A NEW APPROACH TO INTEGRAL REPRESENTATION WITH WEIGHTS

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Abstract. We describe a new approach to the Cauchy-Fantappie-Leray formula, and provide a general method to generate weighted integral formulas on complex manifolds.

1. Introduction

Let D be a bounded open subset of the complex plane $\mathbb{C}$. The Cauchy formula

$$f(a) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)dz}{z-a}, \quad a \in D, \quad f \in \mathcal{O}(\overline{D}) \tag{1.1}$$

which expresses $f(a)$ as an superposition of simple rational functions in terms of the boundary values of f, is one of the basic tools in function theory. The kernel is holomorphic in a and works for all domains in $\mathbb{C}$. In several complex variables things are much more difficult. For the local study the simple product formula obtained from (1.1) will often do, but for global problems one often has to find appropriate global representation formulas. Such a formula is provided for any domain by the The Bochner-Martinelli kernel, but unfortunately it is not holomorphic in a unless $n = 1$, and therefore not adequite for all purposes. Much more flexibility is provided by the Cauchy-Fantappie-Leray integral formula, see (2.9) below.

Many problems in function theory can be reformulated as questions about solutions to the equation $\bar\partial v = f$, where f is a $(0,1)$-form. In a domain D in $\mathbb{C}$ it can always be solved in D by the Cauchy formula

$$v(a) = \frac{1}{2\pi i}\int_D \frac{f(z)\wedge dz}{z-a},$$

provided f is integrable in D. One can obtain weighted solution formulas in the following way. Let

$$u(a) = \frac{1}{2\pi i}\frac{dz}{z-a}$$

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and let $g_0(z,a)$ be a smooth function in $D\times D$ such that

$$g_0(z,z)\equiv 1 \quad\text{and}\quad g_0(z,a) \text{ holomorphic in } a. \tag{1.2}$$

Then

$$\bar\partial\int f\wedge g_0u=f, \tag{1.3}$$

if the integrand is integrable over D (locally uniformly in z). Conversely, if (1.3) holds for all $f\in\mathcal{D}(D)$, then

$$\int f\wedge(g_0-1)u$$

is holomorphic for all f, therefore $(g_0-1)/(z-a)$ must be holomorphic in a which is equivalent to (1.2).

If we consider u as a current in D, then $\bar\partial u=[a]$, where $[a]$ denotes the current of point evaluation at the a. Moreover, if g_0 satisfies (1.2), then

$$\bar\partial(g_0u)=[a]-g_1,$$

where

$$2\pi i(z-a)g_1=dz\wedge\bar\partial g_0.$$

If g_0 is smooth up to the boundary, then we have for functions f,

$$f(a)=\int_{\partial D}fg_0u+\int_D fg_1-\int_D\bar\partial f\wedge g_0u,\quad f\in\mathcal{E}(\overline{D}).$$

In particular we get a weighted representation formula for holomorphic functions with holomorphic kernel.

Solution formulas for $\bar\partial$ in strictly pseudoconvex domains D in $\mathbb{C}^n$ were first obtained by Henkin, [12], and can be constructed from the Cauchy-Fantappie-Leray formula with a suitable choice of section. Similar constructions yield formulas for analytic polyhedra and other domains. Extensions to Stein manifolds were obtained in [15], [10], and [8]. Formulas for subvarietes of domains in $\mathbb{C}^n$ were also obtained in [7].

The first constructions and applications of weighted solution formulas for $\bar\partial$ in several variables appeared in [13] and [17]. In [4] was introduced a quite general method to generate weighted integral formulas in domains in $\mathbb{C}^n$, and it was further developed in [7]. An adaption of this method to the ideas in [15] and [10] appeared in [1].

In this paper we describe a new way to obtain weighted formulas which cover the known cases and also extend to complex manifolds. The method is based on a new way to consider Cauchy-Fantappie-Leray formulas that we describe in Section 2. This approach in turn appeared as a by-product of our efforts to construct Taylor's functional calculus for commuting operators by Cauchy-Fantappie-Leray formulas in a way analogous to the wellknown definition by Cauchy's formula for one single operator, [3].

Contents

2. Representation formulas for holomorphic functions

Let $\mathcal{E}_{p,q}(U)$ denote the space of smooth (p,q) forms in the open set $U \subset \mathbb{C}^n$. Fix a point $a \in \mathbb{C}^n$ and let $\delta_{z-a}\colon \mathcal{E}_{p,q}(U) \to \mathcal{E}_{p-1,q}(U)$ be interior multiplication (contraction) with the vector field

$$2\pi i \sum_1^n (z_k - a_k) \frac{\partial}{\partial z_k}.$$

Then $\delta_{z-a} \circ \delta_{z-a} = 0$ and $\delta\bar\partial = -\bar\partial\delta$. Let $\mathcal{L}^m(U) = \oplus_{k=0}^n \mathcal{E}_{k,k+m}(U)$. If $u \in \mathcal{L}^m(U)$ we say that u has degree m, $\deg u = m$, and we let u_k denote its component of bidegree $(k, k+m)$. Thus, e.g., $u \in \mathcal{L}^0(U)$ is $u = u_0 + u_1 + \ldots + u_n$, where u_j has bidegree (j,j) and $u \in \mathcal{L}^{-1}(U)$ is $u = u_1 + \ldots + u_n$, where u_j has bidegree $(j, j-1)$. We have the following readily verified properties,

$$\deg(f \wedge g) = \deg f + \deg g, \tag{2.1}$$

$$\delta_{z-a} - \bar\partial\colon \mathcal{L}^m(U) \to \mathcal{L}^{m+1}(U), \tag{2.2}$$

$$(\delta_{z-a} - \bar\partial)(f \wedge g) = (\delta_{z-a} - \bar\partial) f \wedge g + (-1)^{\deg f} f \wedge (\delta_{z-a} - \bar\partial) g, \tag{2.3}$$

and

$$(\delta_{z-a} - \bar\partial)^2 = 0. \tag{2.4}$$

Example 1. Let

$$b(z) = \frac{1}{2\pi i} \frac{\partial |z-a|^2}{|z-a|^2}$$

in $\mathbb{C}^n \setminus \{a\}$ and let $u_k = b \wedge (\bar\partial b)^{k-1}$, $k = 1, 2, \ldots, n$. Since $\delta_{z-a}\bar\partial b = -\bar\partial\delta_{z-a} b = -\bar\partial 1 = 0$ it follows from (2.3) that

$$\delta_{z-a} u_1, \quad \delta_{z-a} u_{k+1} = \bar\partial u_k. \tag{2.5}$$

If $u = u_1 + \ldots + u_n$, then u is an element in $\mathcal{L}^0(\mathbb{C}^n \setminus \{a\})$ and (2.5) can be written as

$$(\delta_{z-a} - \bar\partial)u = 1. \tag{2.6}$$

□

Because of (2.4) we have a complex

$$\stackrel{\delta_{z-a}-\bar\partial}{\longrightarrow} \mathcal{L}^{m-1}(U) \stackrel{\delta_{z-a}-\bar\partial}{\longrightarrow} \mathcal{L}^m(U) \to . \tag{2.7}$$

Proposition 2.1. *The complex* (2.7) *is exact if* $a \notin \Omega$.

Proof. If $a \notin U$ then, in view of Example 1, we can find $u \in \mathcal{L}^{-1}(U)$ such that $(\delta_{z-a} - \bar\partial)u = 1$. If $f \in \mathcal{L}^m(U)$ and $(\delta_{z-a} - \bar\partial)f = 0$ it follows from (2.1) to (2.4) that $u \wedge f \in \mathcal{L}^{m-1}(U)$ and $(\delta_{z-a} - \bar\partial)(u \wedge f) = f$. □

We can just as well consider currents instead of smooth forms; if we let $\mathcal{L}^m(U) = \oplus_{k=0}^n \mathcal{D}'_{k,k+m}(U)$, then Proposition 2.1 is still true.

Let a be a fixed point in the domain $\Omega \subset \mathbb{C}^n$ and let $u \in \mathcal{L}^{-1}(\Omega \setminus \{a\})$ be any solution to $(\delta_{z-a} - \bar\partial)u = 1$. Then, cf., (2.5), $\bar\partial u_n = 0$. Moreover, if u' is another solution, then there is $w \in \mathcal{L}^{-2}((\Omega \setminus \{a\})$ such that $(\delta_{z-a} - \bar\partial)w = u - u'$, which means that $-\bar\partial w_n = u_n - u'_n$. Thus u_n defines a Dolbeault cohomology class of bidegree $(n, n-1)$ in in $\Omega \setminus \{a\}$ that we denote ω_{z-a}.

Remark 1. Each representative for the class ω_{z-a} actually occurs as u_n for some solution u to (2.6). In fact, if ϕ is any $(n, n-2)$-form in $\Omega \setminus \{a\}$, and v solves (2.6), then $u = v - (\delta_{z-a} - \bar\partial)\phi$ solves (2.6) as well and $u_n = v_n + \bar\partial\phi$. □

We claim that

$$f(a) = \int_{\partial D} f(z)\omega_{z-a}, \quad f \in \mathcal{O}(\overline{D}), \tag{2.8}$$

if $a \in D$. In fact, (2.8) does not depend on neither the choice of representative of the class ω_{z-a} nor the domain D containing a, according to Stokes' theorem, so we may assume that the form is $u_n = b \wedge (\bar\partial b)^{n-1}$, cf., Example 1, and D is a ball with radius r centered at a. Then we get the integral

$$\frac{1}{(2\pi i)^n} \int_{|z-a|=r} \frac{f(z) \wedge \partial |z|^2 \wedge (\bar\partial\partial |z|^2)^{n-1}}{|z-a|^{2n}} = \frac{c_n}{r^{2n-1}} \int_{|z-a|=r} f(z) d\sigma(z),$$

$c_n = (n-1)!/2\pi^n$, which is equal to $f(a)$ by the mean value property. Thus each representative for ω_{z-a} gives rise to a representation formula.

Example 2. It is enough to define a representative locally in $\Omega \setminus \{a\}$. Suppose that $\Omega \setminus \{a\} = U_1 \cup U_2$ and that we have solutions $(\delta_{z-a} - \bar\partial)u^j = 1$ in U_j. Then we can find a solution in $\mathcal{L}^{-2}(U_1 \cap U_2)$ to $(\delta_{z-a} - \bar\partial)w = u^1 - u^2$ by Proposition 2.1 (e.g., one can simply take $w = u^2 \wedge u^1$). If $\{\chi, 1-\chi\}$ is a partition of unity subordinate the open cover $\{U_1, U_2\}$, then $v = \chi u^1 + (1-\chi)u^2 - \bar\partial\chi \wedge w$ is a welldefined form in $\mathcal{L}^{-1}(\Omega \setminus \{a\})$, and

$$(\delta_{z-a} - \bar\partial)v = \chi + (1-\chi) - \bar\partial\chi \wedge (u^1 - u^2) + \bar\partial\chi \wedge (u^1 - u^2) = 1.$$

Thus v is a new solution to (2.6), obtained by patching together u^1 and u^2. □

Example 3 (The Cauchy-Fantappie-Leray formula). Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary, $a \in D$, and suppose that s is a solution to $\delta_{z-a}s = 1$ on ∂D. Then $\tilde s = s/\delta_{z-a}s$ solves $\delta_{z-a}\tilde s = 1$ in a neighborhood of ∂D, so $u_n = \tilde s \wedge (\bar\partial \tilde s)^{n-1}$ represents ω_{z-a}. However, $u_n = s \wedge (\bar\partial s)^{n-1}$ on ∂D, so (2.8) implies the classical Cauchy-Fantappie-Leray formula

$$f(a) = \int_{\partial D} f(z) s \wedge (\bar\partial s)^{n-1}, \quad f \in \mathcal{O}(\overline{D}). \tag{2.9}$$

Suppose that D admits a holomorphic support function, i.e., a function $\Gamma(z,a) \in C^\infty(\partial D \times D)$ that is holomorphic in a and nonvanishing, and that we can solve $\delta_{z-a}h(z,a) = \Gamma(z,a)$ holomorphically in $a \in D$. If we take $s = h/\delta_{\zeta-a}h$, then (2.9) is a representation formula where the kernel is holomorphic in a. If D is convex and ρ is a convex defining function we can take $\Gamma(z,a) = 2\pi i \delta_{z-a}\partial\rho(z)$. In particular, if D is the unit ball $\{z;\ |z|^2 - 1 < 0\}$, then $s(z) = (2\pi i)^{-1}(|z|^2 - \bar z \cdot a)^{-1}\partial|z|^2$, and we get the Szegő formula

$$f(a) = c_n \int_{|z|=1} \frac{f(z)d\sigma(z)}{(1-\bar z \cdot a)^n}, \quad f \in \mathcal{O}(\overline{D}).$$

□

Example 4. One can ease the condition that u be smooth. Let χ be a smooth cutoff function which is identically 1 in a neighborhood of the point a. From Stokes' theorem and (2.8) we get that

$$f(a) = -\int \bar\partial\chi \wedge f(z)\omega_{z-a}, \tag{2.10}$$

if f is holomorphic in a neighborhood of the support of χ. One can just as well define the cohomology class ω_{z-a} by a sequence u_k of currents in $\Omega\setminus\{a\}$ satisfying (2.5), and then (2.10) still holds, if ω_{z-a} is represented by the current u_n.

For instance, one can take

$$u_k = \left(\frac{1}{2\pi i}\right)^k \frac{dz_k}{z_k - a_k} \wedge \bar\partial \frac{dz_{k-1}}{z_{k-1} - a_{k-1}} \wedge \ldots \wedge \bar\partial \frac{dz_1}{z_1 - a_1},$$

and obtain the representation formula

$$f(a) = -(2\pi i)^{-1} \int_{z_n} \bar\partial_{z_n}\chi(\ldots, a_{n-1}, z_n) \wedge f(\ldots, a_{n-1}, z_n)\frac{dz_n}{z_n - a_n}.$$

□

In the previous example it is easy to show that u has meaning as a current in Ω and that

$$(\delta_{z-a} - \bar\partial)u = 1 - [a] \tag{2.11}$$

in Ω in the current sense, where $[a]$ is the (n,n)-current of point evaluation at a, (i.e., $\delta_{z-a}u_{k+1} = \bar\partial u_k$ for $k \le n-1$ and $\bar\partial u_n = [a]$). This can also be obtained with a u that is smooth outside a, provided it fulfills a certain growth condition:

Proposition 2.2. *Suppose that $u \in \mathcal{L}^{-1}(\Omega \setminus \{a\})$ such that $(\delta_{z-a} - \bar\partial)u = 1$ in $\Omega \setminus \{a\}$ and*

$$|u_k| \lesssim |z-a|^{-(2k-1)}. \tag{2.12}$$

Then u_k are locally integrable in Ω and (2.11) holds in the current sense in Ω.

A form $u \in \mathcal{L}^{-1}(\Omega \setminus a)$ satisfying (2.12) will be referred to as an *admissible* form. For instance, $u = \sum_1^n b \wedge (\bar\partial b)^{k-1}$, cf., Example 1, is admissible. If both u_1 and u_2 are admissible then they can be pieced together to an admissible form v as in Example 2, provided one takes $w = u^1 \wedge u^2$.

Proof. First let $u_k = b \wedge (\bar\partial b)^{k-1}$. Then for $\phi \in \mathcal{D}(\Omega)$,

$$-\int \bar\partial\phi \wedge u_n = -\lim_{\epsilon\to 0}\int_{|z-a|>\epsilon} \bar\partial\phi \wedge u_n = \lim_{\epsilon\to 0}\int_{|z-a|=\epsilon} \phi \wedge u_n =$$

$$= \lim_{\epsilon\to 0} c_n \epsilon^{-2n+1}\int_{|z-a|=\epsilon} \phi\, d\sigma(z) = \phi(a).$$

Thus $\bar\partial u_n = [a]$. If u' is admissible and $w = u \wedge u'$, then $\bar\partial w_n = u'_n - u_n$ and $w_n = \sum u_k \wedge u'_{n-k} = \mathcal{O}(|z-a|^{-(2n-2)})$ so by Stokes' theorem,

$$\int_{|z-a|>\epsilon} \bar\partial\phi \wedge u'_n = \int_{|z-a|>\epsilon} \bar\partial\phi \wedge u_n - \int_{|z-a|=\epsilon} \bar\partial\phi \wedge w_n,$$

and the last integral tends to zero as $\epsilon \to 0$. ☐

Example 5. Let $\xi(z)$ be a smooth $(1,0)$-form in Ω such that $|\xi(z)| \le C|z-a|$ and $|\delta_{z-a}\xi(z)| \ge C|z-a|^2$, and let $s = \xi/\delta_{z-a}\xi$. Then $|s| \lesssim |z-a|^{-1}$ and $|\bar\partial s| \lesssim |z-a|^{-2}$, and hence $u = \sum s \wedge (\bar\partial s)^{k-1}$ is an admissible form. ☐

In the following example we will describe how one can introduce weight factors. In particular, we recover the representation formulas from [4] and [7].

Example 6 (Weighted representation formulas). Let a be a fixed point in $D \subset \mathbb{C}^n$. Suppose that we have a smooth form $g \in \mathcal{L}^0(D)$ such that $(\delta_{z-a} - \bar\partial)g = 0$ and $g_0(a) = 1$. Moreover, assume that $(\delta_{z-a} - \bar\partial)u = 1 - [a]$ in D. Then

$$(\delta_{z-a} - \bar\partial)(g \wedge u) = g \wedge (1 - [a]) = g_n - g_0[a] = g_n - [a]$$

since $g_0(a) = 1$. If

$$K = (g \wedge u)_n = \sum_{k=1}^{n} u_k \wedge g_{n-k}, \quad \text{and} \quad P = g_n,$$

we get that $\bar\partial K = [a] - P$ in the current sense. If u and g are smooth up to the boundary (u except for the point a), then for any function f that is smooth up to the boundary, we have the Koppelman formula

$$f(a) = \int_{\partial D} fK + \int_D fP - \int_D \bar\partial f \wedge K, \qquad f \in \mathcal{E}(\overline{D}). \tag{2.13}$$

If f is holomorphic, then (2.13) is reduced to

$$f(a) = \int_{\partial D} f \wedge K + \int_D fP, \qquad f \in \mathcal{O}(\overline{D}). \tag{2.14}$$

A simple choice of g is as follows. Let q be a smooth $(1,0)$-form in D and assume that $G(\lambda)$ is holomorphic on the image of $z \mapsto \delta_{z-a}q(z)$ and $G(0) = 1$. Then we can take

$$g = \sum_{k=0}^{n} (-1)^k G^{(k)}(\delta_{z-a}q)(\bar\partial q)^k/k!. \tag{2.15}$$

In fact, since $\delta_{z-a}\bar\partial q = -\bar\partial\delta_{z-a}q$ it follows that $(\delta_{z-a} - \bar\partial)g = 0$. One can also have several weights. Given $(\delta_{z-a} - \bar\partial)$-closed forms $g^1, \dots, g^m$ in $\mathcal{L}^0(D)$ one obtain a new one as $g = g^1 \wedge g^2 \wedge \dots \wedge g^m$. □

Remark 2. Note that (2.15) is the formal Taylor expansion of $G(\delta_{z-a}q - \bar\partial q)$ at the point $\delta_{z-a}q$. For instance, if $G(\lambda) = (1+\lambda)^r$, then by the binomial theorem

$$g = \sum_{k=0}^{n} \frac{\Gamma(r-1)}{\Gamma(r-k-1)\Gamma(k-1)}(1 + \delta_{z-a}q)^{r-k}(-\bar\partial q)^k.$$

One can also take $G(\lambda_1, \dots, \lambda_m)$ depending on several variables, with $G(0) = 0$, and get

$$g = G(\delta_{z-a}q^1, \dots, \delta_{z-a}q^m) =$$
$$= \sum_{|\alpha| \le n} G^{(\alpha)}(\delta_{z-a}q^1, \dots, \delta_{z-a}q^m)(-1)^{|\alpha|}(\bar\partial q^1)^{\alpha_1} \wedge \dots \wedge (\bar\partial q^m)^{\alpha_m}/\alpha!.$$

□

Remark 3. In view of Example 2 it is easy to see that in formula (2.14) one can let $K = (u \wedge g)_n$, where u is any solution to $(\delta_{z-a} - \bar\partial)u = 1$ in a neighborhood of ∂D. □

Example 7. Assume that $g \in \mathcal{L}^0(\overline{D})$ is smooth and $(\delta_{z-a} - \bar\partial)g = 0$. If $v \in \mathcal{L}^{-1}$ solves $(\delta_{z-a} - \bar\partial)v = g$ in a neighborhood of ∂D, then

$$g_0(a) = \int_D g_n + \int_{\partial D} v_n. \tag{2.16}$$

In fact, the boundary integral in (2.16) is independent of the choice of solution v by Proposition 2.1 and Stokes' theorem. Therefore we can take $v = u \wedge g$, where $(\delta_{z-a} - \bar\partial)u = 1 - [a]$ on $\overline{D}$, and hence (2.16) follows as in Example 6. In particular, if g has compact support in D, then one can take $v \equiv 0$. □

Example 8 (Weighted formulas in strictly pseudoconvex domains). If $D = \{\rho < 0\}$ is strictly pseudoconvex and ρ is a strictly plurisubharmonic defining function, then one can find a function $\Gamma(z,a)$, see [11], which is smooth in a neighborhood of the closure of $D \times D$, holomorphic in a, and such that moreover $2\mathrm{Re}\,\Gamma(z,a) \geq \rho(z) - \rho(a) + \delta|z-a|^2$ for some $\delta > 0$. Let $v(z,a) = \Gamma(z,a) - \rho(\zeta)$, then

$$2\mathrm{Re}\, v(z,a) \geq -\rho(z) - \rho(a) + \delta|z-a|^2.$$

Moreover, there is a solution h to $\delta_{z-a}h = \Gamma$ on the closure of $D \times D$ that depends holomorphically on a. If we take $q = -h/\rho$ and $G(\lambda) = (\lambda+1)^{-r}$, $r > 0$, in Example 6, then the boundary integral in (2.13) will vanish, and we get the weighted formula

$$f(a) = \int_D P_r(z,a) f(z), \qquad f \in \mathcal{O}(\overline{D}),$$

where

$$P_r = g_n = \begin{pmatrix} -r \\ n \end{pmatrix} (1 + \delta_{z-a}q)^{-r-n}(-\bar\partial q)^n = c_{r,n}\frac{(-\rho(z))^{r-1}\alpha(z,a)}{v(z,a)^{n+r}},$$

and $\alpha = \rho(\bar\partial h)^n - n(\bar\partial h)^{n-1} \wedge \bar\partial\rho \wedge h$ is smooth and holomorphic in a. When $r \searrow 0$ we get back the Cauchy-Fantappie-Leray formula with $s = h/\delta_{z-a}h$. □

Example 9 (Weighted Cauchy-Weyl formulas). Let D be an analytic polyhedron, i.e., assume that we have functions $\psi_1, \dots, \psi_m$, that are holomorphic in a neighborhood Ω of $\overline{D}$ such that $D = \{z \in \Omega;\ |\psi_j(z)| < 1,\ j = 1, \dots, m\}$. By Hefer's theorem we can find $(1,0)$-forms h_j, holomorphic in z as well as a, such that $\delta_{z-a}h_j = \psi_j(z) - \psi_j(a)$ for z and a in a neighborhood of $\overline{\Omega}$. Taking

$$q_j(z) = \frac{\overline{\psi_j(z)}}{1 - |\psi_j(z)|^2} h_j(z,a), \quad \text{thus} \quad \bar\partial q_j = \frac{\overline{\partial\psi_j} \wedge h_j}{(1 - |\psi_j(z)|^2)^2},$$

and $G_j(\lambda_j) = (1+\lambda_j)^{-r_j}$ we get that

$$g^j = \left(\frac{1 - |f_j(z)|^2}{1 - \overline{f_j(z)}f_j(a)}\right)^{r_j} + r_j \frac{(1 - |\psi_j(z)|^2)^{r_j-1}}{(1 - \overline{\psi_j(z)}\psi_j(a))^{r_j+1}} \bar\partial\psi_j(\zeta) \wedge h_j$$

is a $(\delta_{z-a} - \bar\partial)$-closed form in $\mathcal{L}^0(\Omega)$, and $g^1 \wedge \ldots \wedge g^m$ will vanish on ∂D if $r_j > 1$. For $f \in \mathcal{O}(\overline{D})$ we therefore get the weighted Weyl formula

$$f(a) = \int_{\partial D} P(z,a) f(z), \quad a \in D, \tag{2.17}$$

where

$$P = (g^1 \wedge \ldots \wedge g^m)_n =$$

$$c \sum_{|I|=n}' \prod_{j \notin I} \left(\frac{1 - |\psi_j(z)|^2}{1 - \overline{\psi_j(z)}\psi_j(a)} \right)^{r_j} \bigwedge_{\ell \in I} r_\ell \frac{(1 - |\psi_\ell(z)|^2)^{r_j - 1}}{(1 - \overline{\psi_\ell(z)}\psi_\ell(a))^{r_j+1}} \wedge \bar\partial\psi_\ell \wedge h_\ell.$$

By analytic continuation in r_j (2.17) must hold for $r_j > 0$. If $\partial\psi_I \wedge \ldots \wedge \partial\psi_{I_n} \neq 0$ on $D_I = \{z \in \overline{D};\ |\psi_{I_1}(z)| = \cdots = |\psi_{I_n}(z)| = 1\}$ for each multiindex I of length n, then we can let $r_j \searrow 0$ and recover the classical Cauchy-Weyl formula

$$f(a) = c \sum_{|I|=n}' \int_{D_I} f(z) \bigwedge_{\ell \in I} \frac{\bar\partial\psi_\ell \wedge h_\ell}{\psi_\ell(z) - \psi_\ell(a)}.$$

□

For some interesting applications of weighted representation formulas for holomorphic functions, see the survey article [6], and for a historical background and applications of the Cauchy-Fantappie-Leray formula, see [14].

3. Locally defined Cauchy-Fantappie-Leray formulas

It is possible to piece together locally defined CFL formulas to global formulas, see, e.g., Section 3 in [15], or [2]. We shall do this by a generalization of the simple observation in Example 2. Assume that a is a fixed point in $\Omega \subset \mathbb{C}^n$, let $\mathcal{U} = \{U_j\}$ be an open cover of $\Omega \setminus \{a\}$, and let ϵ_j be a formal basis. We consider the exterior algebra over the vector space generated by all covectors and ϵ_j. A section f that can be written as

$$f = \sum_{|I|=k+1}' f_I \wedge \epsilon_I,$$

where $\epsilon_I = \epsilon_{I_0} \wedge \ldots \wedge \epsilon_{I_k}$ and f_I are ordinary forms in $U_I = U_{I_0} \cap \ldots \cap U_{I_k}$, will be referred to as a k-cochain of forms. Let C^k denote the space of all k-cochains; thus C^{-1} is just the space of global forms. The coboundary operator $\rho\colon C^k \to C^{k+1}$ is defined as

$$\rho u = \sum_j \epsilon_j \wedge u.$$

Let ϕ_j be a partition of unity subordinate to the cover $\mathcal{U}$. We can define a mapping $\Phi\colon C^{k+1} \to C^k$ as interior multiplication with $\sum \phi_j \epsilon_j^*$, if ϵ_j^* is the dual basis of ϵ_j. By standard multilinear algebra we have that $\rho \circ \rho = 0$, $\Phi \circ \Phi = 0$, and

$$\rho\Phi + \Phi\rho = I. \tag{3.1}$$

The operators $\bar\partial$ and δ_{z-a} extend to C^k in the obvious way and

$$(\delta_{z-a}-\bar\partial)\rho=-\rho(\delta_{z-a}-\bar\partial). \tag{3.2}$$

Moreover, for the mapping $\bar\partial\Phi$, defined as interior multiplication by $\sum_j\bar\partial\phi_j\wedge\epsilon_j$, we have the relations

$$(\bar\partial\Phi)\rho=\rho(\bar\partial\Phi) \quad \text{and} \quad (\delta_{z-a}-\bar\partial)(\bar\partial\Phi)=(\bar\partial\Phi)(\delta_{z-a}-\bar\partial). \tag{3.3}$$

In fact, $u=\rho\Phi u+\Phi\rho u$ and so $\bar\partial u=-\rho(\bar\partial\Phi)u+\rho\Phi\bar\partial u+(\bar\partial\Phi)\rho u+\Phi\rho\bar\partial u$, and in view of (3.1) we get the first equality in (3.3). The second one is obvious.

Now suppose that we have solutions u_j to $(\delta_{z-a}-\bar\partial)u_j=1$ in U_j, and define the 0-cochain $w_1=-\sum_j u_j\wedge\epsilon_j$ so that $(\delta_{z-a}-\bar\partial)w_1=-\rho 1$ and $\Phi w_1=\sum\phi_j u_j$. Since $(\delta-\bar\partial)$ anticommmutes with ρ, we can recursively find w_k in C^{k-1} such that

$$(\delta_{z-a}-\bar\partial)w_1=-\rho 1, \quad (\delta_{z-a}-\bar\partial)w_{k+1}=-\rho w_k, \quad k\geq 1. \tag{3.4}$$

Proposition 3.1. *If w_k are defined in this way, then*

$$v=\sum_{k=1}^{n}\Phi(\bar\partial\Phi)^{k-1}w_k \tag{3.5}$$

is a global element in $\mathcal{L}^{-1}(\Omega\setminus\{a\})$ and $(\delta_{z-a}-\bar\partial)v=1$.

Proof. Using the relations (3.2) and (3.3) we get

$$(\delta-\bar\partial)\sum_1^\infty\Phi(\bar\partial\Phi)^{k-1}w_k=-\sum_1^\infty\Phi(\bar\partial\Phi)^{k-1}(\delta-\bar\partial)w_k-\sum_1^\infty(\bar\partial\Phi)^k w_k=$$
$$=\Phi\rho 1+\sum_2^\infty\Phi(\bar\partial\Phi)^{k-1}\rho w_{k-1}-\sum_1^\infty(\bar\partial\Phi)^k w_k=1,$$

since

$$\Phi(\bar\partial\Phi)^{k-1}\rho w_{k-1}=\Phi\rho(\bar\partial\Phi)^{k-1}w_{k-1}=(\bar\partial\Phi)^{k-1}w_{k-1}.$$

□

Notice that $v=u_j$ where $\phi_j\equiv 1$, so v is obtained by piecing together the locally defined forms u_j. This will be even more clear if we specify the choices of w_k as in the next example.

Example 10. If $u=-\sum u_j\wedge\epsilon_j$ is given, we can take

$$w_k=u^k/k!=(u\wedge u\wedge\ldots\wedge u)/k!.$$

In fact, since u has even total degree,

$$(\delta-\bar\partial)w_{k+1}=(\delta-\bar\partial)u^{k+1}/(k+1)!=$$
$$=\big((\delta-\bar\partial)u\big)\wedge u^k/k!=-\rho 1\wedge u^k/k!=-\rho u^k/k!=-\rho w_k.$$

By this choice of w_k, a simple computation reveals that

$$v = \sum_{k=1}^{n} \Phi u \wedge (\bar\partial \Phi u)^{k-1}, \tag{3.6}$$

where $\Phi u = \sum_j \phi_j u_j$ and $\bar\partial \Phi u = \sum_j \bar\partial \phi_j \wedge u_j$. Thus we can write

$$v = \sum_{k=1}^{n} \sum_{|J|=k} \pm \phi_{J_1} \bar\partial \phi_{J_2} \wedge \ldots \wedge \bar\partial \phi_{J_k} \wedge u_{J_1} \wedge \ldots \wedge u_{J_k}. \tag{3.7}$$

□

One can specify even further.

Example 11 (CFL formulas with locally defined sections). Suppose we have sections s_j in U_j such that $\delta_{z-a} s_j = 1$. Then we can choose $u_j = \sum_{\ell=1}^{n} s_j \wedge (\bar\partial s_j)^{\ell-1}$ and $w_k = u^k/k!$ as in the previous example. In the final representation formula for holomorphic functions (as long as we do not want to introduce weights as well) only the component v' of v, cf., (3.5), of bidegree $(n, n-1)$ comes into play, and from (3.7) we get that

$$v' = \sum_{k=1}^{n} \sum_{|J|=k} \pm \phi_{J_1} \bar\partial \phi_{J_2} \wedge \ldots \wedge \bar\partial \phi_{J_k} \wedge H(s_{J_1}, \ldots, s_{J_k}), \tag{3.8}$$

where

$$H(s_1, \ldots, s_k) = s_1 \wedge \ldots \wedge s_k \wedge \sum_{|\alpha|=n-k} (\bar\partial s_1)^{\alpha_1} \wedge \ldots \wedge (\bar\partial s_k)^{\alpha_k}.$$

This is precisely the formula obtained in [2]. □

Example 12. Suppose that each s_j in the previous example is holomorphic in U_j. Then, using the same notation, $u_j = s_j$, and from (3.6) we get that

$$v = \sum_{k=1}^{n} s \wedge (\bar\partial s)^{k-1},$$

where $s = \Phi u = \sum_j \phi_j s_j$. Thus we get precisely the CFL formula with the form $s = \sum \phi_j s_j$, obtained by simply patching together the local (holomorphic) forms s_j. The $(n, n-1)$-component of v can be written as

$$v' = \sum_{|J|=n} \pm \phi_{J_1} \bar\partial \phi_{J_2} \wedge \ldots \wedge \bar\partial \phi_{J_n} \wedge s_{J_1} \wedge \ldots \wedge s_{J_n}. \tag{3.9}$$

□

Example 13. Let E be a lineally convex compact set in $\mathbb{C}^n$. This means that through each point outside E there is a complex hyperplane that does not intersect E. Let us furthermore assume that $0 \in E$. We want to obtain a representation for $f \in \mathcal{O}(E)$ as a superposition of functions like $\Pi_1^n(1-\alpha_j\cdot z)^{-1}$, where α_j belongs to the dual complement; this means that the hyperplane $1-\alpha_j\cdot z=0$ does not intersect E.

Locally in $\zeta\in\mathbb{C}^n\setminus E$ we can find a form $\alpha(\zeta)$ such that $\delta_{\zeta-a}\alpha(\zeta)\neq 0$ for all $a\in E$, and after normalization we may assume that $\delta_\zeta\alpha(\zeta)=1$, so that $\delta_{z-a}\alpha(\zeta)=1-\alpha(\zeta)\cdot a$. Now

$$s(\zeta,a)=\alpha(\zeta)/\delta_{\zeta-a}\alpha(\zeta)$$

is defined locally in ζ for all $a\in E$ and $\delta_{\zeta-a}s(\zeta,a)=1$. Hence (3.8) yields a kernel $K(\zeta,a)$ such that

$$f(a)=\int_{\partial D}K(\zeta,a)f(\zeta),\quad a\in E, \tag{3.10}$$

if $D\supset E$ and $f\in\mathcal{O}(\overline{D})$. Moreover, $K(\zeta,z)$ is a locally finite sum of terms like

$$\frac{\gamma(\zeta)}{\Pi_{j=1}^n(1-\alpha_j(\zeta)\cdot a)}$$

so (3.10) indeed provides the desired decomposition of f.

The decomposition is particularily simple to derive if we first notice that $\alpha_j(\zeta)$ can be chosen to be holomorphic in ζ if the open cover is fine enough. In fact, if $\delta_{\zeta^0-a}\alpha^0\neq 0$ for all $a\in E$, then by continuity we can take $\alpha(\zeta)=\alpha^0/\delta_{\zeta^0}\alpha^0$ in a neighborhood of ζ^0. The resulting form $s(\zeta,a)$ is then holomorphic in ζ, and hence we can obtain $K(\zeta,z)$ from (3.9). □

4. Integral representation of higher order forms

We shall now allow more general singular sets than just a point. We start with a quite general setup and let V be the zero set of a holomorphic section η to a holomorphic vector bundle E over a complex manifold X. We let E^* denote the dual bundle and consider the exterior algebra over $E\oplus E^*\oplus T^*$. Suppose that E has rank r and let $e_1,\dots,e_r$ be a local holomorphic frame for E, and let e_j^* be the dual frame for E^*. We let $\mathcal{E}_{p,q}^{k,\ell}(U)$ be the space of smooth sections which locally can be written $f=\sum_{|I|=k,|J|=\ell}f_{I,J}\wedge e_I^*\wedge e_J$, where $f_{I,J}$ are (p,q)-forms in the open subset U of X. Each smooth $A\in\mathcal{E}(U,\mathrm{Hom}\,(E)$ can be identified by a section $\tilde{A}\in\mathcal{E}_{0,0}^{1,1}(U)$ by requiring that $A\xi$ is equal to the contraction of $\tilde{A}$ by the section ξ to E. In particular, the identity mapping $E\to E$ corresponds to the section

$$\tilde{I}=\sum_j e_j^*\wedge e_j,$$

and we have the formula

$$\det(A)\tilde{I}^r=\tilde{A}^r.$$

The $\bar\partial$-operator extends to a mapping $\bar\partial\colon \mathcal{E}^{\ell,k}_{p,q}(U)\to\mathcal{E}^{\ell,k}_{p,q+1}(U)$ in the usual way. If δ_η denotes contraction with the section $2\pi i\eta$, we get a complex

$$\to \mathcal{E}^{\ell,k}_{p,q}(U)\xrightarrow{\delta_\eta}\mathcal{E}^{\ell-1,k}_{p,q}(U)\to . \tag{4.1}$$

Let $\mathcal{L}^m(U)=\oplus_{k=0}^p\mathcal{E}^{k,0}_{0,k+m}(U)$. Analogues of (2.1) to (2.4) clearly hold even in this setting.

Example 14. Let ξ be the dual section of η in E^* with respect to some hermitean metric on E. Then $s=\xi/\delta_\eta\xi$ is defined outside V and $\delta_\eta(s\wedge f)=f$ if $\delta_\eta f=0$. Thus (4.1) is exact if U does not intersect V. Moreover, if $u=\sum_1^r s\wedge(\bar\partial s)^r$, then $u\in\mathcal{L}^{-1}(X\setminus V)$ and $(\delta_\eta-\bar\partial)u=1$. If $f\in\mathcal{L}^m((X\setminus V)$ and $(\delta_\eta-\bar\partial)f=0$, therefore $(\delta_\eta-\bar\partial)(u\wedge f)=f$. $\square$

As before we thus have a complex

$$\xrightarrow{\delta_\eta-\bar\partial}\mathcal{L}^{m-1}(U)\xrightarrow{\delta_\eta-\bar\partial}\mathcal{L}^m(U)\xrightarrow{\delta_\eta-\bar\partial} \tag{4.2}$$

which is exact if $U\subset X\setminus V$. For $u\in\mathcal{L}^m$ we let u_k denote the component in $\mathcal{E}^{k,0}_{0,k+m}$. If $u\in\mathcal{L}^{-1}(X\setminus V)$ and $(\delta_\eta-\bar\partial)u=1$, then u_r defines a Λ^rE^*-valued Dolbeault cohomology class of bidegree $(0,r-1)$ in the same way as before.

For the rest of this section we assume that E is trivial and that e_j is a global holomorphic frame. Let $\eta=\sum\eta_je_j$ and $d\eta=\sum d\eta_j\wedge e_j$. If $u_r\in\mathcal{E}^{p,0}_{0,q}$, then there is a unique (p,q)-form $\hat u_r$ such that

$$\hat u_r\wedge\tilde I^r=(-1)^ru_r\wedge(d\eta)^r. \tag{4.3}$$

Lemma 4.1. *If $u_r=\varphi\wedge e_1^*\wedge\ldots\wedge e_r^*$, where φ is a $(0,q)$-form, then $\hat u_r=\varphi\wedge d\eta_1\wedge\ldots\wedge d\eta_r$.*

Thus $\hat u_r$ is obtained from u_r just by replacing each occurence of e_j^* by $d\eta_j$.

Proof. We have that

$$u_r\wedge(d\eta)^r=\phi\wedge e_1^*\wedge\ldots\wedge e_r^*\wedge r!c_rd\eta_1\wedge\ldots d\eta_r\wedge e_1\wedge\ldots\wedge e_r,$$

where c_r is 1 or -1, only depending on r. However, this is equal to

$$(-1)^r\phi\wedge d\eta_1\wedge\ldots\wedge d\eta_r\wedge r!c_re_1^*\wedge\ldots\wedge e_r^*\wedge e_1\wedge\ldots\wedge e_r$$
$$=(-1)^r\phi\wedge d\eta_1\wedge\ldots d\eta_r\wedge\tilde I^r.$$

$\square$

We can define a cohomology class ω_η in $X\setminus V$ of bidegree $(r,r-1)$ precisely as before by taking a solution $u\in\mathcal{L}^{-1}(X\setminus V)$ to $(\delta_\eta-\bar\partial)u=1$ and let ω_η be defined by $\hat u_r$. In fact, if v is another solution, then we have $w\in\mathcal{L}^{-2}(X\setminus V)$ such that $(\delta_\eta-\bar\partial)w=u-v$, and hence $\bar\partial w_r=v_r-u_r$. This implies, cf., (4.3), that $\bar\partial\hat w_r=\hat v_r-\hat u_r$, since $\bar\partial d\eta=0$. As before, we can have currents instead of smooth forms.

Example 15. Let E be a trivial rank n bundle, $X = \Omega \subset \mathbb{C}^n$, and let $\eta = \sum(z_j - a_j)e_j$. Then the class ω_η coincides with the previously defined class ω_{z-a} in Section 2. □

Lemma 4.2. *Assume that $(d\eta)^r \neq 0$ on V. Let $u \in \mathcal{L}^{-1}(X \setminus V)$ such that $(\delta_\eta - \bar\partial)u = 1$ and*

$$|u_k| = \mathcal{O}(|\eta|^{-(2k-1)}). \tag{4.4}$$

Then

$$\bar\partial \hat u_r = [V] \tag{4.5}$$

in the current sense in X.

Here $[V]$ denotes the current of integration over the submanifold $[V]$.

Proof. The condition means that $d\eta_1 \wedge \ldots \wedge d\eta_r \neq 0$ on V, so V is a regular submanifold and locally $\eta_1, \ldots, \eta_r$ is part of a holomorphic coordinate system. Note that the statement is local. First let $s = \sum_j \bar\eta_j/|\eta|^2$, and $u = \sum s \wedge (\bar\partial s)^{k-1}$. Then $\hat u_r = \hat s \wedge (\bar\partial \hat s)^{r-1}$, where $\hat s = \sum \bar\eta_j d\eta_j/|\eta|^2$, according to Lemma 4.2. One can then proceed exactly as in the proof of Proposition 2.2. □

We say that a solution to $(\delta_\eta - \bar\partial)u = 1$ in $X \setminus V$ is admissible if (4.4) holds. Notice that u in Example 14 is admissible. If u is admissible, then by the lemma

$$(-1)^r[V] \wedge \tilde I^r = \bar\partial \hat u_r \wedge \tilde I^r = \bar\partial(u_r \wedge (d\eta)^r) = \bar\partial u_r \wedge (d\eta)^r, \tag{4.6}$$

and hence

$$\bar\partial u_r \wedge (d\eta)^r = (-1)^r[V] \wedge \tilde I^r. \tag{4.7}$$

Remark 4 (Complete intersections). Suppose that V is a complete intersection, i.e., the codimension of the variety V is r. If u is as in Example 14, then we still have that $\bar\partial \hat u_r = [V]$ if $[V]$ denotes the current of integration over V taking into account the multiplicities of the irreducible components of V. One can also define u analogously to Example 4. For proofs of these facts and a general discussion of residue currents, we refer to [16]. Several formulas below have counterparts for general complete intersections, but in this paper we restrict to the simplest case when V is a regular manifold. □

Example 16 (Weighted integral formulas in $\mathbb{C}^n$). Let Ω be a domain in $\mathbb{C}^n$ and let (ζ, z) be the standard coordinates. Let E be a trivial rank n bundle over $\Omega \times \Omega$ with global frame $e_1, \ldots, e_n$ and $\eta = \sum_j(\zeta_j - z_j)e_j$, so that V is the diagonal $[\Delta]$ in $\Omega \times \Omega$. Assume that u is admissible so that (4.5) holds. Let $g \in \mathcal{L}^0(\Omega \times \Omega)$ be a smooth form such that $(\delta_\eta - \bar\partial)g = 0$ and $g_0 = 1$ on Δ. Then by (4.7),

$$(\delta_\eta - \bar\partial)(u \wedge g \wedge (d\eta)^r) = g \wedge \big((d\eta)^r - (-1)^r[\Delta] \wedge \tilde I^r\big)$$

so if $K = \widehat{(u \wedge g)}_r$ and $P = \hat{g}_r$, we get

$$\bar\partial K = [\Delta] - P. \tag{4.8}$$

As usual (4.8) gives rise to a Koppelman formula for smooth forms f,

$$f(z) = \bar\partial_z \int_D f \wedge K_{p,q-1} + \\ + \int_{\partial D} f \wedge K_{p,q} + \int_D f P_{p,q} - \int_D \bar\partial f \wedge K_{p,q}, \quad f \in \mathcal{E}_{p,q}(\overline{D}), \tag{4.9}$$

where $K_{p,q}$ and $P_{p,q}$ denote the components of K and P which have bidegree (p,q) in dz. Suppose that f is a (p,q)-form, $P_{p,q} = 0$, and that $K_{p,q}$ vanishes for $\zeta \in \partial D$. Then (4.9) becomes a homotopy formula for the $\bar\partial$-complex, i.e., we get operators $\mathcal{K}\colon \mathcal{E}_{p,*+1} \to \mathcal{E}_{p,*}$ and $\mathcal{P}\colon \mathcal{E}_{p,0} \to \Lambda^p \mathcal{O}$, such that

$$\bar\partial \mathcal{K} + \mathcal{K}\bar\partial = I - \mathcal{P}.$$

Concretely this can be achieved in the following way. Let q be a section to E^* and let $G(\lambda)$ be holomorphic on the image of $(\zeta, z) \mapsto \delta_\eta q$. Then, cf., Example 16,

$$g = \sum_{k=0}^{r} (-1)^k G^{(k)}(\delta_{z-a} q)(\bar\partial q)^k / k!$$

is a smooth form in $\mathcal{L}^0(\Omega \times \Omega)$ and $(\delta_\eta - \bar\partial) g = 0$. The resulting Koppelman formula is pecisely the one obtained in [4]. If q is holomorphic in z, then $P_{p,q} = 0$ for $q \geq 1$. If moreover, $u = \sum_1^n s \wedge (\bar\partial s)^{k-1}$ and $s(\zeta, z)$ is holomorpic in z when $\zeta \in \partial D$, then the boundary integral will vanish as well. This can be obtained if D admits a holomorphic support function. Another way to get rid of the boundary integral is to replace g by $g \wedge g^1$ where $g^1 \in \mathcal{L}^0(\Omega \times \Omega)$ is $(\delta_\eta - \bar\partial)$ closed and vanishes on ∂D. For the case when D is strictly pseudoconvex, see, e.g., [4], or Section 6 below. □

5. Integral representation on manifolds

When V is the zero set of some section η to a nontrivial bundle E, then in general a formula like (4.5) cannot hold, because $[V]$ is a representative of the Chern class $c_r(E)$ of top degree, which may be nontrivial. Let D be a holomorphic connection on the bundle E. Then $D = D' + \bar\partial$, where D' maps E-valued (p,q)-forms into E-valued $(p+1,q)$-forms. Moreover, $D^2 = \bar\partial D = \Theta$, the curvature tensor, which is a Hom (E)-valued $(1,1)$-form. Hence it defines a section $\tilde\Theta \in \mathcal{E}^{1,1}_{1,1}$. The connection D has a natural extension to a mapping on $\mathcal{E}^{k,\ell}_{p,q}$ and it is well-known that $D\tilde{\Theta}=0$. Since η is holomorphic, we have that $\bar\partial D\eta=\tilde{\Theta}\eta$, and since $\delta_\eta\tilde{\Theta}=2\pi i\Theta\eta$ we have

$$(\delta_\eta-\bar\partial)(D\eta-\frac{i}{2\pi}\tilde{\Theta})=0,$$

and hence

$$(\delta_\eta-\bar\partial)\big(u\wedge(D\eta-\frac{i}{2\pi}\tilde{\Theta})^r\big)=(D\eta-\frac{i}{2\pi}\tilde{\Theta})^r \tag{5.1}$$

if u is a solution to $(\delta_\eta-\bar\partial)u=1$ in $\mathcal{L}^{-1}(X\setminus V)$ as before. Let us define K such that $(-1)^rK\wedge\tilde{I}^r$ is equal to the component of the form on the left hand side of (5.1) that contains the factor $\tilde{I}^r$; i.e.,

$$(-1)^rK\wedge\tilde{I}^r=\sum_{k=0}^{r-1}\begin{pmatrix} r\\ k\end{pmatrix}u_{r-k}\wedge(D\eta)^{r-k}\wedge(-1)^k\left(\frac{i}{2\pi}\tilde{\Theta}\right)^k. \tag{5.2}$$

Since

$$\left(\frac{i}{2\pi}\tilde{\Theta}\right)^r=\det\left(\frac{i}{2\pi}\Theta\right)\wedge\tilde{I}^r=c_r(D)\wedge\tilde{I}^r,$$

$c_r(D)$ being the Chern form of top degree associated to the connection D, it follows from (5.1) that

$$\bar\partial K=-c_r(D) \tag{5.3}$$

outside V.

Theorem 5.1. *Assume that $(D\eta)^r\neq 0$ on V and let u be an admissible solution to $(\delta_\eta-\bar\partial)u=1$. If K is defined by* (5.2)*, then*

$$\bar\partial K=[V]-c_r(D)$$

in the current sense in X.

Proof. If $\eta=\sum\eta_je_j$ locally, then $D\eta=d\eta+\mathcal{O}(|\eta|)$, where $d\eta=\sum d\eta_j\wedge e_j$. Hence

$$\bar\partial u_r\wedge(D\eta)^r=\bar\partial u_r\wedge(d\eta)^r+(\bar\partial u_r)\wedge\mathcal{O}(|\eta|)=(-1)^r[V]\wedge\tilde{I}^r,$$

according to (4.6). Combined with (5.3) the theorem follows. □

Example 17. Assume that $s=\xi/\delta_\eta\xi$ is a section to E^* over $X\setminus V$ as in Example 14. Then $u=s\wedge(\bar\partial s)^{k-1}$ is admissible. The resulting formula in Theorem 5.1 is a special case of Theorem 2.4 in [8]. The general case of that theorem is covered by Example 20 below. □

It is easy to introduce weight factors even in the manifold case. Let $g\in\mathcal{L}^0(X)$ be smooth and assume that $g_0=1$ on V. Moreover, let $u\in\mathcal{L}^{-1}(X\setminus V)$ be a solution to $(\delta_\eta-\bar\partial)u=1$ in $X\setminus V$. Then

$$(\delta_\eta-\bar\partial)\Big(u\wedge g\wedge(D\eta-\frac{i}{2\pi}\tilde{\Theta})^r\Big)=g\wedge(D\eta-\frac{i}{2\pi}\tilde{\Theta})^r. \tag{5.4}$$

If we assume that u is admissible and identify the components containing the factor $\tilde{I}^r$, we get

Theorem 5.2. *Suppose that $u \in \mathcal{L}^{-1}(X \setminus V)$ is an admissible solution to $(\delta_\eta - \bar\partial)u = 1$. Let K and P be defined by*

$$(-1)^r K \wedge \tilde{I}^r = \sum_{k=1}^{r} \Big(\sum_{\ell+1}^{k} u_\ell \wedge g_{k-\ell} \Big) \wedge \begin{pmatrix} r \\ k \end{pmatrix} (D\eta)^k \wedge (-1)^{r-k} \big(\frac{i}{2\pi} \tilde{\Theta} \big)^{r-k},$$

and

$$(-1)^r P \wedge \tilde{I}^r = \sum_{k=0}^{r} g_k \wedge \begin{pmatrix} r \\ k \end{pmatrix} (D\eta)^k \wedge (-1)^{r-k} \big(\frac{i}{2\pi} \tilde{\Theta} \big)^{r-k}.$$

Then $\bar\partial K = [V] - P$ in the current sense in X.

It is quite easy to find homotopies between various formulas obtained so far. Let us consider a couple of examples.

Example 18. If we have two different holomorphic connections D and D_1 then there is an element $h \in \mathcal{E}_{1,0}(X, \mathrm{Hom}\,(E)) \simeq \mathcal{E}_{1,0}^{1,1}(X)$ such that $D - D_1 = 2\pi i h$, and thus

$$(\delta_\eta - \bar\partial)h = (D\eta - \frac{i}{2\pi}\tilde{\Theta}) - (D_1\eta - \frac{i}{2\pi}\tilde{\Theta}_1).$$

If $g \in \mathcal{L}^0(X)$, $(\delta_\eta - \bar\partial)g = 0$, and

$$\alpha(D, D_1) = \Big(h \wedge \sum_{1}^{r-1} (D\eta - \frac{i}{2\pi}\tilde{\Theta})^j - (D_1\eta - \frac{i}{2\pi}\tilde{\Theta}_1)^{r-1-j} \Big).$$

it follows that

$$(\delta_\eta - \bar\partial)\big(g \wedge \alpha(D, D_1) \big) = g \wedge (D\eta - \frac{i}{2\pi}\tilde{\Theta})^r - g \wedge (D_1\eta - \frac{i}{2\pi}\tilde{\Theta}_1)^r.$$

If β is the form in the left hand side, then

$$(\delta_\eta - \bar\partial)(u \wedge \beta) = u \wedge g \wedge (D\eta - \frac{i}{2\pi}\tilde{\Theta})^r - u \wedge g \wedge (D_1\eta - \frac{i}{2\pi}\tilde{\Theta}_1)^r + \beta$$

in $X \setminus V$. □

Example 19. Suppose that we have two admissible sections u and u' and let K and K' be the corresponding kernels from Theorem 5.2. Then $w = u \wedge u'$ solves $(\delta_\eta - \bar\partial)w = u - u'$ in $X \setminus V$, and thus

$$(\delta_\eta - \bar\partial)\Big(w \wedge g \wedge (D\eta - \frac{i}{2\pi}\tilde{\Theta})^r \Big) = (u - u') \wedge g \wedge (D\eta - \frac{i}{2\pi}\tilde{\Theta})^r,$$

from which we get an explicit form H in $X \setminus V$ such that $\bar\partial H = K - K'$. □

Remark 5. Recall that the Chern form for D is by definition $\det(I + (i/2\pi)\Theta)$. A similar formula as above shows the wellknown fact that the Chern forms for two different holomorphic connections are $\bar\partial$-cohomologous.

In fact, by a similar argument as in Example 18, and with the same h, we have

$$\bar\partial\Big(h\wedge\sum_1^{r-1}(\tilde I+(i/2\pi)\tilde\Theta)^j\wedge(\tilde I+(i/2\pi)\tilde\Theta)^{r-1-j}\Big)=$$

$$\Big(\tilde I+\frac{i}{2\pi}\tilde\Theta\Big)^r-\Big(\tilde I+\frac{i}{2\pi}\tilde\Theta_1\Big)^r.$$

For bidegree reasons one can replace $\bar\partial$ by d. □

We conclude with an example in which we forget about the complex structure and consider a a complex vector bundle over an arbitrary differentiable manifold. We then recover Theorem 2.4 from [8].

Example 20. Let E be complex vector bundle over a differentiable manifold X and let D be any connection on E. Then D has a canonical extension to a mapping $\mathcal{E}_p^{\ell,k}\to\mathcal{E}_{p+1}^{\ell,k}$ (the lower index p denoting the covector degree) such that $D(\phi\wedge\psi)=d\phi\wedge\psi+(-1)^{\deg\phi}\phi\wedge D\psi$ if ϕ is a scalar-valued form and $\psi\in\mathcal{E}_p^{\ell,k}$. Moreover, as before $\Theta=D^2$ defines an element $\tilde\Theta$ in $\mathcal{E}_2^{1,1}$, and the Chern class $c_r(D)$ of top degree is defined by

$$(-1)^r c_r(D)\wedge\tilde I^r=\big(\frac{i}{2\pi}\tilde\Theta\big)^r.$$

Let η be a section to E such that $(D\eta)^r\neq 0$ on $V=\{\eta=0\}$ and let $s=\xi/\delta_\eta\xi$, where ξ is the dual section of ξ with respect to some hermitean metric on E. Moreover, let $u=\sum_1^r s\wedge(Ds)^{k-1}$ and define K by

$$(-1)^r K\wedge\tilde I^r=\sum_{k=1}^r u_k\wedge\left(\begin{array}{c} r\\ k\end{array}\right)(D\eta)^k\wedge(-1)^{r-k}\big(\frac{i}{2\pi}\tilde\Theta\big)^{r-k}.$$

We claim that

$$dK=[V]-c_r(D). \tag{5.5}$$

To see this, first claim that

$$\delta_\eta Ds=-\delta_s D\eta \quad\text{and}\quad 2\pi i D(Ds)=\delta_s\tilde\Theta, \tag{5.6}$$

if δ_s denotes interior multiplication with $2\pi i s$. In fact, since $d\delta_\eta s=d1=0$ we have that $\delta_\eta Ds=-\delta_s D\eta$. Then note that $D^2\eta\cdot s=-\eta\cdot D^2 s$ so $\delta_s\delta_\eta\tilde\Theta=-2\pi i\delta_\eta D(Ds)$, from which the second equality in (5.6) follows. Therefore,

$$(\delta_\eta-D)u=1+\sum_1^r(k-1)s\wedge(Ds)^{k-2}\wedge(\delta_\eta Ds-D(Ds))=$$

$$=1-\alpha\wedge\delta_s(D\eta-\frac{i}{2\pi}\tilde\Theta).$$

Hence,

$$(5.7) \qquad (\delta_\eta - D)(u \wedge (D\eta - \frac{i}{2\pi}\tilde{\Theta})^r) = (D\eta - \frac{i}{2\pi}\tilde{\Theta})^r$$

since $\delta_s \alpha = 0$ and hence

$$\alpha \wedge \delta_s (D\eta - \frac{i}{2\pi}\tilde{\Theta}) \wedge (u \wedge (D\eta - \frac{i}{2\pi}\tilde{\Theta})^r) = \alpha \wedge \frac{1}{r+1}\delta_s (D\eta - \frac{i}{2\pi}\tilde{\Theta})^{r+1} = 0.$$

The equality (5.5) follows from (5.7) outside V. The singularity is handled as before, see also [8]. □

6. Weighted Koppelman formulas on Stein manifolds

Let Ω be a n-dimensional Stein manifold and let E be the pullback of the complex tangent bundle over Ω_ζ to $\Omega_\zeta \times \Omega_z$ under the projection $\pi_\zeta \colon \Omega \times \Omega \to \Omega$, $(\zeta, z) \mapsto \zeta$. It was shown in [15] that E has a holomorphic section η such that

$$\{\eta = 0\} = \Delta \cup F,$$

where Δ is the diagonal in $\Omega \times \Omega$ and F is a closed set disjoint from Δ. In local coordinates near Δ,

$$\eta = \sum_1^n (\zeta_j - z_j) \frac{\partial}{\partial \zeta_j}|_\zeta + \mathcal{O}(|\zeta - z|^2).$$

By Cartan's theorem B there is a holomorphic function ϕ in $\Omega \times \Omega$ such that $\phi = 1$ on Δ and $\phi = \sum \alpha_j \eta_j$ locally, outside Δ, where $\eta = \sum \eta_j e_j$ for some holomorphic frame e_j. As usual we can solve $(\delta_\eta - \bar\partial)u = 1$ in $\mathcal{L}^{-1}(\Omega \times \Omega \setminus (\Delta \cup F))$, and we say that u is admissible if for some integer m, $\phi^m u$ is C^2 outside Δ and (4.4) holds.

Example 21. Let ξ be the dual section in E^* with respect to some metric on E, and let $s = \xi / \delta_\eta \xi$. Since $\delta_\eta \xi \sim |\eta|^2$ and $|\phi| \lesssim |\eta|^2$ it follows that $u = \sum_1^n s \wedge (\bar\partial s)^{k-1}$ is admissible. □

Let $g \in \mathcal{L}^0(\Omega \times \Omega)$ be smooth, and of the form $g = \phi^m g'$, where $g' \in \mathcal{L}^0(\Omega \times \Omega)$ and $(\delta_\eta - \bar\partial)g' = 0$. Furthermore, let u be admissible and let K and P be defined as in Theorem 5.2. Then $\bar\partial K = [\Delta] - P$ in the current sense in $\Omega \times \Omega$, since the factor ϕ^m in g kills the possible singularities on F.

As before, let $K_{p,q}$ and $P_{p,q}$ denote the components of bidegree (p, q) in dz. For a bounded domain $D \subset\subset \Omega$ with smooth boundary we therefore have the Koppelman formula

$$(6.1)\quad f(z) = \bar\partial_z \int_D f \wedge K_{p,q-1} + \\ + \int_{\partial D} f \wedge K_{p,q} + \int_D f P_{p,q} - \int_D \bar\partial f \wedge K_{p,q}, \quad f \in \mathcal{E}_{p,q}(\overline{D}).$$

Let q be a section to E^*, and let $G(\lambda)$ be a function that is holomorphic on the image of $\delta_\eta q$. Then

$$g = \phi^m \sum_{k=0}^{n} G^{(k)}(\delta_\eta q)(\bar\partial q)^k / k!.$$

is in $\mathcal{L}^0(\Omega \times \Omega)$ and $(\delta_\eta - \bar\partial)g = 0$. If q is holomorphic in z, keeping in mind that $\tilde\Theta$ only depends on ζ, it follows that $P_{p,q} = 0$ if $q > 0$. If we can in some way get rid of the boundary integral as well, then we get homotopy formulas for $\bar\partial$ and a representation formula for holomorphic (p,q)-forms with holomorphic kernel.

Remark 6. In [15], as well as in [10], the bundle E is chosen to depend on z instead of ζ. Then $\tilde\Theta$ is a $(1,1)$-form in dz and hence (assuming $g \equiv \phi^m$) $P_{0,q}$ vanishes for all $q > 0$. Thus one obtains homotopy formulas for $\bar\partial$ for $(0,q)$-forms if in addition the boundary integrals vanish. Notice that in this case all terms that involve $\tilde\Theta$ in the definition of K vanishes, so it is not necessary to compute the full formula. It was first computed (without weights) in [8], and so homotopy formulas for (p,q)-forms were obtained. In [10] is used another way to cover the (p,q)-case, see Example 22 below. □

It is now possible to extend a lot of constructions with weighted integral formulas from domains in $\mathbb{C}^n$ to domains in Ω. We will restrict to the case when D is a strictly pseudoconvex domain in Ω with smooth boundary. This is actually a less interesting case beacuse many problems can be localized, and hence handled by formulas in $\mathbb{C}^n$. However, our main objective is to point out a general idea of how one can proceed. We will make use of the following simple observation, which follows from Cartan's theorem B.

Lemma 6.1. *If $\varphi(\zeta, z)$ is a holomorphic function in $\Omega \times \Omega$ which vanishes on Δ, then there is a holomorphic section h to E^* such that $\delta_\eta h = \phi\varphi$.*

By Fornaess' embedding theorem, [11], there is a proper embedding $\psi\colon \Omega \to \mathbb{C}^M_w$ and a strictly convex function $\sigma(w)$ such that $\psi^*\sigma(z)$ is a defining function for D in Ω. Define

$$\Gamma(\zeta, z) = \sum_1^M \frac{\partial\sigma}{\partial w_j}(\psi(\zeta))(\psi_j(z) - \psi_j(\zeta))$$

on $\Omega \times \Omega$. Since σ is strictly convex it follows that

$$2\mathrm{Re}\,\Gamma(\zeta, z) \geq \rho(\zeta) - \rho(z) + \delta|\psi(\zeta) - \psi(z)|^2, \quad (\zeta, z) \in \overline{D} \times \overline{D}.$$

If $v(\zeta, z) = \Gamma(\zeta, z) - \rho(\zeta)$, then $v(\zeta, z)$ is holomorphic in z and

(6.2)

$$2\mathrm{Re}\, v(\zeta, z) \geq -\rho(\zeta) - \rho(z) + \delta|\psi(\zeta) - \psi(z)|^2, \quad v(\zeta, \zeta) = -\rho(\zeta),$$

so $|v|$ defines the usual kind of local Koranyi pseudometric at ∂D. From Lemma 6.1 it follows that we have holomorphic sections h_j to E^* such that $\delta_\eta h_j = \phi(\zeta, z)(\psi_j(\zeta) - \psi_j(z))$. Therefore

$$h(\zeta, z) = \sum_j \frac{\partial \sigma}{\partial w_j}(\psi(\zeta))h_j$$

is a section to E^* which is holomorphic in z and satisfies $\delta_\eta h = \phi\Gamma$, and hence $\delta_\eta h - \phi\rho(\zeta) = \phi v(\zeta, z)$. Let $\check{v}(\zeta, z) = v(z, \zeta)$. In the same way we can find $\check{h}$ such that $\delta_\eta \check{h} - \phi\rho(z) = \phi\check{v}$. Let ξ be the dual section of η in E^* (with respect to some metric on E) and let $s' = |\eta|^2\check{h} - \phi\rho(z)\xi$. Then $\delta_\eta s' = \phi|\eta|^2\check{v}$, so taking $s = s'/\delta_\eta s'$ and $u = \sum s \wedge (\bar\partial s)^{k-1}$ we get an admissible solution to $(\delta_\eta - \bar\partial)u = 1$ outside $\Delta \cup F$. If $G(\lambda) = (1+\lambda)^{-N}$ and $q = -h/\phi\rho(\zeta)$ we have that $(1+\delta_\eta q)^{-1} = -\rho(\zeta)/v(\zeta, z)$. Therefore,

$$g = \phi^M \sum_{k=0}^{n} G^{(k)}(\delta_\eta q) \wedge (\bar\partial q)^k / k!$$

is smooth and $(\delta_\eta - \bar\partial)g = 0$, and if M is large enough we can form the the corresponding kernels K and P and obtain weighted solution formulas for $\bar\partial$ in D, since q is holomorphic in z and the weight kills the boundary integral in (6.1). Moreover we get a weighted representation formula for holomorphic $(p, 0)$-forms with holomorphic kernel.

Example 22 (Solution formulas in vector bundles). Let F be a holomorphic vector bundle over Ω. It is shown in [10] that there is a holomorphic section $\Psi(\zeta, z)$ to the bundle Hom $(\pi_\zeta^* F, \pi_z^* F)$ such that $\Psi(\zeta, \zeta) = Id_\zeta$ for $\zeta \in \Omega$. If K and P are defined as before, then we have that

$$\bar\partial \Psi K = Id_F[\Delta] + \Psi P,$$

which implies a Koppelman formula for F-valued (p, q)-forms in Ω. As before, we obtain homotopy formulas for $\bar\partial$ in domains where we can get rid of the boundary integrals. In particular, one can obtain the case with (p, q)-forms from the case with $(0, q)$-forms by taking F as the bundle of $(p, 0)$ forms. □

7. Further remarks on weight factors

As before let η be a holomorphic section to the bundle E over the complex manifold X. Recall that we have a complex

$$\stackrel{\delta_\eta-\bar\partial}{\longrightarrow} \mathcal{L}^{m-1}(X) \stackrel{\delta_\eta-\bar\partial}{\longrightarrow} \mathcal{L}^{m}(X) \stackrel{\delta_\eta-\bar\partial}{\longrightarrow} \tag{7.1}$$

and let $\mathcal{Z}^0$ denote the kernel in $\mathcal{L}^0(X)$. Clearly $\mathcal{Z}^0$ is an algebra, and we have an injective algebra homomorphism $\mathcal{O}(X) \to \mathcal{Z}^0$. If $g = g_0 + \ldots + g_r \in \mathcal{Z}^0$, then $\bar\partial g_0 = \delta_\eta g_1$ and hence g_0 is holomorphic on $V = \{\eta = 0\}$. Thus we have an algebra homomorphism

$$B\colon \mathcal{Z}^0(X) \to \mathcal{O}(V). \tag{7.2}$$

Moreover, if g is in the image of $(\delta_\eta - \bar\partial)\colon \mathcal{L}^{-1}(X) \to \mathcal{L}^0(X)$, then $g_0 = \delta_\eta u_1$ and hence $Bg = 0$. It follows that B induces a map

$$\tilde B\colon H^0(\mathcal{L}(X)) \to \mathcal{O}(V) \tag{7.3}$$

if $H^0(\mathcal{L}(X)) = \mathcal{Z}^0(X)/\mathrm{Im}\,(\mathcal{L}^{-1}(X) \to \mathcal{L}^0(X))$.

Theorem 7.1. *Suppose that X is a Stein manifold and $(D\eta)^r \neq 0$ on V. Then the mapping (7.3) is an algebra isomorphism.*

Proof. We have a complex

$$0 \longleftarrow \mathcal{O}(X) \stackrel{\delta_\eta}{\longleftarrow} \mathcal{O}(X, E^*) \stackrel{\delta_\eta}{\longleftarrow} \mathcal{O}(X, \Lambda^2 E^*) \stackrel{\delta_\eta}{\longleftarrow}, \tag{7.4}$$

and it follows from Cartan's theorem B that

$$\frac{\mathcal{O}(X)}{\mathrm{Im}\,(\mathcal{O}(X,E^*) \to \mathcal{O}(X))} \simeq \mathcal{O}(V), \tag{7.5}$$

where the isomorphism is induced by the restriction map $r\colon \mathcal{O}(X) \to \mathcal{O}(V)$. Now consider the double complex

$$\mathcal{L}^{\ell,k}(X) = \mathcal{E}^{-\ell,0}_{0,k}(X)$$

with the mappings δ_η and $\bar\partial$. The corresponding total complex with mapping $\delta_\eta - \bar\partial$ is just the complex (7.1). By a standard result in homological algebra it follows that the cohomology group at m of the latter complex is canonically isomorphic to the cohomology at $-m$ of (7.4). In particular we have

$$H^0(\mathcal{L}(X)) \simeq \frac{\mathcal{O}(X)}{\mathrm{Im}\,(\mathcal{O}(X,E^*) \to \mathcal{O}(X))}, \tag{7.6}$$

where the isomorphism is induced by the natural mapping $\mathcal{O}(X) \to H^0(\mathcal{L}(X))$. Therefore the theorem follows from (7.5) and (7.6). $\square$

Corollary 7.2. *If $g \in \mathcal{Z}^0(X)$ and $g_0|V = 0$, then $g = (\delta_\eta - \bar\partial)v$ for some $v \in \mathcal{L}^{-1}(X)$.*

Now let us return to our discussion about weight factors. Let X and η be as in Theorem 7.1 and let g and h be two weight factors, i.e., $g, h, \in \mathcal{Z}^0$ and $g_0|_V = h_0|_V = 1$. It follows from Corollary 7.2 that $g - h = (\delta_\eta - \bar\partial)v$ for some $v \in \mathcal{L}^{-1}(X)$. Thus

$$(\delta_\eta - \bar\partial)v \wedge (D\eta - \frac{i}{2\pi}\tilde\Theta)^r = g \wedge (D\eta - \frac{i}{2\pi}\tilde\Theta)^r - h \wedge (D\eta - \frac{i}{2\pi}\tilde\Theta)^r,$$

cf., Examples 18 and 19. In some concrete situations one can find explicit solutions v.

Example 23. Suppose that

$$g = \sum_0^r G^{(k)}(\delta_\eta q)(-\bar\partial q)^k/k! \quad \text{and} \quad h = \sum_0^r H^{(k)}(\delta_\eta q)(-\bar\partial q)^k/k!,$$

where both G and H are holomorphic on the image of $\delta_\eta q$ and $G(0) = H(0) = 0$. Let $A(\lambda)$ be defined by $\lambda A(\lambda) = G(\lambda) - H(\lambda)$, so that

$$(\lambda A)^{(k)} = \lambda A^{(k)} + kA^{(k-1)}.$$

It is then readily verified that $(\delta_\eta - \bar\partial)v = g - h$ if

$$v = \sum_0^r A^{(k)}(\delta_\eta q)q \wedge (-\bar\partial q)^k/k!.$$

□

Example 24. Let us apply the previous example to derive an expression for the difference of two holomorphic projections with different weights in a strictly pseudoconvex domain D (which we suppose is contained in $\mathbb{C}^n$ for simplicity, the case with a Stein manifold is analogous, cf., Section 6). Let $q = -h/\rho$ as in Example 8 To simplify the computation we take advantage of Remark 2. Since

$$(1+\lambda)^{r+1} - (1+\lambda)^r = \lambda(1+\lambda)^r$$

we have that

$$(1+\delta_{z-a}-\bar\partial q)^{-r+1} - (1+\delta_{z-a}-\bar\partial q)^{-r} = (\delta_{z-a} - \bar\partial)\big(q \wedge (1+\delta_{z-a}-\bar\partial q)^{-r}\big).$$

If P_r is as in Example 8 it follows that $P_r - P_{r-1} = \bar\partial R_r$, where

$$R_r = \binom{-r}{n-1} q \wedge (1+\delta_{z-a}q)^{-r-n+1}(-\bar\partial q)^{n-1}$$
$$= \binom{-r}{n-1} \frac{(-\rho)^{-1+r}}{v^{n-1+r}} h \wedge (-\bar\partial h)^{n-1}.$$

Since the kernels vanish on the boundary we get the formula

$$\int_D P_{r+1}(z,a)f(z) - \int_D P_r(z,a)f(z) = \int_D R_r(z,a) \wedge \bar\partial f, \quad f \in \mathcal{E}(\overline{D}),$$

which expresses the difference of two holomorphic projections with different weights as an integral of $\bar\partial f$. In the ball case this formula was obtained in [6] for integer values of r. □

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