# $\mathcal{Q}_p$ SPACES IN STRICTLY PSEUDOCONVEX DOMAINS

MATS ANDERSSON & HASSE CARLSSON*

ABSTRACT. We extend the definition of $\mathcal{Q}_p$ spaces from the unit disk to a strictly pseudoconvex domain $D$ in $\mathbb{C}^n$ and show that several known properties are true even in the several variable case. We also provide some proofs and examples that are new even when restricted to the one dimensional case.

CONTENTS



## 1. DEFINITION AND MAIN RESULTS

Let $\Delta$ denote the unit disk in $\mathbb{C}$. The space $\mathcal{D}_p$, $-1 < p < \infty$, consists of all holomorphic functions in $\Delta$ such that

$$\int_\Delta (1-|\zeta|^2)^p |f'(\zeta)|^2 d\lambda(\zeta) < \infty.$$

The space $\mathcal{Q}_p$ introduced in [6], consists of all functions $f$ such that $f\circ\phi \in \mathcal{D}_p$ uniformly for all $\phi \in \operatorname{Aut}(\Delta)$. It is well known that $\mathcal{D}_1$ is just $H^2$ and that $\mathcal{Q}_1$ is precisely $BMOA$. It was proved in [5] that if $p > 1$ then $\mathcal{Q}_p = \mathcal{B}$, the Bloch space and in [13] that $\mathcal{Q}_p = \mathbb{C}$ if $p < 0$. If $p = 0$ then $\mathcal{Q}_p = \mathcal{D}_0$, the Dirichlet space. Hence the new interesting cases are $\mathcal{Q}_p$, $0 < p < 1$. These spaces are investigated further in, for

*Date*: April 14, 2000.

1991 *Mathematics Subject Classification.* 32A35, 32A37, 42B30, 47A10.

*Key words and phrases.* $Q_p$-space, corona theorem, Taylor spectrum.

*Both authors partially supported by the Swedish Natural Science Research Council.

example [6] and [7], and in [8] $\mathcal{Q}_p$ is extended to $\mathbb{R}^n$. The aim of this note is to suggest a generalization to several complex variables, and to prove some basic facts. Throughout this paper $D = \{\rho < 0\}$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, and $\rho$ is a smooth strictly plurisubharmonic defining function. We let $\delta(z)$ be the distance from $z$ to $\partial D$ so that $\delta(z) \sim -\rho(z)$. When $D$ is the unit ball we choose $\rho = |\zeta|^2 - 1$. For simplicity, at several points we supply proofs only for the ball and just outline how to obtain the general case.

It is well known that the $\bar\partial$-operator behaves differently in the complex tangential and normal directions near the boundary of a strictly pseudoconvex domain. Therefore it is natural to measure forms with respect to a metric that takes this difference into account, such as

$$\Omega = (-\rho) i\partial\bar\partial \log(1/-\rho).$$

In the ball $\Omega$ is just $-\rho(\zeta)$ times the Bergman metric. If $f$ is a $(q,0)$-form we have that

$$|f|^2_\Omega = \frac{1}{B}(-\rho|f|^2_\beta + |\partial\rho \wedge f|^2_\beta), \tag{1.1}$$

where $|\ \ |_\beta$ denotes the norm induced by the metric form $\beta = (i/2)\partial\bar\partial\rho$, which is equivalent to the Euclidean metric since $\rho$ is strictly plurisubharmonic, and $B = -\rho + |\partial\rho|_\beta \sim 1$. In particular, on the boundary, $|f|^2_\Omega \sim |\partial\rho \wedge f|^2_\beta$, which is the natural norm for the complex tangential boundary values $f|_b$ of $f$.

For $p > 0$ we let $\mathcal{D}_p$ be the space of holomorphic functions in $D$ such that

$$\|f\|^2_{\mathcal{D}_p} = \int_D \delta^{p-1}|\partial f|^2_\Omega + |f(0)|^2 \tag{1.2}$$

is finite.

Our definition of $\mathcal{Q}_p$ will be in terms of Carleson measures defined with respect to the Koranyi balls on $D$. Let $d(p,q)$ be the Koranyi pseudometric on $\partial D$ and $B = B_r = B_r(p) = \{\zeta \in \partial D;\ d(\zeta, p) < r\}$ the corresponding balls. When $D$ is the unit ball $d(\zeta, z) = |1 - \bar\zeta z|$. For $\zeta, z \in D$ we put $d(\zeta, z) = \delta(\zeta) + \delta(z) + d(\zeta', z')$ where $\zeta', z'$ are the projections (defined in some reasonable way) of $\zeta, z$ on $\partial D$. Note that $d$ is not a pseudometric on $D$ since $d(z,z) > 0$ for $z \in D$ but it still satisfies the triangle inequality $d(\zeta, z) \le C(d(\zeta, w) + d(w, z))$. The tent $Q_r(p)$ over $B_r(p)$ is $Q_r(p) = \{z \in D;\ d(z,p) < r\}$. A positive measure $\mu$ in $D$ is a $p$-Carleson measure, $\mu \in \mathcal{C}_p$, if

$$\mu(Q_t(q)) \le Ct^{p+n-1}$$

uniformly for all tents $Q$, and we let $\|\mu\|_{\mathcal{C}_p}$ be the infimum of all possible constants $C$. Thus $\mathcal{C}_1$ is the space of usual Carleson measures.

We define the space $\mathcal{Q}_p$ as the subspace of $H^2$ such that

$$\|f\|^2_{\mathcal{Q}_p} = \|\delta^{p-1}|\partial f|^2_\Omega\|_{\mathcal{C}_p} + \|f\|^2_{H^2}$$

is finite. In Section 3 we prove that when $D$ is the unit ball in $\mathbb{C}^n$, and in particular if $D$ is the unit disk in $\mathbb{C}$, this is equivalent to that $f \circ \phi \in \mathcal{D}_p$ uniformly for all $\phi \in \mathrm{Aut}(B)$, and hence our definition of $\mathcal{Q}_p$ is a natural generalization to several variables. As in one variable, see e.g. [1], Proposition 2.1, $\mathcal{Q}_1 = BMOA$, and we shall see in Section 3 that $\mathcal{Q}_p = \mathcal{B}$, the Bloch space, for $p > 1$. Therefore we will focus on $\mathcal{Q}_p$, $0 < p < 1$.

One can characterize $\mathcal{Q}_p$ in terms of boundary values. It is not hard to prove that $f \in BMOA$ if and only if

$$\sup_{z \in D} \delta(z)^n \int_{\partial D} \frac{|f(\zeta) - f(z)|^2}{d(z,\zeta)^{2n}} < \infty \, ,$$

see Theorem 1.2 in Chapter 6 of [9] for a proof in the classical case. For functions in $\mathcal{Q}_p$ we have the following analogous result.

**Theorem 1.1.** *Suppose that $f \in H^2$ and $0 < p < 1$. Then $f \in \mathcal{D}_p$ if and only if*

$$\int_{\partial D} \int_{\partial D} \frac{|f(\zeta) - f(z)|^2}{d(z,\zeta)^{n+1-p}} < \infty, \tag{1.3}$$

*and $f \in \mathcal{Q}_p$ if and only if*

$$\sup_{B_h} \frac{1}{h^{p+n-1}} \int_{B_h} \int_{B_h} \frac{|f(\zeta) - f(z)|^2}{d(z,\zeta)^{n+1-p}} < \infty. \tag{1.4}$$

Since the condition (1.3) and (1.4) only depend on the boundary values of $f$ it is natural to define the real spaces $\mathcal{D}_p^{\mathbb{R}}(\partial D)$ and $\mathcal{Q}_p^{\mathbb{R}}(\partial D)$, $0 < p < 1$, consisting of $f \in L^2(\partial D)$ such that (1.3) or (1.4) holds. If $0 < p_1 < p_2 < 1$ it is easy to verify that $\mathcal{Q}_{p_1}^{\mathbb{R}}(\partial D) \subset \mathcal{Q}_{p_2}^{\mathbb{R}}(\partial D) \subset BMO(\partial D)$. In §3 we give examples that show that these inclusions are proper. The examples also show that $L^\infty(\partial D)$ is not contained in any $\mathcal{Q}_p^{\mathbb{R}}(\partial D)$. In the one dimensional case, such examples were constructed using gap series, see [6]. Our examples consist of rapidly oscillating functions defined directly on $\partial D$.

It is well-known, see [10] and [1], that the Szegő projection $S\colon L^2(\partial D) \to H^2(D)$ maps $L^p(\partial D)$ into $H^p(D)$ and $BMO(\partial D)$ into $BMOA(D)$. We have an analogous result for $\mathcal{D}_p$ and $\mathcal{Q}_p$.

**Theorem 1.2.** *Assume that $0 < p < 1$. Then the Szegő projection $S$ is a bounded operator from $\mathcal{D}_p^{\mathbb{R}}(\partial D)$ to $\mathcal{D}_p(D)$ and from $\mathcal{Q}_p^{\mathbb{R}}(\partial D)$ to $\mathcal{Q}_p(D)$.*

In the last two sections we discuss the spaces $\mathcal{M}(\mathcal{Q}_p)$ of multipliers on $\mathcal{Q}_p$ spaces. Contrary to the spaces $\mathcal{B}$ and $BMOA$ there is no complete characterization of $\mathcal{M}(\mathcal{Q}_p)$, but the known results extend to the several variable case. Moreover, we extend a result due to Xiao, [21], about corona decomposition of $\mathcal{Q}_p$ functions.

The authors are grateful to J. Xiao for valuable discussions.

## 2. Integral representation in strictly pseudoconvex domains

In the ball we have the wellknown holomorphic projections

$$P_r f(z) = c_r \int_D \frac{\delta^{r-1}(\zeta) f(\zeta)}{v(\zeta,z)^{n+r}} d\lambda(\zeta) \tag{2.1}$$

for $r > 0$, and

$$P_0 f(z) = c \int_{\partial D} \frac{f(\zeta) d\sigma(\zeta)}{v(\zeta,z)^n}, \tag{2.2}$$

where $\delta(\zeta) = 1 - |\zeta|^2$ and $v(\zeta,z) = 1 - \bar{\zeta} \cdot z$. In fact, (2.1) is the orthogonal projection onto the Bergman space with respect to the weight $\delta^{r-1} d\lambda(\zeta)$ and (2.2) is the Szegő projection. In a general strictly pseudoconvex domain $D$ there are similar formulas, see e.g. [2], where $d\lambda(\zeta)$ is replaced by $a(\zeta,z)d\lambda(\zeta)$, $a(\zeta,z)$ and $v(\zeta,z)$ are smooth on the closure of $D \times D$, $v(\zeta,z)$ is holomorphic in $z$, $a(\zeta,z) \sim 1$, $v(\zeta,\zeta) = -\rho(\zeta)$,

$$|v(\zeta,z)| \sim |v(z,\zeta)| \sim d(\zeta,z) , \tag{2.3}$$

and,

$$2\mathrm{Re}\, v(\zeta,z) \geq \delta(z) + \delta(\zeta) + \epsilon|\zeta - z|^2. \tag{2.4}$$

In particular, $\log v$ and $v^\alpha$ are well defined for $\alpha > 0$.

Sometimes we also want $v(\zeta,z)$ to be approximately antiholomorphic in $\zeta$. In fact we can choose $v$ so that $\partial_\zeta v(\zeta,z) = O(|\zeta - z|^2)$ if we only require that $v(\zeta,z)$ is holomorphic for $z$ near the diagonal. In most proofs we will assume that $\bar{\partial}_z v = \partial_\zeta v = 0$ and $a \equiv 1$ as in the ball, and we leave it to the reader to verify that the same arguments work in the general case, though with some extra neglible error terms.

We recall the standard estimates

$$\int_D \frac{\delta(\zeta)^{r-1}}{d(\zeta,z)^{n+r+\gamma}} d\lambda(\zeta) \sim \frac{1}{\delta(z)^\gamma} \tag{2.5}$$

and

$$\int_{\partial D} \frac{1}{d(\zeta,z)^{n+\gamma}} d\lambda(\zeta) \sim \frac{1}{\delta(z)^\gamma} \tag{2.6}$$

if $r, \gamma > 0$. We also have the following estimate.

**Lemma 2.1.** *Assume that* $\alpha > 0$, $\beta > 0$, $r > -1$, $\alpha - r < n+1$, $\beta - r < n+1$, *and* $\alpha + \beta - r > n+1$. *Then*

$$\int_D \frac{\delta(\zeta)^r d\lambda(\zeta)}{d(\zeta,z)^\alpha d(\zeta,w)^\beta} \lesssim \frac{1}{d(z,w)^{\alpha+\beta-r-n-1}}\,. \tag{2.7}$$

*If instead* $\beta - r > n+1$ *we have*

$$\int_D \frac{\delta(\zeta)^r d\lambda(\zeta)}{d(\zeta,z)^\alpha d(\zeta,w)^\beta} \lesssim \frac{1}{d(z,w)^{\alpha}\delta(w)^{\beta-r-n-1}}\,. \tag{2.8}$$

*Proof.* Any point $\zeta$ near the boundary can be identified by a pair $(y,x)$ where $y$ is the distance to the boundary and $x$ is its projection onto $\partial D$. Therefore (2.7) means that

$$\int_{x\in\partial D}\int_{y=0}^{\infty} \frac{y^r dy dx}{(y+d(x,z))^\alpha (y+d(x,w))^\beta} \lesssim \frac{1}{d(z,w)^{\alpha+\beta-r-n-1}}.$$

To prove it we first notice that

$$\int_0^\infty \frac{y^r dy}{(y+a)^\gamma} \sim \frac{1}{a^{\gamma-r-1}}, \quad r > -1, \quad \gamma - r > 1.$$

Let $d = d(z,w)$. Recall that $d(z,w) \le C(d(x,z) + d(x,w))$. If $d(x,z) \le cd$ for some appropriately small $c > 0$, it follows that

$$d(x,w) \ge c'd \ge c''d(x,z).$$

Thus the integral over the set where $d(x,z) \le cd$ can be estimated by

$$\int_{d(x,z)<cd} \frac{1}{d^{\beta-\epsilon}} \int_{y=0}^\infty \frac{y^r dy dx}{(y+d(x,z))^{\alpha+\epsilon}} \lesssim \frac{1}{d^{\beta-\epsilon}} \int_{d(x,z)\le cd} \frac{dx}{d(x,z)^{\alpha+\epsilon-r-1}},$$

if $\epsilon \ge 0$ is chosen so that $n > \alpha + \epsilon - r - 1 > 0$. Now

$$|\{x;\ d(x,z) < s\}| \le |\{s;\ d(x,z') < s\}| \sim s^n, \tag{2.9}$$

and hence the last integral is bounded by $C/d^{\alpha+\epsilon-r-1-n}$. In the same way the set where $d(x,w) \le cd$ is handled. We now consider the set where $d(x,w) \ge d(x,z) \ge cd$. Over this set the integral is estimated by

$$\int_{d(x,z)\ge cd}\int_{y=0}^\infty \frac{y^r dy dx}{(y+d(x,z))^{\alpha+\beta}} \lesssim \int_{d(x,z)\ge cd} \frac{dx}{d(x,z)^{\alpha+\beta-r-1}}.$$

By (2.9) it follows that the last integral is bounded by $1/d^{\alpha+\beta-r-1-n}$. The set where $d(x,z) \ge d(x,w) \ge cd$ is of course handled in the same way and the proof of (2.7) is complete.

To prove (2.8) we just need to modify the estimate of the part where $d(x,w) \le cd$. We have

$$\int_{d(x,w)\le cd} \frac{\delta^r(\zeta)}{d(\zeta,z)^\alpha d(\zeta,w)^\beta} \lesssim \frac{1}{d(w,z)^\alpha}\int_{d(x,w)\le cd} \frac{\delta^r(\zeta)}{d(\zeta,w)^\beta}$$
$$\lesssim \frac{1}{d(w,z)^\alpha}\int_D \frac{\delta^r(\zeta)}{d(\zeta,w)^\beta} \lesssim \frac{1}{d(z,w)^\alpha \delta(w)^{\beta-r-n-1}}\,,$$

where the last inequality follows by (2.5). $\square$

Since $v(\zeta,\zeta) = -\rho(\zeta)$ it follows that $\partial\rho(z) \wedge \partial_z v(\zeta, z) = \mathcal{O}(|\zeta - z|)$ and by (1.1) we get

$$|\partial_z v|_\Omega \sim \sqrt{\delta} + \mathcal{O}(|\zeta - z|) \lesssim \sqrt{|v|}. \tag{2.10}$$

We also have the following useful formula

$$\int_{\partial D} \psi d\sigma = \int_D \psi\beta^n + c_n \int_D \delta\partial\bar{\partial}\psi \wedge \Omega^{n-1}, \tag{2.11}$$

where $d\sigma = d^c\rho \wedge \beta^{n-1}$ which is equivalent to the surface measure on $\partial D$. In fact, Stokes' formula and integration by parts gives that

$$\int_{\partial D} \psi d^c\rho \wedge \beta^{n-1} = \int_D \psi\beta^n + \int_D d\psi \wedge d^c\rho \wedge \beta^{n-1}$$
$$= \int_D \psi\beta^n + \int_D \rho dd^c\psi \wedge \beta^{n-1}.$$

The last integral is equal to

$$\int_D dd^c\psi \wedge d\rho \wedge d^c\rho \wedge \beta^{n-2},$$

and since $\delta dd^c\psi \wedge \Omega^{n-1} = \delta dd^c\psi \wedge \beta^{n-1} + (n-1)dd^c\psi \wedge d^c\rho \wedge \beta^{n-1}$, the equality (2.11) follows.

When $D$ is the unit ball and $f \in \mathcal{L}^1(\partial D)$, then

$$F(z) = \int_{\partial D} \frac{\delta(z)^n f(\zeta)}{|v(\zeta, z)|^{2n}}.$$

is the $\Omega$-harmonic extension (the Poisson-Szegő integral) of $f$, i.e., $F = f$ on $\partial D$ and $dd^cF \wedge \Omega^{n-1} \equiv 0$. In particular, $F = P_0 f$ if $f$ is (the boundary values of) a holomorphic function. In a general strictly pseudoconvex case there is no explicit formula for the $\Omega$-harmonic extension; however, by a similar formula as above, see [4], one get an extension which is almost harmonic in the sense that

$$|\partial\bar{\partial}F \wedge \Omega^{n-1}|_\beta \lesssim \int_{\partial D} \frac{|f(\zeta)|}{|v|^{n+1/2}}$$

which is enough for the purposes of this paper.

## 3. Some basic properties of $\mathcal{Q}_p$ spaces

We begin with a simple but useful reformulation of the $p$-Carleson condition.

**Lemma 3.1.** *For each fixed $\alpha > 0$ we have that*

$$\|\mu\|_{\mathcal{C}_p} \sim \sup_{z\in D} \delta^\alpha(z) \int_D \frac{d\mu(\zeta)}{d(\zeta,z)^{n+\alpha+p-1}} \tag{3.1}$$

*for all measures in $D$.*

**Corollary 3.2.** *Assume that $f \in H^2$ and $\alpha > 0$. Then*

$$\|f\|_{\mathcal{Q}_p} \sim \sup_{z\in D} \delta^\alpha(z) \int_D \frac{\delta^{p-1}(\zeta)|\partial f|_\Omega^2}{d(\zeta,z)^{n+\alpha+p-1}} + \|f\|_{H^2}\,.$$

*Proof of Lemma 3.1.* Take a fixed $z \in D$ and let $A$ denote the right hand side of (3.1). Since $\delta(z) \le d(\zeta,z)$ we have

$$\mu(Q) \lesssim \int_Q \frac{\delta(z)^{n+\alpha+p-1}}{d(\zeta,z)^{n+\alpha+p-1}} d\mu(\zeta)$$
$$\le \delta(z)^{n+p-1}\delta(z)^\alpha \int_D \frac{d\mu(\zeta)}{d(\zeta,z)^{n+\alpha+p-1}} \le A\delta(z)^{n+p-1}.$$

For the opposite estimate note that

$$\delta^\alpha(z) \int_Q \frac{d\mu(\zeta)}{d(\zeta,z)^{n+\alpha+p-1}} \lesssim \frac{1}{\delta(z)^{n+p-1}}\mu(Q) \lesssim \|\mu\|_{\mathcal{C}_p},$$

and

$$\delta(z)^\alpha \int_{D\setminus Q} \frac{d\mu(\zeta)}{d(\zeta,z)^{n+\alpha+p-1}} \lesssim \delta(z)^\alpha \int_{d>\delta(z)} d\mu(\zeta) \int_d^\infty \frac{ds}{s^{p+n+\alpha}}$$
$$= \delta(z)^\alpha \int_{\delta(z)}^\infty \left(\int_{d<s} d\mu(\zeta)\right) \frac{ds}{s^{p+n+\alpha}} \lesssim \|\mu\|_{\mathcal{C}_p}\delta(z)^\alpha \int_{\delta(z)}^\infty \frac{ds}{s^{\alpha+1}} \sim \|\mu\|_{\mathcal{C}_p}\,.$$

This completes the proof. □

*Example* 1. Let $f_w(\zeta) = \log v(w,\zeta)$ for $w \in D$. Then $\|f_w\|_{\mathcal{Q}_p} \le C_p$, where $C_p$ only depends on $p$. In fact, $|\partial f_w|_\Omega \lesssim 1/\sqrt{d(w,\zeta)}$ in view of (2.10) and and thus by (2.7),

$$\delta^\alpha(z) \int_{\zeta\in D} \frac{\delta^{p-1}(\zeta)|\partial f_w|_\Omega^2}{d(z,\zeta)^{n+\alpha+p-1}} \lesssim \delta^\alpha(z) \int_D \frac{\delta^{p-1}}{d(\zeta,z)^{n+\alpha+p-1}d(w,\zeta)} \lesssim 1,$$

and so the claim follows from Lemma 3.1. □

Lemma 3.1 implies the following characterization of $\mathcal{Q}_p(B)$.

**Proposition 3.3.** *Suppose that $D = B$ is the unit ball in $\mathbb{C}^n$. Then $f \in \mathcal{Q}_p$ if and only if $f \circ \phi \in \mathcal{D}_p$ uniformly for all $\phi \in \mathrm{Aut}(B)$.*

*Proof.* Let $\omega = i\partial\bar\partial \log(1/(1-|z|^2))$ be the Bergman metric. If $\phi \in \mathrm{Aut}(B)$ then $\phi^*\omega = \omega$. To see this, we use that, see e.g. [16],

$$1 - |\phi(\zeta)|^2 = \frac{(1-|\zeta|^2)(1-|a|^2)}{|1-\bar a\cdot\zeta|^2} \tag{3.2}$$

for some $a \in B$. Since

$\delta^{p-1}|\partial f|^2_\Omega d\lambda \sim \delta^p|\partial f|^2_\Omega \Omega^n \sim \delta^p \partial f \wedge \overline{\partial f} \wedge \Omega^{n-1} \sim \delta^{n+p-1}\partial f \wedge \overline{\partial f} \wedge \omega^{n-1}$,

we obtain

$$\|f\|^2_{\mathcal{D}_p} \sim \int_B (1-|w|^2)^{p+n-1} i\partial f \wedge \overline{\partial f} \wedge \omega^{n-1}.$$

The requirement that $\phi^* f$ be in $\mathcal{D}_p$ uniformly for all $\phi \in \mathrm{Aut}(B)$ therefore means that

$$\sup_{a\in B}(1-|a|^2)^{p+n-1}\int_B \frac{(1-|\zeta|^2)^{p-1}|\partial f|^2_\Omega}{|1-\bar{a}\cdot\zeta|^{2p+2n-2}} < \infty,$$

which by Lemma 3.1 is equivalent to that $f \in \mathcal{Q}_p$. □

**Lemma 3.4.** *Let $\gamma > 0$, $0 \le \alpha < 1$ and*

$$u(z) = \frac{1}{\delta^\alpha(z)}\int_D \frac{\delta(\zeta)^{\gamma+\alpha}}{d(\zeta,z)^{n+1+\gamma}} d\mu(\zeta)\ .$$

*Then if $\mu \in \mathcal{C}_p$, $u \in \mathcal{C}_p$ as well.*

*Proof.* Let $0 < \beta < 1$. By Fubini's theorem and (2.8) we have

$$\delta^\beta(w)\int_D \frac{u(z)}{d(z,w)^{n+\beta+p-1}} d\lambda(z) \lesssim \delta^\beta(w)\int_D \frac{d\mu(\zeta)}{d(\zeta,w)^{n+\beta+p-1}} \lesssim \|d\mu\|_{\mathcal{C}_p}$$

and hence, by Lemma 3.1, $u \in \mathcal{C}_p$.

□

We note that for holomorphic functions $f$

$$\int_D \delta^{p-1}|\partial f|^2_\Omega \sim \int_D \delta^p |\partial f|^2_\beta \sim \int_D \delta^{p-1}|\partial\rho \wedge \partial f|^2_\beta \tag{3.3}$$

and

$$\int_D |\partial f|^2_\Omega \lesssim \int_{\partial D} |f|^2\ . \tag{3.4}$$

Formula (3.3) follows by an integration by parts,

$$\int_D (-\rho)^p |\partial f|^2_\beta \approx \int_D (-\rho)^p i\partial f \wedge \bar\partial \bar f \wedge i\partial\bar\partial\rho \wedge \beta^{n-2} =$$

$$p\int_D (-\rho)^{p-1} i\partial\rho \wedge \partial f \wedge i\bar\partial\rho \wedge \bar\partial \bar f \wedge \beta^{n-2} = p\int_D (-\rho)^{p-1}|\partial\rho\wedge\partial f|^2_\beta\ .$$

Formula (3.4) is a consequence of $\int_D \delta|\partial f|^2_\beta \lesssim \int_{\partial D}|f|^2$ that follows from Green's identity as $|\partial f|^2_\beta = \Delta|f|^2$.

By (3.3), when defining the $\mathcal{D}_p$ spaces it is enough to have a condition on either the complex normal or the complex tangential components of $\partial f$. A similar result holds for $\mathcal{Q}_p$. Let $L = \frac{1}{2}|\nabla\rho|^{-2}\sum(\partial\rho/\partial\zeta_j)(\partial/\partial\zeta_j)$ be the "complex normal" derivative of $f$ (so that $L\rho \equiv 1$).

**Proposition 3.5.** *If $p > 0$ and $f \in H^2$ the following conditions are equivalent*

(i) $f \in \mathcal{Q}_p$ ,

(ii) $\delta^p |Lf|^2 \in \mathcal{C}_p$

(iii) $\delta^p |\partial f|^2_\beta \in \mathcal{C}_p$

(iv) $\delta^{p-1} |\partial\rho \wedge \partial f|^2_\beta \in \mathcal{C}_p$ .

*Proof.* Since (i) means that (iii) and (iv) holds, it is enough to show that (ii),(iii) and (iv) are equivalent. Clearly (iii) implies (ii) so it remains to prove that (iv) implies (iii) and that (ii) implies both (iii) and (iv).

By Lemma 3.1, to prove that (iv) implies (iii), it is enough to show that $A \lesssim B$, where

$$A = \delta(z)^\alpha \int_D \frac{\delta^p(\zeta)|\partial f|^2_\beta}{d(\zeta, z)^{n+\alpha+p-1}} d\lambda(\zeta)$$

and

$$B = \delta(z)^\alpha \int_D \frac{\delta^{p-1}(\zeta)|\partial\rho \wedge \partial f|^2_\beta}{d(\zeta, z)^{n+\alpha+p-1}} d\lambda(\zeta)$$

for some $\alpha > 0$. Since $|\partial f|^2_\beta \sim \partial f \wedge \overline{\partial f} \wedge \partial\bar\partial\rho \wedge \beta^{n-2}$, Stokes' theorem implies that

$$A \lesssim B + \delta(z)^\alpha \int_D \frac{\delta^p |\partial f \wedge \overline{\partial f} \wedge \partial\bar v \wedge \bar\partial\rho|_\beta}{d(\zeta, z)^{n+\alpha+p}} d\lambda(\zeta) .$$

Since $\partial\bar v = \partial\rho + O(\zeta - z)$ and $\delta \lesssim d$ we get

$$A \lesssim B + \delta(z)^\alpha \int_D \frac{\delta^p(\zeta)|\partial f \wedge \overline{\partial f} \wedge \bar\partial\rho|_\beta}{d(\zeta, z)^{n+\alpha-\frac{1}{2}+p}} d\lambda(\zeta).$$

By the Cauchy-Schwarz' inequality (and again using that $\delta \lesssim d$) we get that $A \lesssim B + \sqrt{A}\sqrt{B}$ and hence $A \lesssim B$ as desired.

To prove the other directions we want to express $f$ in terms of $Lf$. The starting point is (2.1) and the following identity (in the ball),

$$r \int_D \delta^{r-1} \psi d\lambda = (r+n) \int_D \delta^r \psi d\lambda + \int_D \delta^r L\psi d\lambda \ , \tag{3.5}$$

that is obtained by an integration by parts (starting with the last integral). An application of (3.5) to (2.1), holding in mind that $v$ is anti-holomorphic in $\zeta$, then gives

$$f(z) \sim \int_D \frac{\delta(\zeta)^r f(\zeta)}{v(\zeta, z)^{n+r}} d\lambda(\zeta) + Kf(z),$$

where

$$Kf(z) = \int_D \frac{\delta(\zeta)^r Lf(\zeta)}{v(\zeta, z)^{n+r}} d\lambda(\zeta).$$

It is enough to estimate $Kf$ because the first term can be handled in the same way by repeated use of (3.5). Since $|\partial_z v|_\Omega \lesssim \sqrt{d(\zeta,z)}$ we have that

$$|\partial_z Kf(z)|_\Omega \lesssim \int_D \frac{\delta(\zeta)^r |Lf(\zeta)| d\lambda(\zeta)}{d(\zeta,z)^{n+r+1/2}}.$$

Hence, if $0<\epsilon<p<1$ and $r=1$, (for the case $p\geq 1$ choose $r>p$) we obtain

$$\begin{aligned}
&\delta(z)^{p-1}|\partial Kf(z)|^2_\Omega \\
&\qquad\lesssim \delta(z)^{p-1}\left(\int_D \frac{\delta(\zeta)|Lf(\zeta)|}{d(\zeta,z)^{n+3/2}} d\lambda(\zeta)\right)^2 \\
&\lesssim \delta(z)^\epsilon \int_D \frac{d\lambda(\zeta)}{d(\zeta,z)^{n+1+\epsilon}} \cdot \frac{1}{\delta(z)^{1-p+\epsilon}} \int_D \frac{\delta(\zeta)^2|Lf(\zeta)|^2}{d(\zeta,z)^{n+2-\epsilon}} d\lambda(\zeta) \\
&\lesssim \frac{1}{\delta(z)^{1-p+\epsilon}} \int_D \frac{\delta(\zeta)^{1-\epsilon+(1-p+\epsilon)}}{d(\zeta,z)^{n+1+(1-\epsilon)}} \delta(\zeta)^p |Lf(\zeta)|^2 d\lambda(\zeta) \in \mathcal{C}_p\ ,
\end{aligned}$$

by Lemma 3.4. Thus (ii) implies (iii) and (iv). □

**Lemma 3.6.** *If $f\in Q_p$, $p>0$, then*

$$|\partial f(z)|_\Omega \lesssim \frac{1}{\sqrt{\delta(z)}} \quad \textit{and} \quad |f(z)| \lesssim \log(1/\delta(z)).$$

*Proof.* From the proof of Proposition 3.5 we have (with $r=p$) that

$$|\partial f|^2_\Omega \lesssim \left(\int_D \frac{\delta^p|Lf|}{d^{n+p+1/2}} d\lambda\right)^2 \lesssim \int_D \frac{\delta^p}{d^{n+p+3/2}} \int_D \frac{\delta^p |\partial f|^2_\beta}{d^{n+1/2+p-1}} \lesssim \frac{1}{\sqrt{\delta}}\frac{1}{\sqrt{\delta}}$$

since $\delta^p|Lf|^2\in\mathcal{C}_p$. The second statement follows in a similar way using the first one. □

The Bloch space $\mathcal{B}$ consist of all holomorphic functions in $D$ such that $|\partial f|_\Omega \leq C\delta^{-1/2}$. Lemma 3.6 thus means that $\mathcal{Q}_p \subset \mathcal{B}$ for all $p>0$. Just as in one variable we have

**Proposition 3.7.** $\mathcal{Q}_p = \mathcal{B}$ *for* $p>1$.

*Proof.* Let $p>1$. If $|Lf|\leq C\delta^{-1}$, then

$$\delta(z)^\alpha \int \frac{\delta^p|Lf|^2}{d^{n+\alpha+p-1}} \lesssim \delta(z)^\alpha \int \frac{\delta^{p-2}}{d^{n+\alpha+p-1}} \lesssim 1,$$

according to (2.5) since $p-1>0$, and hence, by Lemma 3.1, $\delta^p|Lf|^2 \in \mathcal{C}_p$. It now follows from Proposition 3.5 that $f\in\mathcal{Q}_p$. □

## 4. Boundary characterization of $\mathcal{D}_p$ and $\mathcal{Q}_p$ and the real spaces $\mathcal{D}_p^{\mathbb{R}}(\partial D)$ and $\mathcal{Q}_p^{\mathbb{R}}(\partial D)$

The main objective of this section is to prove Theorem 1.1. In the ball one can use $\mathrm{Aut}(B)$, see [11] or [13] where this is done in the unit disk. However we give a proof that works in general strictly pseudoconvex domains. In one direction it is just an adaption of an argument in [12] and [20] to the several variable case and the non-isotropic structure, which however, for the reader's convenience, we include (Proposition 4.1). The proof of the other direction (Proposition 4.2) is new. The basic ingredient is the estimate (4.7) and earlier this was proved by power series expansions, whereas our proof is based on integral representation and the Carleson-Hörmander inequality for Carleson measures.

**Proposition 4.1.** *Suppose that* $0 < p < 1$ *and that* $f$ *is the boundary values of a function* $F$ *in* $C^\infty(\overline{D})$. *Then*

$$\int_{\partial D}\int_{\partial D}\frac{|f(\zeta)-f(z)|^2}{|v(z,\zeta)|^{n+1-p}} \leq C\int_D \delta^{p-1}|dF|^2_\Omega \tag{4.1}$$

*and*

$$\sup_{B_h}\frac{1}{h^{p+n-1}}\int_{B_h}\int_{B_h}\frac{|f(\zeta)-f(z)|^2}{|v(z,\zeta)|^{n+1-p}} \leq C\|\delta^{p-1}|dF|^2_\Omega\|^2_{\mathcal{C}_p}. \tag{4.2}$$

*Remark* 1. Thus $f \in \mathcal{Q}_p^{\mathbb{R}}$ if for some extension $F$, $\delta^{p-1}|dF|^2_\Omega \in \mathcal{C}_p$. The same conclusion is true even for $p = 1$ if $F$ is the $\Omega$-harmonic extension of $f$. However, it is not true for an arbitrary extension. A counterexample is given by $F(z) = \log\log(1/\delta(z))$. Then $|dF|^2_\Omega$ is a Carleson measure but the boundary values are even not in $H^2$. □

*Proof of Proposition 4.1.* It is enough to prove the last estimate. Recall that $B_h(p) = \{z;\ d(p,z) < h\}$. Since $d$ is a pseudometric we have

$$\begin{aligned}\int_{d(z,p)<h}\int_{d(\zeta,p)<h}&\frac{|f(\zeta)-f(z)|^2}{d(\zeta,z)^{n+1-p}}d\sigma(\zeta)d\sigma(z)\\ &\leq \int_{d(z,p)<Ch}\int_{d(\zeta,z)<h}\frac{|f(\zeta)-f(z)|^2}{d(\zeta,z)^{n+1-p}}d\sigma(\zeta)d\sigma(z)\\ \lesssim \int_{d(z,p)<Ch}&\left(\int_{t=0}^{h}\left(\int_{d(\zeta,z)=t}|f(\zeta)-f(z)|^2 d\sigma_t(\zeta)\right)\frac{dt}{t^{n+1-p}}\right)d\sigma(z),\end{aligned} \tag{4.3}$$

where $d\sigma_t$ is the surface measure on $\{\zeta; d(\zeta,z)=t\}$ induced by $d\sigma(\zeta)$. Since the problem is local, we can choose local coordinates $z = (y,x)$ such that $y = \delta(z)$. To each $x$ we have a unique orthogonal decomposition $a(x) = a_2(x) + a'(x)$ of each $a \in S^{2n-2}$ such that $a'(x)$ is complex tangential. After possibly modifying the definition of $d$ slightly, we may assume that

$$\{\xi;\ d(x,\xi) = t\} = \{x + ta_2(x) + \sqrt{t}a'(x);\ a \in S^{2n-2}\}.$$

Therefore the right hand side of (4.3) is bounded by

(4.4)

$$\int_{d(x,p)<Ch}\left(\int_{t=0}^{h}\left(\int_{|a|=1}|f(x+ta(x))-f(x)|^2 d\sigma(a)\right)t^{p-2}dt\right)d\lambda(x)$$

where $ta(x)=ta_2(x)+\sqrt{t}a'(x)$. Now,

$$\begin{aligned}(4.5)\quad f(x+ta(x))-f(x) &= f(x+ta(x))-F(t,x+ta(x))\\ &\quad +F(t,x)-f(x)+F(t,x+ta(x))-F(t,x)\\ &= \int_0^t \partial_y F(u,x+ta(x))du+\int_0^t \partial_y F(u,x)du\\ +\int_{s=0}^{t}&\left(a_2(x)\cdot\partial_x F(t,x+sa)+\frac{1}{2\sqrt{s}}a'(x)\cdot\partial_x F(t,x+sa(x))\right)ds\;.\end{aligned}$$

If we plug the right hand side of (4.5) into the right hand side of (4.4) we get three terms $I_1$, $I_2$ and $I_3$. By an application of Minkowski's inequality in $I_3$ it can be estimated by

$$\int_{|a|=1}\int_{t=0}^{h}\left(\int_{s=0}^{t}\Big(\int_{d(x,p)<Ch}|a_2\cdot\partial_x F(t,x+sa_2+\sqrt{s}a')|^2+\right.$$
$$\left.\frac{1}{4s}|a'\cdot\partial_x F(t,x+sa_2+\sqrt{s}a')|^2 d\lambda(x)\Big)^{1/2}ds\right)^2 t^{p-2}dt.$$

Since the mapping $x\mapsto x+sa_2(x)+\sqrt{s}a'(x)$ is invertible (for small $s$), and $d$ is a pseudometric, we can replace the expression $x+sa_2(x)+\sqrt{s}a'(x)$ in the inner integral by $x$ if we just blow up the constant $C$ appropriately. Therefore we just get

$$\int_0^h\int_{d(x,p)<Ch}\left(t|a_2(x)\cdot\partial_x F(t,x)|^2+|a'(x)\cdot\partial_x F(t,x)|^2\right)t^{p-1}dtd\lambda(x)$$

which is bounded by $t^{p+n-1}$ by assumption. By a similar argument the two remaining terms $I_1$ and $I_2$ are both estimated by

$$\int_{t=0}^{h}\left(\int_0^t g(u)du\right)^2 t^{p-2}dt, \tag{4.6}$$

where

$$g(u)=\left(\int_{d(x,p)<Ch}|\partial_y F(u,x)|^2 d\lambda(x)\right)^{1/2}.$$

Since $p-2\neq -1$, by Hardy's inequality (4.6) is bounded by

$$\int_{u=0}^{h} g(u)^2u^p du\lesssim h^{p+n-1}\;.$$

Thus the proposition is proved. $\square$

**Proposition 4.2.** *Suppose that $f \in L^2(\partial D)$, $0 < p < 1$. Then $f$ has an extension $F$ to $D$ such that*

$$\int_D \delta^{p-1}|dF|^2_\Omega \le C \int_{\partial D}\int_{\partial D} \frac{|f(\zeta)-f(z)|^2}{d(z,\zeta)^{n+1-p}}, \tag{4.7}$$

*and*

$$\|\delta^{p-1}|dF|^2_\Omega\|^2_{\mathcal{C}_p} \le C \sup_{B_h} \frac{1}{h^{p+n-1}} \int_{B_h}\int_{B_h} \frac{|f(\zeta)-f(z)|^2}{d(z,\zeta)^{n+1-p}}. \tag{4.8}$$

If $f$ is the boundary values of a holomorphic function one can let $F$ be just the holomorphic extension. Thus the proposition implies the other direction of Theorem 1.1. For an arbitrary $f$ we let $F$ be the $\Omega$-harmonic extension in the ball case and the approximate harmonic extension mentioned in Section 2 in the general case. Summing up we have

**Corollary 4.3.** *Let $f \in L^2(\partial D)$ and $0 < p < 1$. Then $f \in \mathcal{D}_p^{\mathbb{R}}$ if and only if there is an extension $F$ such that $\delta^{p-1}|dF|^2_\Omega$ has finite mass, and $f \in \mathcal{Q}_p^{\mathbb{R}}$ if and only if $\delta^{p-1}|dF|^2_\Omega \in \mathcal{C}_p$.*

*Proof of Proposition 4.2.* As usual we only prove this for the ball. The crucial point is to prove (4.7). By (2.6),

$$\int_{z\in D} \delta(z)^{p-1}|dF(z)|^2_\Omega \lesssim \int_{z\in D} \partial|F(z)|^2_\Omega \int_{\zeta\in\partial D} \frac{1}{d(\zeta,z)^{n+1-p}}$$
$$\sim \int_{\zeta\in\partial D}\int_{z\in D} \frac{|dF(z)|^2_\Omega}{d(\zeta,z)^{n+1-p}}.$$

Thus it is enough to prove that

$$I(\zeta) = \int_{z\in D} \frac{|dF(z)|^2_\Omega}{d(\zeta,z)^{n+1-p}} \lesssim \int_{z\in\partial D} \frac{|f(z)-f(\zeta)|^2}{d(\zeta,z)^{n+1-p}} = Q(\zeta). \tag{4.9}$$

For simplicity let us first consider the case when $F$ is holomorphic. We can then write

$$\frac{|\partial F|^2_\Omega}{d^{n+1-p}} \sim \left|\frac{\partial(F-f(z))}{v^{\frac{n+1-p}{2}}}\right|^2_\Omega$$
$$\lesssim \left|\partial\left(\frac{(F-f(z))}{v^{\frac{n+1-p}{2}}}\right)\right|^2_\Omega + \left|\frac{(F-f(z))}{v^{\frac{n+1-p}{2}}}\right|^2 |\partial\log v|^2_\Omega.$$

By (3.4) and the Carleson-Hörmander inequality, keeping in mind that $|\partial\log v|^2_\Omega$ is a Carleson measure, we get that $I(z) \lesssim Q(\zeta)$. If $F$ is just $\Omega$-harmonic, the proof is somewhat more involved. We have that

$$\frac{|dF|^2_\Omega}{|v|^{n+1-p}} = \delta\frac{dd^c|F|^2}{|v|^{n+1-p}} \wedge \Omega^{n-1} \lesssim \delta\, dd^c \frac{|F-f(z)|^2}{|v|^{n+1-p}} \wedge \Omega^{n-1}$$
$$+ \frac{|F-f(z)|^2}{|v|^{n+3-p}}|\partial\log v|^2_\Omega + \frac{|F-f(z)|}{|v|^{n+2-p}}|\partial\log v|_\Omega.$$

By (2.11) and the Cauchy-Schwarz' inequality we thus have $I \lesssim Q + R + \sqrt{I}\sqrt{R}$, where

$$R = \int_D \frac{|F - f(z)|^2}{|v|^{n+3-p}} |\partial \log v|^2_\Omega.$$

It is thus enough to verify that $R \lesssim Q$. By (2.10) $|\partial \log v|^2_\Omega \lesssim |v|$. Furthermore, since the Poisson-Szegő integral is positive and maps 1 to 1, it follows that $|F - f(z)|^2 \le U$ where $U$ is the Poisson-Szegő integral of $|f - f(z)|^2$. Thus

$$\begin{aligned} R \le \int_D \frac{U(\zeta)}{|v|^{n+2-p}} &= \int_{w\in\partial D} |f(w) - f(z)|^2 \int_D \frac{\delta(\zeta)^n d\lambda(\zeta)}{|v(\zeta, z)|^{n+2-p}|v(\zeta, w)|^{2n}} \\ &\lesssim \int_{w\in\partial D} |f(w) - f(z)|^2 \int_D \frac{d\lambda(\zeta)}{|v(\zeta, z)|^{n+2-p-\alpha}|v(\zeta, w)|^{n+\alpha}} \\ &\lesssim \int_{w\in\partial D} \frac{|f(w) - f(z)|^2}{|v(z, w)|^{n+1-p}} \end{aligned}$$

by (2.7), if $\alpha$ is chosen so that $1 - p < \alpha < 1$. It follows that $R \lesssim Q$ and hence (4.7) is proved.

In the proof of (4.8) we assume that $F = f$ is holomorphic. The general case is completely analogous. We first note that $\|f\|_{BMO(\partial D)} \le \|f\|_{Q_p^{\mathbb{R}}(\partial D)}$, since

$$\begin{aligned} &\left(\frac{1}{|B_h|}\int_{B_h} |f - f_{B_h}|\right)^2 \\ &\le \frac{1}{|B_h|}\int_{B_h} |f - f_{B_h}|^2 \le \frac{1}{|B_h|^2}\int_{B_h}\int_{B_h} |f(z) - f(\zeta)|^2 \\ &\lesssim \frac{h^{n+1-p}}{h^{2n}} \int_{B_h}\int_{B_h} \frac{|f(z) - f(\zeta)|^2}{d(\zeta, z)^{n+1-p}} \le \|f\|^2_{Q_p^{\mathbb{R}}(\partial D)}. \end{aligned}$$

Fix a Carleson cube $Q_h$. We need to estimate $\int_{Q_h} \delta^p |\partial f|^2_\beta$. Decompose $f$ on the boundary as $f = f_1 + f_2 + f_3$, where $f_1 = f_{B_h}$, $f_2 = \chi(f - f_1)$, $f_3 = (1 - \chi)(f - f_1)$ and $\chi$ is the characteristic function of $B_h$. Extend $f_i$ to $D$ as $f_i = P_0 f_i$, so that $f = f_1 + f_2 + f_3$ also in $D$. Of course $f_1$ gives no contribution. For $f_2$ we have by (4.7),

$$\int_{Q_h} \delta^p |\partial f_2|^2_\beta \le \int_D \delta^p |\partial f_2|^2_\beta \le \int_{B_h}\int_{B_h} \frac{|f(z) - f(\zeta)|^2}{|v(\zeta, z)|^{n+1-p}} \le \|f\|^2_{Q_p} d^{n+p-1}.$$

Finally, for $f_3$ we have by [1, p. 272], $|\partial f_3|_\beta \lesssim \|f\|_{BMO}/h$. Hence

$$\int_{Q_h} \delta^p |\partial f_3|^2 \lesssim \frac{h^{p+n+1}}{h^2} \|f\|^2_{BMO} \lesssim h^{n+p-1} \|f\|^2_{Q_p^{\mathbb{R}}}.$$

□

## 5. Inclusions between $Q_p^{\mathbb{R}}(\partial D)$-spaces

If $0 < p_1 < p_2 < 1$ it is easy to verify that $Q_{p_1}^{\mathbb{R}}(\partial D) \subset Q_{p_2}^{\mathbb{R}}(\partial D) \subset BMO(\partial D)$. In this section we will construct functions that show that these inclusions are all strict. The examples also show that $L^\infty(\partial D)$ is not contained in any $Q_p^{\mathbb{R}}(\partial D)$.

To simplify the presentation we first consider the Euclidean space $\mathbb{R}^n$. We say that $f \in Q_p(\mathbb{R}^n)$ if and only if

$$\sup \frac{1}{h^{n+p-1}} \int_{B_h} \int_{B_h} \frac{|f(x) - f(y)|^2}{|x-y|^{n+1-p}} dxdy < +\infty \,, \tag{5.1}$$

where the supremum is taken over all balls in $\mathbb{R}^n$ with radius $h$.

Let $B_i = \{x; |x| \le 1/i\}$. Decompose $\mathbb{R}^n \setminus \{0\}$ as $\cup \Lambda_i$ where $\Lambda_i = B_{5i-2} \setminus B_{5i+3}$ and let $A_i = B_{5i} \setminus B_{5i+1}$. Choose a smooth function $\chi_i$ with $\chi_i = 1$ on $A_i$, $\operatorname{supp} \chi_i \subset B_{5i-1} \setminus B_{5i+1}$ and $\|d\chi_i\|_\infty \lesssim i^2$. This is possible since $d(B_i^c, B_{i+1}) = \frac{1}{i} - \frac{1}{i+1} \approx \frac{1}{i^2}$. Define $f = f_\alpha$ by

$$f_\alpha(x) = \sum_{i=1}^{\infty} i^{(\alpha-1)/2} \chi_i(x) \;.$$

**Proposition 5.1.** *Assume that* $0 < \alpha < 1$. *Then* $f_\alpha \in Q_p$ *if and only if* $p \ge \alpha$. *If* $\alpha = 1$, $f_\alpha$ *is in no* $Q_p$, $0 < p < 1$.

*Proof.* Assume first that $p \ge \alpha$. To check (5.1), it is enough to consider balls centered at 0 with radius $1/5k$ for some $k$. By symmetry we may also assume that $|x| \ge |y|$. We want to show that

$$I(k) = k^{n+p-1} \sum_{i \ge k} \int_{x \in \Lambda_i} dx \int_{|y| \le |x|} \frac{|f(x) - f(y)|^2}{|x-y|^{n+1-p}} dy$$

is bounded as $k \to \infty$. Divide the range of integration in the inner integral depending on wheter $|x - y| \lesssim 1/i^2$ or not. If $x \in \Lambda_i$ we always have $|f(x) - f(y)| \le |f(x)| + |f(y)| \lesssim i^{(\alpha-1)/2}$. If furthermore $|x - y| \lesssim 1/i^2$, then in order that $f(x) - f(y) \ne 0$ also $y \in \Lambda_i$ and hence

$$|f(x) - f(y)|^2 = i^{\alpha-1} |\chi_i(x) - \chi_i(y)|^2 \lesssim i^{\alpha+3} |x-y|^2 \;.$$

This implies

$$\int_{|y| \le |x|} \frac{|f(x) - f(y)|^2}{|x-y|^{n+1-p}} dy$$

$$\lesssim i^{\alpha+3} \int_{|x-y| \lesssim 1/i^2} \frac{dy}{|x-y|^{n-1-p}} + i^{\alpha-1} \int_{|x-y| \gtrsim 1/i^2} \frac{dy}{|x-y|^{n+1-p}} \lesssim i^{\alpha-2p+1}$$

and we get

$$I(k) \lesssim k^{n+p-1} \sum_{i \ge k} i^{\alpha-2p+1} |\Lambda_i| \approx k^{n+p-1} \sum_{i \ge k} i^{\alpha-2p-n} \approx k^{\alpha-p} < \infty$$

as $\alpha - p \le 0$.

Conversely, if $p<\alpha$ it is enough to show that

$$I(k)=k^{n+p-1}\sum_{i\geq k}\int_{x\in A_i}dx\int_{y\in B_k}\frac{|f(x)-f(y)|^2}{|x-y|^{n+1-p}}dy\to\infty$$

as $k\to\infty$. For $x\in A_i$ there is a ball $B_x$ with center on the ray from the origin through $x$ with radius $\sim 1/i^2$ contained in $B_{5i+2}\setminus B_{5i+3}$. Then on $B_x$, $f(y)=0$ and $|x-y|\sim 1/i^2$. Thus

$$\begin{aligned}I(k)&\geq k^{n+p-1}\sum_{i\geq k}\int_{x\in A_i}dx\int_{B_x}\frac{|f(x)-f(y)|^2}{|x-y|^{n+1-p}}dy\\&\gtrsim k^{n+p-1}\sum_{i\geq k}i^{2(n+1-p)+\alpha-1}\int_{x\in A_i}dx\int_{y\in B_x}dy\\&\approx k^{n+p-1}\sum_{i\geq k}i^{\alpha-2p-n}\approx k^{\alpha-p}\to\infty\ ,\end{aligned}$$

as $\alpha-p>0$.

□

When we try to transfer this example to $\mathbb{C}^n$, there is an additional difficulty due to the non-isotropic distance. Therefore we modify the example slightly and let $A_i$ be a certain part of the annulus (but with volume comparable to it) where only the distance in the "long" directions are important. The details are as follows: Fix a point $p\in\partial D$ and choose coordinates $x=(x_1,x')\in\mathbb{R}\times\mathbb{R}^{2n-2}$ so that $d(x)=d(x,p)\sim|x_1|+|x'|^2$. In analogy with the Euclidean case we let $B_i=\{x;|x_1|\leq 1/i,|x'|\leq 1/\sqrt{i}\}$, $\Lambda_i=B_{5i-2}\setminus B_{5i+3}$ but now we let $A_i$ be that part of $\Lambda_i$ defined by

$$A_i=\left\{x;|x_1|<\frac{1}{3i},\ \frac{1}{\sqrt{5i+1}}<|x'|<\frac{1}{\sqrt{5i}}\right\}\ .$$

Note that $|A_i|\sim|\Lambda_i|\sim 1/i^{n+1}$. We also let

$$\tilde{A}_i=\left\{x;|x_1|<\frac{1}{2i},\frac{1}{\sqrt{5i+2}}<|x'|<\frac{1}{\sqrt{5i-1}}\right\}$$

and choose $\chi_i$ with $\operatorname{supp}\chi_i\subset\tilde{A}_i$, $\chi_i=1$ on $A_i$ and $\|d\chi_i\|_\infty\lesssim i^{3/2}$. Note that $|\chi_i(x)-\chi_i(y)|\lesssim i^{3/2}|x-y|\lesssim\sqrt{i^3d(x,y)}$. Define $f=f_\alpha$ by

$$f_\alpha(x)=\sum_{i=1}^{\infty}i^{\alpha-1}\chi_i(x)\ .$$

**Proposition 5.2.** *Assume that $0<\alpha<1$. Then $f_\alpha\in Q_p^{\mathbb{R}}$ if and only if $p\geq\alpha$. If $\alpha=1$, $f_\alpha$ is in no $Q_p^{\mathbb{R}}$, $0<p<1$.*

*Proof.* The proof is very similar to that of Proposition 5.1. If $p \geq \alpha$, we want to estimate

$$I(k) = k^{n+p-1} \sum_{i \geq k} \int_{x \in \Lambda_i} dx \int_{d(y) \leq d(x)} \frac{|f(x) - f(y)|^2}{d(x,y)^{n+1-p}} dy \ .$$

If $x \in \Lambda_i$ and $\chi_j(y) \neq 0$ for some $j \neq i$ then $|x - y| \gtrsim \frac{1}{i^{3/2}}$ and hence $d(x,y) \gtrsim \frac{1}{i^{3/2}}$. Thus if $x \in \Lambda_i$ and $d(x,y) \lesssim \frac{1}{i^{3/2}}$ we have $|f(x) - f(y)| \lesssim i^{\alpha-1} |\chi_i(x) - \chi_i(y)| \lesssim i^{\alpha-1} \sqrt{i^3 d(x,y)}$. Otherwise we use that $|f(x) - f(y)| \leq i^{\alpha-1}$ to obtain

$$\begin{aligned} I(k) \lesssim k^{n+p-1} \sum_{i \geq k} i^{2(\alpha-1)} \int_{x \in \Lambda_i} dx \\ \left\{ \int_{d(x,y) \lesssim i^{-3/2}} \frac{i^3 dy}{d(x,y)^{n-p}} + \int_{d(x,y) \gtrsim i^{-3/2}} \frac{dy}{d(x,y)^{n-p+1}} \right\} \\ \lesssim k^{n+p-1} \sum_{i \geq k} i^{2(\alpha-1)+3-3p} |\Lambda_i| \lesssim k^{2(\alpha-p)} \ , \end{aligned}$$

and hence $I(k)$ is bounded as desired.

When $p < \alpha$ it is enough to show that

$$I(k) = k^{n+p-1} \sum_{i \geq k} \int_{x \in A_i} dx \int_{B_{5k}} \frac{|f(x) - f(y)|^2}{d(x,y)^{n+1-p}} dy$$

is unbounded. For $x \in A_i$ there is an Euclidean ball of radius $\frac{1}{i^{3/2}}$ contained in $\Lambda_i \setminus \tilde{A}_i$, and hence also a Koranyi ball $B_x$ with radius $1/i^3$. For $y \in B_x$, $d(x,y) \gtrsim 1/i^3$ and hence

$$\begin{aligned} I(k) \gtrsim k^{n+p-1} \sum_{i \geq k} i^{2(\alpha-1)} \int_{x \in A_i} dx \int_{B_x} i^{3(n+p-1)} dy \\ \gtrsim k^{n+p-1} \sum_{i \geq k} i^{2\alpha-3p+1} \gtrsim k^{2(\alpha-p)} \to \infty \end{aligned}$$

as $k \to \infty$. □

## 6. Boundedness of the Szegő projection

Our aim now is to prove Theorem 1.2, and to begin with we prove the similar statement for the explicit operator $P_0$ in (2.2). In the ball case $P_0 f$ coincides with the Szegő integral $Sf$ and so we are done. In the general case it was proved in [10], see also [2], that $Sf = P_0 f + Rf$, where $Rf$ is somewhat regularizing; in particular it maps $L^p(\partial D)$ into $C^\epsilon(\partial D)$ for some $\epsilon$ if $p$ is large enough, and Theorem 1.2 follows.

**Proposition 6.1.** *Assume that* $0 < p < 1$. *The explicit projection* $P_0$ *maps* $\mathcal{D}_p^{\mathbb{R}} \to \mathcal{D}_p$ *and* $\mathcal{Q}_p^{\mathbb{R}} \to \mathcal{Q}_p$.

*Proof.* Let $F$ be the extension of $f$ from Proposition 4.2; thus the $\Omega$-harmonic extension in the ball case. Letting $\psi = F/v^n$ in (2.11) we get that

$$P_0 f(z) = \int_{\partial D} \frac{F}{v^n} = \text{good terms} + \int_D \delta \partial F \wedge \bar{\partial} \frac{1}{v^n} \wedge \Omega^{n-1}$$

so

$$\partial_z P_0 f(z) = \text{good terms} + \int_D \delta \partial F \wedge \partial_z \bar{\partial}_\zeta \frac{1}{v^n} \wedge \Omega^{n-1}.$$

It follows from (2.10) that $|\partial_z \bar{\partial}_\zeta v^{-n}|_\Omega \lesssim |v|^{-(n+1)}$ so

$$|\partial_z P_0 f(z)|_\Omega \lesssim \text{good terms} + \int_D |\partial F|_\Omega \frac{1}{d^{n+1}} \ .$$

Hence if $0 < \epsilon < 1$ we have

$$\begin{aligned} \delta^{p-1} |\partial P_0 f|_\Omega^2 &\lesssim \text{good terms} + \delta^{p-1} \int_D |\partial F|_\Omega^2 \frac{\delta^\epsilon}{d^{n+1}} \int_D \frac{\delta^{-\epsilon}}{d^{n+1}} \\ &\lesssim \text{good terms} + \delta^{p-1-\epsilon} \int_D |\partial F|_\Omega^2 \frac{\delta^\epsilon}{|v|^{n+1}}. \end{aligned} \tag{6.1}$$

If $\epsilon$ is chosen so that $-1 < p - 1 - \epsilon < 0$ and we integrate with respect to $z$ we get

$$\int_D \delta^{p-1} |\partial P_0 f|_\Omega^2 \lesssim \int_D \delta^{p-1} |\partial F|_\Omega^2,$$

which proves that $P_0$ maps $\mathcal{D}_p^{\mathbb{R}}$ onto $\mathcal{D}_p$.

To prove the $\mathcal{Q}_p$ boundedness, we obtain from (6.1) (ignoring the good terms) and Fubini's theorem that

$$\begin{aligned} &\delta^\alpha(w) \int_{z \in D} \frac{\delta^{p-1}(z) |\partial P_0 f|_\Omega^2}{d(z,w)^{n+\alpha+p-1}} \\ &\lesssim \delta^\alpha(w) \int_{\zeta \in D} \delta^\epsilon(\zeta) |\partial F(\zeta)|_\Omega^2 \int_{z \in D} \frac{\delta^{p-1-\epsilon}(z)}{d(z,w)^{n+\alpha+p-1} d(\zeta, z)^{n+1}} \\ &\lesssim \delta^\alpha(w) \int_{\zeta \in D} \frac{\delta^{p-1}(\zeta) |\partial F(\zeta)|_\Omega^2}{d(\zeta, w)^{n+\alpha+p-1}} \ , \end{aligned}$$

where the last inequality follows from (2.8) if $\alpha$ is small enough. Hence by Lemma 3.2, $P_0 f \in \mathcal{Q}_p$. □

## 7. Multipliers on $\mathcal{D}_p$ and $\mathcal{Q}_p$

There is a precise characterization of the multipliers $\mathcal{M}(BMO)$ and $\mathcal{M}(BMOA)$, due to Stegenga [17] for $n = 1$ and [15] in a general strictly pseudoconvex domain, and for the Bloch space, see [14]. In one variable there is also a characterization of the multipliers $\mathcal{M}(\mathcal{D}_p)$ of $\mathcal{D}_p$ due to Taylor, [18]. For the $\mathcal{Q}_p$ spaces, $0 < p < 1$, no such characterization is known. As in one variable, [21] we have though the following two partial results.

**Proposition 7.1.** *If $g \in \mathcal{M}(\mathcal{Q}_p)$, then $g \in H^\infty$ and*

$$\delta(z)\int_D \frac{\delta^{p-1}|\log(1/v(\zeta,z)|^2|\partial g|^2_\Omega}{d(\zeta,z)^{n+p}} \lesssim 1. \tag{7.1}$$

Note that in particular, by Lemma 3.1, $\mathcal{M}(\mathcal{Q}_p) \subseteq \mathcal{Q}_p$.

*Proof.* For any $f \in \mathcal{Q}_p$ we have by Lemma 3.6 that

$$|g(z)f(z)| \le \|gf\|_{\mathcal{Q}_p}\log(1/\delta(z)) \le \|M_g\|\|f\|_{\mathcal{Q}_p}\log(1/\delta(z)).$$

If we take $f_w(z) = \log v(w,z)$ and let $z = w$ we get by Example 1 that

$$|g(z)|\log(1/\delta(z)) \le \|M_g\|\log(1/\delta(z))$$

and hence $g \in H^\infty$. Furtermore, using

$$|\partial g|^2_\Omega|f|^2 \lesssim |\partial(gf)|^2_\Omega + |\partial f|^2_\Omega,$$

we obtain

$$\delta(z)\int_D \frac{\delta^{p-1}|f|^2|\partial g|^2_\Omega}{|v|^{n+p}} \lesssim \|gf\|^2_{\mathcal{Q}_p} + \|f\|^2_{\mathcal{Q}_p} \lesssim \|f\|^2_{\mathcal{Q}_p}.$$

Taking again $f(\zeta) = f_z(\zeta)$, we get the desired conclusion. □

A similar argument shows that any $g \in \mathcal{M}(\mathcal{D}_p)$ must be bounded.

**Proposition 7.2.** *If $g \in H^\infty$ and*

$$\delta(z)\int_D \frac{\delta^{p-1}(\log(1/\delta))^2|\partial g|\Omega^2}{d(\zeta,z)^{n+p}} \lesssim 1, \tag{7.2}$$

*then $g \in \mathcal{M}(\mathcal{Q}_p)$.*

*Proof.* By Lemma 3.6 we have for $f \in \mathcal{Q}_p$

$$\delta(z)\int_D \frac{\delta^{p-1}|\partial(gf)|^2_\Omega}{d(\zeta,z)^{n+p}}$$
$$\lesssim \delta(z)\int_D \frac{\delta^{p-1}|\partial f|^2_\Omega}{d(\zeta,z)^{n+p}} + \delta(z)\int_D \frac{\delta^{p-1}(\log(1/\delta)^2|\partial g|^2_\Omega}{d(\zeta,z)^{n+p}} \lesssim 1.$$

Thus by Corollary 3.2, $gf \in \mathcal{Q}_p$.

□

One can obtain similar results for the real spaces $\mathcal{Q}_p^{\mathbb{R}}$. The lack of a complete characterization of $\mathcal{M}(\mathcal{Q}_p)$ causes us some problems in proving corona type theorems for $\mathcal{Q}_p$, or more generally for computing the joint spectrum for an $m$-tuple of multipliers on $\mathcal{Q}_p$. For technical reasons we must assume that the multipliers $g_j \in \mathcal{M}(\mathcal{Q}_p)$ actually are multipliers on the real space $\mathcal{Q}_p^{\mathbb{R}}$. We do not know if this extra assumption is necessary in general. Indeed if $n = 1$ it is not since we have

**Proposition 7.3.** *Assume that $0 < p < 1$ and $n = 1$. If $g \in \mathcal{M}(\mathcal{Q}_p)$, then it is also a multiplier on the space $\mathcal{Q}_p^{\mathbb{R}}$.*

*Proof.* Assume that $g \in \mathcal{M}(\mathcal{Q}_p)$ and $u \in \mathcal{Q}_p^{\mathbb{R}}$. Since the Szegő projector $S$ is bounded on $\mathcal{Q}_p$ it follows that $u$ is the boundary values of $f + \bar{f}$ for some function $f \in \mathcal{Q}_p$. Since $gf$ is in $\mathcal{Q}_p$ by assumption it is enough to check that $g\bar{f}$ is in $\mathcal{Q}_p^{\mathbb{R}}$. However,

$$|d(f\bar{g})|_\beta \lesssim |\partial f|_\beta + |\partial(gf)|_\beta$$

and hence by Corollary 4.3 $f\bar{g} \in \mathcal{Q}_p$. $\square$

## 8. Spectral properties of multipliers on $\mathcal{D}_p$ and $\mathcal{Q}_p$

Finally we consider corona type theorems for $\mathcal{Q}_p$. Let $\widetilde{\mathcal{M}}(\mathcal{Q}_p)$ denote the space of holomorphic multipliers on $\mathcal{Q}_p^{\mathbb{R}}$, i.e., the set of functions $b \in H^2$ such that $f \mapsto bf$ is bounded on $\mathcal{Q}_p^{\mathbb{R}}$. If $n = 1$ we have seen that $\widetilde{\mathcal{M}}(\mathcal{Q}_p) = \mathcal{M}(\mathcal{Q}_p)$.

**Theorem 8.1.** *Let $0 < p < 1$ and assume that $g_1, \ldots, g_m \in \widetilde{\mathcal{M}}(\mathcal{Q}_p)$ satisfy*

$$|g| \geq \delta > 0. \tag{8.1}$$

*If $\phi \in \mathcal{Q}_p$, there are $u_j \in \mathcal{Q}_p$ such that $\sum g_j u_j = \phi$ and $\|u\|_{\mathcal{Q}_p} \leq C_\delta \|\phi\|_{\mathcal{Q}_p}$, where $C_\delta$ is independent of $m$. The same result holds for $\mathcal{D}_p$ instead of $\mathcal{Q}_p$.*

The case $p = 1$ was proved in [15]. For analogous statements for other functions spaces, see [14] and [3] and the references given there. With a small generalization we get a statement about the joint spectrum (Taylor spectrum) of multipliers on $\mathcal{Q}_p$ and $\mathcal{D}_p$.

**Theorem 8.2.** *Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and assume that $g_1, \ldots, g_m \in \widetilde{\mathcal{M}}(\mathcal{Q}_p)$. Then $\sigma(g, \mathcal{Q}_p) = \overline{g(D)}$. Again one can replace $\mathcal{Q}_p$ by $\mathcal{D}_p$.*

For the definition of Taylor spectrum and related results see [3] and the references given there. Following the set up (and notation) in [3], the proofs of Theorem 8.1 and Theorem 8.2 boil down to get $\mathcal{D}_p^{\mathbb{R}}$ and $\mathcal{Q}_p^{\mathbb{R}}$ estimates of each of the terms $(\delta_g K)^k(\gamma \cap (\bar\partial\gamma)^k \cap \phi$ given that $\phi$ is in $\mathcal{D}_p$ or $\mathcal{Q}_p$, where $\gamma = \sum \gamma_j e_j = \sum \bar g_j/|g|^2 e_j$ and $K$ is an appropriate solution operator for the $\bar\partial$- equation. We concentrate on the case with $\mathcal{Q}_p$, $0 < p < 1$ ; the estimates for $\mathcal{D}_p$ follow in a similar but simpler way.

If $k = 0$ then we have to show that the boundary values of $u = \bar g_j \phi/|g|^2$ is in $\mathcal{Q}_p^{\mathbb{R}}$. By Corollary 4.3 it is enough to check that $\delta^{p-1}|du|_\Omega^2$ is in $\mathcal{C}_p$. However, using that $|g_j| \lesssim 1$ we get

$$|du|_\Omega \lesssim |\partial g_j|_\Omega |\phi| + |g||\partial\phi|_\Omega \lesssim |\partial\phi|_\Omega + |\partial(g_j\phi)|_\Omega$$

and since both $\phi$ and $g_j\phi$ are in $\mathcal{Q}_p$ it follows that $\delta^{p-1}|du|_\Omega^2$ is in $\mathcal{C}_p$.

For $k \geq 1$ we need estimates of the operator $K$.

**Theorem 8.3.** *Let $0 < p < 1$. There is an operator*

$$K\colon C^\infty_{0,q+1}(\overline{D}) \to C_{0,q}(\overline{D}),$$

*such that $\bar\partial K f = f$ if $\bar\partial f = 0$, and which satisfies the estimates*

$$\|\delta^\tau |Kf|_\Omega\|_{\mathcal{C}_p} \lesssim \|\delta^{\tau+1/2}|f|_\Omega\|_{\mathcal{C}_p} \tag{8.2}$$

*for any (not necessarily $\bar\partial$ closed) $f$ and $\tau > p-2$.*

*In the limit case $\tau = p-2$ we have, for the complex tangential boundary values of $Kf$,*

$$\|Kf\|_{\mathcal{Q}_p^{\mathbb{R}}} \lesssim \|\delta^{p-3/2}|f|_\Omega\|_{\mathcal{C}_p} \tag{8.3}$$

*and*

$$\|Kf\|^2_{\mathcal{Q}_p^{\mathbb{R}}} \lesssim \|\delta^{p-1}|f|_\Omega^2\|^2_{\mathcal{C}_p}. \tag{8.4}$$

We can now conclude the proof of Theorem 8.2. Let $k = 1$ and put $f = \gamma \cap \bar\partial\gamma \cap \phi$. As before one verifies that

$$|f|_\Omega^2 \lesssim |\partial\phi|_\Omega^2 + |\partial(g\phi)|_\Omega^2 .$$

Hence $\delta^{p-1}|f|_\Omega^2 \in \mathcal{C}_p$ and by (8.4), $Kf$ is in $\mathcal{Q}_p^{\mathbb{R}}$.

Finally we consider $k \geq 2$. We have proved that $\delta^{p-1}|\partial g \cap \phi|_\Omega^2 \in \mathcal{C}_p$. Also, by Theorem 7.1, $g \in H^\infty \cap \mathcal{Q}_p$. Hence $|\partial g|_\Omega \lesssim \delta^{-1/2}$ and $\delta^{p-1}|\partial g|_\Omega^2 \in \mathcal{C}_p$. Thus

$$\delta^{(k-2)/2+p-1}|\gamma \cap (\bar\partial\gamma)^k \cap \phi|_\Omega$$
$$\lesssim \delta^{p-1}|\partial g|_\Omega |\partial g \cap \phi|_\Omega \lesssim \delta^{p-1}(|\partial g|_\Omega^2 + |\partial g \cap \phi|_\Omega^2) \in \mathcal{C}_p .$$

By $k-1$ applications of (8.2) and one of (8.3) we obtain $(\delta_g K)^k(\gamma \cap (\bar\partial\gamma)^k \cap \phi) \in \mathcal{Q}_p^{\mathbb{R}}$.

*Sketch of proof of Theorem 8.3.* We shall choose $K$ as the operator $K_\alpha$ in [2], with $\alpha$ large enough. It then satisfies the estimate

$$|Kf(z)| \lesssim \int_D \frac{\delta(\zeta)^{r-1}|f|_\Omega}{d(\zeta,z)^{n+r-1/2}} \left(\frac{d(\zeta,z)}{\sigma}\right)^{2n-1}, \tag{8.5}$$

where $r$ is some large number, and $\sigma(\zeta,z)$ is like $d(\zeta',z') + |d(\zeta) - d(z)| + \sqrt{d(\zeta)+d(z)}c(\zeta',z')$, where $c(\zeta',z')$ is the distance between $\zeta'$ and $z'$ in the complex tangential directions. Moreover, the boundary values of $Kf$ has an extension $u$ to the interior such that

$$|du|_\Omega \lesssim \int_D \frac{\delta^{r-1}|f|_\Omega}{d(\zeta,z)^{n+r}}. \tag{8.6}$$

We first consider (8.2). We decompose the integral in (8.5) into two parts; one where $d(\zeta,z) \le c\sigma(\zeta,z)$, and one where $d(\zeta,z) > c\sigma(\zeta,z)$. The first part immediately is handled by Lemma 3.4. In the second case $d(\zeta,z) \sim \delta(\zeta) \sim \delta(z)$, and using this the desired estimate follows by a straight forward calculation.

Next we prove (8.3). Since $\delta(\zeta) \le d(\zeta,z)$ we may assume that $r$ is suitably small in (8.6). By Schwarz' inequality we obtain

$$|du(z)|_\Omega^2 \lesssim \int_{\zeta\in D} \frac{\delta(\zeta)^{p-3/2}|f|_\Omega}{d(\zeta,z)^{n+r+p-1}} \int_{\zeta\in D} \frac{\delta(\zeta)^{2r-p-1/2}|f|_\Omega}{d(\zeta,z)^{n+r+1-p}}\,.$$

By Lemma 3.1 the first integral is bounded by $\delta^{-r}$ and hence

$$\begin{aligned}\delta^\alpha(w)\int_{z\in D} &\frac{\delta(z)^{p-1}|\partial u|_\Omega^2}{d(w,z)^{n+\alpha+p-1}} \\ &\lesssim \delta^\alpha(w)\int_{\zeta\in D} \delta(\zeta)^{p-3/2}|f(\zeta)|_\Omega \\ &\qquad\times \int_{z\in D} \frac{\delta(z)^{p-1-r}\delta(\zeta)^{2(r-p)+1}}{d(w,z)^{n+\alpha+p-1}d(\zeta,z)^{n+r+1-p}}\,.\end{aligned}$$

Now, if $r = p - \epsilon$ and $\epsilon$ and $\alpha$ are small enough (2.8) implies

$$\delta^\alpha(w)\int_{z\in D} \frac{\delta(z)^{p-1}|\partial u|_\Omega^2}{d(w,z)^{n+\alpha+p-1}} \lesssim \delta^\alpha(w)\int_{\zeta\in D} \frac{\delta(z)^{p-3/2}|f|_\Omega}{d(w,z)^{n+\alpha+p-1}}\,,$$

and (8.3) follows by Lemma 3.4.

To prove (8.4) we use (8.6) and proceed exactly as in the proof of Proposition 6.1, and conclude that the $\mathcal{C}_p$ norm of $\delta^{p-1}|\partial u|^2$ is bounded by a constant times the $\mathcal{C}_p$ norm of $\delta^{p-1}|f|^2$.

□

*Remark* 2. To prove Theorem 8.1 when $p = 1$ we need a somewhat deeper result than Theorem 8.3. The estimates (8.2) and (8.3) hold for $p = 1$ as well, but (8.4) has to be replaced by the Wolff type estimate

$$\|Kf\|_{BMO}^2 \lesssim \||f|_\Omega^2\|_{\mathcal{C}_1} + \||\partial f|_\Omega\|_{\mathcal{C}_1}^2. \tag{8.7}$$

The extra condition on $\partial f$ in the last estimate cannot be dispensed with. In fact, let $f$ be $\bar\partial u$, where $u$ is the function from Remark 1. If there were a solution $v \in BMO$ to $\bar\partial v = f$, then also the $L^2(\partial D)$ minimal solution $v - Sv$ would do since $S$ is bounded on $BMO$; but one can check that $v - Sv = Kf$ is not in $BMO$.

In [3], (8.7) is proved by duality and the $T1$ theorem. However in this connection it is natural to suggest another argument. It is shown in [2] that there is a decomposition $Kf = Tf + M\partial f +$ error terms, where $Tf$ is anti-holomorphic. Moreover, $\bar\partial Tf$ satisfies the same estimate as $Kf$, namely (8.5), and as in the proof of (8.4) we get that $|\bar\partial Tf|^2_\Omega \in \mathcal{C}_1$. Since $Tf$ is anti-holomorphic it follows that $Tf|_{\partial D}$ is in $BMO$. The boundary values of $M(\partial f)$ has an extension $v$ to $D$ such that

$$|dv|_\Omega \lesssim \int_D \frac{\delta^{r-1/2}|\partial f|_\Omega}{d(\zeta, z)^{n+r+1/2}}.$$

Using Lemma 3.4 with $\alpha = -1/2$ we get that $\delta^{-1/2}|dv| \in \mathcal{C}_1$, and $\delta^{-1/2}|dv|_\Omega \lesssim \delta^{-1}$, and by an easy argument, cf. [19], it then follows that $M\partial f|_{\partial D}$ is in $BMO$. □

## References


[1] M. Andersson and H. Carlsson: *Wolff-type estimates for $\bar\partial_b$ and the $H^p$-corona problem in strictly pseudoconvex domains*, Ark. Math.,**32** (1994), 255–276.

[2] M. Andersson and H. Carlsson: *Formulas for approximate solutions of the $\partial\bar\partial$ equation in a strictly pseudoconvex domain*, Revista Mat. Iberoamericana **11** (1995), 67–101.

[3] M. Andersson, and H. Carlsson: *Estimates of solutions of the $H^p$ and $BMOA$ corona problem*, Math. Ann. **316**, 83-102 (2000).

[4] M. Andersson, H. Carlsson, M. Roginskaya and S. Sandberg: *An approximate Poisson integral in a strictly pseudoconvex domain*, In preparation.

[5] R. Aulaskari and P. Lappan: *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal*, Complex analysis and its applications, Pitman Research Notes in Math. **305** (1994), 136-146.

[6] R. Aulaskari, J. Xiao and R. Zhao: *On subspaces and subsets of BMOA and UBC*, Analysis **15** (1995), 101-121.

[7] M. Essén and J. Xiao: *Some results on $\mathcal{Q}_p$ spaces,* $0 < p < 1$, J. reine angew. Math. **485** (1997), 173-195.

[8] M. Essén, S. Jansson, L. Peng and J. Xiao: *$Q_p$ spaces of several real variables*, Preprint.

[9] J. Garnett: *Bounded analytic functions*, Academic Press,1981.

[10] N. Kerzman & E. Stein: *The Szegő kernel in terms of Cauchy-Fantappiè kernels*, Duke Math. J. **45** (1978), 197–224.

[11] X. Massaneda: *Private communication*, .

[12] A. Nicolau: *The corona property for bounded analytic functions in some Besov spaces*, Proc AMS **110** (1990), 135-140.

[13] A. Nicolau and J. Xiao: *Bounded functions in Möbius invariant Dirichlet-type spaces*, J. Funct. Anal. **150** (1997), 383-425.

[14] J.M. Ortega & J. Fabrega: *Corona type decomposition in some Besov spaces*, Math. Scand. **78** (1996), 93–111.

[15] J.M. Ortega & J. Fabrega: *Pointwise multipliers and corona type decomposition in $BMOA$*, Ann. Inst. Fourier **46** (1996), 111–137 .

[16] W. Rudin: *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag (1980).

[17] D. Stegenga: *Bounded Toeplitz operators on $H^1$ and applications of the duality between $H^1$ and the functions of bounded mean oscillation.*, Amer. J. Math. **98** (1976), 573–589.

[18] G. D. Taylor: *Multipliers of the Dirichlet space*, TAMS., **123** (1996), 229-240.

[19] N. Varopoulos: *BMO functions and the $\bar{\partial}$-equation*, Pacific J. Math., **71** (1977), 221-273.

[20] J. Xiao: *The $\bar{\partial}$-problem for the multipliers of the Sobolev space*, Manuscripta Math., **97** (1998, 217–232).

[21] J. Xiao: *The $Q_p$ corona theorem*, To appear in Pacific J. Math., **94** (2000).

Department of Mathematics, Chalmers University of Technology and Göteborg University, S-412 96 GÖTEBORG, SWEDEN

*E-mail address*: matsa@math.chalmers.se, hasse@math.chalmers.se