j \in \Omega_m$. We say that a cube $Q \in \mathcal{D}_m$ is good (i.e. $Q \in \mathcal{D}_m^G$) if

$$Q \subset \bigcup_j \frac{3}{2} S_{j,m},$$

and we say that it is bad (i.e. $Q \in \mathcal{D}_m^B$) if it is not good and

$$Q \subset \bigcup_j 2S_{j,m}.$$

Both good and bad cubes are contained in $\bigcup_j 2S_{j,m}$. Roughly speaking, the difference between good and bad cubes is that bad cubes may be supported near the boundary of $\bigcup_j 2S_{j,m}$, while the good ones are far from the boundary.

Now we define g_m and b_m:

$$g_m = \sum_{i:\, Q_{i,m} \in \mathcal{D}_m^G} w_{i,m}\, m_{Q_{i,m}}(f_m),$$

$$b_m = \sum_{i:\, Q_{i,m} \in \mathcal{D}_m^B} w_{i,m}\, m_{Q_{i,m}}(f_m).$$

Because there is some overlapping among the cubes in $\mathcal{D}_m$, we have used the weights $w_{i,m}$ in the definition of these functions. However one should think that g_m and b_m are approximations of the mean of f over the cubes of $\mathcal{D}_m^G$ and $\mathcal{D}_m^B$, respectively.

The following remark will be useful.

Claim 1. *Let $Q_{h,m} \in \mathcal{D}_m$ be such that either $g_m \not\equiv 0$, $b_m \not\equiv 0$ or $U_m \not\equiv 0$ on $Q_{h,m}$. Then there exists some j such that $\widehat{Q}^3_{h,m} \subset 4S_{j,m}$ and so $Q_{h,m} \subset 4S_{j,m}$.*

Proof. In the first two cases $Q_{h,m} \cap 2S_{j,m} \neq \varnothing$ for some j. In the latter case by (a) of Lemma 7.8 and our construction, there exists some j such that $\widehat{Q}^3_{h,m} \cap 2S_{j,m} \neq \varnothing$.

So in any case $\widehat{Q}^3_{h,m} \cap 2S_{j,m} \neq \varnothing$ for some j. Arguing as in Lemma 6.3, for α_3 big enough, it is easily checked that $\ell(\widehat{Q}^3_{h,m}) \leq \ell(S_{j,m})/4$, and so $\widehat{Q}^3_{h,m} \subset 4S_{j,m}$. □

Let us see now that (e) is satisfied.

Claim 2. *If $Q \in \mathcal{D}_m$ and $\delta(Q, 2R_0) \leq (m - \frac{1}{10})\, A$ (so $\ell(Q) = 0$), then $U_m \equiv g_m \equiv b_m \equiv 0$ on Q and $Q \notin \mathcal{D}_m^G \cup \mathcal{D}_m^B$.*

Proof. Assume that $Q \equiv \{x\}$ and that either $g_m \not\equiv 0$, $b_m \not\equiv 0$ or $U_m \not\equiv 0$ on Q, or $Q \in \mathcal{D}_m^G \cup \mathcal{D}_m^B$. By the preceeding claim, $Q \subset 4S_{j,m}$ for some j. Then,

$$\begin{aligned} \delta(x, 2R_0) &= \delta(x, 4S_{j,m}) + \delta(4S_{j,m}, 2R_0) \pm \varepsilon_0 \\ &\geq \delta(4S_{j,m}, 2R_0) - \varepsilon_0 \\ &\geq \delta(S_{j,m}, 2R_0) - 8^n\, C_0 - \varepsilon_0 > \left(m - \frac{1}{10}\right) A. \end{aligned}$$

□

The following estimate will be necessary in many steps of our construction.

Claim 3. *Let Q be some cube of the m-th generation and $x, y \in 2Q$. Then, if $g_1, \ldots, g_m$ and $b_1, \ldots, b_m$ satisfy (a), then*

$$\sum_{k=1}^{m} |U_k(x) - U_k(y)| \leq \frac{A}{100}\, \|f\|_*.$$

We postpone the proof of Claim 3 until Subsection 8.5. Let us see that (a) holds.

Claim 4. *If $Q \in \mathcal{D}_m^G \cup \mathcal{D}_m^B$, then $|m_Q f_m| \leq C_9\, A\, \|f\|_*$. Also, $|g_m|, |b_m| \leq C_8\, A\, \|f\|_*$.*

Proof. First we will prove the first statement. By Claim 2, we know that $\delta(Q, 2R_0) > (m - \frac{1}{10})\, A$. Let $R \in \mathcal{D}_{m-1}$ be such that $Q \cap R \neq \varnothing$. We must have $\ell(R) > 0$. Otherwise, $Q \equiv R$ and $\delta(R, 2R_0) > (m - \frac{1}{10})\, A > (m-1)\, A + \varepsilon_1$, which is not possible.

Since $\ell(Q) \leq \ell(R)/10$, we have $Q \subset 2R$. We know $|m_R f_m| \leq A\,\|f\|_*$ because (b) holds for $m-1$. By Claim 3 (for $m-1$ and R) we get

$$\begin{aligned} |m_Q f_m| &\leq |m_R f_m| + |m_Q f_m - m_R f_m| \\ &\leq |m_R f_m| + |m_Q f - m_R f| + \Big| m_Q\Big(\sum_{k=1}^{m-1} U_k\Big) - m_R\Big(\sum_{k=1}^{m-1} U_k\Big)\Big| \\ &\leq C\,A\,\|f\|_* + |m_Q f - m_R f|. \end{aligned}$$

The term $|m_Q f - m_R f|$ is also bounded above by $C\,A\,\|f\|_*$ because Q and R are doubling, $f \in RBMO(\mu)$, and it is easily checked that $\delta(Q,R) \leq C\,A$.

The estimates on g_m and b_m follow from from the definition of these functions and the estimate $|m_Q f_m| \leq C_9\,A\,\|f\|_*$ for $Q \in \mathcal{D}_m^G \cup \mathcal{D}_m^B$. □

Let us prove (d) now.

Claim 5. *If* $Q \in \mathcal{D}_m$ *and* $|m_Q f_m| \leq \frac{8}{20} A\,\|f\|_*$, *then* $U_m \equiv 0$ *and* $g_m \equiv b_m \equiv 0$ *on* Q.

Proof. Suppose that $Q \equiv Q_{h,m} \in \mathcal{D}_m$ is such that either $g_m \not\equiv 0$, $b_m \not\equiv 0$ or $U_m \not\equiv 0$ on $Q_{h,m}$. By Claim 1 we have $Q_{h,m} \subset 4S_{j,m}$ for some j. By construction, the center of $S_{j,m}$ belongs to some cube $Q_{i,m}$ with $|m_{Q_{i,m}} f_m| \geq \frac{3}{4} A\,\|f\|_*$. It is easily seen that $\delta(Q_{h,m}, 4S_{j,m})$, $\delta(Q_{i,m}, 4S_{j,m}) \leq C' + \alpha_1 + \alpha_2 + \alpha_3$. Thus

$$|m_{Q_{i,m}} f - m_{Q_{h,m}} f| \leq (C'' + 2\alpha_1 + 2\alpha_2 + 2\alpha_3)\,\|f\|_*.$$

Since $Q_{i,m}$ and $Q_{h,m}$ are contained in a common cube of the generation $m-1$, by Claim 3 we get

$$\begin{aligned} |m_{Q_{i,m}} f_m - m_{Q_{h,m}} f_m| &\leq |m_{Q_{i,m}} f - m_{Q_{h,m}} f| \\ &\quad + \Big| m_{Q_{i,m}}\Big(\sum_{k=1}^{m-1} U_k\Big) - m_{Q_{h,m}}\Big(\sum_{k=1}^{m-1} U_k\Big)\Big| \\ &\leq (C'' + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + A/100)\,\|f\|_* \\ &\leq \frac{1}{10} A\,\|f\|_*, \end{aligned}$$

and so

$$|m_{Q_{h,m}} f_m| \geq \left(\frac{3}{4} - \frac{1}{10}\right) A\,\|f\|_* > \frac{8}{20} A\,\|f\|_*.$$

□

The statement (c) is a consequence of the fact that if $Q \in \mathcal{D}_m^G$, then Q is far from the boundary of $\bigcup_j 2S_{j,m}$. Then U_m is very close to $m_Q f_m$ on Q, since we only integrate over cubes of $\mathcal{D}_m^G \cup \mathcal{D}_m^B$ in order to obtain $U_m(x)$ for $x \in Q$. On the other hand, if $Q \in \mathcal{D}_m^B$, this argument does not work because Q may be near the boundary of $\bigcup_j 2S_{j,m}$, and so it may happen that we integrate on some cubes from $\mathcal{D}_m \setminus (\mathcal{D}_m^G \cup \mathcal{D}_m^B)$ for obtaining $U_m(x)$, $x \in Q$.

Let us see (c) in detail.

Claim 6. *If $Q \in \mathcal{D}_m^G$ and $\ell(Q) > 0$, then $|m_Q f_{m+1}| \leq \frac{7}{20} A \|f\|_*$.*

Proof. Consider $Q_{i,m} \in \mathcal{D}_m^G$. We want to see that U_m is very close to $m_{Q_{i,m}} f_m$ on this cube. By (a) of Lemma 7.8 we have to deal with the cube $\widehat{Q}^3_{i,m}$.

Let us see that if $P \in \mathcal{D}_m$ is such that $P \cap \widehat{Q}^3_{i,m} \neq \varnothing$, then $P \in \mathcal{D}_m^G \cup \mathcal{D}_m^B$. Notice that $P \subset \widehat{\widehat{Q}}{}^3_{i,m}$. Now, by the definition of good cubes, there exists some j such that $Q_{i,m} \cap \frac{3}{2} S_{j,m} \neq \varnothing$, which implies $\widehat{\widehat{Q}}{}^3_{i,m} \cap \frac{3}{2} S_{j,m} \neq \varnothing$. For α_3 big enough, we have $\ell(\widehat{\widehat{Q}}{}^3_{i,m}) \ll \ell(S_{j,m})$, and then $\widehat{\widehat{Q}}{}^3_{i,m} \subset 2 S_{j,m}$. So $P \in \mathcal{D}_m^G \cup \mathcal{D}_m^B$.

Let us estimate the term

$$\sup_{y \in \widehat{Q}^3_{i,m}} |(g_m(y) + b_m(y)) - m_{Q_{i,m}} f_m|.$$

Recall that

$$g_m(y) + b_m(y) = \sum_{h:\, Q_{h,m} \in \mathcal{D}_m^G \cup \mathcal{D}_m^B} w_{h,m}(y)\, m_{Q_{h,m}} f_m.$$

By the arguments above, if $y \in \widehat{Q}^3_{i,m}$ and $w_{h,m}(y) \neq 0$, then $Q_{h,m}$ has been chosen for supporting g_m or b_m, i.e. $Q_{h,m} \in \mathcal{D}_m^G \cup \mathcal{D}_m^B$. Then,

$$g_m(y) + b_m(y) - m_{Q_{i,m}} f_m = \sum_{h:\, Q_{h,m} \in \mathcal{D}_m} w_{h,m}(y)\, (m_{Q_{h,m}} f_m - m_{Q_{i,m}} f_m).$$

By Claim 3 we obtain

$$\begin{aligned} |m_{Q_{h,m}} f_m - m_{Q_{i,m}} f_m| &\leq \frac{1}{100} A \|f\|_* + |m_{Q_{h,m}} f - m_{Q_{i,m}} f| \\ &\leq \left(\frac{1}{100} A + C + 2\,\delta(Q_{h,m}, Q_{i,m}) \right) \|f\|_* \\ &\leq \frac{1}{50} A \, \|f\|_* \end{aligned}$$

(we have used that $\delta(Q_{h,m}, Q_{i,m}) \leq C$, with C depending on α_1, α_2). Then we get

$$|g_m(y) + b_m(y) - m_{Q_{i,m}} f_m| \leq \frac{1}{50} A \, \|f\|_*. \tag{8.10}$$

For $x \in Q_{i,m}$, we have

$$\begin{aligned} |U_m(x) - m_{Q_{i,m}} f_m| &\leq \left| U_m(x) - m_{Q_{i,m}} f_m \int \varphi_{y,m}(x)\, d\mu(y) \right| \\ &\quad + |m_{Q_{i,m}} f_m| \left| 1 - \int \varphi_{y,m}(x)\, d\mu(y) \right|. \end{aligned} \tag{8.11}$$

Let us estimate the first term on the right hand side. By (8.10) and (7.8) we obtain

$$\begin{aligned}
&\left|U_m(x) - m_{Q_{i,m}} f_m \int \varphi_{y,m}(x)\, d\mu(y)\right| \\
&\quad= \left|\int_{\widehat{Q}^3_{i,m}} \varphi_{y,m}(x)\,(g_m(y) + b_m(y) - m_{Q_{i,m}} f_m)\, d\mu(y)\right| \\
&\quad\le (1+\varepsilon_3)\,\frac{1}{50}\,A\,\|f\|_*.
\end{aligned}$$

On the other hand, by (7.8), (7.9) and Claim 4, the second term on the right hand side of (8.11) is bounded above by $\varepsilon_3\, C_8\, A\,\|f\|_*$. Thus we have

$$|m_{Q_{i,m}} f_{m+1}| \le \left((1+\varepsilon_3)\,\frac{1}{50} + \varepsilon_3\, C_8\right) A\,\|f\|_* \le \frac{7}{20}\,A\,\|f\|_*,$$

if we choose ε_3 small enough. □

Now we are going to show that (b) also holds.

Claim 7. *If $Q \in \mathcal{D}_m$ and $\ell(Q) > 0$, then $|m_Q f_{m+1}| \le A\|f\|_*$.*

Proof. If $Q \in \mathcal{D}^G_m$, we have already seen that $|m_Q f_{m+1}| \le \frac{7}{20}\,A\,\|f\|_*$.

If $Q \in \mathcal{D}_m \setminus \mathcal{D}^G_m$, then $Q \cap \bigcup_j S_{j,m} = \varnothing$ (because $\ell(Q) \ll \ell(S_{j,m})$ and $Q \not\subset \bigcup_j \frac{3}{2} S_{j,m}$). By construction, we have

$$|m_Q f_m| \le \frac{3}{4}\,A\,\|f\|_*. \tag{8.12}$$

If $U_m \equiv 0$ on Q, then $|m_Q f_{m+1}| = |m_Q f_m| \le \frac{3}{4}\,A\,\|f\|_*$.

Now we consider the case $Q \equiv Q_{h,m} \cap \bigcup_j S_{j,m} = \varnothing$ such that $U_m \not\equiv 0$ on Q. By Claim 1 there exists some j with $\widehat{Q}^3_{h,m} \subset 4S_{j,m}$. Recall that by (a) of Lemma 7.8, if $x \in Q_{h,m}$, we have

$$U_m(x) = \int_{\widehat{Q}^3_{h,m}} \varphi_{y,m}(x)\,(g_m(y) + b_m(y))\, d\mu(y).$$

So if $\varphi_{y,m}(x) \ne 0$ and $y \in Q_{i,m}$, we have $Q_{i,m} \cap \widehat{Q}^3_{h,m} \ne \varnothing$. Therefore, $Q_{i,m} \subset \widehat{\widehat{Q}}{}^3_{h,m}$. Then,

$$\delta(Q_{i,m}, Q_{h,m}) \le C + \delta(Q_{i,m}, \widehat{\widehat{Q}}{}^3_{h,m}) + \delta(Q_{h,m}, \widehat{\widehat{Q}}{}^3_{h,m}) \le C + 2\alpha_1 + 2\alpha_2 \le \frac{A}{400}.$$

Therefore, $|m_{Q_{i,m}} f - m_{Q_{h,m}} f| \le \frac{A}{100}\,\|f\|_*$. By Claim 1 we get

$$\begin{aligned}
|m_{Q_{i,m}} f_m - m_{Q_{h,m}} f_m| &\le |m_{Q_{i,m}} f - m_{Q_{h,m}} f| \\
&\quad + \left| m_{Q_{i,m}}\Big(\sum_{k=1}^{m-1} U_k\Big) - m_{Q_{h,m}}\Big(\sum_{k=1}^{m-1} U_k\Big)\right| \\
&\le \frac{1}{10}\,A\,\|f\|_*. \qquad (8.13)
\end{aligned}$$

Recall also that, by (d),

$$|m_{Q_{h,m}} f_m| \geq \frac{8}{20}\, A\, \|f\|_*. \tag{8.14}$$

From the definition of g_m, b_m and (8.13), (8.14), we derive that $m_{Q_{h,m}} f_m$ and $U_m(x)$ have the same sign.

On the other hand, from (8.12) and (8.13) we get

$$|m_{Q_{i,m}} f_m| \leq \frac{34}{40}\, A\, \|f\|_*.$$

So by the definition of g_m anb b_m we have

$$\|g_m + b_m\|_{L^\infty(\mu)} \leq \frac{34}{40}\, A\, \|f\|_*,$$

and by (7.8) we obtain

$$|U_m(x)| \leq \frac{34}{40}\, A\, \|f\|_* \int \varphi_{y,m}(x)\, d\mu(y) \leq (1+\varepsilon_3)\, \frac{34}{40}\, A\, \|f\|_* \leq A\, \|f\|_* \tag{8.15}$$

(assuming ε_3 small enough). By (8.12), (8.15) and since $m_{Q_{h,m}} f_m$ and $U_m(x)$ have the same sign, (b) holds also in this case. □

Therefore, (a)–(e) are satisfied.

8.4. **Proof of (f), (g) and (h).**

The statement (f) is a direct consequence of the following.

Claim 8. *If $\delta(x, 2R_0) < \infty$, and if $Q = \{x\} \in \mathcal{D}_m$ (i.e. $\ell(Q) = 0$), then $h_0(x) = f_{m+1}(x)$ and $|h_0(x)| \leq C_9\, A\, \|f\|_*$.*

Proof. Take m such that $(m-1)\, A < \delta(x, 2R_0) \leq m\, A$. By (e) we get $U_{m+k}(x) = 0$ for $k \geq 1$. Therefore, $f_{m+1}(x) = f_{m+2}(x) = \cdots = h_0(x)$. By (a) we have

$$|f_{m+1}(x)| \leq |f_m(x)| + |U_m(x)| \leq |f_m(x)| + 2\, C_8\, (1+\varepsilon_3)\, A\, \|f\|_*.$$

So we only have to estimate $|f_m(x)|$.

Take $Q_{i,m-1} \in \mathcal{D}_{m-1}$ with $x \in Q_{i,m-1}$. Since $\ell(Q_{i,m-1}) > 0$, by (b) we have $|m_{Q_{i,m-1}} f_m| \leq A\, \|f\|_*$. Applying Claim 3 we get

$$\begin{aligned} |m_{Q_{i,m-1}} f_m - f_m(x)| &\leq |m_{Q_{i,m-1}} f - f(x)| + \frac{A}{100}\, \|f\|_* \\ &\leq C \left(1 + \delta(x, Q_{i,m-1}) + \frac{A}{100}\right) \|f\|_*. \end{aligned}$$

It is easily checked that $\delta(x, Q_{i,m-1}) \leq A + \varepsilon_0 + \varepsilon_1$. Then we get $|f_m(x)| \leq C\, A\, \|f\|_*$. □

Now we turn our attention to (g). Given some good cube $Q_{i,m} \in \mathcal{D}_m^G$ with $\ell(Q_{i,m}) > 0$, we denote

$$Z_{i,m} := Z(Q_{i,m}, A\, \|f\|_*/30)$$

(see Definition 5.2; roughly speaking $Z_{i,m}$ is the part of $Q_{i,m}$ where f does not oscillate too much with respect to $m_{Q_{i,m}}f$). If $Q_{i,m} \in \mathcal{D}_m^G$ and $\ell(Q_{i,m}) = 0$, we set $Z_{i,m} = Q_{i,m}$. The set $Z_{i,m}$ has a very nice property:

Claim 9. *Let $k > m$ and $Q_{i,m} \in \mathcal{D}_m^G$. If $P \in \mathcal{D}_k$ is such that $P \cap Z_{i,m} \neq \varnothing$, then $g_k \equiv b_k \equiv 0$ on P and $P \notin \mathcal{D}_k^G \cup \mathcal{D}_k^B$.*

Proof. Consider first the case $\ell(Q_{i,m}) = 0$. If $P \in \mathcal{D}_k$ is such that $P \cap Q_{i,m} \neq \varnothing$, then $\ell(P) \leq \ell(Q_{i,m})/10 = 0$ and so $P \equiv Q_{i,m}$. Therefore,

$$\delta(P, 2R_0) \leq m\,A \leq \left(k - \frac{1}{10}\right) A.$$

By (e), we get $b_k \equiv g_k \equiv 0$ on P.

Assume now $\ell(Q_{i,m}) > 0$. Let $x \in P \cap Z_{i,m}$. From the definition of $Z_{i,m}$, we have

$$|m_{Q_{i,m}}f - m_S f| \leq \frac{A}{30}\,\|f\|_* \tag{8.16}$$

for any $S \in \mathcal{D}_{m+j}$, $j \geq 1$, with $x \in S$. Also, by Claim 6 we have

$$|m_{Q_{i,m}}f_{m+1}| \leq \frac{7}{20}A\,\|f\|_*.$$

Consider now $P_{m+1} \in \mathcal{D}_{m+1}$ with $x \in P_{m+1}$. Observe that $\ell(P_{m+1}) \leq \ell(Q_{i,m})/10$ and $P_{m+1} \subset 2Q_{i,m}$. We have

$$\begin{aligned}
|m_{P_{m+1}}f_{m+1}| &\leq |m_{Q_{i,m}}f_{m+1}| + |m_{Q_{i,m}}f_{m+1} - m_{P_{m+1}}f_{m+1}| \\
&\leq \frac{7}{20}A\,\|f\|_* + |m_{Q_{i,m}}f - m_{P_{m+1}}f| \\
&\quad + \left|m_{Q_{i,m}}\Big(\sum_{k=1}^{m} U_k\Big) - m_{P_{m+1}}\Big(\sum_{k=1}^{m} U_k\Big)\right|.
\end{aligned}$$

By (8.16) and Claim 3 we obtain $|m_{P_{m+1}}f_{m+1}| \leq \frac{8}{20}A\,\|f\|_*$. By (d), on P_{m+1} we have $g_{m+1} \equiv b_{m+1} \equiv 0$ and also $U_{m+1} \equiv 0$. Thus,

$$f_{m+2} \equiv f_{m+1}$$

on any cube $P_{m+1} \in \mathcal{D}_{m+1}$ containing x.

Take now $P_{m+2} \in \mathcal{D}_{m+2}$ with $x \in P_{m+2}$. On this cube $f_{m+2} \equiv f_{m+1}$, and then we have

$$\begin{aligned}
|m_{P_{m+2}}f_{m+2}| &\leq |m_{Q_{i,m}}f_{m+1}| + |m_{Q_{i,m}}f_{m+1} - m_{P_{m+2}}f_{m+1}| \\
&\leq \frac{7}{20}A\,\|f\|_* + |m_{Q_{i,m}}f - m_{P_{m+2}}f| \\
&\quad + \left|m_{Q_{i,m}}\Big(\sum_{k=1}^{m} U_k\Big) - m_{P_{m+2}}\Big(\sum_{k=1}^{m} U_k\Big)\right|.
\end{aligned}$$

Again by (d), we get $g_{m+2} \equiv b_{m+2} \equiv U_{m+2} \equiv 0$ on P_{m+2}. Thus, $f_{m+3} = f_{m+1}$ on P_{m+2}.

Going on, we will obtain $g_{m+j} \equiv b_{m+j} \equiv U_{m+j} \equiv 0$ for all $j \geq 1$ on any cube $P_{m+j} \in \mathcal{D}_{m+j}$ containing x. □

As a consequence of Claim 9, $Z_{i,m}$ is a good place for supporting g_m. If, for each m, g_m were supported on $\bigcup_i Z_{i,m}$, then the supports of g_m, $m \geq 1$, would be disjoint for different m's. This is the idea that Carleson used in [Ca].

So we are going to make some "corrections" according to this argument. We have

$$U_m^G(x) = \sum_{i \in I_m} \varphi_{y_i,m}(x) \int w_{i,m}(y)\, g_m(y)\, d\mu(y).$$

For each $Q_{i,m}$ with $\ell(Q_{i,m}) > 0$ we set

$$u_{i,m}(y) = \int w_{i,m}\, g_m\, d\mu \cdot \frac{\chi_{Z_{i,m}}(y)}{\mu(Z_{i,m})}.$$

If $\ell(Q_{i,m}) = 0$, we set $u_{i,m}(y) = w_{i,m}(y)\, g_m(y) \equiv g_m(y)$ (we do not change anything in this case). Then U_m^G can be written as

$$U_m^G(x) = \sum_{i \in I_m} \varphi_{y_i,m}(x) \int u_{i,m}(y)\, d\mu(y).$$

As in the case of U_m^B in Subsection 8.2, if we set $\mathcal{D}_m^G = \mathcal{D}_m^{1,G} \cup \cdots \cup \mathcal{D}_m^{N,G}$ where each subfamily $\mathcal{D}_m^{p,G}$ is disjoint, we can write U_m^G in the following way:

$$U_m^G(x) = \sum_{p=1}^{N} \int \varphi_{y,m}^p(x)\, g_m^p(y)\, d\mu(y)$$

with

$$g_m^p(y) = \sum_{i:\, Q_{i,m} \in \mathcal{D}_m^{p,G}} u_{i,m}(y)$$

and

$$\varphi_{y,m}^p(x) = \varphi_{y_i,m}(x)$$

if $y \in Q_{i,m}$ and $Q_{i,m} \in \mathcal{D}_m^p$.

By Proposition 5.3, if A is big enough we have $\mu(Z_{i,m}) \geq \mu(Q_{i,m})/2$ (if $\ell(Q_{i,m}) > 0$). Then it easily checked that $\|u_{i,m}\|_{L^\infty(\mu)} \leq 2\, \|g_m\|_{L^\infty(\mu)}$ for all i. Thus, from (a), (g.2) follows. Moreover, because of Claim 9, (g.3) also holds.

One of the differences between our construction and Carleson's one is that, because of the regularity of Lebesgue measure, Carleson can treat the bad cubes in a way very similar to the way for the good ones. We have not been able to operate as Carleson. However, as it has been shown in Subsection 8.2, the packing condition (8.2) is also a good solution. Let us prove that this condition is satisfied.

Claim 10. *For any $R \in \mathcal{D}_m$ with $\ell(R) > 0$, the bad cubes satisfy the packing condition*

$$\sum_{\substack{Q:\, Q\cap R\neq\varnothing\\ Q\in D_k^B,\, k>m}} \mu(Q) \leq C\,\mu(R).$$

Proof. Let $k > m$ be fixed. We are going to estimate the sum

$$\sum_{\substack{Q:\, Q\cap R\neq\varnothing\\ Q\in D_k^B}} \mu(Q).$$

Let $Q \in \mathcal{D}_k^B$ be such that $Q \cap R \neq \varnothing$. Since Q is a bad cube, there exists some j such that $2S_{j,k} \cap Q \neq \varnothing$. Then we have $Q \subset 4S_{j,k}$. Since $A \gg \alpha_1 + \alpha_2 + \alpha_3$ and $4S_{j,k} \cap R \neq \varnothing$, we get $\ell(S_{j,k}) \leq \ell(R)/20$, and so $4S_{j,k} \subset 2R$.

By the finite overlapping of the cubes Q in $\mathcal{D}_k$, we have

$$\begin{aligned}\sum_{\substack{Q:\, Q\cap R\neq\varnothing\\ Q\in D_k^B}} \mu(Q) &\leq C\,\mu\Bigl(\bigcup_{j:\, S_{j,k}\subset 2R} 2S_{j,k}\Bigr)\\ &\leq C \sum_{j:\, S_{j,k}\subset 2R} \mu(2S_{j,k}) \leq C \sum_{j:\, S_{j,k}\subset 2R} \mu(S_{j,k}).\end{aligned}$$

Now, from the construction of g_k^p, it is easy to check that $\mu(S_{j,k}) \leq C\,\mu\Bigl(S_{j,k} \cap \bigl\{\sum_{p=1}^N |g_k^p| \neq 0\bigr\}\Bigr)$. This fact and the bounded overlapping of the cubes $S_{j,k}$ give

$$\sum_{\substack{Q:\, Q\cap R\neq\varnothing\\ Q\in D_k^B}} \mu(Q) \leq C\,\mu\Bigl(2R \cap \Bigl\{\sum_{p=1}^N |g_k^p| \neq 0\Bigr\}\Bigr).$$

Summing over $k > m$, as the supports of the functions g_k^p are disjoint for different k's, we obtain

$$\sum_{\substack{Q:\, Q\cap R\neq\varnothing\\ Q\in D_k^B,\, k>m}} \mu(Q) \leq C \sum_{k>m} \mu\Bigl(2R \cap \Bigl\{\sum_{p=1}^N |g_k^p| \neq 0\Bigr\}\Bigr) \leq C\,\mu(2R) \leq C\,\mu(R).$$

□

8.5. **Proof of Claim 3.** We only need to check that

$$\sum_{k=1}^m C_8\, A \int |\varphi_{z,k}(x) - \varphi_{z,k}(y)|\, d\mu(z) \leq \frac{A}{100}.$$

Let $x_0 \in \operatorname{supp}(\mu)$ be such that $x, y \in 2Q_{x_0,m}$. Obviously, we can assume $\ell(Q_{x_0,m}) > 0$. For each $k \leq m$ we set

$$\int |\varphi_{z,k}(x) - \varphi_{z,k}(y)|\, d\mu(z) = \int_{\mathbb{R}^d \setminus \check{Q}^1_{x_0,k}} + \int_{\check{Q}^1_{x_0,k}} = I_{1,k} + I_{2,k}.$$

Let us estimate the integrals $I_{1,k}$. Notice that if $x, y \in 2Q_{x_0,m}$, then $x, y \in 2Q_{x_0,k} \subset \frac{1}{2}\check{Q}^1_{x_0,k}$. Thus $|x-z| \approx |y-z| \approx |x_0-z|$ for $z \in \mathbb{R}^d \setminus \check{Q}^1_{x_0,k}$. So by (d) of Lemma 7.8 we have

$$\begin{aligned} I_{1,k} &\leq C\,\alpha_2^{-1} \int_{\mathbb{R}^d \setminus \check{Q}^1_{x_0,k}} \frac{|x-y|}{|x-z|^{n+1}}\,d\mu(z) \\ &\leq C\,\alpha_2^{-1}\,\frac{\ell(Q_{x_0,m})}{\ell(\check{Q}^1_{x_0,k})}. \qquad (8.17) \end{aligned}$$

In case $k > m$, by Lemma 6.4 we get

$$I_{1,k} \leq C\,\alpha_2^{-1}\,\frac{\ell(Q_{x_0,m})}{\ell(Q_{x_0,k})} \leq C_{13}\,\alpha_2^{-1}\,2^{-\gamma\,(m-k)\,A}.$$

Therefore,

$$C_8\,A \sum_{k=1}^{m} I_{1,k} \leq C_8\,\alpha_2^{-1}\,A \sum_{k=1}^{m-1} 2^{-\gamma\,(m-k)\,A} + C_8\,C_{13}\,\alpha_2^{-1}\,A\,\frac{\ell(Q_{x_0,m})}{\ell(\check{Q}^1_{x_0,m})}. \qquad (8.18)$$

The first sum on the right hand side is $\leq C\,\alpha_2^{-1}\,A\,2^{-\gamma\,A}$, and for A big enough and $\alpha_2 > 1$ is $\leq 1 \leq A/400$. The second term on the right hand side is also $\leq A/400$ if we choose α_2 big enough (or α_1 big enough since then $\ell(\check{Q}_{x_0,m}) \gg \ell(\check{Q}^1_{x_0,m})$). Thus

$$C_8\,A \sum_{k=1}^{m} I_{1,k} \leq \frac{A}{200}.$$

We consider now the integrals $I_{2,k}$. By Lemma 7.8,

$$|\varphi'(u)| \leq C\,\frac{\alpha_2^{-1}}{\ell(\check{Q}^1_{x_0,k})^{n+1}}$$

for all $u \in Q_{x_0,k}$. Therefore,

$$I_{2,k} \leq C\,\alpha_2^{-1} \int_{\check{Q}^1_{x_0,k}} \frac{|x-y|}{\ell(\check{Q}^1_{x_0,k})^{n+1}}\,d\mu(z) \leq C\,\alpha_2^{-1}\,\frac{\ell(Q_{x_0,m})}{\ell(\check{Q}^1_{x_0,k})}.$$

This is the same estimate that we have obtained for $I_{1,k}$ in (8.17), and then we also have

$$C_8\,A \sum_{k=1}^{m} I_{2,k} \leq \frac{A}{200},$$

if we choose A and α_2 (or α_1) big enough. □

9. Appendix

In this section we will prove the following result, which is used in Section 4 to show that Theorem 1.2 follows from the Main Lemma.

Lemma 9.1. *Consider $f \in L^1(\mu)$ with $\int f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Then there exists a sequence of functions f_k, $k \geq 1$, bounded with compact support such that $\int f_k\,d\mu = 0$, $f_k \to f$ in $L^1(\mu)$ and $\|M_\Phi(f - f_k)\|_{L^1(\mu)} \to 0$.*

So if we consider the space

$$H^1_\Phi(\mu) = \Big\{ f \in L^1(\mu) : \textstyle\int f\, d\mu = 0,\ M_\Phi f \in L^1(\mu) \Big\},$$

with norm $\|f\|_{H^1_\Phi(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}$, then Lemma 9.1 asserts that functions in $H^1_\Phi(\mu)$ which are bounded and have compact support are dense in $H^1_\Phi(\mu)$. In particular, $H^1_\Phi(\mu) \cap H^{1,\infty}_{atb}(\mu)$ is dense in $H^1_\Phi(\mu)$.

In this section we will assume that the center of any square Q may be any point of $\mathbb{R}^d$, not necessarily belonging to $\operatorname{supp}(\mu)$. As in the previous sections, the sides of the squares are parallel to the axes and they are closed.

Let us introduce some additional notation. For $\rho > 1$, we set

$$M_{(\rho)} f(x) = \sup_{Q \ni x} \frac{1}{\mu(\rho Q)} \int_Q |f|\, d\mu.$$

This non centered maximal operator is bounded above by the operator defined as

$$M^{(\rho)} f(x) = \sup_{\rho^{-1} Q \ni x} \frac{1}{\mu(Q)} \int_Q |f|\, d\mu.$$

This is the version of the Hardy-Littlewood operator that one obtains taking supremums over cubes Q which may be non centered at x but such that $x \in \rho^{-1} Q$. Recall that since $0 < \rho^{-1} < 1$, one can apply Besicovich's Covering Theorem and then one gets that $M^{(\rho)}$ is of weak type $(1,1)$ and bounded in $L^p(\mu)$, $p \in (1,\infty]$. As a consequence, $M_{(\rho)}$ is also of weak type $(1,1)$ and bounded in $L^p(\mu)$, $p \in (1,\infty]$

Remark 9.2 (Whitney covering). Let $\Omega \subset \mathbb{R}^d$ be open, $\Omega \neq \mathbb{R}^d$. Then Ω can be decomposed as $\Omega = \bigcup_{i \in I} Q_i$, where Q_i, $i \in I$, are squares with disjoint interiors, with $20 Q_i \subset \Omega$ and such that, for some constants $\beta > 20$ and $D \geq 1$, $\beta Q_k \cap \Omega^c \neq \varnothing$ and for each square Q_k there are at most D squares Q_i with $10 Q_k \cap 10 Q_i \neq \varnothing$ (in particular, the family of squares $\{10 Q_i\}_{i \in I}$ has finite overlapping).

In [To3] a decomposition of Calderón-Zygmund type adapted for non doubling measures was introduced. This decomposition was used to prove an interpolation theorem between $(H^1_{atb}(\mu), L^1(\mu))$ and $(L^\infty(\mu), RBMO(\mu))$. In [To4] it was shown that this decomposition was also useful for proving that CZO's bounded in $L^2(\mu)$ are of weak type $(1,1)$ too, as in the doubling case (this result had been proved previously in [NTV2] using different techniques). To prove Lemma 9.1 we will use the following variant of the Calderón-Zygmund decomposition of [To3].

Lemma 9.3. *Let $f \in L^1(\mu)$ with $\int f\, d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. For any $\lambda > 0$, let $\Omega_\lambda = \{x \in \mathbb{R}^d : M_{(2)} f(x) > \lambda\}$. Then Ω_λ is open and $|f| \leq 2^{d+1} \lambda$ μ-a.eq. in $\mathbb{R}^d \setminus \Omega_\lambda$. Moreover, if we consider a Whitney decomposition of Ω_λ into cubes Q_i (as in Remark 9.2), then we have:*

(a) *For each i there exists a function $w_i \in \mathcal{C}^\infty(\mathbb{R}^d)$ with* $\operatorname{supp}(w_i) \subset \frac{3}{2} Q_i$, *$0 \leq w_i \leq 1$, $\|w_i'\|_\infty \leq C\, \ell(Q_i)^{-1}$ such that $\sum_i w_i(x) = 1$ if $x \in \Omega_\lambda$.*

(b) *For each i, let R_i be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_i$, $k \geq 1$, with $R_i \cap \Omega_\lambda^c \neq \varnothing$. Then there exists a family of functions α_i with* $\operatorname{supp}(\alpha_i) \subset R_i$ *satisfying*

$$\int \alpha_i \, d\mu = \int f \, w_i \, d\mu, \tag{9.1}$$

$$\|\alpha_i\|_{L^\infty(\mu)} \, \mu(R_i) \leq C \, \|\alpha_i\|_{L^1(\mu)} \tag{9.2}$$

and

$$\sum_i |\alpha_i| \leq B \, \lambda \tag{9.3}$$

(where B is some constant).

(c) *f can be written as $f = g + b$, with*

$$g = f \Big(1 - \sum_i w_i\Big) + \sum_i \alpha_i$$

and

$$b = \sum_i (f \, w_i - \alpha_i),$$

and then $\|g\|_{L^\infty(\mu)} \leq C \, \lambda$ and $\operatorname{supp}(b) \subset \Omega_\lambda$.

Proof. The set Ω_λ is open because $M_{(2)}$ is lower semicontinuous. Since for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence of $(2, 2^{d+1})$-doubling cubes centered at x with side length tending to zero, it follows that for μ-a.e. $x \in \mathbb{R}^d$ such that $|f(x)| > 2^{d+1}\lambda$ there exists some $(2, 2^{d+1})$-doubling cube Q with $\int_Q |f| \, d\mu / \mu(Q) > 2^{d+1}\lambda$ and so $M_{(2)} f(x) > \lambda$.

The existence of the functions w_i of (a) is a standard known fact. The assertion (c) follows from the other statements in the lemma. So the only question left is the statement (b).

Notice that, since $R_i \cap \Omega_\lambda^c \neq \varnothing$, we have

$$\int_{R_i} |f| \, d\mu \leq \lambda \, \mu(2R_i) \tag{9.4}$$

for each i.

To construct the functions α_j we would like to start by the smallest cube R_i, and go on with the bigger cubes R_j following an order of non decreasing sizes. Since in general there does not exist a cube R_i with minimal side length in the family $\{R_i\}_{i=1}^\infty$, we will have to modify a little the argument. For each fixed N we will construct functions α_i^N, $1 \leq i \leq N$, with $\operatorname{supp}(\alpha_i^N) \subset R_i$, satisfying (9.1), (9.2) and (9.3). Finally, applying weak limits when $N \to \infty$, we will get the functions α_i.

The functions α_i^N that we will construct will be of the form $\alpha_i^N = a_i^N \, \chi_{A_i^N}$, with $a_i^N \in \mathbb{R}$ and $A_i^N \subset R_i$. To avoid a complicate notation, suppose that the cubes R_i, $1 \leq i \leq N$, satisfy $\ell(R_i) \leq \ell(R_{i+1})$ (we can assume this because we are taking a finite number of cubes). We set $A_1^N = R_1$ and

$$\alpha_1^N = a_1^N \chi_{R_1},$$

where the constant a_1^N is chosen so that $\int_{Q_1} f\, w_1\, d\mu = \int \alpha_1^N\, d\mu$.

Suppose that α_1^N, $\alpha_2^N \ldots, \alpha_{k-1}^N$ (for some $k \leq N$) have been constructed, satisfy (9.1) and $\sum_{i=1}^{k-1} |\alpha_i| \leq B\,\lambda$, where B is some constant (which will be fixed below).

Let $R_{s_1}, \ldots, R_{s_m}$ be the subfamily of cubes R_i, $1 \leq i \leq k-1$, such that $R_{s_j} \cap R_k \neq \varnothing$. As $l(R_{s_j}) \leq l(R_k)$ (because of the non decreasing sizes of R_i), we have $R_{s_j} \subset 3R_k$. Taking into account that for $i = 1, \ldots, k-1$

$$\int |\alpha_i^N|\, d\mu \leq \int |f\, w_i|\, d\mu$$

by (9.1), and using that R_k is $(6, 6^{n+1})$-doubling and (9.4), we get

$$\begin{aligned} \sum_j \int_{R_{s_j}} |\alpha_{s_j}^N|\, d\mu &\leq \sum_j \int |f\, w_{s_j}|\, d\mu \\ &\leq C \int_{3R_k} |f|\, d\mu \leq C\lambda\mu(6R_k) \leq C_{14}\lambda\, \mu(R_k). \end{aligned}$$

Therefore,

$$\mu\left\{\textstyle\sum_j |\alpha_{s_j}^N| > 2C_{14}\lambda\right\} \leq \frac{\mu(R_k)}{2}.$$

So we set

$$A_k^N = R_k \cap \left\{\textstyle\sum_j |\alpha_{s_j}^N| \leq 2C_{14}\lambda\right\},$$

and then $\mu(A_k^N) \geq \mu(R_k)/2$.

The constant a_k^N is chosen so that for $\alpha_k^N = a_k^N\, \chi_{A_k^N}$ we have $\int \alpha_k^N\, d\mu = \int f\, w_k\, d\mu$. Then we obtain

$$\begin{aligned} |a_k^N| &\leq \frac{1}{\mu(A_k^N)} \int |f\, w_k|\, d\mu \leq \frac{2}{\mu(R_k)} \int |f\, w_k|\, d\mu \\ &\leq \frac{2}{\mu(R_k)} \int_{\frac{1}{2}R_k} |f|\, d\mu \leq C_{15}\lambda \end{aligned}$$

(this calculation also applies to $k = 1$). Thus,

$$|\alpha_k^N| + \sum_j |\alpha_{s_j}^N| \leq (2C_{14} + C_{15})\,\lambda.$$

If we choose $B = 2C_{14} + C_{15}$, (9.3) follows for the cubes $R_1, \ldots, R_n$.

Now it is easy to check that (9.2) also holds. Indeed we have

$$\|\alpha_i^N\|_{L^\infty(\mu)}\, \mu(R_i) \leq C\, |a_i^N|\, \mu(A_i^N) = C \left| \int_{Q_i} f\, w_i\, d\mu \right| \leq C\, \|\alpha_i^N\|_{L^1(\mu)}.$$

Finally, taking weak limits in the weak-$*$ topology of $L^\infty(\mu)$, one easily obtains the required functions α_i. The details are left to reader. A similar argument can be found in the proof of Lemma 7.3 of [To3]. ☐

Using the decomposition above we can prove Lemma 9.1 partially. This will be the first step of its proof.

Lemma 9.4. *The subspace $H^1_\Phi(\mu)\cap L^\infty(\mu)$ is dense in $H^1_\Phi(\mu)$.*

Proof. Given $f\in H^1_\Phi(\mu)$, for each integer $k\geq 0$, we consider the generalized Calderón-Zygmund decomposition of f given in the preceeding lemma, with $\lambda=2^k$. We will adopt the convention that all the elements of that decomposition will carry the subscript k. Thus we write $f=g_k+b_k$, as in (c) of Lemma 9.1. We know that g_k is bounded and satisfies $\int g_k\,d\mu=0$ (because $\int b_k\,d\mu=0$). We will show that $g_k\to f$ in $L^1(\mu)$ and $\|M_\Phi(g_k-f)\|_{L^1(\mu)}\to 0$ as $k\to\infty$ too.

It is not difficult to check that b_k tends to 0 in $L^1(\mu)$. Indeed, if we set $\Omega_k=\{M_{(2)}f(x)>2^k\}$, then $\mu(\Omega_k)\to 0$ as $k\to\infty$, because f, $M_\Phi f\in L^1(\mu)$. Thus

$$\int|b_k|\,d\mu\leq 2\sum_i\int|f\,w_{i,k}|\,d\mu\leq C\int_{\Omega_k}|f|\,d\mu\xrightarrow{k\to\infty}0,$$

and so $g_k\to f$ in $L^1(\mu)$.

Let us see that $\|M_\Phi b_k\|_{L^1(\mu)}\to 0$ as $k\to\infty$. We denote $b_{i,k}=f\,w_{i,k}-\alpha_{i,k}$. Then we have

$$\|M_\Phi b_k\|_{L^1(\mu)}\leq\sum_i\|M_\Phi b_{i,k}\|_{L^1(\mu)}.$$

The estimates for each term $\|M_\Phi b_{i,k}\|_{L^1(\mu)}$ are (in part) similar to the ones in Lemma 3.1 for estimating M_Φ over atomic blocks. We write

$$\begin{aligned}\|M_\Phi b_{i,k}\|_{L^1(\mu)}\;&\leq\;\int_{\mathbb{R}^d\setminus 2R_{i,k}}M_\Phi b_{i,k}\,d\mu\\&\quad+\int_{2R_{i,k}}M_\Phi(f\,w_{i,k})\,d\mu+\int_{2R_{i,k}}M_\Phi\alpha_{i,k}\,d\mu\end{aligned}\tag{9.5}$$

Taking into account that $\int b_{i,k}\,d\mu=0$, it is easily seen that

$$\int_{\mathbb{R}^d\setminus 2R_{i,k}}M_\Phi b_{i,k}\,d\mu\leq C\,\|b_{i,k}\|_{L^1(\mu)}\leq C\,\|f\,w_{i,k}\|_{L^1(\mu)}$$

(the calculations are similar to the ones in (3.1) and (3.2)).

Let us consider the last term on the right hand side of (9.5) now. By (9.1) and (9.2) we get

$$\int_{2R_{i,k}}M_\Phi\alpha_{i,k}\,d\mu\leq\|\alpha_{i,k}\|_{L^\infty(\mu)}\mu(2R_{i,k})\,d\mu\leq C\,\|f\,w_{i,k}\|_{L^1(\mu)}.$$

We split the second integral on the right hand side of (9.5) as follows:

$$\int_{2R_{i,k}} M_\Phi(f\,w_{i,k})\,d\mu = \int_{2R_{i,k}\setminus 2Q_{i,k}} + \int_{2Q_{i,k}}.$$

As in (3.4), we have

$$\begin{aligned}\int_{2R_{i,k}\setminus 2Q_{i,k}} M_\Phi(f\,w_{i,k})\,d\mu &\leq C\,\|f\,w_{i,k}\|_{L^1(\mu)} \int_{2R_{i,k}\setminus 2Q_{i,k}} \frac{1}{|x-z_{Q_{i,k}}|^n}\,d\mu(x)\\ &\leq C\,\|f\,w_{i,k}\|_{L^1(\mu)}\,(1+\delta(Q_{i,k},R_{i,k}))\\ &\leq C\,\|f\,w_{i,k}\|_{L^1(\mu)}.\end{aligned}$$

Finally we have to deal with $\int_{2Q_{i,k}} M_\Phi(f\,w_{i,k})\,d\mu$. Consider $x\in 2Q_{i,k}$ and $\varphi\sim x$. Then

$$\left|\int \varphi\,(f\,w_{i,k})\,d\mu\right| = \left|\int (\varphi\,w_{i,k})\,f\,d\mu\right| \leq C\,M_\Phi f(x), \tag{9.6}$$

because $C\,\varphi\,w_{i,k}\sim x$ for some constant $C>0$, since for $y\in\mathbb{R}^d$ we have

$$0\leq w_{i,k}\varphi(y)\leq\varphi(y)\leq\frac{1}{|y-x|^n}$$

and

$$\begin{aligned}|(\varphi\,w_{i,k})'(y)| &\leq |\varphi'(y)\,w_{i,k}(y)|+|\varphi(y)\,w'_{i,k}(y)|\\ &\leq \frac{1}{|y-x|^{n+1}}+\frac{C}{|y-x|^n}\,|w'_{i,k}(y)|.\end{aligned}$$

Recall that $|w'_{i,k}(y)|\leq C\ell(Q_{i,k})^{-1}$ and $\operatorname{supp}(w_{i,k})\subset 2Q_{i,k}$. Then we get $|w'_{i,k}(y)|\leq C\,|y-x|^{-1}$ for all $y\in\mathbb{R}^d$. Thus $|(\varphi\,w_{i,k})'(y)|\leq C\,|y-x|^{-n-1}$. So (9.6) holds and then

$$\int_{2Q_{i,k}} M_\Phi(f\,w_{i,k})\,d\mu\leq C\int_{2Q_{i,k}} M_\Phi f\,d\mu.$$

When we gather the previous estimates, we obtain

$$\|M_\Phi b_{i,k}\|_{L^1(\mu)}\leq C\,\|f\,w_{i,k}\|_{L^1(\mu)}+C\int_{2Q_{i,k}} M_\Phi f\,d\mu.$$

Taking into account the finite overlap of the cubes $2Q_{i,k}$ (recall that they are Whitney cubes covering Ω_k), we get

$$\|M_\Phi b_k\|_{L^1(\mu)}\leq C\int_{\Omega_k}(|f|+M_\Phi f)\,d\mu \xrightarrow{k\to\infty} 0,$$

and we are done. □

Proof of Lemma 9.1. Take $f\in H^1_\Phi(\mu)\cap L^\infty(\mu)$. Consider the infinite increasing sequence of the cubes $Q_k=4^{N_k}\,[-1,1]^d$ that are $(4,4^{n+1})$-doubling.

Let w be a $\mathcal{C}^\infty$ function such that $\chi_{[-1,1]^d}(x) \leq w(x) \leq \chi_{[-2,2]^d}(x)$ for all x. We denote $w_k(x) = w(4^{-N_k}x)$ (so $\chi_{Q_k}(x) \leq w_k(x) \leq \chi_{2Q_k}(x)$) and we set

$$f_k = w_k\, f - \frac{\chi_{Q_k}}{\mu(Q_k)} \int w_k\, f\, d\mu.$$

It is clear that f_k is bounded, has compact support and converges to f in $L^1(\mu)$ as $k \to \infty$. We will prove that

$$\begin{aligned}(9.7)\qquad \|M_\Phi(f - f_k)\|_{L^1(\mu)} \ &\leq\ C\left|\int w_k\, f\, d\mu\right| + C\int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi f\, d\mu \\ &\quad + \int_{4Q_k} M_\Phi((1-w_k)\, f)\, d\mu.\end{aligned}$$

Finally we will show that the terms on the right hand side of (9.7) tend to 0 as $k \to \infty$ and we will be done.

Let us consider first the integral of $M_\Phi(f - f_k)$ over $\mathbb{R}^d \setminus 4Q_k$. We set

$$\int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi(f - f_k)\, d\mu \leq \int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi f\, d\mu + \int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi f_k\, d\mu.$$

We only have to estimate the last integral on the right hand side. Take $x \in \mathbb{R}^d \setminus 4Q_k$, $\varphi \sim x$ and let $y_0 \in 2Q_k$ be the point where φ attains its minimum over $2Q_k$ (recall that we assume $\varphi \geq 0$ and $\varphi \in \mathcal{C}^1$). We denote $c_k = \int w_k\, f\, d\mu/\mu(Q_k)$ and then we set

$$\begin{aligned}\int f_k\, \varphi\, d\mu \ &=\ \int f(y)\,(\varphi(y) - \varphi(y_0))\, d\mu(y) \\ &=\ \int w_k(y)\, f(y)\,(\varphi(y) - \varphi(y_0))\, d\mu(y) \\ &\quad - c_k \int_{Q_k} (\varphi(y) - \varphi(y_0))\, d\mu(y) \ =\ I_1 - I_2.\end{aligned}$$

Let us consider the function $\psi(y) = w_k(y)\,(\varphi(y) - \varphi(y_0))$. This function satisfies

$$0 \leq \psi(y) \leq \varphi(y)$$

and

$$\begin{aligned}|\psi'(y)| \ &\leq\ |w_k(y)\, \varphi'(y)| + |w_k'(y)|\, |\varphi(y) - \varphi(y_0)| \\ &\leq\ \frac{1}{|y-x|^{n+1}} + C\,\ell(Q_k)^{-1}\, \frac{\ell(Q_k)}{|y-x|^{n+1}} \ =\ C\, \frac{1}{|y-x|^{n+1}}.\end{aligned}$$

Therefore $C\,\psi \sim x$ for some constant $C > 0$ and so $|I_1| \leq C\, M_\Phi f(x)$. For I_2 we use a cruder estimate:

$$|I_2| \leq C\, |c_k|\, \mu(Q_k)\, \frac{\ell(Q_k)}{|y_0 - x|^{n+1}}.$$

Thus we obtain

$$M_\Phi f_k(x) \leq C\, M_\Phi f(x) + C\, |c_k|\, \mu(Q_k)\, \frac{\ell(Q_k)}{|y_0 - x|^{n+1}}.$$

Since

$$\int_{\mathbb{R}^d\setminus 4Q_k} \frac{1}{|y_0-x|^{n+1}}\, d\mu(x) \leq C\,\ell(Q_k)^{-1},$$

we get

$$\begin{aligned}\int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi f_k\, d\mu &\leq C\int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi f\, d\mu + C\,|c_k|\,\mu(Q_k)\\ &= C\int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi f\, d\mu + C\left|\int w_k\, f\, d\mu\right|. \qquad (9.8)\end{aligned}$$

Now we have to deal with $\int_{4Q_k} M_\Phi(f-f_k)\, d\mu$. For $x\in 4Q_k$ we write

$$M_\Phi(f-f_k)(x) \leq M_\Phi((1-w_k)\,f)(x) + M_\Phi\left(\frac{|c_k|}{\mu(Q_k)}\,\chi_{Q_k}\right)(x). \qquad (9.9)$$

Since $M_\Phi\chi_{Q_k}(x)\leq 1$ and Q_k is $(4,4^{n+1})$-doubling, we get

$$\int_{4Q_k} M_\Phi\left(\frac{|c_k|}{\mu(Q_k)}\,\chi_{Q_k}\right)(x)\, d\mu(x) \leq C\,|c_k| = C\left|\int w_k\, f\, d\mu\right|. \qquad (9.10)$$

From (9.8), (9.9) and (9.10) we derive (9.7).

Now we have to see that the terms on the right hand side of (9.7) tend to 0 as $k\to\infty$. Since f, $M_\Phi f\in L^1(\mu)$, by the dominated convergence theorem

$$\lim_{k\to\infty}\left|\int w_k\, f\, d\mu\right| + \int_{\mathbb{R}^d\setminus 4Q_k} M_\Phi f\, d\mu = 0.$$

Let us turn our attention to the third term on the right hand side of (9.7). Take $x\in 4Q_k$ and $\varphi\sim x$. It is easily seen that $C\,w_k\,\varphi\sim x$ for some constant $C>0$. So we get $M_\Phi(w_k\, f)(x)\leq C\, M_\Phi f(x)$ and then for any $x\in\mathbb{R}^d$,

$$\chi_{4Q_k}(x)\, M_\Phi((1-w_k)\, f)(x) \leq \chi_{4Q_k}(x)\,(M_\Phi f(x)+M_\Phi(w_k\, f)(x)) \leq C\, M_\Phi f(x).$$

Therefore, if we show that $\chi_{4Q_k}(x)\, M_\Phi((1-w_k(x))\, f)(x)$ tends to 0 pointwise as $k\to\infty$, we will be done by a new application of the dominated convergence theorem.

For a fixed $x\in\mathbb{R}^d$, let k_0 be such that $x\in\frac{1}{2}Q_k$ for $k\geq k_0$. Notice that if $\varphi\sim x$ and $y\notin Q_k$, then $|\varphi(y)|\leq C/\ell(Q_k)^n$. Thus

$$\begin{aligned}\left|\int \varphi(x)(1-w_k(x))\, f(x)\, d\mu(x)\right| &\leq \|f\|_{L^1(\mu)}\,\|(1-w_k)\,\varphi\|_{L^\infty(\mu)}\\ &\leq C\,\frac{\|f\|_{L^1(\mu)}}{\ell(Q_k)^n}.\end{aligned}$$

Then we get

$$\chi_{4Q_k}(x)\, M_\Phi((1-w_k(x))\, f)(x) \leq C\,\frac{\|f\|_{L^1(\mu)}}{\ell(Q_k)^n} \xrightarrow{k\to\infty} 0.$$

$\square$

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DEPARTMENT OF MATHEMATICS, CHALMERS, 412 96 GÖTEBORG, SWEDEN
E-mail address: `xavier@math.chalmers.se`