Generalized permutation patterns and a classification of the Mahonian statistics

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Abstract

We introduce generalized permutation patterns, where we allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We show that essentially all Mahonian permutation statistics in the literature can be written as linear combinations of such patterns. Almost all known Mahonian permutation statistics can be written as linear combinations of patterns of length at most 3. There are only fourteen possible such Mahonian statistics, which we list. Of these, eight are known and we give proofs for another three. The remaining three we conjecture to be Mahonian. We also give an explicit numerical description of the combinations of patterns a Mahonian statistic must have, depending on the maximal length of its patterns.

1 Introduction and preliminaries

The simplest, and best known, Mahonian permutation statistic is the number of inversions. Its distribution, which is the defining criterion of a Mahonian statistic, was given already in 1839, by Rodriguez [20]. However, it was with MacMahon [16], almost a century ago, that the systematic study of permutation statistics saw the light of day and it is his name that the Mahonian ones bear. Among other things, MacMahon defined the major index of a permutation, and showed that it is equidistributed with the number of inversions.

Since then, and in particular in the last decade, many new Mahonian statistics have been described in the literature. Apart from "pure" permutation statistics, they have arisen in different contexts, such as the study of Motzkin paths, orthogonal polynomials and algebra, and they also have strong connections to rook theory.

Seemingly, these statistics have very different character, which is underscored by their disparate definitions. However, we shall show in this paper that almost all Mahonian permutation statistics in the literature essentially belong to a class of statistics containing only fourteen different such.

This class of statistics can be seen as the next step in complexity after the simply defined number of inversions. For each integer $n \geq 2$ there is a corresponding class of Mahonian statistics, whose complexity in definition grows with n. For each such class we give strong numerical conditions that the definitions must satisfy in order to give Mahonian statistics.

We define a permutation in the symmetric group $\mathcal{S}_n$ to be a word (or sequence) $a_1a_2\cdots a_n$ of length n consisting of all the elements of $\{1,2,\ldots,n\}$. It is convenient to define $\mathcal{S}_\infty$ as the disjoint union of the $\mathcal{S}_n$ for $n=1,2,3,\ldots$.

A *k-pattern* is a function from $\mathcal{S}_\infty$ to $\mathbb{N}$ that counts the number of occurrences of certain subsequences (not necessarily contiguous) of length k in a permutation in $\mathcal{S}_\infty$. We write our patterns as words in the alphabet $a,b,c,\ldots$, where two adjacent letters may or may not be separated by a dash. The absence of a dash between two adjacent letters in a pattern indicates that the corresponding letters in the permutation must be adjacent, and in the order given by the pattern. Also, the ordering (by size) of the letters in a subword matching a certain pattern (and thus counted by that pattern) must be the same as the ordering of the letters in the pattern, which is based on the usual ordering of the alphabet $a,b,c,\ldots$. Here are some examples:

- The pattern $(a\text{-}b\text{-}c)$ counts increasing subsequences of length 3 in a permutation. This is a "classical" permutation pattern (see below).
- The pattern $(b\text{-}a)$ is the well known number of inversions in a permutation (denoted INV here).
- The pattern (ba) counts the *descents* in a permutation $\pi=a_1a_2\cdots a_n$, that is, the number of i's such that $a_i>a_{i+1}$. (We frequently also refer to the descent i as consisting of the two letter subword a_ia_{i+1}.)
- The pattern $(b\text{-}ca\text{-}d)$ counts the number of occurrences of letters a_i,a_k,a_{k+1},a_j with $i<k<j$ and $a_{k+1}<a_i<a_k<a_j$. Thus, the permutation $\pi=314265$ has two occurrences of $(b\text{-}ca\text{-}d)$, namely 3-42-6 and 3-42-5, so we write $(b\text{-}ca\text{-}d)\pi=2$.

The pattern $(a\text{-}b\text{-}c)$, and any pattern that in our notation has dashes between every pair of adjacent letters, is of a type that might be called classical. These patterns, usually written with the positive integers and without the (implicit) dashes, have mostly been studied with respect to *avoidance*, that is, how many permutations in $\mathcal{S}_n$ have *no* occurrence of the pattern in question. For example, the number of 132-avoiding permutations π in $\mathcal{S}_n$ is known to be the n-th Catalan number $\binom{2n}{n}/(n+1)$. In our notation, this is the cardinality of the set $\{\pi\in\mathcal{S}_n \mid (a\text{-}c\text{-}b)\pi=0\}$.

Although the study of pattern avoidance is scarcely more than a decade old, there is already a sizable, and rapidly growing, literature on the subject. In recent years, this has also been extended to counting the permutations with a given number of occurrences of a pattern. For some background on this, and for more references, see [2, 18, 21, 23]. Pattern avoidance for our generalized patterns has been studied by Claesson [4].

Another type of patterns implicitly present in the literature is the set of patterns of length 3 with no dashes in our notation. These are the *valleys* ((bac) and (cab)), the *peaks* ((acb) and (bca)), the *double ascents* (abc) and the *double descents* (cba) in a permutation, the study of which was pioneered by Françon and Viennot [11], and which is intimately related to Flajolet's [8] generation of Motzkin paths by means of certain continued fractions.

A *pattern function* is a linear combination of patterns and a *d-function* is a linear combination of patterns that have length at most d. The length of a pattern is its number of letters, disregarding dashes.

In this paper we show that most known Mahonian permutation statistics can be written as linear combinations of patterns and that there is a finite number of Mahonian d-functions for each d. In particular, we show that, up to some simple equivalences, there are (at most) fourteen different Mahonian 3-functions. Eight of these are known to be Mahonian (and these include almost all Mahonian statistics in the literature) and we provide proofs for three more. For the remaining three, which we conjecture to be Mahonian, there is overwhelming evidence that they are.

2 Mahonian statistics and pattern functions

A permutation statistic is *Mahonian* if it has the same distribution as INV, the number of inversions. It is easy to see, and was proved by Rodriguez [20], that the distribution of INV is given by the generating function

$$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}\,\pi} = [n]! := [n][n-1]\cdots[1], \tag{1}$$

where $[k] = 1 + q + q^2 + \cdots + q^{k-1}$.

Clearly, INV is identical with the pattern $(b\text{-}a)$. MacMahon [16] showed that the *major index* of a permutation, MAJ, is Mahonian. The usual definition of MAJ is the sum of the descents in a permutation. For example, $\mathrm{MAJ}\, 41523 = 1 + 3 = 4$, since π has descents in positions 1 and 3.

A naive way of computing MAJ is to count, for each descent in π, the letters in π preceding the latter of the two letters constituting the descent. If a letter thus preceding a descent is smaller than both letters in the descent it will be counted by the pattern $(a\text{-}cb)$. If the size of the letter lies between that of the descent letters it will be counted by $(b\text{-}ca)$, and if it is larger than both, then it is counted by $(c\text{-}ba)$. Finally, we need to count the first letter in the descent, which is done by the pattern (ba), which of course counts the descents in a permutation.

Thus, we can write MAJ as a combination of patterns:

$$\mathrm{MAJ} = (a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba).$$

Another Mahonian statistic is MAK, introduced by Foata and Zeilberger [10]. It was essentially defined as the pattern $(b\text{-}ca)$ plus the sum of the *descent bottoms* in π. A descent bottom is simply the smaller (rightmost) letter in a descent. It is easy to see that the sum of descent bottoms in π equals the sum of patterns $(a\text{-}cb) + (cb\text{-}a) + (ba)$. Thus, we can write MAK as follows:

$$\mathrm{MAK} = (b\text{-}ca) + (cb\text{-}a) + (a\text{-}cb) + (ba).$$

The Mahonian statistic MAD introduced in [5] is obtained from MAK by replacing descent bottoms with *descent differences*, that is, the sum of the differences in size between the two letters of a descent. Thus,

$$\mathrm{MAD} = (b\text{-}ca) + (b\text{-}ca) + (ca\text{-}b) + (ba).$$

In [22], Simion and Stanton defined 16 different Mahonian statistics, each of which is a combination of the patterns $(b\text{-}ca)$, $(ca\text{-}b)$, (ab) and (ba). (One of these statistics equals MAD on permutations, but not on words (permutations of multisets), where MAD is still Mahonian.) As it turns out, these are 4 genuinely different statistics, the others being images under the "trivial" bijections from the symmetric group to itself. These trivial bijections will be treated later on in this paper.

All the Mahonian statistics mentioned above, except for INV, are *descent-based*, that is, they are defined in terms of the descents (or ascents) in a permutation and the number of descents appears "transparently" in the definition. There are some Mahonian statistics in the literature that are based instead on *excedances*. An excedance in a permutation $\pi = a_1 a_2 \cdots a_n$ is an i such that $a_i > i$, and the number of excedances in a permutation is denoted exc. The first of these statistics was *Denert's statistic*, DEN, introduced by Denert [6]. It was shown by Foata and Zeilberger [10] that the pair (exc, DEN) has the same distribution as (des, MAJ). In particular, DEN is Mahonian.

Several authors, namely Biane [1], de Médicis and Viennot [17], and Foata and Zeilberger [10], have defined bijections from $\mathcal{S}_n$ to the set of *labeled Motzkin* *paths* in order to prove, among other things, equidistribution results for Mahonian statistics. It was shown by Clarke, Steingrímsson and Zeng [5] that these bijections are all essentially equivalent and based on that a bijection from $\mathcal{S}_n$ to itself was given. This bijection was shown to prove not only the equidistribution of (exc, DEN) and (des, MAK) but also the equidistribution of (exc, INV) and (des, MAD).

In fact, this bijection can even be used to translate an excedance-based statistic of Haglund [13, Theorem 5], which we call HAG, into a descent-based statistic DAG, which then can be written as a combination of patterns. The statistic DAG has patterns of length up to 4, and this is the only Mahonian statistic we are aware of in the literature that has patterns of length greater than 3. In Section 6 we show how to rewrite Haglund's original statistic into an excedance-based form and then translate it, using the bijection in [5], into a descent-based statistic, which then is written as a pattern-function.

We know of no excedance-based Mahonian statistic in the literature that can not be translated into a descent-based Mahonian statistic via the bijection in [5]. Moreover, *all* descent-based Mahonian statistics defined directly on permutations that we are aware of can be written as combinations of patterns.

However, there are two families of statistics, due to Dworkin [7] and Haglund [13], respectively, that are Mahonian, but these statistics are defined relative to arbitrary *boards* considered in rook theory. Thus, the definitions must vary as the length of the permutations varies. Some families of boards, nevertheless, give coherent definitions of the statistics for all n. One of Dworkin's statistics, based on the triangular board for each n, turns out to be equivalent to DEN, as shown by Haglund [13]. Haglund's statistic for the same boards is the statistic HAG mentioned above. There may well be other families of statistics among these that can be defined directly on the permutations, but we have not made a systematic study of this.

There are also many Mahonian statistics that interpolate betwen known Mahonian statistics, or otherwise are defined on subwords of a permutation (or word) [3, 12, 14, 15, 19]. We will not treat these here.

Apart from these execptions, it seems that all Mahonian permutation statistics in the literature can be written as pattern functions or else are equivalent, via the bijection in [5], to such functions.

This leads to an obvious question: How many Mahonian d-functions are there, in particular, is there a finite number for each d? Moreover, *which* pattern functions are Mahonian? We answer these questions (almost) completely for 3-functions and we give an explicit numerical description of the combinations of patterns a Mahonian d-function must have for all d.

3 The main results

So far, our patterns have had an implicit dash at the beginning and the end, in the sense that they have been allowed to begin, and end, anywhere in a permutation. Strictly speaking, we should write $(\text{-}ba\text{-})$ instead of (ba). The generalization that consists of allowing patterns to have or not to have a dash at the beginning/end is worth studying, and causes slight changes in the results presented in Section 4, as will be mentioned later. However, we relegate this generalization to the sidelines and treat (a limited part of) it separately in Section 5.

Nevertheless, in this section we will consider patterns that are required to begin at the first letter in a permutation and/or end at the last letter. We write such patterns with square brackets to indicate this. For example, the pattern $[b\text{-}a)$ counts the number of letters in a permutation that are smaller than the first letter, and $[c\text{-}b\text{-}a]$ counts decreasing subsequences of length three that contain both the first and last letter in a permutation. Most importantly here, a pattern that has no dashes (not even implicit ones at the beginning or end) is identically zero on $\mathcal{S}_n$ except when n is the length of the pattern. For example, $[bac]$ is zero except on the single permutation 213.

In this section, a *k-pattern with i dashes*, or *(k,i)-pattern*, is a pattern of length k with i dashes, where we count implicit dashes at the beginning or end. Thus, $[bac]$ is a $(3,0)$-pattern and $[b\text{-}a)$ is a $(3,2)$-pattern.

Using this, we can show that any d-function, when restricted to $\mathcal{S}_n$ for $n \geq d$, can be written as a linear combination of d-patterns. As an example, if a 4-function contains the (3,1)-pattern $[ba\text{-}c]$, that pattern can be rewritten as a combination of four $(4,1)$-patterns and a $(3,0)$-pattern:

$$[ba\text{-}c] = [ba\text{-}dc] + [ba\text{-}cd] + [ca\text{-}bd] + [cb\text{-}ad] + [bac]. \tag{2}$$

Namely, any occurrence of the pattern $[ba\text{-}c]$ in a permutation π will be detected by exactly one of the patterns in the RHS of (2). Which of the patterns in the RHS will detect this depends on the size of the letter in π preceding the letter corresponding to the c, relative to the size of other letters in the pattern $[ba\text{-}c]$. If there is no letter in π between those corresponding to the a and the c this will be detected by the pattern $[bac]$. Conversely, any pattern in π detected by the RHS of (2) must correspond to a unique occurrence of the pattern $[ba\text{-}c]$.

Now, $[bac]$ is 0 except on $\mathcal{S}_3$, so we have written $[ba\text{-}c]$ as a linear combination of 4-patterns, when considered as a function on $\mathcal{S}_n$ for $n \geq 4$. In general, any k-pattern with i dashes can be written in terms of $(k+1)$-patterns with i

dashes, and one k-pattern with $i-1$ dashes. Given a d-function, we can thus successively strip each of its k-patterns, for $k < d$, of its dashes, and end up with a function whose k-patterns for $k < d$ have no dashes and thus vanish on $\mathcal{S}_d$ and higher. We record this as follows.

Proposition 1 *Any d-function, when restricted to $\mathcal{S}_n$ for $n \geq d$, can be written as a linear combination of d-patterns.* $\square$

We call the rewriting of which (2) is an example *upgrading*. Observe that the number of dashes never increases in an upgrading of a pattern. Thus, for any $d \geq 2$, the above procedure can be used to write the Mahonian pattern $(b\text{-}a)$, that is, the number of inversions, as a combination of d-patterns with one, two or three dashes plus some shorter patterns with no dashes. A simple inductive argument then yields the following lemma.

Lemma 2 *The statistic* INV*, when restricted to $\mathcal{S}_n$ for $n \geq d$, can be written as a combination of d-patterns, of which $d!/2$ have three dashes, $(d-2)d!/2$ have two dashes, and $\binom{d-1}{2}d!/2$ have one dash.*

We now wish to determine which linear combinations of d-patterns can be Mahonian (on $\mathcal{S}_n$ for $n \geq d$). First a definition and a proposition.

Definition 3 *The* weight on $\mathcal{S}_n$ *of a function f is the sum* $\sum_{\pi \in \mathcal{S}_n} f(\pi)$.

To compute the weight of $(b\text{-}a)$, the number of inversions, on $\mathcal{S}_n$, we proceed as follows. We wish to count the total number of inversions in all permutations in $\mathcal{S}_n$. Each inversion consists of two letters x and y in a permutation, where $x < y$ and y precedes x in π. There are $\binom{n}{2}$ such pairs and it suffices to count how many inversions one such pair is involved in, over all permutations in $\mathcal{S}_n$. There are $\binom{n}{2}$ ways to choose the two places in a permutation where we put the x and the y. The remaining $(n-2)$ letters in the permutation can be arranged arbitrarily, in $(n-2)!$ ways, as we are only counting the inversions involving x and y. Thus, the total number of inversions, that is the weight of $(b\text{-}a)$, is

$$\binom{n}{2} \cdot \binom{n}{2}(n-2)! = \frac{n!}{2}\binom{n}{2}.$$

A simple generalization of the above argument yields the following proposition. Note that a pattern with $k+1$ dashes has k blocks of letters separated by the dashes and recall that we are counting dashes at the beginning/end of a pattern. Also, we define $\binom{m}{-1}$ to be 1 if $m = -1$ and 0 otherwise.

Proposition 4 *The weight on $\mathcal{S}_n$ of a d-pattern with $k+1$ dashes is given by*

$$W_n(d,k) = \frac{n!}{d!}\binom{n-d+k}{k}.$$

In particular, the weight of a d-pattern with no dashes ($k=-1$) is 1 if $n=d$ and 0 otherwise.

Now, two functions with the same distribution must have the same weight so, by definition, a Mahonian function must have the same weight as $(b\text{-}a)$, the number of inversions. We record this for later use.

Corollary 5 *The weight of a Mahonian function on $\mathcal{S}_n$ is $\frac{n!}{2}\binom{n}{2}$.*

Clearly, the weight of a sum of patterns is the the sum of the respective weights. As it turns out, this gives significant restrictions on the possible combinations of patterns in order for a function to be Mahonian.

Theorem 6 *Let f be an arbitrary Mahonian d-function, written so that all of its k-patterns, for $k < d$, have no dashes. Then f has $d!/2$ $(d,3)$-patterns, $(d-2)d!/2$ $(d,2)$-patterns, and $\binom{d-1}{2}d!/2$ $(d,1)$-patterns.*

Proof: Observe first that each d-pattern has value 1 on precisely one permutation in $\mathcal{S}_d$ and 0 on the others. Thus, we can only have positive integral combinations of patterns. Therefore a Mahonian linear combination of patterns cannot contain any patterns with more than three dashes. Namely, the weight of such a pattern is $P(n)\cdot n!/d!$ where P is a polynomial in n of degree greater than two, whereas the weight of $\textsc{inv} = (b\text{-}a)$ is $n!/2$ times a polynomial in n of degree two.

It therefore suffices to consider the possible combinations of d-patterns with 0, 1, 2 or 3 dashes. If the respective numbers of these patterns are x, y, z and w then, by Proposition 4 and Corollary 5, they must satisfy the equation

$$\binom{n-d-1}{-1}x + \binom{n-d}{0}y + \binom{n-d+1}{1}z + \binom{n-d+2}{2}w = \frac{n!}{2}\binom{n}{2}$$

for all $n \geq d$, where, as in Proposition 4, $\binom{k}{-1}$ is 1 if $k=-1$ and 0 otherwise. Solving this equation for $n = d, \ldots, d+3$ is equivalent to solving a system of linear equations corresponding to the following matrix.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{pmatrix}$$

This matrix is easily seen to be invertible, so there is a unique possible solution to the above equation. By Lemma 2 this solution is as claimed, and holds for all $n \geq d$. $\square$

Let f be a Mahonian d-function. If we allow k-patterns with any number of dashes for $k < d$ then such a function can be written in more than one way as a d-function, possibly including combinations of patterns with arbitrary real coefficients. However, any such combination can be put into the standard form of Theorem 6 (while remaining the same function) so the theorem essentially rules out anything but positive integral combinations.

Even if we consider the more natural situation with k-patterns, for $k < d$, that don't vanish above $\mathcal{S}_k$, we can give a unique standard way of writing any Mahonian d-function, if we add a modest requirement.

Namely, if we demand that all patterns in f contain at least two dashes (in particular if a pattern is required to have dashes at the beginning and end), then we can again write f in a certain standard form and say exactly how many patterns of each type there must be in f.

Corollary 7 *Let f be a Mahonian d-function whose patterns all have at least two dashes. Then f can be written as a sum of $k!/2$ patterns of length k with two dashes, for $2 \leq k < d$, and $d!/2$ patterns of length d with three dashes.*

Proof: If we repeatedly upgrade all the $(k,3)$-patterns for $k < d$, we are left with a combination of $(d,3)$-patterns and $(k,2)$-patterns for $k = 2, 3, \ldots, d$. It follows from Theorem 6 that there must be exactly $d!/2$ $(d,3)$-patterns. Now, f must contain exactly one 2-pattern (in order to be Mahonian on $\mathcal{S}_2$), and this 2-pattern has two dashes, by hypothesis. That is not enough weight to make a function Mahonian on $\mathcal{S}_3$, so f must contain some 3-patterns. In fact, f must contain exactly three 3-patterns in order to have the weight of a Mahonian function on $\mathcal{S}_3$. An easy induction argument shows that, under the assumption that all the k-patterns have two dashes, there must be exactly $k!/2$ such patterns for each $k < d$. That, in turn, leaves no room for d-patterns, other than the $d!/2$ $(d,3)$-patterns already shown to be present. $\square$

4 A classification of the Mahonian 3-functions

According to Corollary 7, a Mahonian pattern function all of whose patterns have dashes at the beginning and end must contain one of the patterns (ba) and (ab). In order to find all such Mahonian functions, however, we can restrict to the pattern (ba). In fact, for each pattern function f with a given distribution, there are three others that obviously have the same distribution. These are functions obtained from f by one of the three *trivial* bijections of $\mathcal{S}_n$ to itself, namely *reversion*, R, *complementation*, C, and the composition $R \circ C$.

The reverse of a permutation $\pi = a_1a_2 \cdots a_n$ is the permutation $\pi^r = a_na_{n-1} \cdots a_1$ and the complement of π is the permutation $\pi^c = b_1b_2 \cdots b_n$ where $b_i = n + 1 - a_i$. As an example, since

$$\text{MAJ} = (a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba),$$

reversing each of the patterns in MAJ yields the function

$$\text{MAJ}^r = (bc\text{-}a) + (ac\text{-}b) + (ab\text{-}c) + (ab)$$

and clearly $\text{MAJ}^r\, \pi^r = \text{MAJ}\, \pi$ for any permutation π.

In what follows, we will make use of this, and we will in particular only consider pattern functions whose 2-pattern is (ba). A Mahonian 3-function all of whose patterns have dashes at the beginning and the end must thus consist of the pattern (ba) and three 3-patterns with one "internal" dash each. Allowing the pattern $[b\text{-}a)$ instead of (ba) yields a few more Mahonian statistics different from those with 2-pattern (ba) but we will treat this separately in Section 5.

The number of ways of combining one of the 2-patterns (ba) and (ab) and three 3-patterns with one internal dash each is $2 \cdot \binom{14}{3} = 728$ and only $728/4 = 182$ if we take into account the trivial bijections mentioned above. Computer-aided calculations show that of these 182 3-functions, all but 14 fail to have the Mahonian distribution already on $\mathcal{S}_5$. Among these fourteen statistics, eight are known, but six seem to be new. For three of those six we prove in Proposition 9 that they are Mahonian.

This leaves three possible Mahonian statistics for which proofs are missing. However, we have verified by computer that they have the Mahonian distribution for $n \leq 11$ so the following conjecture is a safe bet.

Conjecture 8 *The following statistics (number 6,11,13 in Table 1) are Mahonian:*

$$(ac\text{-}b) + (ba\text{-}c) + (c\text{-}ba) + (ba),$$
$$(a\text{-}cb) + (b\text{-}ca) + (b\text{-}ca) + (ba),$$
$$(bc\text{-}a) + (ca\text{-}b) + (ca\text{-}b) + (ba).$$

In Table 1 we give a list of all fourteen (possible) Mahonian 3-functions. We group them into the seven equivalence classes induced by the relation $\sim$, where two statistics S and T satisfy $S \sim T$ if the distribution of the bistatistics (des, S) and (des, T) is the same. Here des is the number of descents, and thus equals (ba).

The equidistributions of such bistatistics have been much studied (see e.g. [5, 7, 9, 10, 13, 17]) and the fact that all Mahonian 3-functions must contain the pattern $(ba) = \text{des}$ "explains" why this is a natural classification. Note that if $S = S' + (ba)$ and $T = T' + (ba)$ are two equidistributed functions and (des, S) and (des, T) are also equidistributed then so are S' and T'. The converse of this is not true, of course, so stripping two statistics with different distributions of the pattern (ba) may result in statistics with the same distribution. It is easily checked, however, that this does not happen with any two different classes in Table 1. Because of this, and for simplicity, we omit writing the pattern (ba) in the statistics in Table 1. In Table 2 we give the distribution of the bistatistic (des, S) for the statistics S in each of the seven equivalence classes in Table 1. Observe that in Table 2 the statistics S *do contain* the pattern (ba).

We now prove that the statistics number 5, 10 and 12 in Table 1 are Mahonian.

Proposition 9 *The following statistics are Mahonian:*

$$\begin{aligned}
\textsc{stat} &= (ac\text{-}b) + (ba\text{-}c) + (cb\text{-}a) + (ba), \\
\textsc{stat}' &= (ac\text{-}b) + (ca\text{-}b) + (cb\text{-}a) + (ba), \\
\textsc{stat}'' &= (a\text{-}cb) + (c\text{-}ab) + (c\text{-}ba) + (ba).
\end{aligned}$$

Proof: We prove, by induction on the length of a permutation, that $\textsc{stat}$ is Mahonian. The proofs for the other two statistics are similar and are omitted. We analyze how the value of $\textsc{stat}$ changes as we prepend $k = 1, 2, \ldots, n$ to a permutation $\pi \in \mathcal{S}_{n-1}$ and add one to those letters in π that are greater than or equal to k. For example, prepending 3 to the permutation 4132 we get 35142. (In the case of $\textsc{stat}''$, the letter should be appended to the end of π.)

Let the first letter of π be m and suppose we are prepending k to π. There are two cases, depending on whether k is greater than m or not.

If k is smaller than or equal to m, then the only pattern in $\textsc{stat}$ that is affected is $(ac\text{-}b)$. If $k = m$ then there is no effect. If $k = m - 1$ then the value of $(ac\text{-}b)$ will increase by one, since the one letter between k and $m' = m + 1$ (the "new value" of m) in size will appear after m' in the resulting permutation. In general, if $k = m - i$, where $0 \leq i < m$, then the value of $(ac\text{-}b)$ will increase by i.

1	$(ac{-}b)$	+	$(ac{-}b)$	+	$(b{-}ac)$	
2	$(ac{-}b)$	+	$(ac{-}b)$	+	$(b{-}ca)$	
3	$(ac{-}b)$	+	$(b{-}ca)$	+	$(b{-}ca)$	
4	$(b{-}ca)$	+	$(b{-}ca)$	+	$(ca{-}b)$	MAD
5	$(ac{-}b)$	+	$(ba{-}c)$	+	$(cb{-}a)$	STAT
6	$(ac{-}b)$	+	$(ba{-}c)$	+	$(c{-}ba)$	
7	$(a{-}cb)$	+	$(b{-}ca)$	+	$(cb{-}a)$	MAK
8	$(a{-}cb)$	+	$(b{-}ca)$	+	$(c{-}ba)$	MAJ
9	$(a{-}cb)$	+	$(ca{-}b)$	+	$(cb{-}a)$	MAKL
10	$(ac{-}b)$	+	$(ca{-}b)$	+	$(cb{-}a)$	STAT$'$
11	$(a{-}cb)$	+	$(b{-}ca)$	+	$(b{-}ca)$	
12	$(a{-}cb)$	+	$(c{-}ab)$	+	$(c{-}ba)$	STAT$''$
13	$(bc{-}a)$	+	$(ca{-}b)$	+	$(ca{-}b)$	
14	$(bc{-}a)$	+	$(ca{-}b)$	+	$(cb{-}a)$	INV

Table 1: All Mahonian 3-functions (omitting the pattern (ba)), up to trivial bijections. The first four belong to those defined by Simion and Stanton [22]. The statistic MAKL appears in [5, Prop. 13].

1	0	0	0	0	0	0	0	0	0	0
0	4	3	5	5	5	3	1	0	0	0
0	0	6	6	11	12	12	9	6	3	1
0	0	0	4	3	5	5	5	3	1	0
0	0	0	0	1	0	0	0	0	0	0

1	0	0	0	0	0	0	0	0	0	0
0	4	9	9	4	0	0	0	0	0	0
0	0	0	6	14	22	14	6	0	0	0
0	0	0	0	0	0	4	9	9	4	0
0	0	0	0	0	0	0	0	0	0	1

1	0	0	0	0	0	0	0	0	0	0
0	4	9	5	6	1	1	0	0	0	0
0	0	0	10	14	20	12	7	3	0	0
0	0	0	0	0	1	7	8	6	4	0
0	0	0	0	0	0	0	0	0	0	1

1	0	0	0	0	0	0	0	0	0	0
0	4	3	8	4	5	1	1	0	0	0
0	0	6	3	14	12	14	8	6	2	1
0	0	0	4	1	5	5	6	3	2	0
0	0	0	0	1	0	0	0	0	0	0

1	0	0	0	0	0	0	0	0	0	0
0	4	6	8	7	1	0	0	0	0	0
0	0	3	7	12	20	14	10	0	0	0
0	0	0	0	1	1	6	5	9	4	0
0	0	0	0	0	0	0	0	0	0	1

1	0	0	0	0	0	0	0	0	0	0
0	4	3	5	3	5	2	3	1	0	0
0	0	6	6	13	9	14	7	7	3	1
0	0	0	4	3	8	4	5	1	1	0
0	0	0	0	1	0	0	0	0	0	0

1	0	0	0	0	0	0	0	0	0	0
0	4	6	6	6	2	2	0	0	0	0
0	0	3	9	12	18	12	9	3	0	0
0	0	0	0	2	2	6	6	6	4	0
0	0	0	0	0	0	0	0	0	0	1

Table 2: Distribution of (des, S) on $\mathcal{S}_5$ for the seven different equivalence classes of statistics S (together with (ba)) in Table 1, in the same order. Rows are indexed by number of descents, columns by value of S; both start at 0.

If k is greater than m then we are creating a descent at the beginning of the permutation, thus increasing (ba) by one. Also, each letter in π that is smaller than m (and thus to the right of m) will contribute an increase of one to $(cb\text{-}a)$. Therefore, the total increase to $(cb\text{-}a)$ and (ba) will be precisely m. In addition, if $k = m + i$, where $i \geq 1$, then $(ba\text{-}c)$ will increase by $n - m - i$. The pattern $(ac\text{-}b)$ is not affected in this case.

Thus, prepending the letters $1, 2, \ldots, n$ to π will increase the value of STAT by $0, 1, \ldots, n-1$, respectively (but not necessarily in this order). Given that the distribution of STAT on $\mathcal{S}_1$ is 1, this implies, by induction, that its distribution on $\mathcal{S}_n$ is the same as that of INV, given in (1). $\square$

We conjecture that STAT belongs to the same equivalence class as MAJ and MAK. As is the case for all conjectures in this paper, this has been verified by computer for $n \leq 11$.

Conjecture 10 *The distribution of the bistatistic* (des, STAT) *is equal to that of* (des, MAJ)*.*

5 A generalization

If we allow patterns with no (implicit) dash at the beginning, as in Section 3, we find several candidates for Mahonian statistics among the 3-functions (where we only consider the 2-pattern $[b\text{-}a)$ and we have restricted the 3-patterns to have dashes at the beginning and end). All but four of these can be shown to equal some of the ones in Table 1, as functions, and so are not new. We conjecture that the remaining four are Mahonian and that they belong to the equivalence class of (des, MAJ).

Conjecture 11 *The following statistics are Mahonian, where, as in Section 3, a square bracket* [*at the beginning of a pattern means that the pattern must begin at the first letter of the permutation. Furthermore, the bistatistics* (des, S_i)*, for* $i = 1, 2, 3, 4$*, are equidistributed with* (des, MAJ)*.*

$$\begin{aligned}
S_1 &= (a\text{-}cb) + (b\text{-}ac) + (cb\text{-}a) + [b\text{-}a), \\
S_2 &= (a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + [b\text{-}a), \\
S_3 &= (a\text{-}cb) + (b\text{-}ca) + (cb\text{-}a) + [b\text{-}a), \\
S_4 &= (a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + [b\text{-}a).
\end{aligned}$$

6 A Mahonian 4-function

We now show, without giving all details, how the statistic HAG of Haglund [13, Thm. 5] can be rewritten to make it suitable for "translation" by the bijection in [5] into a statistic DAG, which in turn can be written in terms of patterns of lengths up to four.

We use here terminology from [5], where $\mathrm{Ddif}\,\pi$ is the sum of descent differences $(a_i - a_{i+1})$ over all descents i in π, and Edif is the coresponding sum of excedance differences $(a_i - i)$ over all excedances i. Moreover, $\mathrm{exc}(\pi)$ is the subword of π consisting of those letters a_i for which $a_i > i$ and $\mathrm{nex}(\pi)$ is the complementary subword of $\mathrm{exc}(\pi)$ in π.

Haglund calls his statistic simply *stat* and defines it as follows, where $\pi = a_1 a_2 \cdots a_n$ and we take i to be less than j in all sets:

$$\mathrm{Edif}\,\pi + \sum_{a_i \le i} (1 - a_i) + \textsc{inv}(\mathrm{exc}(\pi)) + \#\{a_i \le j < a_j\} + \#\{a_i < a_j \le j\}.$$

This can be rearranged and then rewritten as follows:

$$\begin{aligned}
&\mathrm{Edif}\,\pi + \textsc{inv}(\mathrm{exc}(\pi)) + \sum_{a_i \le i} (1 - a_i) + \#\{a_i < a_j \le j\} + \#\{a_i \le j < a_j\} \\
&= \mathrm{Edif}\,\pi + \textsc{inv}(\mathrm{exc}(\pi)) - \#\{a_j < a_i \le i\} + \#\{a_i \le j < a_j\} \\
&= \mathrm{Edif}\,\pi + \textsc{inv}(\mathrm{exc}(\pi)) - \textsc{inv}(\mathrm{nex}(\pi)) + E,
\end{aligned}$$

where E is the sum of numbers defined for each excedance bottom k as the number of letters a_i with $i < k$ and $a_i \le k$.

We define a descent-based version of this statistic, DAG by

$$\textsc{dag}\,\pi = \mathrm{Ddif}\,\pi + \mathrm{Res}(\mathrm{Destops}) - \mathrm{Res}(\mathrm{NonDestops}) + D, \tag{3}$$

where D is the sum of numbers defined for each descent bottom a_i as the number of descent tops smaller than or equal to a_i and non-descent bottoms smaller than or equal to a_i. Moreover, Res is a function equal to the pattern $(b\text{-}ca)$, and Res(Destops) is the number of occurrences of that pattern where the letter corresponding to the b is a descent top, that is, the first letter a_i in a descent $a_i > a_{i+1}$. The term Res(NonDestops) is the corresponding number for the non-descent tops.

Applying the bijection Φ in [5, Section 3] to a permutation π we have that $\textsc{dag}\,\pi = \textsc{hag}\,\Phi(\pi)$. Thus DAG is Mahonian since HAG is.

With some work, it is possible to write (3) as follows:

$$(ba) + [a\text{-}cb) + (cba) + (ca\text{-}b) \tag{4}$$
$$+2 \cdot (ca\text{-}db) + 2 \cdot (cb\text{-}da) + (ab\text{-}dc) + (ba\text{-}dc) + (dc\text{-}ab) + (dc\text{-}ba).$$

Using the identity

$$[a\text{-}cb) = (a\text{-}cb) - [(da\text{-}cb) + (ca\text{-}db) + (ba\text{-}dc) + (ab\text{-}dc)]$$

(obtained by upgrading $(a\text{-}cb)$) we can then rewrite (4) as follows:

$$(ba) + (cab) + (cba) + (a\text{-}cb)$$
$$+2 \cdot (ca\text{-}db) + 2 \cdot (cb\text{-}da) + (dc\text{-}ab) + (dc\text{-}ba) + (da\text{-}bc) + (db\text{-}ac).$$

Finally, to get this into the standard form of Corollary 7, we upgrade $(a\text{-}cb)$ and obtain

$$\begin{aligned}\textsc{dag} = \quad & (ba) + (cab) + (cba) + (acb) \\ & +2 \cdot (ca\text{-}db) + 2 \cdot (cb\text{-}da) + (dc\text{-}ab) + (dc\text{-}ba) + (da\text{-}bc) \\ & +(db\text{-}ac) + (ad\text{-}cb) + (ac\text{-}db) + (ab\text{-}dc) + (ba\text{-}dc).\end{aligned}$$

We have compared, with the aid of a computer, the statistic DAG to all the statistics in Table 1 (and the statistics obtained from these by the bijection $R \circ C$) and found that it is not equal, as a function, to any of them. Thus, DAG is genuinely a 4-function. It was shown by Haglund [13, Thm. 5] that (exc, HAG) is equidistributed with (des, MAJ). Appealing to the properties of the bijection Φ in [5, Prop. 3], it follows that (des, DAG) also has the same distribution.

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