On the growth of harmonic and $\mathcal{M}$-harmonic functions near to a thin set on the boundary of the unit ball

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1 Introduction

In the article our aim is to describe the singular part of a measure on the unit sphere in $\mathbb{C}^n$ in terms of behaviour of its harmonic and $\mathcal{M}$-harmonic extension near to the boundary. The description involves characterisation of the "level of singularity" of the singular part of the measure. As an application by an interaction of the description for harmonic and $\mathcal{M}$-harmonic functions we obtain a new proof of a weaker variant of the Aleksandrov theorem about singular part of pluri-harmonic measure (see [1], Theorem 3.2.1). This new proof shows a new reasons for the theorem to be true and as it doesn't use any properties of pluri-harmonic functions, except the one that they are both harmonic and $\mathcal{M}$-harmonic, the approach can give some possibilities for generalisations.

Though the results seems to be new in application to the complex sphere, they are analogies (together with the main ideas of the proof) of the similar results for a real hyperplane (see [6, 2, 3]). So the article should be considered rather as a remark than as a new result.

The structure of the article is following. In the first section we make it clear what we mean as "the level (dimension) of singularity" (our definitions differ from the similar ones in [4]), and introduce the characteristic of a measure in this scale. In the second and third sections we show the connection of this characteristic and the speed of the growth of the $\mathcal{M}$-harmonical and the harmonical extensions of the measure near to the boundary. In the last section we apply the results to prove that the dimension of singular part of a pluriharmonic measure is at least $2n-2$ both in Euclidean and Koranyi sense (a weaker variant of the Aleksandrov theorem). We also prove for pluriharmonic measures that their distribution in "level of singularity" coinsides for both Koranyi and Euclidean sense, which seems to be a new property (though intuitively equivalent to the Aleksandrov theorem).

2 "The level of singularity"

First we define more exactly what we mean as "the level of singularity" of a measure. As we compair singularity with a Hausdorff measure of some order on the ball, the result depends of the choice of metric on the ball. There are two possibilities: isotropic (Euclidean: $d_E(x,y) = |x-y|$) or non-isotropic (Koranyi: $d_K(x,y) = |1- < x,y > |^{\frac{1}{2}}$). All the results of this chapter is true for the both of the metrics. We will denote a ball in Koranyi metric as Q or B_K, and a ball in Euclidian metric as B or B_E.

Definition 1 *Let us say that a measure μ on a metric space M has dimension less than α if $\limsup_{\delta\to 0} \frac{|\mu|(\overline{B_M}(x,\delta))}{\delta^\alpha} = \infty$ for μ-a.e. x.*
Let us say that a measure μ on a metric space M has dimension less or equal than α if $\limsup_{\delta\to 0} \frac{|\mu|(\overline{B_M}(x,\delta))}{\delta^\alpha} > 0$ for μ-a.e. x.
Let us say that a measure μ on the metric space has dimension larger than α if $\limsup_{\delta\to 0} \frac{|\mu|(\overline{B_M}(x,\delta))}{\delta^\alpha} = 0$ for μ-a.e. x.
Let us say that a measure μ on the metric space has dimension larger or equal than α if $\limsup_{\delta\to 0} \frac{|\mu|(\overline{B_M}(x,\delta))}{\delta^\alpha} < \infty$ for μ-a.e. x. In all cases $B_M(x,\delta)$ is a ball with respect to the metric.
We will say that a measure μ is less singular than the Hausdorff measure of dimension α if for any Borel set E of σ-finite Hausdorff measure of order α, $|\mu|(E) = 0$.

We say that a Hausdorff measure of a set A is σ-finite if $A = \cup A_j$ where each A_j in the countable union has finite Hausdorff measure. The restriction of a measure μ to a Borel set K (not necessary compact) we will denote as μ_K ($\mu_K(E) = \mu(E \cap K)$). We will say that a measure is supported on a Borel set K if $\mu = \mu_K$. If a statement and the proof is similar for both Euclidean and Koranii metrics, we denote the metric as M. Let us note that 5-covering lemma is true for both metrics. We will denote the Hausdorff measure of order α as h_α. By N we denote an index which does through natural numbers.

Lemma 1 *If one has a finite regular Borel measure μ on a metric space there exist decomposition $\mu = \mu_s + \mu_a$, where the measure μ_a has dimension larger than α and μ_s is supported on some set of σ-finite Hausdorff measure of order α.*

Proof (in short). Let us introduce $D_\alpha(\mu)(x) = \limsup_{\delta\to 0} \frac{|\mu|(\overline{B_M}(x,\delta))}{\delta^\alpha}$. The set K on which $D_\alpha(\mu)(x) > 0$ is a Borel set of σ-finite Hausdorff measure of order α, as $h_\alpha(\{D_\alpha(\mu) > t\}) \leq c\frac{\|\mu\|}{t}$. Let $\mu_s = \mu|_K$. Then the rest is μ_a, and $D_\alpha(\mu_a) = 0$ outside of K, which means μ_a-a.e.

Lemma 2 *If a measure μ has dimension large than α then the measure μ is less singular than the Hausdorff measure of order α.*

Proof. Fix $\varepsilon > 0$ and an arbitrary set K of finite Hausdorff measure of order α. Let denote as E_N the Borel set of points x for which $|\mu|(B_M(x,r)) < \varepsilon r^\alpha$

for all $r < \frac{1}{N}$. Then $|\mu|(K \setminus (\bigcup_N E_N)) = 0$ as μ-a.e. point belongs to some E_N according to the definition. Consider $K \cap E_N$. This set is of finite Hausdorff measure of order α and so can be covered by a collection of open balls, such that $\sum_\gamma r_\gamma^\alpha < 2h_\alpha(K \cap E_N) \leq 2h_\alpha(K)$ and $r_\gamma < \frac{1}{4N}$ for all γ. If a ball from the covering doesn't include any points of $K \cap E_N$ we will remove it (and the rest will be still a covering). If it includes a point from $K \cap E_N$, we will replace it by a ball of twice larger radius with centre in the point. The obtained collection of the balls $\{B_{M,\gamma}\}$ is a covering. As the centres of the balls belong to E_N and radii are less then $\frac{1}{2N}$, we have the estimate

$$|\mu|(K \cap E_N) \leq \sum |\mu|(B_{M,\gamma}) \leq \sum \varepsilon(2r)^\alpha \leq \varepsilon 2^{\alpha+1} h_\alpha(K).$$

As $(K \cap E_N) \subset (K \cap E_{N+1})$, and the right side of the estimate doesn't depend from N, $|\mu|(K \cap (\bigcup_N E_N)) \leq \varepsilon 2^{\alpha+1} h_\alpha(K)$. So $|\mu|(K) \leq \varepsilon 2^{\alpha+1} h_\alpha(K)$. And as ε is arbitrary small we have proved that $|\mu|(K) = 0$. Now it remains to notice that a set of σ-finite Hausdorff measure of order α is a union of countable collection of sets of finite Hausdorff measure of order α. And as the measure $|\mu|$ of each of the set in the union is zero the measure $|\mu|$ of the hole union is also zero.

Corollary 1 *The decomposition in lemma 1 is unique.*

Proof. Let us have two decompositions $\mu = \mu_s + \mu_a = \mu_s' + \mu_a'$. Then $\mu_s - \mu_s' = \mu_a' - \mu_a$. The measure $\mu_s - \mu_s'$ is supported on some set K of σ-finite Hausdorff measure of order α, so $\|\mu_s - \mu_s'\| = |\mu_s - \mu_s'|(K)$. At the same time the same measure $\mu_a' - \mu_a$ is less singular than the Hausdorff measure of order α (as difference of two measures with this property), so $|\mu_s - \mu_s'|(K) = |\mu_a' - \mu_a|(K) = 0$. Thus $\mu_s = \mu_s'$, $\mu_a = \mu_a'$ and the decomposition is unique.

Lemma 3 *If a measure is less singular then the Hausdorff measure of order α then the measure has dimension large then α.*

Proof. Let $K_N = \{x : D_\alpha(\mu) > \frac{1}{N}\}$. It is a Borel set and $h_\alpha(K_N) \leq cN\|\mu\| < \infty$. So $|\mu|(K_N) = 0$. $\{x : D_\alpha \neq 0\} = \cup K_N$, so its measure is also 0.

Lemma 4 *If a finite measure is supported by a set of σ-finite Hausdorff measure of order α then it has dimension less or equal to α.*

Proof. Let K is the Borel set of σ-finite measure which supports μ. We will suppose first that the set K is of finite Hausdorff measure of order α. Now we will consider the set $E_N = \{x : D_\alpha(\mu)(x) < \frac{1}{N}\}$. This set is union of sets $E_{N,l}$ where $E_{N,l}$ is the set of points for which $|\mu|(B_M(x,r)) < \frac{1}{N} r^\alpha$ as soon as $r < \frac{1}{l}$. Now, as $h_\alpha(E_{N,l} \cap K) < \infty$, as it was done before one can cover the set $E_{N,l} \cap K$ with balls of radius less than $\frac{1}{l}$ and centre in some point of the set in such a way that $\sum r_\gamma^\alpha < C_\alpha h_\alpha(E_{N,l} \cap K) \leq C_\alpha h_\alpha(K)$. So,

$$|\mu|(E_{N,l}) \leq \sum |\mu|(B_{M,\gamma}) < \sum \frac{1}{N} r_\gamma^\alpha \leq \frac{1}{N} C_\alpha h_\alpha(K).$$

As the estimate doesn't depend from l and $E_{N,l} \subset E_{N,l+1}$, one has that $|\mu|(E_N) \leq C_\alpha \frac{1}{N} h_\alpha(K)$. Now $E_{N+1} \subset E_N$, $\{x : D_\alpha(\mu)(x) = 0\} = \cap E_N$, and the right part of the estimate for $|\mu|(E_N)$ tends to zero, thus $|\mu|(\{x : D_\alpha(\mu)(x) = 0\}) = 0$ i.e. the measure μ has dimension less or equal to α. If the set K has σ-finite Hausdorff measure of order α, the we can consider μ_j, a restriction of μ to the sets component K_j of finite Hausdorff measure of order α. $D_\alpha(\mu_j) \leq D_\alpha(\mu)$, so the set $E = \{D_\alpha(\mu) = 0\} \subset \bigcup_j \{D_\alpha(\mu_j) = 0\}$ and $|\mu|(E) \leq \sum_j |\mu_j|(E) = 0$.

Summarising all the lemmas we have proved the following.

Proposition 1 *Each finite regular Borel measure μ on a metric space can be in a unique way decomposed in a sum of two finite regular Borel measures $\mu = \mu_s + \mu_a$, such that the measure μ_s is supported on a set of σ-finite Hausdorff measure of order α and so has dimension at least α, and the measure μ_a has dimension large than α, and so is less singular then the Hausdorff measure of order α.*

Now we a in a position to introduce the function which characterise singularity of a given measure.

Definition 2 *For a finite regular Borel measure we will consider a "scaling in dimension function" $d_{\mu,M} : \mathbb{R} \to \mathbb{R}_+$, which is defined in a point α as the variation $\|\mu_s\|$, where μ_s is the measure from the proposition 1 (for negative values of α one also can formally define the function, though it equals zero).*

3 Estimate for the $\mathcal{M}$-harmonic extension.

One call a function u in the unit ball $\mathcal{M}$-harmonic if it is a solution of the equation $\widetilde{\Delta} u = 0$, for the Laplace-Beltrami operator $\widetilde{\Delta}$.

Definition 3 *We will denote $\mathcal{P}[\mu]$ the $\mathcal{M}$-harmonic extension of the measure μ from the sphere inside the ball and $\mathcal{P}_{+,\alpha}[\mu](\xi) = \limsup_{r\to 1}(1-r^2)^\alpha |\mathcal{P}(r\xi)|$.*

Remark. The similar limit one can consider not for radial approach, but for approach, for example, in an admissible region.

Lemma 5 $|\mu|(\{\mathcal{P}_{+,n-\frac{\alpha}{2}}[\mu](\xi) > 0\}) \leq d_{\mu,K}(\alpha)$.

Proof. Let us for a fixed measure μ introduce a function $M_r^\alpha(\xi) = \sup\limits_{\delta<r}\{\frac{|\mu|(Q(\xi,\delta))}{\delta^\alpha}\}$.

The integral representation gives us,

$$\mathcal{P}[\mu](r\xi) = \int_S \frac{(1-r^2)^n}{|1- < \omega, r\xi > |^{2n}} d\mu(\omega).$$

It is clear, as the Poisson-Szegö kernel is positive, that if we can prove the estimate for $|\mu|$ then it is true for the measure μ itself. So we will suppose the measure to be positive.

Decompose the sphere S in the union of following sets. $S_0 = \{\omega \in S : d_K(\omega,\xi) < (1-r^2)^{\frac{1}{2}}\}$, $S_k = \{\omega \in S : 2^{k-1}(1-r^2)^{\frac{1}{2}} \le d_K(\omega,\xi) < 2^k(1-r^2)^{\frac{1}{2}}\}$, for all positive integers $k < \frac{1}{4}(-\log_2(1-r^2))$ and the rest points S_∞. We will estimate the integral over each of these sets via $M^\alpha_{(1-r^2)^{\frac{1}{4}}}(\xi)$.

$$\int_{S_0} \frac{(1-r^2)^n}{|1-<\omega, r\xi>|^{2n}} d\mu(\omega) = \int_{S_0} \frac{(1-r^2)^n}{|1-r+r(1-<\omega,\xi>)|^{2n}} d\mu(\omega) \le$$

$$\int_{S_0} \frac{(1-r^2)^n}{((1-r)+rRe(1-<\omega,\xi>))^{2n}} d\mu(\omega) \le \int_{S_0} \frac{(1-r^2)^n}{(1-r)^{2n}} d\mu(\omega) \le$$

$$2^{2n}\frac{|\mu|(Q(\xi,(1-r^2)^{\frac{1}{2}})}{(1-r^2)^n} \le C(n)(1-r^2)^{\frac{\alpha}{2}-n} M^\alpha_{(1-r^2)^{\frac{1}{2}}}(\xi) \le$$

$$C(n)(1-r^2)^{\frac{\alpha}{2}-n} M^\alpha_{(1-r^2)^{\frac{1}{4}}}(\xi).$$

If $r > \frac{1}{2}$,

$$\int_{S_k} \frac{(1-r^2)^n}{|1-<\omega, r\xi>|^{2n}} d\mu(\omega) = \int_{S_k} \frac{(1-r^2)^n}{|1-r+r(1-<\omega,\xi>)|^{2n}} d\mu(\omega) \le$$

$$\int_{S_k} \frac{(1-r^2)^n}{(((1-r)+rRe(1-<\omega,\xi>))^2 + rIm(1-<\omega,\xi>)^2)^n} d\mu(\omega) \le$$

$$2^{2n}\int_{S_k} \frac{(1-r^2)^n}{|1-<\omega,\xi>|^{2n}} d\mu(\omega) \le C(n)2^{-4nk}\frac{|\mu|(Q(\xi,2^k(1-r^2)^{\frac{1}{2}})}{(1-r^2)^n} \le$$

$$C(n)2^{-4nk}2^{k\alpha}(1-r^2)^{\frac{\alpha}{2}-n} M^\alpha_{2^k(1-r^2)^{\frac{1}{2}}}(\xi) = C(n)2^{(\alpha-4n)k}(1-r^2)^{\frac{\alpha}{2}-n} M^\alpha_{(1-r^2)^{\frac{1}{4}}}(\xi).$$

And for the rest, if $r > \frac{1}{2}$, is valid the estimate

$$\int_{S_\infty} \frac{(1-r^2)^n}{|1-<\omega, r\xi>|^{2n}} d\mu(\omega) = \int_{S_\infty} \frac{(1-r^2)^n}{|1-r+r(1-<\omega,\xi>)|^{2n}} d\mu(\omega) \le$$

$$\int_{S_\infty} \frac{(1-r^2)^n}{(((1-r)+rRe(1-<\omega,\xi>))^2 + rIm(1-<\omega,\xi>)^2)^n} d\mu(\omega) \le$$

$$2^{2n}\int_{S_\infty} \frac{(1-r^2)^n}{|1-<\omega,\xi>|^{2n}} d\mu(\omega) \le C(n)\|\mu\|.$$

Summing all the estimates we obtain,

$$(1-r^2)^{n-\frac{\alpha}{2}}\mathcal{P}[\mu](r\xi) \le C(n)(M^\alpha_{(1-r^2)^{\frac{1}{4}}}(\xi) + (1-r^2)^{n-\frac{\alpha}{2}}\|\mu\|).$$

As soon as we take the $\limsup_{r\to 1}$ from both parts of the inequality, we obtain that $\mathcal{P}_{+,n-\frac{\alpha}{2}}[\mu](\xi) \le C(n)D_\alpha(\mu)(\xi)$, where the derivate corresponds to the Koranii metric. So, $\{\mathcal{P}_{+,n-\frac{\alpha}{2}}[\mu](\xi) > 0\} \subset \{D_\alpha(\mu)(\xi) > 0\}$.

By the construction in the lemma 1, $\mu_s = \mu|_K$, where the set K is the set of all points on which $D_\alpha(\mu) \neq 0$, so, $|\mu|(\{\mathcal{P}_{+,n-\frac{\alpha}{2}}[\mu](\xi) > 0\}) \le |\mu|(\{D_\alpha(\mu)(\xi) > 0\}) = |\mu_s|(\{D_\alpha(\mu)(\xi) > 0\}) = \|\mu_s\| = d_{\mu,K}(\alpha)$.

4 Estimates for the harmonic extension.

Definition 4 *We will denote $P[\mu]$ the harmonic extension of the measure μ from the sphere inside the ball and $P_{+,\alpha}[\mu](\xi) = \limsup_{r\to 1}(1-r^2)^\alpha |P(r\xi)|$.*

Remark. Again the similar limit one can consider not for radial approach, but for approach, for example, in an nontangential region.

A lemma similar to the lemma 5 is true for the harmonic extension.

Lemma 6 $|\mu|(\{P_{+,2n-1-\alpha}[\mu](\xi) > 0\}) \le d_{\mu,E}(\alpha)$.

Proof. Let us introduce a function $M^\alpha_r(\mu)(\xi) = \sup\limits_{0<\delta<r} \frac{|\mu|(B(\xi,\delta))}{\delta^\alpha}$. Again as the kernel is positive it is enough to prove the estimate for the positive measures.

The integral representation gives

$$P[\mu](r\xi) = \int_S \frac{(1-r^2)}{|\omega - r\xi|^{2n}} d\mu(\omega).$$

Again we will decompose the sphere. $S_0 = \{\omega \in S : |\omega - \xi| < 2(1-r)\}$, $S_k = \{\omega \in S : 2^{k+1}(1-r) \le |\omega - \xi| < 2^{k+2}(1-r)\}$ for $k < -\frac{2n-1}{2n}\log_2(1-r)$ and the rest is S_∞. We will estimate the integral over each of these sets via $M^\alpha_{4(1-r)^{\frac{1}{2n}}}(\xi)$.

$$\int_{S_0} \frac{1-r^2}{|r\xi - \omega|^{2n}} d\mu(\omega) \le \int_{S_0} \frac{1-r^2}{|\xi - r\xi|^{2n}} d\mu(\omega) \le \frac{2}{(1-r)^{2n-1}} |\mu|(S_0) \le$$

$$C(n)(1-r)^{\alpha-(2n-1)} M^\alpha_{2(1-r)}(\xi) \le C(n)(1-r)^{\alpha-(2n-1)} M^\alpha_{4(1-r)^{\frac{1}{2n}}}(\xi).$$

$$\int_{S_k} \frac{1-r^2}{|r\xi - \omega|^{2n}} d\mu(\omega) \le \int_{S_k} \frac{1-r^2}{(|\xi - \omega| - (1-r))^{2n}} d\mu(\omega) \le$$

$$2^{2n} \int_{S_k} \frac{1-r^2}{|\xi - \omega|^{2n}} d\mu(\omega) \le 2^{4n} \cdot 2^{-2kn}(1-r)^{1-2n} |\mu|(S_k) \le$$

$$C(n)2^{(\alpha-2n)k}(1-r)^{(\alpha-(2n-1))} M^\alpha_{4\cdot 2^k(1-r)}(\xi) \le C(n)2^{(\alpha-2n)k}(1-r)^{(\alpha-(2n-1))} M^\alpha_{4(1-r)^{\frac{1}{2n}}}(\xi).$$

And for the rest

$$\int_{S_\infty} \frac{1-r^2}{|r\xi - \omega|^{2n}} d\mu(\omega) \le \int_{S_\infty} \frac{1-r^2}{(|\xi - \omega| - (1-r))^{2n}} d\mu(\omega) \le$$

$$2^{2n} \int_{S_\infty} \frac{1-r^2}{|\xi - \omega|^{2n}} d\mu(\omega) \le 2^{2n} \int_{S_\infty} \frac{1-r^2}{((1-r)^{\frac{1}{2n}})^{2n}} d\mu(\omega) \le C(n)|\mu|(S_\infty) \le C(n)\|\mu\|.$$

Summing all the estimates,

$$(1-r)^{2n-1-\alpha} P[\mu](r\xi) \le C(n)(M^\alpha_{4(1-r)^{\frac{1}{2n}}}(\mu)(\xi) + (1-r)^{2n-1-\alpha}\|\mu\|),$$

As soon as we take the $\limsup_{r\to 1}$ from both parts of the inequality, we obtain that $P_{+,n-\frac{\alpha}{2}}[\mu](\xi) \le C(n)D_\alpha(\mu)(\xi)$, where the derivate corresponds to the Euclidean metric. So, $\{P_{+,n-\frac{\alpha}{2}}(\xi)[\mu] > 0\} \subset \{D_\alpha(\mu)(\xi) > 0\}$.

By the construction in the lemma 1, $\mu_s = \mu|_K$, where the set K is the set of all points on which $D_\alpha(\mu) \ne 0$, so, $|\mu|(\{P_{+,n-\frac{\alpha}{2}}[\mu](\xi) > 0\}) \le |\mu|(\{D_\alpha(\mu)(\xi) > 0\}) = |\mu_s|(\{D_\alpha(\mu)(\xi) > 0\}) = \|\mu_s\| = d_{\mu,E}(\alpha)$.

The proof of the following theorem for a signed measures is based on the Besicovitch covering theorem. As the sphere with the Eucleadian metric on it is a submanifold of $\mathbb{R}^n$, the balls on the sphere correspond to the balls in $\mathbb{R}^n$, and so the Besicovitch covering theorem is valid in this case. The similar result for $\mathcal{M}$-harmonic extension can be produced by the same proof in case the measure is positive. In case when the measure is absolutely continuous with respect to some Hausdorff measure of order α on a set of finite Hausdorff measure of order α (or countable sum of such measures) the similar result is also valid (see [7]). In the general case for the Koranyi metric Besicovitch theorem is not valid (see [5]), and so the similar result is doubtfull.

Lemma 7 $|\mu|(\{P_{+,2n-1-\alpha}[\mu](\xi) > 0\}) \ge d_{\mu,E}(\alpha)$.

We are going to prove that for some small enough $c > 0$ the estimate $P_{+,2n-1-\alpha}[\mu](\xi) \ge cD^\alpha(\mu)(\xi)$, is valid $|\mu|$-a.e. (the derivative D^α is in sense of the Euclidean metric). To prove this let us consider the decomposition of μ in a sum of the positive and negative parts μ_+ and μ_-. We are going to prove that the inequality $P_{+,2n-1-\alpha}[\mu](\xi) \ge cD^\alpha(\mu)(\xi)$ is valid μ_+-a.e.

We will use the following lemma. Though it is a weaker variant of one of Besicovitch theorems, we will prove it for the sake of completeness.

Lemma 8 *Let μ_1 and μ_2 are two positive mutually singular measures. Then for any fixed constants $C, c > 0$, for μ_1-a.e. point ξ the inequality $c\mu_1(B(\xi,r)) - C\mu_2(B(\xi,r)) \ge \frac{c}{2}(\mu_1(B(\xi,r) + \mu_2(B(\xi,r)))$ is valid for all balls with centre in the point ξ and small enough radius r.*

Let us consider the set K of all points for which the statement is not true. For each point with centre in this set one can choose a ball B with the centre in this point and arbitrary small radius such that $C'\mu_2(B) > \mu_1(B)$, where C' depends only from c and C. As both measures are finite regular Borel measures for any $\varepsilon > 0$, one can choose two compact subsets K_1 and K_2 of the disjoint supports of μ_1 and μ_2 such that $(\mu_1 + \mu_2)(S \backslash (K_1 \cup K_2)) < \varepsilon$.

Consider $\mu' = \mu_2|_{S\backslash K_2}$. By the choice of the set K_2, $\|\mu'\| < \varepsilon$. For each point of $K \cap K_1$ we can choose a ball B such that $C'\mu_2(B) > \mu_1(B)$ with the centre in the point and such a small radius that the ball doesn't intersect K_2. By the Besicovitch theorem, there exists such a covering of $K \cap K_1$ by these balls $\{B_\gamma\}$, that each point is covered not more then by $C(n)$ balls.

$$\mu_1(K \cap K_1) \le \mu_1(\cup B_\gamma) \le \sum \mu_1(B_\gamma) \le \sum C'\mu_2(B_\gamma) = \sum C'\mu'(B_\gamma) \le$$
$$C'C(n)\mu'(\cup B_\gamma) \le C'C(n)\|\mu'\| \le C'C(n)\varepsilon.$$

As, by the choice of K_1, $\mu_1(K \setminus K_1) \leq \varepsilon$, we have $\mu_1(K) \leq (C'C(n)+1)\varepsilon$. And, as far as we can choose arbitrary small ε, we have proved that $\mu_1(K) = 0$.

To prove the lemma 7, first we will decompose the sphere as before.

$$P[\mu](r\xi) = \int_S \frac{1-r^2}{|\omega - r\xi|^{2n}} d\mu(\omega) = \int_{S_0} \frac{1-r^2}{|\omega - r\xi|^{2n}} d\mu(\omega) + \sum_k \int_{S_k} \frac{1-r^2}{|\omega - r\xi|^{2n}} d\mu(\omega) +$$

$$\int_{S_\infty} \frac{1-r^2}{|\omega - r\xi|^{2n}} d\mu(\omega) \geq \int_{S_0} \frac{1-r^2}{|\omega - r\xi|^{2n}} (d\mu_+(\omega) - d\mu_-(\omega)) +$$

$$\sum_k \int_{S_k} \frac{1-r^2}{|\omega - r\xi|^{2n}} (d\mu_+(\omega) - d\mu_-(\omega)) - C(n)\|\mu\|.$$

Now let us find that by the choice of decomposition there exist two positive constants $c(n)$ and $C(n)$, such that

$$c(n)2^{-kn}(1-r)^{1-2n} \leq \inf_{S_k}\{\frac{1-r^2}{|\omega - r\xi|^{2n}}\} \leq \sup_{S_k}\{\frac{1-r^2}{|\omega - r\xi|^{2n}}\} \leq C(n)2^{-kn}(1-r)^{1-2n}.$$

So, as for some constant $0 < c'(n) \leq \ frac12$ is true that $1 > \frac{1}{2} + c'(n) \sum_k 2^{-kn}$, we can estimate

$$P[\mu](r\xi) \geq (1-r)^{1-2n}(c(n)\mu_+(S_0) - C(n)\mu_-(S_0)) +$$

$$\sum_k (c(n)2^{-kn}(1-r)^{1-2n}\mu_+(S_k) - C(n)2^{-kn}(1-r)^{1-2n}d\mu_-(S_k)) - C(n)\|\mu\| \geq$$

$$(1-r)^{1-2n}(\frac{c(n)}{2}\mu_+(S_0) - C(n)\mu_+(S_0) +$$

$$\sum_k (c(n)c'(n)2^{-kn}\mu_+(\cup_{m\leq k}S_m) - C(n)2^{-kn}\mu_-(S_k))) - C(n)\|\mu\|.$$

According to the lemma 8 for μ_+-a.e. ξ for small values $(1-r)^{\frac{1}{2n}}$ all the differences $c(n)c'(n)2^{-kn}\mu_+(\cup_{m\leq k}S_m) - C(n)2^{-kn}\mu_-(S_k) \geq c|\mu|(S_k) > 0$, so for these r, close enough to 1,

$$P[\mu](r\xi) \geq (1-r)^{1-2n}c|\mu|(S_0) - C(n)\|\mu\|.$$

So, the function $P_{+,\alpha}[\mu](\xi) = \limsup_{r\to 1}(1-r^2)^\alpha P[\mu](r\xi)$ satisfies $P_{+,2n-1-\alpha}[\mu](\xi) \geq \limsup_{r\to 1}(1-r^2)^{(2n-1)-\alpha}((1-r)^{1-2n}c|\mu|(B(\xi, 2(1-r))) - C(n)\|\mu\|) = cD_\alpha\mu(\xi)$.

For the set $K = \{D_\alpha\mu > 0\} \setminus \{P_{+,(2n-1)-\alpha}[\mu] > 0\}$ we have proved that $\mu_+(K) = 0$. Similarly one proves that $\mu_-(K) = 0$. Together it give us that $|\mu|(K) = 0$ and by the construction of lemma 1 $|\mu|(\{P_{+,(2n-1)-\alpha}[\mu] > 0\}) \geq |\mu|(\{D_\alpha\mu > 0\}) = d_{\mu,E}(\alpha)$. The lemma 7 is proved.

5 Application

Let us suppose that a measure is the boundary value of some pluriharmonic in the ball function. Then this function is both harmonic and $\mathcal{M}$-harmonic. So $P_{+,\alpha}[\mu] = \mathcal{P}_{+,\alpha}[\mu]$ in any point of the sphere. Which according to the lemmas gives us that $d_{\mu,E}((2n-1)-\alpha) \le d_{\mu,K}(2n-2\alpha)$. Now we will compare the distributional functions according to their definitions. As any Koranyi ball can be covered by an Euclidean one of the same radius, one find that a set of zero (or σ-finite) measure with respect to Hausdorff measure of order α in Koranyi metric is the set of zero (or σ-finite) measure with respect to the Hausdorff measure in Euclidean metric of the same order α. So, if measure is less singular than Hausdorff measure of order α with respect to Euclidean metric it is less singular than Hausdorff measure of order α with respect to Koranii metric and $d_{\mu,K}(\alpha) \le d_{\mu,E}(\alpha)$. Combining the two obtained inequalities one finds that $d_{\mu,K}((2n-1)-\alpha) \le d_{\mu,K}(2n-2\alpha)$. As the functions $d_{\mu,K}$ growth, for $(2n-1)-\alpha > 2n-2\alpha$ (for $\alpha > 1$), this can be true only if $d_{\mu,K}((2n-1)-\alpha) = d_{\mu,K}(2n-2\alpha)$. So for $\beta < 2n-2$, $d_{\mu,K}(\beta) = 0$, and thus $d_{\mu,E}(\beta) = 0$ for all $\beta < 2n-2$. This shows us that a pluriharmonic measure on the sphere can't have dimension less than $2n-2$.

As any Koranyi ball of radius r can be covered by $([\frac{1}{r}]+1)^{2n-2}$ balls of radius r^2, for any measure is valid the inequality $d_{\mu,K}(2n-2\alpha) \le d_{\mu,E}(2n-1-\alpha)$. This gives the equality $d_{\mu,K}(2n-2\alpha) = d_{\mu,E}(2n-1-\alpha)$ for any pluriharmonic measure.

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References

[1] Aleksandrov, A.B. : Function Theory in the Ball, Springer-Verlag, 1994

[2] Besikovitch, A.S.: A general form of the covering principle and relative differentiation of additive functions. - Proc.Cambridge Phil.Soc. 41, 1945, 103-110

[3] Besikovitch, A.S.: A general form of the covering principle and relative differentiation of additive functions. - Proc.Cambridge Phil.Soc. 42, 1946, 1-10

[4] Kahane,J.-P.: Désintegration des mesures selon la demension. - C.R.Acad.Sci., Paris,Ser.I,306:107-110,1988

[5] Korányi, A.; Reimann,H.M.: Foundations for the theory of Quasiconformal mappings on the Heisenberg Group. - Advances in Mathematics, Vol. 111, No.1, 1995, 1-88

[6] Watson, N.A.: Applications of geometric measure theory to the study of Gauss-Weierstrass and Poisson integrals. - Annales Academiæ Scientiarum Fennicæ, Series A.I.Mathematica, Vol.19, 1994, 115-132

[7] Ph.Jaming;M.Roginskaya (in preparation)