

# BOUNDARY BEHAVIOUR OF $\mathcal{M}$ -HARMONIC FUNCTIONS AND NON-ISOTROPIC HAUSDORFF MEASURE

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ABSTRACT. In this paper we consider weighted boundary behaviour of  $\mathcal{M}$ -harmonic functions on the unit ball of  $\mathbb{C}^n$ . In particular, we show that a measure on the unit sphere of  $\mathbb{C}^n$  admits a component along the non-isotropic Hausdorff measure on that sphere and that this component gives rise to certain weighted boundary behaviour of the  $\mathcal{M}$ -harmonic extension of the original measure.

## 1. INTRODUCTION

In this paper, we study weighted boundary behaviour of invariant-harmonic ( $\mathcal{M}$ -harmonic) extensions of measures on the unit sphere  $\mathbb{S}^n$  of  $\mathbb{C}^n$  to its unit ball  $\mathbb{B}_n$ .

In a first step, we show that, for every  $0 \leq \alpha \leq n$ , a finite Radon measure on  $\mathbb{S}^n$  decomposes into a sum of two mutually singular measures  $\nu_0$  and  $\nu_1$  where  $\nu_1$  is a component along the Hausdorff measure  $m_\alpha$ . Note that decompositions of measures along Hausdorff measures already appear in Kahane and Katznelson's work [7, 5, 6] but their decomposition does not suit our needs.

The Hausdorff measure we consider here is the non-isotropic Hausdorff measure associated to the distance  $d(\zeta, \xi) = |1 - \langle \zeta, \xi \rangle|^{\frac{1}{2}}$  on  $\mathbb{S}^n$ . This distance is the most adapted to invariant complex analysis on  $\mathbb{B}_n$  and relations between the corresponding Hausdorff measure and boundary behaviour of  $\mathcal{M}$ -harmonic functions already occurs in several places (see *e.g.* [4, 2, 1, 3]).

Now, let  $P[\nu]$  denote the  $\mathcal{M}$ -harmonic extension (or the Poisson-Szegő integral) of a measure  $\nu$ . We next show that the component  $\nu_1$  of  $\nu$  along the Hausdorff measure  $m_\alpha$  is essentially recovered by a suitably weighted boundary limit of  $P[\nu]$ .

These results may be summarized in the following theorem :

**Theorem.** *Let  $0 \leq \alpha \leq n$  and let  $m_\alpha$  be the non-isotropic Hausdorff measure of order  $\alpha$ . There exist constants  $C, c > 0$  such that, for every finite complex valued Radon measure  $\nu$  on  $\mathbb{S}^n$ , there exist an  $m_\alpha$ -summable function  $f$ , and a measure  $\nu_0$  mutually singular to  $m_\alpha$  such that  $\nu = f m_\alpha + \nu_0$ , and*

$$c|f(\zeta)| \leq \limsup_{\substack{z \in \Gamma_\gamma(\zeta) \\ z \rightarrow \zeta}} (1 - |z|)^{n-\alpha} |P[\nu](z)| \leq C|f(\zeta)|, \quad m_\alpha\text{-a.e on } \mathbb{S}^n$$

where  $\Gamma_\gamma(\zeta)$  is an admissible approach region at  $\zeta \in \mathbb{S}^n$ .

Note that the extreme cases  $\alpha = 0$  and  $\alpha = n$  are already known (see [10], section 7).

The article is organised as follows: in the next section, we present the precise setting for our problem. The following section is devoted to the decomposition of measures along Hausdorff measures and some related lemmas. Finally, we apply these results to the study of weighted boundary behaviour of  $\mathcal{M}$ -harmonic functions and prove our main theorem.

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## 2. SETTING

**2.1.  $\mathcal{M}$ -harmonic functions.** In this paper,  $\mathbb{B}_n$  will be the unit ball of  $\mathbb{C}^n$  and  $\mathbb{S}^n$ , its boundary. Denote by  $|\cdot|$  the euclidean norm on  $\mathbb{C}^n$  and let  $d\sigma$  be the usual surface measure on  $\mathbb{S}^n$ .

Let  $\mathcal{M}$  denote the group of holomorphic automorphisms of  $\mathbb{B}_n$ . The Laplace operator on  $\mathbb{B}_n$  that is invariant under  $\mathcal{M}$  is given by

$$\tilde{\Delta}f = \frac{4(1-|z|^2)}{n+1} \sum_{i,j=1}^n [\delta_{i,j} - \bar{z}_i z_j] \frac{\partial^2 f}{\partial z_j \partial \bar{z}_i},$$

where  $\delta_{i,j}$  is the Kronecker symbol.

**Definition.** A function  $f \in C^2(\mathbb{B}_n)$  is said to be  $\mathcal{M}$ -harmonic if  $\tilde{\Delta}f = 0$ .

The invariant Poisson kernel  $P$  on  $\mathbb{B}_n \times \mathbb{S}^n$  is given by

$$P(z, \xi) = \frac{(1-|z|^2)^n}{|1-\langle z, \xi \rangle|^{2n}} \quad z \in \mathbb{B}_n, \quad \xi \in \mathbb{S}^n.$$

It is well known that if  $\nu$  is a finite measure on  $\mathbb{S}^n$ , then

$$P[\nu](z) := \int_{\mathbb{S}^n} P(z, \xi) d\nu(\xi)$$

satisfies  $\tilde{\Delta}P[\nu] = 0$ . As standard reference on  $\mathcal{M}$ -harmonic functions we will use [9, 10].

**2.2. Non-isotropic Hausdorff measure.** For the statements and the proofs of the main results of this paper, we recall the notion of “non-isotropic” Hausdorff measure.

We will consider  $d(\zeta, \omega) = |1 - \langle \zeta, \omega \rangle|^{\frac{1}{2}}$ , the non-isotropic distance on  $\mathbb{S}^n$ . To keep with the habits of complex analysts, for  $\delta \geq 0$ , we denote by

$$Q(\zeta, \delta) := \{\omega \in \mathbb{S}^n : d(\omega, \zeta)^2 < \delta\} = \{\omega \in \mathbb{S}^n : |1 - \langle \zeta, \omega \rangle| < \delta\}$$

the corresponding non-isotropic balls of radius  $\sqrt{\delta}$  (or Koranyi balls). If  $K$  is a compact subset of  $\mathbb{S}^n$ ,  $0 < \alpha \leq n$ , the non-isotropic Hausdorff measure of  $K$  is defined by

$$m_\alpha(K) = \liminf_{\Delta \rightarrow 0} \sum \delta_j^\alpha,$$

where the infimum is over all covers  $\{Q(\zeta_j, \delta_j)\}$  of  $K$  by Koranyi balls with radii  $\delta_j < \Delta$ . If  $A$  is an arbitrary subset of  $\mathbb{S}^n$ , then

$$m_\alpha(A) = \sup\{m_\alpha(K) : K \text{ compact, } K \subset A, m_\alpha(K) < +\infty\},$$

(the fact that one may restrict attention to subsets  $K$  with  $m_\alpha(K) < \infty$  results from [8], chapter 6).

The non-isotropic Hausdorff-dimension is then defined in the usual way.

We will use the following definition.

**Definition.** Let  $\nu$  be a finite measure on  $\mathbb{S}^n$ , define the upper  $\alpha$ -derivate of  $\nu$  as

$$\overline{D}_\alpha \nu(\zeta) = \limsup_{\delta \rightarrow 0^+} \frac{\nu(Q(\zeta, \delta))}{\delta^\alpha},$$

and the lower  $\alpha$ -derivate of  $\nu$  as

$$\underline{D}_\alpha \nu(\zeta) = \liminf_{\delta \rightarrow 0^+} \frac{\nu(Q(\zeta, \delta))}{\delta^\alpha}.$$

If the limit exists we will call it  $\alpha$ -derivate and denote it as  $D_\alpha \nu(\zeta)$ .

One can then state Frostman’s Theorems (Theorem 6.9 in [8]) as:

For  $\nu$  a finite positive measure on  $\mathbb{S}^n$  and  $E \subset \mathbb{S}^n$  a Borel set,

- if  $\overline{D}_\alpha \nu \leq C$  on  $E$  then  $m_\alpha(E) \geq \frac{\nu(E)}{KC}$ , for some absolute constant  $K > 0$ ,
- if  $\overline{D}_\alpha \nu \geq C$  on  $E$  then  $m_\alpha(E) \leq K \frac{\nu(E)}{C}$ , for some absolute constant  $K > 0$ .

*Remark.* Since  $\sigma(Q(\zeta, \delta)) \simeq \delta^n$ , when  $\alpha = n$ ,  $m_n$  is equivalent to the Lebesgue measure on  $\mathbb{S}^n$ .

When  $n = 1$ , the non-isotropic Hausdorff measure corresponds to the usual Hausdorff measure on the boundary.

Note that the non-isotropic Hausdorff measure depends on “directional” considerations. For instance, let  $\mathbb{S}^{n,k} = \{(z_1, \dots, z_k, 0, \dots, 0) \in \mathbb{S}^n\}$  be the intersection of  $\mathbb{S}^n$  with  $\mathbb{C}^k$  (included in  $\mathbb{C}^n$  in a standard way). Then, Frostman’s Lemma with  $\mu = \sigma_k$ , Lebesgues’ measure on  $\mathbb{S}^{n,k}$  (extended to  $\mathbb{S}^n$  in a standard way) shows that  $\mathbb{S}^{n,k}$  has non-isotropic dimension  $k$ .

Let  $\mathbb{S}_{\mathbb{R}}^{n,k} = \text{Re } \mathbb{S}^{n,k} = \{(x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{S}^n : x_1, \dots, x_k \in \mathbb{R}\}$ . Then  $Q(\zeta, \delta) \cap \mathbb{S}_{\mathbb{R}}^{n,k}$  are the usual euclidean balls with radius  $\sqrt{\delta}$ . Thus Frostman’s Lemma with  $\mu = \sigma_k$ , Lebesgues’ measure on  $\mathbb{S}_{\mathbb{R}}^{n,k}$  shows that  $\mathbb{S}_{\mathbb{R}}^{n,k}$  has dimension  $\frac{k}{2} - 1$ . In particular  $\mathbb{S}_{\mathbb{R}}^{n,2k}$  and  $\mathbb{S}^{n,k}$  do not have same dimension, though they are isometric in Euclidian metric.

### 3. DENSITY ESTIMATES

**Definition.** Let  $\nu$  be a measure on  $\mathbb{S}^n$  and  $0 \leq \alpha \leq n$ . We will say that  $\nu$  is mutually singular to  $m_\alpha$  if for every Borel set  $K$  such that  $0 < m_\alpha(K) < \infty$  there exists a set  $E$  such that  $m_\alpha(E) = 0$  and  $|\nu|(K \setminus E) = 0$ .

**Lemma 3.1.** Let  $\alpha > 0$ . There exists a constant  $C$ , depending only on  $\alpha$ , such that for any complex measure  $\nu$  on  $\mathbb{S}^n$ , and for every  $t > 0$ ,

$$m_\alpha(\{\zeta \in \mathbb{S}^n : \overline{D}_\alpha |\nu|(\zeta) > t\}) \leq C \frac{\|\nu\|}{t}.$$

*Proof.* It is enough to prove the lemma for positive measures. This is then just Frostman’s Lemma applied to  $E = \{\zeta \in \mathbb{S}^n : \overline{D}_\alpha |\nu|(\zeta) > t\}$ .  $\square$

**Lemma 3.2.** Let  $0 \leq \alpha \leq n$  and let  $\nu$  be a Radon measure on  $\mathbb{S}^n$ . If  $\nu$  is mutually singular to  $m_\alpha$ , then

$$m_\alpha(\{\zeta \in \mathbb{S}^n : \overline{D}_\alpha |\nu|(\zeta) > 0\}) = 0.$$

*Proof.* It is enough to prove the statement for positive measures.

Denote by  $K_a = \{\zeta \in \mathbb{S}^n : \overline{D}_\alpha |\nu|(\zeta) > a\}$ . As, for fixed  $\delta > 0$ ,  $\zeta \mapsto Q(\zeta, \delta)$  is lower semi-continuous,  $K_a$  is of course a Borel set. Further, by Lemma 3.1,  $m_\alpha(K_a) < +\infty$ . Finally, as

$$\{\zeta \in \mathbb{S}^n : \overline{D}_\alpha |\nu|(\zeta) > 0\} = \bigcup_{a \in \mathbb{Q}_+} K_a,$$

it is enough to prove that  $m_\alpha(K_a) = 0$  for every  $a > 0$ .

Fix  $a > 0$ . As  $\nu$  is mutually singular to  $m_\alpha$  we can also assume that  $\nu(K_a) = 0$ .

Let  $\delta_0$  be small enough to have

$$(1) \quad m_\alpha(K_a) < 2 \inf \sum r_i^\alpha$$

where the infimum runs over all covers  $\{Q(\zeta_i, r_i)\}$  of  $K_a$  by balls of radius  $r_i < 9\delta_0$ .

Let  $\varepsilon > 0$ . As  $\nu(K_a) = 0$  and  $\nu$  is a Radon measure there exists an open set  $\Omega$ , such that  $K_a \subset \Omega$  and  $\nu(\Omega) < \varepsilon$ .

Since, for each  $\zeta \in K_a$ ,  $\overline{D}_\alpha \nu(\zeta) > a$ , there exists a ball  $Q(\zeta, r_\zeta) \subset \Omega$  of radius  $r_\zeta < \delta_0$  such that  $\nu(Q(\zeta, r_\zeta)) > ar_\zeta^\alpha$ .

By the “5-covering Lemma” adapted to Koranyi balls (see [10] Lemma 7.5), there exists a countable disjoint subcollection  $\{Q_i\}_{i \in \tau}$  of  $\{Q(\zeta, r_\zeta)\}_{\zeta \in K_a}$  such that  $K_a \subset \bigcup_{i \in \tau} 9Q_i$ . But then with (1),

$$m_\alpha(K_a) \leq 2 \cdot 9^\alpha \sum_{i \in \tau} r_i^\alpha \leq \frac{2 \cdot 9^\alpha}{a} \sum_{i \in \tau} \nu(Q_i) \leq C\nu(\Omega) \leq C\varepsilon.$$

As  $\varepsilon$  is arbitrary small,  $m_\alpha(K_a) = 0$ . □

**Lemma 3.3.** *Let  $0 \leq \alpha \leq n$ . There exists a constant  $C > 0$ , which depends only on  $\alpha$ , such that, for every Borel set  $K$  such that  $0 < m_\alpha(K) < \infty$ , the measure  $\nu = m_\alpha|_K$  satisfies*

$$C \leq \overline{D}_\alpha \nu(\zeta) \leq 1,$$

*$m_\alpha$ -almost everywhere on  $K$ , and*

$$\overline{D}_\alpha \nu(\zeta) = 0,$$

*$m_\alpha$ -almost everywhere on  $\mathbb{S}^n \setminus K$ .*

*Proof.* This is *mutadis mutandis* the same proof as in [8], Theorem 6.2. □

**Proposition 3.4.** *Let  $\nu$  be a finite Radon measure on the sphere  $\mathbb{S}^n$  and let  $0 \leq \alpha \leq n$ . Then there exists a Borel function  $f$  summable with respect to  $m_\alpha$  and a finite Borel measure  $\nu_0$  mutually singular to  $m_\alpha$ , such that  $\nu = fm_\alpha + \nu_0$ .*

*Proof.* Let  $K \subset \mathbb{S}^n$  be such that  $0 < m_\alpha(K) < +\infty$ . Then, by Radon-Nikodým,  $\nu = \nu_K + f_K m_\alpha|_K$ , where  $\nu_K$  is mutually singular to  $m_\alpha|_K$ . By construction, if  $K' \subset K$  is such that  $0 < m_\alpha(K') < +\infty$ , then  $f_{K'} = f_K$ ,  $m_\alpha$  and  $\nu$ -almost everywhere on  $K'$ .

This allows to define  $m_\alpha$ -almost everywhere on  $E = \{\zeta \in \mathbb{S}^n : \overline{D}_\alpha |\nu|(\zeta) > 0\}$  a function  $f$  such that  $f|_K = f_K$  for any  $K \subset E$ . Extend this function by zero to the rest of the sphere and let  $\nu_0 = \nu - fm_\alpha$ . Then

*i. the function  $f$  is  $m_\alpha$ -summable, i.e.  $\int |f| dm_\alpha < +\infty$ . Indeed*

$$\int |f| dm_\alpha = \sup_K \int |f| dm_\alpha = \sup_K \int |f_K| dm_\alpha \leq \|\nu\|.$$

*ii. The measure  $\nu_0$  is mutually singular to  $m_\alpha$  :* Let  $K$  be a Borel subset of  $\mathbb{S}^n$  such that  $0 < m_\alpha(K) < +\infty$ . By construction,  $f = f_{K \cap E}$   $m_\alpha$ -almost everywhere on  $K \cap E$ , thus  $\nu_0|_{K \cap E} = \nu_{K \cap E}$ . By the Radon-Nykodim construction of  $\nu_{K \cap E}$ , there is a set  $F_K$ , such that  $m_\alpha(F_K) = 0$  and  $|\nu_0|((K \cap E) \setminus F_K) = |\nu_{K \cap E}|((K \cap E) \setminus F_K) = 0$ .

On the set  $K \setminus E$  we have  $\overline{D}_\alpha |\nu| = 0$  and applying the first part of Frostman’s Lemma for an arbitrary small constant  $C$ , we get  $|\nu|(K \setminus E) = 0$ . As  $\nu_0|_{\mathbb{S}^n \setminus E} = \nu|_{\mathbb{S}^n \setminus E}$ , we have  $|\nu_0|(K \setminus E) = 0$ . Finally,

$$|\nu_0|(K \setminus F_K) = |\nu_0|(((K \cap E) \setminus F_K) \cup (K \setminus E)) = 0,$$

and as  $m_\alpha(F_K) = 0$  we get that  $\nu_0$  is mutually singular to  $m_\alpha$ .

*iii. The measure  $\nu_0$  is finite :*  $\nu$  and  $fm_\alpha$  are both finite, so  $\nu_0 = \nu - fm_\alpha$  is finite.

So  $f$  and  $\nu_0$  have the desired properties and the proof is complete. □

**Lemma 3.5.** *Let  $\nu$  be a finite Radon measure and let  $\nu = \nu_0 + fm_\alpha$  be the decomposition of  $\nu$  given by Proposition 3.4. Then*

$$(2) \quad C|f(\zeta)| \leq \overline{D}_\alpha |\nu|(\zeta) \leq |f(\zeta)|$$

*$m_\alpha$ -almost everywhere.*

*Proof.* It is enough to prove the lemma for positive measures. Moreover, by Lemma 3.2,  $\overline{D}_\alpha|\nu_0| = 0$   $m_\alpha$ -almost everywhere, so it is enough to consider the case  $\nu = fm_\alpha$  with  $f \geq 0$ .

We will call a function  $g \in L^1(m_\alpha)$  “simple” if there exists a countable collection of disjoint Borel sets  $\{K_n\}_{n \in \mathbb{Z}}$  and real numbers  $\{a_n\}_{n \in \mathbb{Z}}$ , such that  $m_\alpha(K_n) < \infty$  for all  $n \in \mathbb{Z}$  and  $g = \sum_{n \in \mathbb{Z}} a_n \chi_{K_n}$ , where  $\chi_{K_n}$  is the characteristic function of the set  $K_n$ . By Lemma 3.3, the statement (2) is true for simple functions.

Let us consider the sequence of simple functions  $\{g_n\}_{n \in \mathbb{N}}$ , given by  $g_n(\zeta) = \frac{[nf(\zeta)]}{n}$  (where  $[nf(\zeta)]$  denotes the integer part of  $nf(\zeta)$ ). It is easy to see that  $g_n \in L^1(m_\alpha)$  and  $g_n \rightarrow f$  both pointwise ( $m_\alpha$ -a.e.) and in  $L^1(m_\alpha)$ -sense. But  $|\overline{D}_\alpha(fm_\alpha) - \overline{D}_\alpha(g_n m_\alpha)| \leq \overline{D}_\alpha(|f - g_n|m_\alpha)$ . By Lemma 3.1,

$$m_\alpha(\{|\overline{D}_\alpha(fm_\alpha) - \overline{D}_\alpha(g_n m_\alpha)| > t\}) \leq m_\alpha(\{\overline{D}_\alpha(|f - g_n|m_\alpha) > t\}) \leq \frac{\|f - g_n\|_{L^1(m_\alpha)}}{t} \rightarrow 0,$$

when  $n \rightarrow \infty$ , i.e.  $\overline{D}_\alpha(g_n m_\alpha)$  converges to  $\overline{D}_\alpha(fm_\alpha)$  in  $m_\alpha$ . So there exists a subsequence  $\{g_{n_j}\}$ , for which  $\overline{D}_\alpha(g_{n_j} m_\alpha)$  converges to  $\overline{D}_\alpha(fm_\alpha)$  pointwise  $m_\alpha$ -a.e. Then, passing to the limit in

$$C|g_{n_j}(\zeta)| \leq \overline{D}_\alpha|g_{n_j} m_\alpha|(\zeta) \leq |g_{n_j}(\zeta)|$$

(which is true  $m_\alpha$ -a.e.), we obtain Inequality (2) for  $f$ .  $\square$

#### 4. NONTANGENTIAL LIMITS

*Notation.* For  $\zeta \in \mathbb{S}^n$  and  $\gamma > \frac{1}{2}$ , we will consider the usual approach regions

$$\Gamma_\gamma(\zeta) = \{z \in B_n : |1 - \langle z, \zeta \rangle| < \gamma(1 - |z|^2)\}.$$

As usual,  $C$  will denote some absolute constant that may change from one inequality to the next one.

**Lemma 4.1.** *Let  $\gamma > \frac{1}{2}$  and let  $\alpha \in \mathbb{R}$  be such that  $0 \leq \alpha \leq n$ . Then there exists a constant  $C = C(\alpha, \gamma)$  such that for every finite complex measure  $\nu$  on  $\mathbb{S}^n$ ,*

$$(3) \quad \limsup_{\substack{z \rightarrow \zeta \\ z \in \Gamma_\gamma(\zeta)}} (1 - |z|)^{n-\alpha} P[\nu](z) \leq C \overline{D}_\alpha|\nu|(\zeta)$$

for every  $\zeta \in \mathbb{S}^n$ .

*Proof.* As the kernel  $P$  is positive,  $|P[\nu]| \leq P[|\nu|]$ , so it is enough to prove the lemma for positive measures. By standard consequences of mean-value properties, it is also enough to consider the case  $\gamma = 1$ . Write  $\Gamma(\zeta)$  for  $\Gamma_1(\zeta)$ .

Fix  $\zeta \in \mathbb{S}^n$  such that  $\overline{D}_\alpha|\nu|(\zeta)$  is finite. Let  $z \in \Gamma(\zeta)$  and set  $\delta = \frac{1}{2}(1 - |z|)$ .

Let  $\varepsilon > 0$  and take  $\Delta < 2$  small enough to have  $\frac{\nu(Q(\zeta, r))}{r^\alpha} < \overline{D}_\alpha|\nu|(\zeta) + \varepsilon$  for all  $r < \Delta$ . Let  $N$  be the greatest integer such that  $2^N \delta < \Delta$  (we may assume that  $\delta$  is small enough to have  $N > 1$ , i.e.  $\delta < \frac{1}{4}\Delta$ ). Set

$$V_0 = \{\omega \in \mathbb{S}^n : |1 - \langle \omega, \zeta \rangle| < \delta\} = Q(\zeta, \delta),$$

for  $k = 1, 2, \dots, N$ , set

$$V_k = \{\omega \in \mathbb{S}^n : 2^{k-1}\delta \leq |1 - \langle \omega, \zeta \rangle| < 2^k\delta\} \subset Q(\zeta, 2^k\delta),$$

and set  $V_\infty = \mathbb{S}^n \setminus (\cup_{k=0}^N V_k)$ . Then

$$\begin{aligned}
(1 - |z|)^{n-\alpha} P[\nu](z) &= (1 - |z|)^{n-\alpha} \int_{\mathbb{S}^n} P(z, \omega) d\nu(\omega) \\
(4) \quad &= (1 - |z|)^{n-\alpha} \int_{V_0} P(z, \omega) d\nu(\omega) + (1 - |z|)^{n-\alpha} \sum_{k=1}^N \int_{V_k} P(z, \omega) d\nu(\omega) \\
&\quad + (1 - |z|)^{n-\alpha} \int_{V_\infty} P(z, \omega) d\nu(\omega).
\end{aligned}$$

For the first term, if  $\omega \in V_0$  then one has

$$P(z, \omega) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \omega \rangle|^{2n}} \leq \frac{2^n}{(1 - |z|)^n} = \frac{2^{2n}}{\delta^n}.$$

So,

$$\begin{aligned}
(1 - |z|)^{n-\alpha} \int_{V_0} P(z, \omega) d\nu(\omega) &\leq \frac{1}{2^{n-\alpha}} \delta^{n-\alpha} \frac{2^{2n}}{\delta^n} \nu(V_0) = C \frac{\nu(V_0)}{\delta^\alpha} = C \frac{\nu(Q(\zeta, \delta))}{\delta^\alpha} \\
&\leq C \overline{D}_\alpha[\nu](\zeta) + C\varepsilon.
\end{aligned}$$

The sum in (4) of the integrals over  $V_k$  can be estimated in a similar way: if  $z \in \Gamma(\zeta)$  and  $\xi \in \mathbb{S}^n$ , then

$$P(z, \xi) \leq \frac{32^n (1 - |z|)^n}{|1 - \langle \xi, \zeta \rangle|^{2n}},$$

(see *e.g.* [10], page 90). In particular, for  $\xi \in V_k$ ,

$$P(z, \xi) \leq \frac{32^n 2^{-2n(k-1)}}{\delta^n} = \frac{C}{2^{2nk} \delta^n}.$$

We thus get

$$\begin{aligned}
(1 - |z|)^{n-\alpha} \int_{V_k} P(z, \xi) d\nu(\xi) &\leq C \frac{1}{2^{2nk}} \frac{\nu(V_k)}{\delta^\alpha} \leq C \frac{1}{2^{(2n-\alpha)k}} \frac{\nu(Q(\zeta, 2^k \delta))}{(2^k \delta)^\alpha} \\
&\leq C \frac{1}{2^{(2n-\alpha)k}} (\overline{D}_\alpha[\nu](\zeta) + \varepsilon).
\end{aligned}$$

As  $2n - \alpha > 0$ , the sum of all corresponding integrals can be estimated by  $C \overline{D}_\alpha[\nu](\zeta) + C\varepsilon$ .

Finally, in (4), the integral over  $V_\infty$  can be estimated by  $\frac{2^{2n} (1 - |z|)^n}{\Delta^{2n}} |\nu|(\mathbb{S}^n \setminus Q(\zeta, \frac{\Delta}{2}))$ . Summerizing the previous estimates, (4) gives

$$(1 - |z|)^{n-\alpha} P[\nu](z) \leq C \overline{D}_\alpha[\nu](\zeta) + C\varepsilon + \frac{2^{2n} (1 - |z|)^n}{\Delta^{2n}} |\nu|(\mathbb{S}^n \setminus Q(\zeta, \frac{\Delta}{2})).$$

As  $\Delta$  is fixed, passing to the lim sup, we get

$$\limsup_{z \rightarrow \zeta, z \in \Gamma_\gamma(\zeta)} (1 - |z|)^{n-\alpha} P[\nu](z) \leq C \overline{D}_\alpha[\nu](\zeta) + C\varepsilon.$$

As  $\varepsilon > 0$  is arbitrary, we get the desired estimate (3).  $\square$

Combining this with Lemma 3.2, it follows that

**Corollary 4.2.** *Let  $0 \leq \alpha \leq n$  and  $\gamma > \frac{1}{2}$ . Let  $\nu$  be a finite measure on  $\mathbb{S}^n$ , then if  $\nu$  is mutually singular with  $m_\alpha$ ,*

$$\lim_{\substack{z \in \Gamma_\gamma(\zeta) \\ z \rightarrow \zeta}} (1 - |z|)^{n-\alpha} P[\nu](z) = 0 \quad m_\alpha - \text{almost everywhere.}$$

The next lemma provides the estimate from below :

**Lemma 4.3.** *Let  $0 \leq \alpha \leq n$ . There exists a constant  $C > 0$  such that for every positive measure  $\nu$  and every  $\zeta \in \mathbb{S}^n$  there exists a sequence  $z_i \in \Gamma(\zeta)$  such that  $R_i = 1 - |z_i| \rightarrow 0$  and,*

$$(1 - |z_i|)^{n-\alpha} P[\nu](z_i) \geq C \overline{D}_\alpha(\nu)(\zeta).$$

*Proof.* Fix  $\zeta \in \mathbb{S}^n$  and let  $\delta > 0$ . Then,

$$(1 - |z|)^{n-\alpha} P[\nu](z) = (1 - |z|)^{n-\alpha} \int_{\mathbb{S}^n} P(z, \xi) d\nu(\xi) \geq (1 - |z|)^{n-\alpha} \int_{Q(\zeta, \delta)} P(z, \xi) d\nu(\xi).$$

But, for  $z \in \Gamma(\zeta)$  such that  $1 - |z| = \delta$ ,

$$P(z, \xi) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}} \geq \frac{(1 - |z|^2)^n}{(|1 - \langle \zeta, \xi \rangle|^{\frac{1}{2}} + |1 - \langle z, \zeta \rangle|^{\frac{1}{2}})^{4n}} \geq C \frac{\delta^n}{\delta^{2n}} = C \frac{1}{\delta^n}$$

so that

$$(1 - |z|)^{n-\alpha} \int_{Q(\zeta, \delta)} P(z, \xi) d\nu(\xi) \geq C \frac{\nu(Q(\zeta, \delta))}{\delta^\alpha}.$$

The result follows directly by the definition  $\overline{D}_\alpha(\nu)(\zeta) = \limsup_{r \rightarrow 0} \frac{\nu(Q(\zeta, r))}{r^\alpha}$ .  $\square$

We are now in position to prove the main theorem :

**Theorem 4.4.** *Let  $0 \leq \alpha \leq n$  and  $\gamma > \frac{1}{2}$ . There exist constants  $C, c > 0$  such that, for every complex valued Radon measure  $\nu$ , there exist an  $m_\alpha$ -summable function  $f$ , and a measure  $\nu_0$  mutually singular to  $m_\alpha$  such that  $\nu = f m_\alpha + \nu_0$ , and*

$$(5) \quad c|f(\zeta)| \leq \limsup_{\substack{z \in \Gamma_\gamma(\zeta) \\ z \rightarrow \zeta}} (1 - |z|)^{n-\alpha} |P[\nu](z)| \leq C|f(\zeta)|$$

$m_\alpha$ -almost everywhere.

*Proof.* The decomposition of  $\nu$  is given in Proposition 3.4 and Corollary 4.2 shows that the theorem is valid for  $\nu_0$ . So, it is enough to consider  $\nu = f m_\alpha$ . Further, the upper estimate is then just a combination of Inequality (3) and Lemma 3.5.

Let us first assume that  $\nu$  is a positive measure, *i.e.*  $f$  is positive. Then the lower estimate in (5) results from Lemma 4.3.

Next, assume that  $\nu$  thus  $f$  are real. Write  $f = f^+ - f^-$  where  $f^+$  and  $f^-$  are respectively the positive and the negative parts of  $f$ , in particular  $|f(\zeta)| = \max\{f^+(\zeta), f^-(\zeta)\}$ . As the kernel  $P$  is positive,

$$(6) \quad |P[\nu]| = |P[(f^+ - f^-)m_\alpha]| = |P[f^+m_\alpha] - P[f^-m_\alpha]|.$$

Now, for each  $\zeta \in \mathbb{S}^n$ , at most one of  $f^+(\zeta)$  and  $f^-(\zeta)$  is non zero. Thus, by the upper estimate, for  $m_\alpha$ -almost every  $\zeta \in \mathbb{S}^n$ , at most one of  $(1 - |z|)^{n-\alpha} P[f^+m_\alpha]$  and  $(1 - |z|)^{n-\alpha} P[f^-m_\alpha]$  has non

zero limit as  $z \rightarrow \zeta$ ,  $z \in \Gamma_\gamma(\zeta)$ . It follows from the lower estimate for positive measures and (6) that  $m_\alpha$ -almost every  $\zeta \in \mathbb{S}^n$ ,

$$\begin{aligned} \limsup_{\substack{z \in \Gamma_\gamma(\zeta) \\ z \rightarrow \zeta}} (1 - |z|)^{n-\alpha} |P[\nu]| &= \limsup_{\substack{z \in \Gamma_\gamma(\zeta) \\ z \rightarrow \zeta}} |(1 - |z|)^{n-\alpha} P[f^+ m_\alpha] - (1 - |z|)^{n-\alpha} P[f^- m_\alpha]| \\ &\geq c \max\{f^+(\zeta), f^-(\zeta)\} = c|f(\zeta)|. \end{aligned}$$

To conclude, if  $\nu$  is a complex measure, then  $\nu = \nu_{\mathbb{R}} + i\nu_{i\mathbb{R}}$  where  $\nu_{\mathbb{R}} = (\operatorname{Re} f)m_\alpha$ ,  $\nu_{i\mathbb{R}} = (\operatorname{Im} f)m_\alpha$  are real measures. As the kernel  $P$  is real,  $\operatorname{Re} P[\nu] = P[\nu_{\mathbb{R}}]$  and  $\operatorname{Im} P[\nu] = P[\nu_{i\mathbb{R}}]$ . The lower estimate is true for both  $\operatorname{Re} P[\nu]$  and  $\operatorname{Im} P[\nu]$ , thus

$$\begin{aligned} \limsup_{\substack{z \in \Gamma_\gamma(\zeta) \\ z \rightarrow 0}} (1 - |z|)^{n-\alpha} |P[\nu]| &\geq \limsup_{\substack{z \in \Gamma_\gamma(\zeta) \\ z \rightarrow \zeta}} (1 - |z|)^{n-\alpha} \max\{|P[\nu_{\mathbb{R}}]|, |P[\nu_{i\mathbb{R}}]|\} \\ &\geq c \max\{|\operatorname{Re} f(\zeta)|, |\operatorname{Im} f(\zeta)|\} \geq c \frac{|\operatorname{Re} f(\zeta)| + |\operatorname{Im} f(\zeta)|}{2} \\ &\geq \frac{c}{4} |f(\zeta)|, \end{aligned}$$

which completes the proof.  $\square$

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