TENSOR PRODUCTS OF MINIMAL HOLOMORPHIC REPRESENTATIONS

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ABSTRACT. Let $D = G/K$ be an irreducible bounded symmetric domain with genus p and $H^\nu(D)$ the weighted Bergman spaces of holomorphic functions for $\nu > p - 1$. The spaces $H^\nu(D)$ form unitary (projective) representations of the group G and have analytic continuation in ν; they give also unitary representations when ν in the Wallach set, consisting of a continuous part and a discrete part of r points. The first non-trivial discrete point $\nu = \frac{a}{2}$ gives the minimal highest weight representation of G. We give the irreducible decomposition of tensor product $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$. As a consequence we discover some new spherical unitary representations of G and find the expansion of the corresponding spherical functions in terms of the K-invariant (Jack symmetric) polynomials, the coefficients being continuous dual Hahn polynomials.

INTRODUCTION

Let $D = G/K$ be an irreducible bounded symmetric domain of rank r in a complex vector space V with Lebesgue measure $dm(z)$. The Bergman reproducing kernel of D is of the form $h(z, w)^{-p}$, where p is the genus of D. Let $\nu > p - 1$ and consider the weighted Bergman space H^ν with the weighted measure $h(z, z)^{\nu - p} dm(z)$. They give naturally unitary representations of the group G and have analytic continuation in the parameter ν. The set of those ν for which H^ν still form unitary representations is called the Wallach set and has been determined by various methods ([24], [29] and [5]). It is a union of an open interval and a discrete set, the last point in the discrete set is $\nu = 0$ and corresponds to the trivial representation. Suppose that the rank r of D is bigger than 2. The other points in the discrete Wallach set correspond to some singular representations of G; the K-types appearing in the representations form some lower dimensional lattices. The first discrete point $\nu = \frac{a}{2}$ above the trivial point $\nu = 0$ gives the minimal representation and the lattice of K-types is one dimensional. Minimal and singular representations are of considerable interests since they normally can not be constructed by standard methods. One may well expect that the representations appearing in the tensor product decomposition are also some minimal (singular) representations, thus it is worthwhile to study. Indeed

we discover some new irreducible unitary (minimal) representations that appear in the decomposition. We also find the annihilating invariant differential operators of the tensor product; we find the expansion of a family of spherical functions in terms of the K-invariant polynomials, which are the Jack symmetric polynomials, the coefficient being the continuous dual Hahn polynomials.

To give a brief background we consider first the case of the unit disk $D = G/K = SU(1,1)/SO(2)$ in the complex plane. The Hilbert space H^ν in question are the ones with reproducing kernels $(1-z\bar{w})^{-\nu}$, $\nu \geq 0$, with $\nu = 0$ giving the trivial (and minimal) representation; so the problem of tensor product of the minimal representations in this case is trivial, however the consideration for other parameter of ν will give us ideas for treating higher rank cases. In our earlier papers [32] and [21] we studied the explicit spectral decomposition of the tensor products. The main idea there is to study restriction operator R, $Rf(z) = (1 - |z|^2)^\nu f(z)$ from the tensor product to the space $C^\infty(D)$, which was considered earlier also by Repka [22] and [23] (an idea due to Howe, see loc. cit.); the operator R intertwines the tensor product action with the regular action of G. We consider further its polar decomposition, $R = |R|U$. The operator R is bounded and has dense range into the space $L^2(D)$ with the G-invariant measure, for $\nu > 1$. Thus for those ν the operator U is a unitary intertwining operator onto the space $L^2(D)$, whose decomposition is given by the known spherical transform ([7], Introduction). However for smaller values of ν, the above polar decomposition does not make sense. Let $\phi_\lambda(z)$ be the spherical function on the unit disk. Our idea is simply to consider the power series expansion of the function $(1 - |z|^2)^{-\nu}\phi_\lambda(z)$,

$$(1 - z\bar{z})^{-\nu}\phi_\lambda(z) = \sum_{m=0}^{\infty} p_{\nu,m}(\lambda)(z\bar{z})^{2n}.$$

Conceptually the l.h.s. is restriction to diagonal (z, z) of the eigenfunction $R^{-1}\phi_\lambda$ of the Casimir element on the tensor product, and the formula is its expansion in terms of the $K = SO(2)$-invariant elements $(z\bar{w})^{2n}$ in the tensor products. The action of the Casimir element is equivalent to a multiplication by $(\lambda^2 + \frac{1}{4})$ on the coefficients $p_{\nu,m}(\lambda)$. It turns out that $p_{\nu,m}(\lambda)$ are the continuous dual Hahn polynomials, whose orthogonality relation has been proved by Wilson (see [30], [1] and [14]). So by using the orthogonality relation, we found in [19] the irreducible decomposition of the tensor product $H^\nu \otimes \overline{H^\nu}$ for all $\nu > 0$.

Consider general irreducible bounded symmetric domain $D = G/K$ of rank $r \geq 2$. By a general consideration we know that the representations appearing in the decomposition of the tensor product $H^\nu \otimes \overline{H^\nu}$ are spherical, for all ν in the Wallach set. Our interests will be the tensor product when ν is in the discrete Wallach set. So let $\nu = \frac{a}{2}(j - 1)$ be such a point. We consider the expansion of the functions $R^{-1}\phi_\lambda$

in terms of the K-invariant polynomials.; more precisely

$$(0.1) \qquad h^{-\nu}(z,z)\phi_{\underline{\lambda}}(z) = \sum_{\underline{\mathbf{m}}} p_{\underline{\mathbf{m}}}(\underline{\lambda})K_{\underline{\mathbf{m}}}(z,z),$$

in a neighborhood of $z = 0$. (See Section 1 for the definition of $K_{\underline{\mathbf{m}}}$.) This formula makes sense for all ν. For any invariant differential operator $\mathcal{M} \in \mathcal{D}^G(D)$ with eigenvalue $\mathcal{M}(\underline{\lambda})$ on the spherical function $\phi_{\underline{\lambda}}$, the invariant differential operator $R^{-1}\mathcal{M}R$ on the basis vectors $\{K_m(z,w)\}$ is then unitarily equivalent to the multiplication operator by $\mathcal{M}(\underline{\lambda})$ on the coefficients $\{p_{\underline{\mathbf{m}}}(\underline{\lambda})\}$, see Proposition 2.4. Our problem will be partly to identify those polynomials.

The algebra $\mathcal{D}^G(D)$ is commutative with r generators. In his paper [26] Shimura constructs an r-tuple of generators $(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_r)$. Later the author [35] has found the eigenvalues of the generators on the spherical functions. When $\nu = \frac{a}{2}$ we prove that the image $R(H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}})$ of the tensor product under R is annihilated by the operators $\mathcal{L}_2, \ldots, \mathcal{L}_r$; see Proposition 4.1. By our early results [35] on the eigenvalues of the Shimura operator we know that a spherical function $\phi_{\underline{\lambda}}$ is annihilated by those Shimura operators if and only $\underline{\lambda}$ is in certain one dimensional hyperplane. For those $\phi_{\underline{\lambda}}$ we find the above expansion, where only those $\underline{\mathbf{m}} = (m, 0, \ldots, 0)$ appear. The coefficients turns out also to be the continuous dual Hahn polynomials, their orthogonality relation then gives the spectrum of the multiplication operator by $\mathcal{L}_1(\underline{\lambda})$ and thus the operator of $R^{-1}\mathcal{L}_1 R$; see Theorem 6.1. For type I domain $SU(2,2)/(S(U(2) \times U(2))$, this has been done in [20] by using the explicit (Berezin's) formula for spherical functions [9].

It turns out that when (and only when) D is a non-tube domain of type one $SU(r, r+b)/S(U(r) \times U(r+b))$ with $b \geq 2$ there are discrete parts, to be called complementary series, appearing; and they naturally deserve further study. Viewing the tensor product as the space of Hilbert-Schmidt operators we thus get a *quantization* of the complementary series, see Theorem 7.1. In fact, we find that a larger family of spherical representations can be quantized as Schatten von-Newmann $\mathfrak{S}_q$ operators on $H^{\frac{a}{2}}$, so that we get some invariant Banach spaces (instead of Hilbert spaces) generated by the spherical functions. See also [4] for the rank one case.

We mention that the tensor product decomposition has been an important methods in producing new representations, and has been studied extensively in the literature, in particularly in its relation to dual pairs; see [11]. In their paper [3] Sahi and Dvorsky studied the tensor products $H_1 \otimes \cdots \otimes H_l$ of several more general singular representations and construct dual pairs. See also [8], [16], [17] and references therein. Also, there are some renewed interests in complementary series in connection with some other analytical problems; see e.g. [12].

The paper is organized as follows. In Section 1 we recall some well-known results on holomorphic spaces on bounded symmetric domains, and we prove a decomposition

result on the point-wise product of two irreducible polynomial spaces. In Section 2 we incorporate the known results on tensor product of $H^{\nu} \otimes \overline{H^{\nu}}$ for regular parameter ν. Section 3 and 4 are devoted to Shimura invariant differential operators and their annihilating property. In Sections 5 and 6 we find the irreducible decomposition of the tensor product. In Section 7 we study in particular the complementary series appearing in the decomposition. Section 8 is devoted to the proof of positive definiteness of the spherical functions, which, with the help of our explicit formula for the Clebsch-Gordan coefficients and their orthogonality relation, is straightforward and of an abstract nature. For the convenience of the reader we list some known property of the continuous dual Hahn polynomials in the last section.

1. Bounded Symmetric Domains and the Polynomial Spaces

In this section we fix notations and recall some necessary results on bounded symmetric domains. The notation and setup will be the same as in [35] and [34], so that we will be very brief.

Let G/K be an irreducible Hermitian symmetric space. It can be realized as a bounded convex domain D in a complex n-dimensional space V with G realized as the identity component of the group of biholomorphic mappings and K the isotropic subgroup of $0 \in \Omega$. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be its Cartan decomposition. The Lie algebra $\mathfrak{k}$ has one-dimensional center. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{+} \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{-}$ be the corresponding eigenspace decomposition of $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$. The vector space $V = \mathfrak{p}^{+}$ can be identified with the holomorphic tangent space.

The vector space V has a structure of a Jordan triple system so that $\mathfrak{p} = \{\xi_v(z) = v - Q(z)\bar{v}; v \in V\}$ where $Q(z) \in Aut(\bar{V}, V)$ is a quadratic operator. We normalize the K-invariant inner product $\langle z, w \rangle$ on V by as in [34], so that a minimal tripotent has norm 1.

We fix $\{e_1, \ldots, e_r\}$ a frame of V and $\mathfrak{a} = \mathbb{R}\xi_{e_1} + \cdots + \mathbb{R}\xi_{e_r}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ with basis vectors $\xi_{e_1}, \xi_{e_2}, \cdots, \xi_{e_r}$. Let $\{\beta_j\}_{j=1}^r \subset \mathfrak{a}^*$ be the basis of $\mathfrak{a}^*$ determined by

$$\beta_j(\xi_{e_k}) = 2\delta_{j,k}, \ 1 \le j, k \le r,$$

and define an ordering on $\mathfrak{a}^*$ via

$$\beta_r > \beta_{r-1} > \cdots > \beta_1 > 0.$$

We will write an element $\underline{\lambda} \in (\mathfrak{a}^*)$ as

$$\underline{\lambda} = \sum_{j=1}^{r} \lambda_j \beta_j.$$

and identify $\underline{\lambda}$ with $(\lambda_1, \lambda_2, \cdots, \lambda_r)$. The half sum of the positive roots is given by

$$(1.1) \qquad \underline{\rho} = \sum_{j=1}^{r} \rho_j \beta_j = \sum_{j=1}^{r} \frac{b + 1 + a(j-1)}{2} \beta_j,$$

where a is the root multiplicity of $\frac{\beta_j \pm \beta_k}{2}$ and $2b$ the root multiplicity of $\frac{\beta_j}{2}$.

Let $\mathcal{P}(V)$ be the space of all holomorphic polynomials on V. The group K acts naturally on $\mathcal{P}(V)$ induced from its regular action on V. Its irreducible decomposition is now well known; see [10], [25] and [5]. To state the result we let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$ that contains the elements $D(e_j, e_j)$, $j = 1, 2, \ldots, r$. Let $\gamma_1 > \gamma_2 \cdots > \gamma_r$ be the corresponding Harish-Chandra strongly orthogonal roots. Thus $\gamma_k(D(e_j, e_j)) = 2\delta_{jk}$. The space $V = \mathfrak{p}^+$ is now of highest weigh γ_1 with highest weight vector e_1; and dual space $V' = \mathfrak{p}^-$ is of lowest weight $-\gamma_1$. The subspace $\mathcal{P}_m$ of homogeneous polynomial of degree m is decomposed into irreducible subspaces with multiplicity one as

$$\mathcal{P}_m = \sum_{\underline{\mathbf{m}}} \mathcal{P}_{\underline{\mathbf{m}}}$$

where each $\mathcal{P}_{\underline{\mathbf{m}}}$ is of lowest weight $-\underline{\mathbf{m}} = -(m_1 \gamma_1 + \cdots + m_r \gamma_r)$ with $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$ being integers, and the summation is over all $\underline{\mathbf{m}}$ with $|\underline{\mathbf{m}}| = m_1 + m_2 + \cdots + m_r = m$.

We define a K-invariant function $h(z)$ on D so that

$$h(c_1 e_1 + c_2 e_2 + \cdots + c_r e_r) = (1 - |c_1|^2)(1 - |c_2|^2) \ldots (1 - |c_r|^2)$$

and let $h(z, w)$ be its polarization, holomorphic in z and antiholomorphic in w so that $h(z, z) = h(z)$. Consider the weighted probability measure

$$(1.2) \qquad d\mu_\nu(z) = c_\nu h(z)^{\nu - p} dm(z)$$

with $\nu > p - 1$ and c_ν a normalization constant. We denote H^ν the corresponding weighted Bergman space; it has reproducing kernel $h(z, w)^{-\nu}$.

The group G acts unitarily on H^ν via the following

$$(1.3) \qquad \pi(\nu)(g)f(z) = (J_{g^{-1}}(z))^{\frac{\nu}{p}} f(g^{-1} z), \quad g \in G,$$

and it gives irreducible unitary (projective) representation of G. One may also consider more generally the actions of G on vector-valued C^∞-functions on D; see (3.1) below.

We recall now the Faraut-Koranyi expansion of the reproducing kernel $h(z, w)^{-\nu}$. Let $K_{\underline{\mathbf{m}}}$ be as in [5] the reproducing kernel of the subspace $\mathcal{P}_{\underline{\mathbf{m}}}$ with the Fock norm.

Theorem 1.1. *(Faraut and Koranyi* [5]*, Theorem 3.8) The function $h(z, w)^{-\nu}$ has the following expansion*

(1.4) $$h^{-\nu}(z, w) = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(z, w)$$

for all $\nu \in \mathbb{C}$, and the convergence is uniform on compact subsets of $D \times D$. Here

$$(\nu)_{\mathbf{m}} = \prod_{j=1}^{r} (v - \frac{a}{2}(j - 1))_{m_j} = \prod_{j=1}^{r} \prod_{k=1}^{m_j} (v - \frac{a}{2}(j - 1) + k - 1).$$

It follows from this expansion that the kernel $h^{-\nu}(z, w)$ is positive definite and defines a Hilbert space if and only if ν is in

(1.5) $$W(D) = \{0, \frac{a}{2}, \dots, \frac{a}{2}(r - 1)\} \cup (\frac{a}{2}(r - 1), \infty),$$

also called Wallach set.

If $\nu = \frac{a}{2}(j - 1)$ in the discrete Wallach set, only certain subspaces $\mathcal{P}_{\mathbf{m}}$ are in the Hilbert space $H^\nu = H^{\frac{a}{2}(j-1)}$; more precisely

(1.6) $$H^{\frac{a}{2}(j-1)} = \sum_{\mathbf{m}:m_j=0} \mathcal{P}_{\mathbf{m}}.$$

Moreover it forms an irreducible representation of G with the action $\pi(\nu)$. In particular, the algebraic sum of all $\mathcal{P}_{\mathbf{m}}$ with $m_j = 0$ forms an irreducible representation of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

The next lemma will be used in the proof of Theorem 5.2.

Lemma 1.2. *Let $1 \le j \le r - 1$. Consider the product $\mathcal{P}_{\mathbf{m}} \cdot \mathcal{P}_{\mathbf{m}'}$ (consisting of sum of point-wise products two polynomials in the respective spaces) of a subspace $\mathcal{P}_{\mathbf{m}}$ with signature $\mathbf{m} = (m_1, 0, \dots, 0)$ and $\mathcal{P}_{\mathbf{m}'}$ with $\mathbf{m}' = (m'_1, \dots, m'_{j-1}, 0, \dots, 0)$. Let*

$$\mathcal{P}_{\mathbf{m}} \cdot \mathcal{P}_{\mathbf{m}'} = \sum_{\mathbf{n}} \mathcal{P}_{\mathbf{n}}$$

be its irreducible decomposition under K (which is multiplicity free by Theorem 1.1). Then the signatures $\mathbf{n} = (n_1, \dots, n_r)$ that appear in the decomposition satisfy $n_{j+1} = 0$.

To prove the lemma we give a more general result on tensor products of Hilbert spaces with reproducing kernels; it might have been proved previously, but we include here a simple proof. (Similar arguments have also been used in studying tensor product of a holomorphic Hilbert space with its conjugate; see e.g. [4].)

Let temporarily $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ be two Hilbert space of holomorphic functions on a bounded domain Ω in $\mathbb{C}^n$ with reproducing kernels $K_1(z, \xi)$ and $K_1(z, \xi)$, so that the point evaluation is a continuous linear functional. Thus the reproducing kernels $K_1(z, \xi)$ and $K_1(z, \xi)$ are positive definite. The point-wise (Schur) product $K_1 K_2(z, \xi) = K_1(z, \xi) K_2(z, \xi)$ is again semipositive definite. Let $\mathcal{H}(K_1 K_2)$ denote

the corresponding Hilbert space determined by the reproducing kernel and $H(D)$ the space of all holomorphic functions on D. The tensor product $\mathcal{H}(K_1) \otimes \mathcal{H}(K_2)$ can be realized as a space of holomorphic functions $F(z, w)$ in two variables.

Lemma 1.3. *Consider the operator* $R : \mathcal{H}(K_1) \otimes \mathcal{H}(K_2) \to H(D)$ *by the restriction to the diagonal* $Rf(z) = f(z, z)$. *Then* R *extends to an isometry from* $(\mathrm{Ker}\, R)^{\perp}$ *onto* $\mathcal{H}(K_1 K_2)$.

Proof. Clearly $\mathrm{Ker}\, R$ is a closed subspace of $\mathcal{H}(K_1) \otimes \mathcal{H}(K_2)$. Thus R defines a one-to-one map from $(\mathrm{Ker}\, R)^{\perp}$ into $H(D)$. Consider the elements in $\mathcal{H}(K_1) \otimes \mathcal{H}(K_2)$ of the form $g = \sum c_j K_1(z, \xi_j) K_2(w, \xi_j)$. Firstly these elements are in $(\mathrm{Ker}\, R)^{\perp}$; indeed for any $f \in \mathrm{Ker}\, R$, by the reproducing property,

$$\langle f, g \rangle = \sum \bar{c}_j f(\xi_j, \xi_j) = 0.$$

Secondly they are dense in $(\mathrm{Ker}\, R)^{\perp}$, since if $g_0 \in (\mathrm{Ker}\, R)^{\perp}$ is orthogonal to all g, it is in particular orthogonal to $g = g = K_1(z, \xi) K_2(w, \xi)$ for any fixed ξ, and

$$Rg_0(\xi) = g_0(\xi, \xi) = \langle g_0, g \rangle = 0,$$

namely $g_0 \in \mathrm{Ker}\, R \cap (\mathrm{Ker}\, R)^{\perp} = \{0\}$.

The images of g are

$$Rg(z) = \sum c_j K_1(z, \xi_j) K_2(z, \xi_j),$$

and they form a dense subspace of $\mathcal{H}(K_1 K_2)$. Moreover R is an isometry, again by the reproducing property. Thus R extends to an isometry from the closure of those g, which is $(\mathrm{Ker}\, R)^{\perp}$, onto $\mathcal{H}(K_1 K_2)$. $\qquad\square$

Specifying the above result with K_1 and K_2 replace by $H^{-\frac{a}{2}}$ respectively $H^{-\frac{a}{2}(j-1)}$ we see that the products $\mathcal{P}_{\mathbf{m}} \cdot \mathcal{P}_{\mathbf{m}'}$ two irreducible subspaces in the $H^{-\frac{a}{2}}$ and $H^{-\frac{a}{2}(j-2)}$ is a subspace in $H^{-\frac{a}{2}(j-1)}$, whose decomposition under K is given by the Faraut-Koranyi expansion in Theorem 1.1.

2. Tensor product of $\pi(\nu) \otimes \overline{\pi(\nu)}$ for $\nu > p - 1$ and Berezin transform

The tensor product of $\pi(\nu) \otimes \overline{\pi(\nu)}$ for the large parameter $\nu > p - 1$ have been studied in several contexts; see [18], [36] and reference therein. We recall some of the results in the literature. Consider the tensor product $H^{\nu} \otimes \overline{H^{\nu}}$, realized as the space of Hilbert-Schmidt operators T on H^{ν} with kernel $T(z, w)$ holomorphic in z and anti-holomorphic in w. The group G acts on the tensor product via $\pi(\nu) \otimes \overline{\pi(\nu)}$, and it gives a (genuine) representation. To study the irreducible decomposition we consider the map $R : H^{\nu} \otimes \overline{H^{\nu}} \to C^{\infty}(D)$, defined by

$$(2.1) \qquad\qquad RF(z) = F(z, z) h(z, z)^{\nu}.$$

Then R intertwines the action $\pi(\nu) \otimes \overline{\pi(\nu)}$ with the regular action $\pi(0)$ on $C^\infty(D)$. Clearly R is one-to-one. Note that the inverse operator R^{-1} is defined on the space of all real analytic functions on D and is essentially the so called polarization. (However generally, $R^{-1}f(z, w)$ for a real analytic function f is defined only on a small set near the diagonal, so it is not a function on $D \times D$; we thank Bo Berndtsson and Miroslav Engliš for some kind correspondence. In our case we shall only concern the operator R^{-1} as defined on $R(H^\nu \otimes \overline{H^\nu})$.) When considering functions $f(z, w)$ holomorphic in z and anti-holomorphic in w we will frequently identify f with its restriction $f(z, z)$ to the diagonal and simply write $f(z)$.

Let $L^2(D)$ be the L^2-space on D with respect to the G-invariant measure $\frac{dm(z)}{h(z,z)^p}$. When $\nu > p-1$ the operator R is an injective bounded operator into the space $L^2(D)$ with dense image; the decomposition of the later space is well known by the theory of spherical transform. Moreover the operator RR^* is actually the Berezin transform on $L^2(D)$. Denote $(H^\nu \otimes \overline{H^\nu})_0$ the subspace of K-invariant elements. In that subspace there is an orthogonal basis given by $K_{\mathbf{m}}(z, w)$, the spherical transforms of their images $R(K_{\mathbf{m}})$ under R provide then the Clebsch-Gordan coefficients; moreover they give also the coefficients in the coefficients of the expansion of the spherical function in terms of $K_{\mathbf{m}}$.

We let

$$(2.2) \qquad E_{\mathbf{m}}(z, z) = E_{\mathbf{m},\nu}(z, z) = (\nu)_{\mathbf{m}} K_{\mathbf{m}}(z, z)$$

$$(2.3) \qquad e_{\mathbf{m}}(z) = e_{\mathbf{m},\nu}(z, z) = \frac{E_{\mathbf{m},\nu}}{d_{\mathbf{m}}^{\frac{1}{2}}} = \frac{(\nu)_{\mathbf{m}} K_{\mathbf{m}}(z, z)}{d_{\mathbf{m}}^{\frac{1}{2}}},$$

where $d_{\mathbf{m}} = \dim \mathcal{P}_{\mathbf{m}}$. The following result is then a direct consequence of (and in fact is equivalent to) the expansion (1.4).

Lemma 2.1. *If $\nu > \frac{a}{2}(r-1)$, then the functions $e_{\mathbf{m},\nu}$ form an orthonormal basis of $(H_\nu \otimes \overline{H_\nu})_0$; if $\nu = \frac{a}{2}(j-1)$ for some $j = 1, \ldots, r$, then the functions $e_{\mathbf{m},\nu}$ for $\mathbf{m} = (m_1, \ldots, m_{j-1}, 0, \ldots 0)$ form an orthonormal basis of $(H_\nu \otimes \overline{H_\nu})_0$.*

The next result is proved in [34] (see (5.5) there). Let $\phi_{\underline{\lambda}}(z)$ be the spherical function D. We denote $\widehat{f}(\underline{\lambda})$ the spherical transform of a K-invariant function f on D,

$$\widehat{f}(\underline{\lambda}) = \int_D f(z) \phi_{\underline{\lambda}}(z) \frac{dm(z)}{h(z, z)^p}.$$

The constant c_ν below is that in (1.2). Note first that the function $h^{-\nu}(z)\phi_{\underline{\lambda}}(z)$ is formally the restriction to the diagonal of the polarization $R^{-1}\phi_{\underline{\lambda}}$ of $\phi_{\underline{\lambda}}$.

Lemma 2.2. *Consider the power series expansion of $h^{-\nu}(z)\phi_{\underline{\lambda}}(z)$*

$$(2.4) \qquad (R^{-1}\phi_{\underline{\lambda}})(z) = h^{-\nu}(z)\phi_{\underline{\lambda}}(z) = \sum_{\mathbf{m}} \mathcal{E}_{\mathbf{m},\nu}(\underline{\lambda}) E_{\mathbf{m}}(z, z)$$

in terms of the K-invariant polynomials $E_{\mathbf{m}}(z, z)$. Let

$$(2.5) \qquad \varepsilon_{\mathbf{m}}(\underline{\lambda}) = \varepsilon_{\mathbf{m},\nu}(\underline{\lambda}) = d_{\mathbf{m}}^{\frac{1}{2}} \mathcal{E}_{\mathbf{m},\nu}(\underline{\lambda}).$$

Suppose $\nu > p - 1$. Then the coefficients $\mathcal{E}_{\mathbf{m},\nu}(\underline{\lambda})$ can be obtained by the spherical transform,

$$(2.6) \qquad b_\nu(\underline{\lambda}) d_{\mathbf{m}} \mathcal{E}_{\mathbf{m},\nu}(\underline{\lambda}) = (\widehat{c_\nu h^\nu K_{\mathbf{m}}})(\underline{\lambda}).$$

and

$$(2.7) \qquad b_\nu(\underline{\lambda}) \varepsilon_{\mathbf{m},\nu}(\underline{\lambda}) = (\widehat{c_\nu h^\nu e_{\mathbf{m}}})(\underline{\lambda})$$

Remark 2.3. The above expansion (2.4) is valid a priori only in a neighborhood of $z = 0$. There arises naturally an interesting question as whether it holds for all $z \in D$. We

Proposition 2.4. *Suppose $\nu \in \{0, \frac{a}{2}, \ldots, \frac{a}{2}(r-1)\} \cup (\frac{a}{2}(r-1), \infty)$. Let $\mathcal{M}$ be an invariant differential operator on $C^\infty(D)$ with eigenvalue $\mathcal{M}(\underline{\lambda})$ on the spherical function $\phi_{\underline{\lambda}}$. Then the invariant differential operator $R^{-1}\mathcal{M}R$ on the orthonormal basis $\{e_{\mathbf{m}}(z)\}$ of $(H^\nu \otimes \overline{H^\nu})_0$ and the multiplication operator $\mathcal{M}(\underline{\lambda})$ on system of polynomials $\varepsilon e_{\mathbf{m}}(\underline{\lambda})$ have the same matrix form. Namely if*

$$(2.8) \qquad (R^{-1}\mathcal{M}R) e_{\mathbf{m}}(z) = \sum_{\mathbf{m}'} a_{\mathbf{m}}(\mathbf{m}') e_{\mathbf{m}'}(z),$$

then

$$(2.9) \qquad \mathcal{M}(\underline{\lambda}) \varepsilon_{\mathbf{m},\nu}(\underline{\lambda}) = \sum_{\mathbf{m}'} a_{\mathbf{m}}(\mathbf{m}') \varepsilon_{\mathbf{m}',\nu}(\underline{\lambda})$$

Proof. Suppose $\nu > p - 1$. The matrix form (2.8) can also be written as, after multiplifying the constant c_ν,

$$\mathcal{M}(c_\nu h^\nu e_{\mathbf{m}})(z) = \sum_{\mathbf{m}'} a_{\mathbf{m}}(\mathbf{m}') c_\nu h^\nu(z) e_{\mathbf{m}'}(z).$$

We perform the spherical transform on the equation and use (2.6). The LHS then becomes $\mathcal{M}(\underline{\lambda}) b_\nu(\underline{\lambda}) \varepsilon_{\mathbf{m},\nu}(\underline{\lambda})$ and the r.h.s. is

$$\sum_{\mathbf{m}'} a_{\mathbf{m}}(\mathbf{m}') b_\nu(\underline{\lambda}) \varepsilon_{\mathbf{m}',\nu}(\underline{\lambda});$$

dividing the both sides by $b_\nu(\underline{\lambda})$ proves the claim for $\nu > p - 1$. It is easily seen that the matrix forms (2.8) and (2.9) depend polynomially on the parameter ν. Thus it holds for all ν and $\mathbf{m}$ whenever $e_{\mathbf{m},\nu}$ is well-defined. $\square$

Remark 2.5. The above Proposition is a justification of a simple formal calculation. Let L be a self-adjoint operator on a Hilbert space, and $\{e_m\}$ an orthogonal basis so that each vector is in the domain of L. Suppose

$$Le_m = a_n e_{m-1} + b_m e_m + c_m e_{m+1}$$

and that $\psi = \sum_{m=0}^{\infty} f_m e_m$ is an eigenvector of L with eigenvalue λ (in some proper sense). Then the coefficients f_n satisfy the following recurrence relation

$$\lambda f_m = a_{m+1} f_{m+1} + b_m f_m + c_{m-1} f_{m-1}$$

which we may refer as the dual relation to the former. On the other hand, if the two relations hold then the series $\psi = \sum_{m=0}^{\infty} f_m e_m$ is an eigenvector of L whenever it makes sense. If moreover $\{e_m\}$ is an orthonormal basis then $a_{m+1} = c_m$ since L is a self-adjoint, and the two relations are exactly the same.

3. INVARIANT DIFFERENTIAL OPERATORS

We introduce the Shimura system of generators of all invariant differential operators on D. Let (X, τ) be a holomorphic representation of $K^{\mathbb{C}}$ on a finite dimensional vector space X. This then induces a homogeneous bundle W on D. A smooth section of the line bundle will be identified with an element in $C^{\infty}(D, X)$, the space of X-valued C^{∞}-function on $D = G/K$. The induced action of G on $C^{\infty}(D, X)$ is

$$(3.1) \qquad \pi_X(g)f(z) = \tau(dg^{-1}(z))^{-1} f(g^{-1}z).$$

In particular, if $X = \mathbb{C}$ is one dimensional with τ on $K^{\mathbb{C}}$ being $(\det(k))^{-\frac{\nu}{p}}$, we get the action (1.3) on $C^{\infty}(D)$.

We let $\mathcal{D} = \mathcal{D}_X$ be as in [35] the holomorphic covariant differentiation operator on $C^{\infty}(D, X)$. It maps $C^{\infty}(D, X)$ into $C^{\infty}(D, V' \otimes X)$, where V' is the dual space of V, viewed as the space of holomorphic cotangent space. More importantly, it intertwines the corresponding actions of G,

$$\mathcal{D}_X(\pi_X(g)f) = \pi_{V' \otimes X}(g)(\mathcal{D}_X f).$$

Let $\mathcal{D}_X^m = \mathcal{D}_{\otimes^{m-1} V' \otimes X} \ldots D_{V' \otimes X} D_X$ be the iterate of $\mathcal{D}$. It has been proved in [35] that $\mathcal{D}^m$ actually maps $C^{\infty}(D, X)$ into $C^{\infty}(D, S_m(V') \otimes X)$, where $S_m(V')$ stands for the subspace of symmetric tensors of $\otimes^m V'$. The space $S_m(V')$, as a K-module, is equivalent to the space of all homogeneous polynomials of degree m, thus can be decomposed under K as irreducible subspaces $S_{\underline{\mathbf{m}}}(V')$, of signature $\underline{\mathbf{m}} = (m_1, \ldots, m_r)$; we let $P_{\underline{\mathbf{m}}}$ be the orthogonal projection onto the corresponding subspace. The operator $P_{\underline{\mathbf{m}}} \mathcal{D}^m$ thus maps into the subbundle $S_{\underline{\mathbf{m}}}(V') \otimes X$. The Shimura invariant differential operators on $C^{\infty}(D, X)$ are then defined by

$$\mathcal{L}_{\underline{\mathbf{m}}} = (\mathcal{D}_X^m)^* P_{\underline{\mathbf{m}}} \mathcal{D}_X^m,$$

where $(\mathcal{D}_X^m)^*$ is the Hilbert space adjoint. In the present paper we will only consider the operator $\mathcal{L}_{\underline{\mathbf{m}}}$ on the trivial line bundle on D.

The operator $\mathcal{D}$ has been previously studied by Shimura for classical domains. In particular, Shimura [27] has given easier formula of the operators $P_{\underline{\mathbf{m}}} \mathcal{D}^m$ in terms of the Cayley-Capelli type operators, when $\underline{\mathbf{m}}$ are the fundamental representations

$\underline{\mathbf{m}} = \underline{\mathbf{1}}^j = (1, \ldots, 1, 0, \ldots, 0)$. We specify his results (see Theorem 4.7 loc. cit.) to the special case of scalar-valued functions; it is proved there for classical domains and is generalized in [35] for all bounded symmetric domains.

Theorem 3.1. *Let $1 \leq j \leq r$ and $\varepsilon = \frac{a}{2}(j-1)$. The operator $P_{\underline{\mathbf{1}}^j} \partial^j$ has the following intertwining property*

$$(3.2) \qquad P_{\underline{\mathbf{1}}^j} \partial^j (J_g^\varepsilon(z) f(gz)) = J_g(z)^{\frac{\varepsilon}{p}} (\otimes^j dg'(z)) (P_{\underline{\mathbf{1}}^j} \partial^j f)(gz);$$

the operator $P_{\underline{\mathbf{1}}^j} \mathcal{D}^j$ on the space $C^\infty(D)$ with the regular action of G can be expressed in terms of $P_{\underline{\mathbf{1}}^j} \partial^j$ as

$$(3.3) \qquad P_{\underline{\mathbf{1}}^j} \mathcal{D}^j f = h^{\frac{a}{2}(j-1)} P_{\underline{\mathbf{1}}^j} \partial^j (h^{-\frac{a}{2}(j-1)} f).$$

Consider the trivial line bundle on D. It has been proved by Shimura [26] that the operators

$$(3.4) \qquad \mathcal{L}_j = \mathcal{L}_{\underline{\mathbf{1}}^j}, \quad j = 1, \ldots, r$$

form a system of generators of the algebra of all invariant differential operators on $C^\infty(D)$.

Recall the intertwining operator R. Via conjugation by R we get r invariant differential operators

$$(3.5) \qquad \mathcal{L}_{j,\nu} = R^{-1} \mathcal{L}_{\underline{\mathbf{1}}^j} R, \quad j = 1, \ldots, r$$

on the tensor product $H^\nu \otimes \overline{H^\nu}$.

4. The invariant annihilating differential operators of the spaces H^ν and of the tensor products $H^\nu \otimes \overline{H^\nu}$ at the discrete points.

We shall prove in this section that the Shimura operators on $C^\infty(D)$ are annihilating differential operators of the image under R of the $H^\nu \otimes \overline{H^\nu}$ at the reducible points $\nu = \frac{a}{2}(j-1)$, $j = 1, \ldots, r$. We mention that holomorphic differential operators that annihilate the H^ν have been studied in [2] and [31].

Proposition 4.1. *Let $1 \leq j_0 \leq r$. Consider the intertwining operator $R = R_{\frac{a}{2}(j_0-1)}$ from the tensor product $H^{\frac{a}{2}(j_0-1)} \otimes \overline{H^{\frac{a}{2}(j_0-1)}}$ into $C^\infty(D)$. Its image $R(H^{\frac{a}{2}(j_0-1)} \otimes \overline{H^{\frac{a}{2}(j_0-1)}})$ is annihilated by the Shimura invariant differential operators $\mathcal{L}_j$ for $j_0 \leq j \leq r$.*

To prove the proposition we need the following

Lemma 4.2. *Let $1 \leq j \leq r$. The operator $P_{\underline{\mathbf{1}}^j} \partial^j$ annihilates all polynomial spaces $\mathcal{P}_{\underline{\mathbf{m}}}$ with $\underline{\mathbf{m}} = (m_1, \ldots, m_r)$ and $m_j = 0$.*

Proof. The space $H^{\frac{a}{2}(j-1)}$ is an irreducible representation of G with the action $\pi(\frac{a}{2}(j-1))$. Consider its subspace of the algebraic sum of all polynomial, by (1.6) it is a direct sum of those $\mathcal{P}_{\underline{\mathbf{m}}}$ so that $m_j = 0$. It forms an irreducible representation of of $\mathfrak{g}^{\mathbb{C}}$. By the intertwining property of $P_{\underline{\mathbf{1}}j}\partial^j$ stated in Theorem 3.1 we know that the kernel $\text{Ker}\, P_{\underline{\mathbf{1}}j}\partial^j$ is an $\mathfrak{g}^{\mathbb{C}}$-invariant subspace. Clearly the constant function $f_0 = 1$ is in the kernel. By the irreduciblility we know that $\text{Ker}\, P_{\underline{\mathbf{1}}j}\partial^j$ is the whole polynomial space. $\square$

Proof of Proposition 4.1. The function $1 \otimes 1$ is a cyclic vector of the $H^\nu \otimes \overline{H^\nu}$ under the action of G as easily seen using the reproducing property (see e.g. [4]), thus the function $R(1 \otimes 1) = h^\nu$ is a cyclic vector of $R(H^\nu \otimes \overline{H^\nu})$. Let $\nu = \frac{a}{2}(j_0 - 1)$. We prove $\mathcal{L}_j R(1 \otimes 1) = \mathcal{L}_j h^{\frac{a}{2}(j_0-1)} = 0$, the result then follows by the cyclic property of $R(1 \otimes 1)$. Now $\mathcal{L}_j = (\mathcal{D}^j)^* P_{\underline{\mathbf{1}}j} \mathcal{D}^j$ and by Theorem 3.1 we have

$$P_{\underline{\mathbf{1}}j} \mathcal{D}^j h^{\frac{a}{2}(j_0-1)} = h^{\frac{a}{2}(j-1)} P_{\underline{\mathbf{1}}j} \partial^j h^{-\frac{a}{2}(j-j_0)}.$$

We claim that $P_{\underline{\mathbf{1}}j} \partial^j h^{-\frac{a}{2}(j-j_0)} = 0$. Indeed consider the Faraut-Koranyi expansion of $h^{-\frac{a}{2}(j-j_0)}$:

$$h^{-\frac{a}{2}(j-j_0)}(z,z) = \sum_{\underline{\mathbf{m}}:m_{j-j_0+1}=0} (\frac{a}{2}(j-j_0))_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z,z).$$

Being the power series expansion of $h^{-\frac{a}{2}(j-j_0)}(z,z)$ it is absolutely convergent in a neighborhood of $z = 0$. Each term $K_{\underline{\mathbf{m}}}(z,z)$ is a sum of holomorphic polynomials in $\mathcal{P}_{\underline{\mathbf{m}}}$ with $m_{j-j_0+1} = 0$, consequently $m_j = 0$, with coefficients being anti-holomorphic polynomials. Thus $P_{\underline{\mathbf{1}}j} \partial^j K_{\underline{\mathbf{m}}}(z,z) = 0$ by Lemma 4.2. Consequently $P_{\underline{\mathbf{1}}j} \partial^j h^{-\frac{a}{2}(j-j_0)} = 0$, by the commutativity of the differentiation and summation in a power series expansion. $\square$

Proposition 4.3. *The spherical function $\phi_{\underline{\lambda}}$ is annihilated by all $\mathcal{L}_j$, $j = j_0, \ldots, r$, if and only if $\underline{\lambda}$ is in the Weyl group orbit of $(i\rho_1, \ldots, i\rho_{r-j_0}, i\rho_{r-j_0+1}s_{r-j_0+2}, \ldots, \underline{\mathbf{s}}_r)$ for some $(s_{r-j_0+2} \ldots, s_r) \in \mathbb{C}^{j_0-1}$.*

To prove the result we recall the formula for the eigenvalues of $\mathcal{L}_j$ obtained in [35], Theorem 5.3.

Lemma 4.4. *Consider the Shimura operators $\mathcal{L}_j$ on the trivial line bundle on D. Their eigenvalues on the spherical functions $\phi_{\underline{\lambda}}$ are given by*

$$\begin{aligned}
\mathcal{L}_j(\underline{\lambda}) = C_j \sum_{k=0}^{j} h_{j-k}\left(\rho_2^2 - \rho_1^2, \cdots, \rho_{r-j+1}^2 - \rho_1^2\right) \\
\times m_k\left(\lambda_1^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2\right).
\end{aligned} \tag{4.1}$$

Here C_j is a positive constant, h_k is the complete symmetric function of degree k and m_k the kth elementary symmetric function.

Proof of Proposition 4.3. The result will follow somewhat easier if we follow the proof of Theorem 5.3 in [35]. Suppose that $\underline{\lambda}$ is as in the Proposition. We may assume, up to the Weyl group action, that $\lambda_{j+1} = \rho_2, \ldots, \lambda_r = i\rho_{r-j+1}$, since $r - j + 1 \leq r - j_0 + 1$ and the set $\{\rho_1, \rho_2, \ldots, \rho_{r-j+1}\}$ is a subset of $\{\rho_1, \rho_2, \ldots, \rho_{r-j_0+1}\}$. It is proved in [35] (see formula (5.20) and (5.25) there) that in this case the eigenvalue of $\mathcal{L}_j$ is of the form

$$\mathcal{L}_j(\underline{\lambda}) = C_j \prod_{s=1}^{j} (\lambda_s^2 + (\frac{1+b}{2})^2) = C_j \prod_{s=1}^{j} (\lambda_s^2 + \rho_1^2)$$

Thus if in addition $\lambda_{j_0+1} = i\rho_1$, then $\mathcal{L}_j(\underline{\lambda}) = 0$.

Conversely, suppose now that $\underline{\lambda}$ is such that $\mathcal{L}_j(\underline{\lambda}) = 0$ for all $j_0 \leq j \leq r$. We start with the condition $\mathcal{L}_j(\underline{\lambda}) = 0$ for $j = r$ and prove that after a signed permutation we have $\lambda_1 = i\rho_1$, and prove successively by using the result by the condition $\mathcal{L}_j(\underline{\lambda}) = 0$ for $j = r - 1, r - 2, \ldots, j_0$.

Indeed, the eigenvalue $\mathcal{L}_r(\underline{\lambda})$ is

$$\mathcal{L}_r(\underline{\lambda}) = C_r \prod_{j=1}^{r} (\lambda_j^2 + \rho_1^2).$$

Thus $\mathcal{L}_r(\underline{\lambda}) = 0$ implies that one of the λ_k is $\pm i\rho_1$. We may assume $\lambda_1 = i\rho_1$. Fix this value of λ_1. We study the condition $\mathcal{L}_j(\underline{\lambda}) = 0$ for $j = r - 1$. The eigenvalue of $\mathcal{L}_j(\underline{\lambda})$ for $j = r - 1$ is then, by Lemma 4.4

$$
\begin{aligned}
\mathcal{L}_j(\underline{\lambda}) &= C_{r-1} \sum_{k=0}^{r-1} h_{r-1-k}(\rho_2^2 - \rho_1^2) m_k(0, \lambda_2^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2) \\
&= C_{r-1} \sum_{k=0}^{r-1} (\rho_2^2 - \rho_1^2)^{r-1-k} m_k(\lambda_2^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2) \\
&= C_{r-1}(\rho_2^2 - \rho_1^2)^{r-1} \sum_{k=0}^{r-1} (\rho_2^2 - \rho_1^2)^{-k} m_k(\lambda_2^2 + \rho_1^2, \cdots, \lambda_r^2 + \rho_1^2) \\
&= C_{r-1}(\rho_2^2 - \rho_1^2)^{r-1} \prod_{k=2}^{r} (1 + (\rho_2^2 - \rho_1^2)^{-1}(\lambda_k^2 + \rho_1^2))
\end{aligned}
$$

(4.2)

where in the last equality we have used the formula

$$\sum_{k=0}^{l} m_k(x_1, \ldots, x_l) t^k = \prod_{k=1}^{l} (1 + x_k t)$$

for the generating function of m_k. The product is then, disregarding a positive factor,

$$\prod_{k=2}^{r} (\lambda_k^2 + \rho_2^2).$$

Thus if $\mathcal{L}_j(\underline{\lambda}) = 0$ for $j = r - 1$, then one of λ_k, $k = 2, \ldots, r$, is $i\rho_2$; we may assume $\lambda_2 = i\rho_2$.

We claim that generally if $\lambda_s = i\rho_s$, for all $1 \le s \le j$, for a fixed $1 \le j \le r$, then the eigenvalue of $\mathcal{L}_{r-j}(\underline{\lambda})$ is

$$(4.3) \qquad C \prod_{k=j+1}^{r} (\lambda_k^2 + \rho_{j+1}^2).$$

Accepting temporarily the claim, the eigenvalue of $\mathcal{L}_j(\underline{\lambda}) = 0$ for $j = r - 2$ implies that $\lambda_3 = i\rho_3$, and the result is then proved by repeating the argument.

The eigenvalue $\mathcal{L}_{r-j}(\underline{\lambda})$ is, disregarding the nonzero constant,

$$\sum_{k=0}^{r-j} h_{r-j-k} \left(\rho_2^2 - \rho_1^2, \ldots, \rho_{j+1}^2 - \rho_1^2 \right) m_k \left(-\rho_2^2 + \rho_1^2, \cdots, -\rho_j^2 + \rho_1^2, \lambda_{j+1}^2 + \rho_1^2, \lambda_r^2 + \rho_1^2 \right)$$

To simplify the nation we write $c_k = \rho_k^2 - \rho_1^2$ for $k = 2, \ldots, j+1$. The above formula is

$$\sum_{k=0}^{r-j} h_{r-j-k}(c_2, \ldots, c_{j+1}) \, m_k \left(-c_2, \ldots, -c_j, \lambda_{j+1}^2 + \rho_1^2 \ldots, \lambda_r^2 + \rho_1^2 \right)$$

As a polynomial of λ_{j+1}^2, the point $\lambda_{j+1}^2 = -\rho_{j+1}^2$ is its zero. Indeed, denoting temporarily $\mathbf{d} = (\lambda_{j+2} + \rho_1^2, \ldots, \lambda_r^2 + \rho_1^2) \in \mathbb{C}^{r-j-1}$, at the point $\lambda_{j+1}^2 = -\rho_{j+1}^2$ the polynomial is

$$\sum_{k=0}^{r-j} h_{r-j-k}(c_2, \ldots, c_{j+1}) \, m_k(-c_2, \cdots, -c_j, -c_{j+1}, \mathbf{d})$$

$$= \sum_{k=0}^{r-j} h_{r-j-k}(c_2, \ldots, c_{j+1}) \sum_{l=0}^{k} m_{k-l}(-c_2, \cdots, -c_j, -c_{j+1}) m_l(\mathbf{d})$$

$$(4.4) \qquad = \sum_{l=0}^{r-j} m_l(\mathbf{d}) \sum_{k=l}^{r-j} h_{r-j-k}(c_2, \ldots, c_{j+1}) \, m_{k-l}(-c_2, \cdots, -c_j, -c_{j+1})$$

$$= \sum_{l=0}^{r-j} m_l(\mathbf{d}) \sum_{k=0}^{r-j-l} h_k(c_2, \ldots, c_{j+1}) \, m_{r-j-k}(-c_2, \cdots, -c_j, -c_{j+1})$$

$$= \sum_{l=0}^{r-j} m_l(\mathbf{d}) \delta_{r-j-l,0} = m_{r-j}(\mathbf{d})$$

where $\delta_{r-j-l,0}$ is the Kronecker symbol; here we have use the formula

$$(4.5) \qquad \sum_{l=0}^{k} (-1)^s m_s h_{k-s} = \delta_{k,0};$$

see [15], Section I.2. But the dimensionality of $\mathbf{d}$ is $r - j - 1$ which is less than $r - j$, thus $m_{r-j}(\mathbf{d}) = 0$.

Consequently $\mathcal{L}_{r-j}(\underline{\lambda})$ as a polynomial of λ_{j+1}^2 has a factor $\lambda_{j+1}^2 + \rho_{j+1}^2$. Being a symmetric polynomial of $\lambda_{j+1}^2, \ldots, \lambda_r^2$, it has a factor $\lambda_k^2 + \rho_{j+1}^2$ for all $k = j+1, \ldots, r$; thus it is a nonzero constant multiple of their product, thereby the claim (4.3). $\qquad\square$

Remark 4.5. The above proof has in effect solved a system of polynomial equations $\mathcal{L}_j(\underline{\lambda}) = 0$, $j_0 \le j \le r$.

5. Matrix form of the operator $R_{\frac{a}{2}}^{-1} \mathcal{L}_1 R_{\frac{a}{2}}$ and explicit formulas for the spherical functions

We consider now that $\nu = \frac{a}{2}$. As is proved in the previous section the invariant differential operators $\mathcal{L}_j$ for $j \ge 2$ act as zero operators on the image of tensor product $H^\nu \otimes \overline{H^\nu}$ under R. We thus consider now the spectral decomposition of the operator $\mathcal{L}_{1,\frac{a}{2}} = R^{-1} \mathcal{L}_1 R$. Moreover, the representations appearing in the decomposition are spherical; any spherical representation is uniquely determined by the spherical function, which in turn is determined by the corresponding eigenvalues of the Shimura operators $\mathcal{L}_1, \ldots, \mathcal{L}_r$. The operators $\mathcal{L}_2, \ldots, \mathcal{L}_r$ annihilate the spherical functions appearing in the representation. We need only determine the eigenvalues of $\mathcal{L}_1$ on the spherical functions that appear in the decomposition, which in turn is given by the spectrum of $\mathcal{L}_{1,\frac{a}{2}}$.

We will find the matrix form of the $\mathcal{L}_{j,\frac{a}{2}}$ acting on the basis $K_{\mathbf{m}}$ of $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$. As remarked earlier, the functions $K_{\mathbf{m}}$ are polynomials in (z, w), holomorphic in z and anti-holomorphic in w, so are also their actions under $\mathcal{L}_{j,\frac{a}{2}}$, thus they are uniquely determined by their restriction on the diagonal; the restrictions are then K-invariant, and are determined by their restriction on $\exp(\mathfrak{a}^+) \cdot 0 = \{(s_1, \ldots, s_r); 0 \le s_1 \le \cdots \le s_r < 1\}$.

Write $s_j = \exp(t_j \xi_j) \cdot 0 = \tanh t_j$, $j = 1, \ldots, r$. In terms of the coordinates $(t_1, \ldots, t_r)$, the radial part of the Laplace Beltrami operator is

$$\mathcal{L}_1 = \frac{1}{4}\Big(\sum_{j=1}^r \partial_j^2 + a \sum_{r \ge i > j \ge 1} \coth(t_i \pm t_j)(\partial_i \pm \partial_j) + 2\sum_{j=1}^r \coth 2t_j \partial_j + 2b \sum_{j=1}^r \coth t_j \partial_j\Big);$$

see [7], Chapter II, Proposition 3.9. However it is more convenient to use the coordinates $x_j = s_j^2$, it is

(5.1)
$$\begin{aligned}
\mathcal{L}_1 &= \sum_{j=1}^r x_j(1-x_j)^2 \partial_j^2 + \sum_{j=1}^r (1-x_j)^2 \partial_j + b \sum_{j=1}^r (1-x_j)^2 \partial_j \\
&\quad + a \sum_{j \ne k} \frac{(1-x_j)(1-x_k)}{x_j - x_k} x_j \partial_j
\end{aligned}$$

We can then find a formula for the operator $\mathcal{L}_{1,\nu} = R_\nu^{-1} \mathcal{L}_1 R_\nu$. When acting on K-invariant functions in (z, w) and restricting to the diagonal (z, z), $z = (s_1, \ldots, s_r)$,

with coordinates $(x_1, \ldots, x_r) = (s_1^1, \ldots, s_r^2)$

$$\mathcal{L}_{1,\nu} = R_\nu^{-1} \mathcal{M}_1 R_\nu$$

$$= \sum_{j=1}^r x_j \left((1-x_j)^2 \partial_j^2 - 2\nu(1-x_j)\partial_j + \nu(\nu-1) \right)$$

(5.2)

$$+ \sum_{j=1}^r \left((1-x_j)^2 \partial_j - \nu(1-x_j) \right) + b \sum_{j=1}^r \left((1-x_j)\partial_j - \nu \right)$$

$$+ a \sum_{j \neq k} \frac{(1-x_k)}{x_j - x_k} x_j \left((1-x_j)\partial_j - \nu \right)$$

As is proved in the previous section the spherical functions that appear in the decomposition of the tensor product satisfy, up to the Weyl group action,

(5.3) $$\underline{\lambda} = (i\rho_1, \ldots, i\rho_{r-1}, \lambda),$$

for some (with some abuse of notation) $\lambda = \lambda_r \in \mathbb{C}$. We will find the recurrence relations (2.8) and (2.9) with $\underline{\lambda}$ as above.

Notice first that when $\nu = \frac{a}{2}$ the expansion (1.4) becomes

$$h^{-\frac{a}{2}}(z) = \sum_{m=0}^\infty (\frac{a}{2})_{\underline{m}} K_{\underline{m}}(z, z)$$

with $\underline{m} = (m, 0, \ldots, 0)$, and for $z = s_1 e_1 + \cdots + s_r e_r$

$$K_{\underline{m}}(z, z) = \frac{1}{(\frac{a}{2})_m} \sum_{k_1 + \cdots + k_r = m} \frac{(\frac{a}{2})_{k_1} \cdots (\frac{a}{2})_{k_r}}{k_1! \cdots k_r!} s_1^{2k_1} \cdots s_r^{2k_r}$$

(5.4) $$E_{\underline{m}}(z, z) = \sum_{k_1 + \cdots + k_r = m} \frac{(\frac{a}{2})_{k_1} \cdots (\frac{a}{2})_{k_r}}{k_1! \cdots k_r!} s_1^{2k_1} \cdots s_r^{2k_r}.$$

For simplicity we write hereafter $\underline{m} = m$.

Lemma 5.1. *With the above notation,*

(5.5) $$\mathcal{L}_{1, \frac{a}{2}} E_m = A_m E_{m-1} + B_m E_m + C_m E_{m+1}$$

where

$$A_m = (m + \frac{a}{2} r)(m + 1 + b + \frac{a}{2}(r-1)),$$

$$B_m = - \left(m(2m + ar + b) + \frac{a}{2} r(1 + b + \frac{a}{2}(r-1)) \right)$$

and

$$C_m = (m+1)(m + \frac{a}{2})$$

Before performing the calculation we note that by invariance the operator $\mathcal{L}_{1, \frac{a}{2}}$ maps each vectors E_m into a unique linear combination of themself. By the formula (5.2) we see easily that $\mathcal{L}_{1, \frac{a}{2}} E_m$ is actually a linear combination of E_{m-1}, E_m and E_{m+1}. To find the coefficients we rewrite $\mathcal{L}_{1,\nu}$ (for general ν) as a sum of three operators that

are rising, lowering and respectively keeping the degrees of a homogeneous symmetric polynomials. First consider the last term there,

$$\sum_{j \neq k} \frac{(1-x_k)}{x_j - x_k} x_j = \sum_{j<k} \left(\frac{(1-x_k)}{x_j - x_k} x_j + \frac{(1-x_j)}{x_k - x_j} x_k \right) = \sum_{j<k} \frac{(x_j - x_k)}{x_j - x_k} = \binom{r}{2},$$

and

$$\sum_{j \neq k} \frac{(1-x_k)}{x_j - x_k}(1-x_j)x_j \partial_j = \sum_{j<k} \frac{(1-x_k)(1-x_j)}{x_j - x_k}(x_j \partial_j - x_k \partial_k).$$

Thus

$$\mathcal{L}_{1,\nu} = \mathcal{L}^+ + \mathcal{L}^0 + \mathcal{L}^-,$$

where

(5.6)
$$\mathcal{L}^+ = \sum_{j=1}^r x_j \left(\nu(\nu-1) + 2\nu x_j \partial_j + x_j^2 \partial_j^2 \right) + \sum_{j=1}^r (\nu x_j + x_j^2 \partial_j)$$
$$+ a \sum_{j<k} x_j x_k \frac{x_j \partial_j - x_k \partial_k}{x_j - x_k},$$

(5.7)
$$\mathcal{L}^0 = -2(\nu + 1 + b) \sum_{j=1}^r x_j \partial_j - 2 \sum_{j=1}^r x_j^2 \partial_j^2 - a \sum_{j<k} (x_j + x_k) \frac{x_j \partial_j - x_k \partial_k}{x_j - x_k}$$
$$- \nu r(1+b) - \nu a \binom{r}{2}$$

and

(5.8)
$$\mathcal{L}^- = \sum_{j=1}^r x_j \partial_j^2 + (1+b) \sum_{j=1}^r \partial_j + a \sum_{j<k} \frac{x_j \partial_j - x_k \partial_k}{x_j - x_k}.$$

To treat the last term in $\mathcal{L}^-$ we observe further that the operator $\frac{x \partial_x - y \partial_y}{x-y}$ acting on a symmetric polynomial $f(x,y) = x^c y^d + y^c x^d$ $(c \geq d \geq 0)$ is

(5.9)
$$\frac{x \partial_x - y \partial_y}{x-y} f(x,y) = (c-d)(xy)^d \frac{x^{c-d} - y^{c-d}}{x-y}$$
$$= (c-d)(xy)^d (x^{c-d-1} + \cdots + y^{c-d-1});$$

if we evaluate the result at $y = 0$, it is possibly nonzero only if $c > d = 0$, in which case it is $c x^{c-1}$; if it is further evaluated at $x = y = 0$, it is zero unless $c = 1$.

Proof. Recall that $\nu = \frac{a}{2}$. We calculate $\mathcal{L}^+ E_m$ and evaluate it at $(x_1, x_2, \dots) = (x_1, 0, \dots, 0)$, which we write as $(x_1, 0)$. We find

$$\mathcal{L}^+ E_m(x_1, 0) = (\frac{a}{2})_m \frac{1}{m!} \left(\frac{a}{2}(\frac{a}{2} - 1) + am + m(m-1) + \frac{a}{2} + m \right) x_1^{m+1}$$

$$= (\frac{a}{2})_m \frac{1}{(m)!}(\frac{a}{2} + m)^2 x_1^{m+1}$$

$$= (m+1)(\frac{a}{2} + m)(\frac{a}{2})_{m+1} \frac{1}{(m+1)!} x_1^{m+1}$$

$$= (m+1)(\frac{a}{2} + m) E_{m+1}(x_1, 0).$$

So that

$$\mathcal{L}^+ E_m(x_1, \dots, x_r) = (m+1)(\frac{a}{2} + m) E_{m+1}(x_1, \dots, x_r)$$

by the uniqueness of of the recurrence (5.5).

Next we calculate $\mathcal{L}^- E_m(x_1, 0)$. Clearly

$$(\sum_{j=1}^r x_j \partial_j^2) E_m(x_1, 0) = x_1^{m-1} m(m-1) \frac{(\frac{a}{2})_m}{m!} = (m-1)(\frac{a}{2} + m - 1) E_{m-1}(x_1, 0).$$

To deal with the differentiation $\sum_{j=1}^r \partial_j = \partial_1 + \sum_{j=2}^r \partial_j$ we write

$$E_{\underline{m}}(x) = \frac{(\frac{a}{2})_m}{m!} x_1^m + \sum_{k_1 + \dots + k_r = m, k_1 < m} \frac{(\frac{a}{2})_{k_1} \dots (\frac{a}{2})_{k_r}}{k_1! \dots k_r!} x_1^{k_1} \dots x_r^{k_r},$$

the differentiation ∂_1 on the second term in E_m, when evaluated at $(x_1, 0)$, is clearly zero. Therefore

$$\partial_1 E_m(x_1, 0) = \partial_1 \frac{(\frac{a}{2})_m}{m!} x_1^m = \frac{(\frac{a}{2})_m}{(m-1)!} x_1^{m-1}.$$

Using the observation (5.9) we see that

$$(\sum_{j=2}^r \partial_j) E_m(x_1, 0) = (\sum_{j=2}^r \partial_j) \left(\sum_{j=2}^r \frac{(\frac{a}{2})_{m-1}(\frac{a}{2})_1}{(m-1)!} x_1^{m-1} x_j \right) = (r-1) \frac{\frac{a}{2}(\frac{a}{2})_{m-1}}{(m-1)!}.$$

So that

$$\mathcal{L}^- E_m(x_1, 0) = C \frac{(\frac{a}{2})_{m-1}}{(m-1)!} x_1^{m-1} = C E_{m-1}(x_1, 0)$$

with the constant C given by

$$C = (m-1)(\frac{a}{2} + m - 1)) + (1+b)(\frac{a}{2} + m - 1 + (r-1)\frac{a}{2}) + a(r-1)(\frac{a}{2} + m - 1)$$

$$+ a\frac{a}{2} \frac{(r-1)(r-2)}{2}$$

$$= (m - 1 + \frac{a}{2}r)(m + b + \frac{a}{2}(r-1)),$$

which is the A_m claimed in the Lemma. Finally $\mathcal{L}^0 E_m(x_1, 0)$ can be calculated by similar method. $\square$

We consider the dual relation of (5.7):

$$(5.10) \qquad -(\lambda^2 + \rho_r^2)\mathcal{E}_m(\underline{\lambda}) = A_{m+1}\mathcal{E}_{m+1}(\underline{\lambda}) + B_m\mathcal{E}_m(\underline{\lambda}) + C_{m-1}\mathcal{E}_{m-1}(\underline{\lambda})$$

with $\mathcal{E}_0(\underline{\lambda}) = 1$. This relation (5.10) is exactly (9.3) in the appendix, (with α there being ρ_r, $\beta = \frac{1+b}{2}$ and $\gamma = -\frac{1+b}{2} + \frac{a}{2}$) which along with the given $\mathcal{E}_0(\underline{\lambda})$ uniquely determines the polynomials; therefore

$$(5.11) \qquad \mathcal{E}_m(\underline{\lambda}) = {}_3F_2(-m, \rho_r + i\lambda_r, \rho_r - i\lambda_r; 1 + b + \frac{a}{2}(r-1), \frac{a}{2}r; 1).$$

Consequently, by the argument in Remark 2.5, the series

$$(5.12) \qquad \psi(z) = \sum_m \mathcal{E}_m(\underline{\lambda})E_m(z)$$

is an eigenfunction of $\mathcal{L}_{1,\frac{a}{2}}$ with eigenvalue $-(\lambda^2 + \rho_r)^2$, whenever the power series is uniformly convergent.

Theorem 5.2. *The function $h^{-\frac{a}{2}}(z)\phi_{\underline{\lambda}}(z)$ for $\underline{\lambda}$ as in (5.3) has the following expansion*

$$(5.13) \qquad h(z)^{-\frac{a}{2}}\phi_{\underline{\lambda}}(z) = \sum_m \mathcal{E}_m(\underline{\lambda})E_m(z),$$

and the series converges uniformly on compact subset of D.

To prove the theorem we need a technical estimate.

Lemma 5.3. *Suppose $\alpha, \beta \in \mathbb{C}$ and $\gamma > 0, \delta > 0$. Let $M \geq 0$ be such that $|\alpha| + M \geq \gamma$, $|\beta| + M \geq \delta$. Then*

$$|{}_3F_2(-m, \alpha, \beta; \delta, \gamma; 1)| \leq (m+1)!\frac{(|\alpha| + M)_m(|\beta| + M)_m}{(\delta)_m(\gamma)_m}$$

Proof. By its definition

$$_3F_2\left(-m, \alpha \, \beta; \gamma, \delta; 1\right) = \sum_{j=0}^m (-1)^j \binom{m}{j}\frac{(\alpha)_j(\beta)_j}{(\gamma)_j(\delta)_j}.$$

We observe that

$$\binom{m}{j} \leq m!,$$

and for any two positive numbers $a \geq b > 0$

$$1 \leq \frac{a}{b} \leq \frac{a(a+1)}{b(b+1)} \leq \cdots \leq \frac{(a)_j}{(b)_j}$$

for any $j \geq 0$. Using these two inequalities we have

$$|{}_3F_2(-m, \alpha, \beta; \gamma, \delta; 1)|$$

$$\leq m! \sum_{j=0}^{m} \frac{(|\alpha|)_j (|\beta|)_j}{(\gamma)_j (\delta)_j}$$

(5.14)

$$\leq m! \sum_{j=0}^{m} \frac{(|\alpha| + M)_j (|\beta| + M)_j}{(\gamma)_j (\delta)_j}$$

$$\leq m! \sum_{j=0}^{m} \frac{(|\alpha| + M)_m (|\beta| + M)_m}{(\gamma)_m (\delta)_m}$$

$$= (m + 1)! \frac{(|\alpha| + M)_m (|\beta| + M)_m}{(\gamma)_m (\delta)_m}.$$

$\square$

Proof of Theorem 5.2. We prove that the function $\psi(z)$ defined in (5.12) is convergent and that $\phi_{\underline{\lambda}}(z) = h^{\frac{a}{2}}(z)\psi(z)$. Notice that $E_m(s_1 e_1 + \cdots + s_r e_r) \leq \frac{(\frac{a}{2}r)_m}{m!} s^{2m}$, where $s = \max\{s_1, \ldots, s_r\}$. Using the previous lemma, with α, β, γ and δ replaced by the corresponding numbers in (5.11), and a fixed large M, we get

$$|E_m(\underline{\lambda})| E_m(s_1 e_1 + \cdots + s_r e_r) \leq (m + 1)! \frac{(|\alpha| + M)_m (|\beta| + M)_m}{(\gamma)_m (\delta)_m} \frac{(\frac{a}{2}r)_m}{m!} s^{2m}$$

By the ratio test the series $\psi(z)$ is convergent if $s < 1$, we thus get the uniformly convergent on compact sets of D. The function $\psi(z)$ is an eigenfunction of $\mathcal{L}_{1, \frac{a}{2}}$ with eigenvalue $-(\lambda^2 + \rho_r^2)$, that is, the function $h^{\frac{a}{2}}(z)\psi(z)$ is an eigenfunction of the Laplace-Beltrami operator $\mathcal{L}_1$. We prove that $h^{\frac{a}{2}}(z)\psi(z)$ is also eigenfunction of the higher order Shimura operators $\mathcal{L}_j$ for $j = 2, \ldots, r$ with eigenvalue 0, namely they are K-invariant eigenfunctions of a system of generators. The result follows then by the uniqueness of the spherical function. By its definition and Theorem 3.1, $\mathcal{L}_{\underline{1}^j} = (\mathcal{D}^j)^* P_{\underline{1}^j} \mathcal{D}^j$ and $P_{\underline{1}^j} \mathcal{D}^j = h^{\frac{a}{2}(j-1)} P_{\underline{1}^j} \partial^j (h^{-\frac{a}{2}(j-1)})$, so that $P_{\underline{1}^j} \mathcal{D}^j$ acting on the function $h^{\frac{a}{2}}(z)\psi(z)$ is

$$P_{\underline{1}^j} \mathcal{D}^j (h^{\frac{a}{2}}(z)\psi(z)) = h^{\frac{a}{2}(j-1)}(z) P_{\underline{1}^j} \partial^j (h^{-\frac{a}{2}(j-1)} h^{\frac{a}{2}}(z)\psi(z)) = P_{\underline{1}^j} \partial^j (h^{-\frac{a}{2}(j-2)}(z)\psi(z)).$$

The function $\psi(z)$ is a sum of $K_{\mathbf{m}}$ with $m_2 = 0$, namely the expansion (5.11), and the convergence is uniform on compact subset of D, as just proved; the function $h^{-\frac{a}{2}(j-2)}(z)$ a sum of $K_{\mathbf{m}'}$ with $n_{j-1} = 0$, also with the same uniform convergence (see Theorem 1.1). The product of two such $K_{\mathbf{m}}$ and $K_{\mathbf{m}'}$ is a finite sum of $K_{\mathbf{n}}$ with $n_j = 0$, by Lemma 1.2. That is, $h^{-\frac{a}{2}(j-2)}(z)\psi(z)$ is a sum of $K_{\mathbf{m}}$ with $m_j = 0$, all of which are annihilated by $P_{\underline{1}^j} \partial^j$, by Lemma 4.2. Thus $P_{\underline{1}^j} \mathcal{D}^j (h^{\frac{a}{2}}\psi) = 0$, consequently $\mathcal{L}_{\underline{1}^j}(h^{\frac{a}{2}}\psi) = 0$.

$\square$

Proposition 4.3 and Theorem 5.2 can be then summarized in the following corollary; that (a) implies (c) is in Theorem 5.2, and that (c) implies (a) can be proved using the same method as in the above proof (which will not be used in the present paper).

Corollary 5.4. *The following assertions are equivalent for $\underline{\lambda} \in (\mathfrak{a}^*)^{\mathbb{C}}$*

(a) *$\phi_{\underline{\lambda}}$ is annihilated by all $\mathcal{L}_j$, $2 \leq j \leq r$.*

(b) *$\underline{\lambda}$ is in the Weyl group orbit of $(i\rho_1, i\rho_2, \ldots, i\rho_{r-1}, \lambda)$ for some $\lambda \in \mathbb{C}$.*

(c) *In the expansion of $h^{-\frac{a}{2}}(z)\phi_{\underline{\lambda}}(z)$ in terms of $K_{\underline{\mathbf{m}}}(z, z)$ all the coefficients of $K_{\underline{\mathbf{m}}}$ are 0 if $m_2 > 0$.*

Remark 5.5. By the formula (1.3.16) in [14] we see that for $z = s_1 e_1$, the function $\psi(z)$ is

$$\psi(z) = (1 - s_1^2)^{-\frac{a}{2}} {}_3F_2\left(\frac{a}{2}, \rho_r + i\lambda, \rho_r - i\lambda; \frac{a}{2}r, 1 + b + \frac{a}{2}(r-1); \frac{s_1^2}{s_1^2 - 1}\right),$$

and consequently

$$\phi_{\underline{\lambda}}(z) = {}_3F_2\left(\frac{a}{2}, \rho_r + i\lambda, \rho_r - i\lambda; \frac{a}{2}r, 1 + b + \frac{a}{2}(r-1); \frac{s_1^2}{s_1^2 - 1}\right).$$

If we formally put $r = 1$ and $a = 0$, namely when D is the unit ball in $\mathbb{C}^{1+b}$, then above formula reduces to

$$\phi_{\underline{\lambda}}(z) = {}_2F_1\left(\rho + i\lambda, \rho - i\lambda; 1 + b; \frac{|z|^2}{|z|^2 - 1}\right);$$

this is the known formula for the spherical function on rank one domains (see e.g. [7], p.484 or more explicitly [33]).

6. IRREDUCIBLE DECOMPOSITION OF THE TENSOR PRODUCT $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$

In terms of the orthonormal basis e_m of $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$, the matrix form of $\mathcal{L}_{1,\frac{a}{2}}$ is

$$(6.1) \qquad \mathcal{L}_{1,\frac{a}{2}} e_{m,\frac{a}{2}} = A_m \left(\frac{d_{m-1}}{d_m}\right)^{\frac{1}{2}} e_{m-1} + B_m e_m + C_m \left(\frac{d_{m+1}}{d_m}\right)^{\frac{1}{2}} e_{m+1}$$

On the other hand, recalling Proposition 2.3 with $\varepsilon_m(\underline{\lambda})$ defined in (2.5), we have

(6.2)

$$-(\lambda^2 + \rho_r^2)\varepsilon_m(\underline{\lambda}) = A_m \left(\frac{d_{m-1}}{d_m}\right)^{\frac{1}{2}} \varepsilon_{m-1}(\underline{\lambda}) + B_m \varepsilon_m(\underline{\lambda}) + C_m \left(\frac{d_{m+1}}{d_m}\right)^{\frac{1}{2}} \varepsilon_{m+1}(\underline{\lambda}).$$

(This can also be proved by direct calculation by using (5.10), noticing that $\frac{A_{m+1}}{C_m} = \frac{d_{m+1}}{d_m}$.) Namely the operator $\mathcal{L}_{1,\frac{a}{2}}$ on e_m and the multiplication operator $-(\lambda^2 + \rho_r^2)$ on ε_m has the same matrix form. However, in view of the formula (5.11) and Theorem 9.1 in Section 9, ε_m are orthonormal basis for a L^2-space with respect to certain measure, and consequently it gives the spectrum of the multiplication operator by $-(\lambda^2 + \rho_r^2)$ and further the spectrum of $\mathcal{L}_{1,\frac{a}{2}}$.

We summarize our results in the following theorem. Observe that discrete parts appear if $a < 1 + b$; this happens precisely when D is the Type I non-tube domain $SU(r,s)$ $(r \leq s)$ with $b = s - r \geq 2$. Thus for other irreducible domains there are only continuous spectra. We let $\Sigma = \mathbb{R}^+$ for other domains and

$$\Sigma = \mathbb{R}^+ \cup \{i(\frac{a}{2} - \frac{1+b}{2} + k); \ k = 0, \ldots, k_0\}$$

if D is the type I domain $SU(r, r+b)/S(U(r) \times U(r+b))$, and where k_0 is the largest nonnegative integer such that $\frac{a}{2} - \frac{1+b}{2} + k < 0$. Let μ be the measure given in Section 9 with

$$(6.3) \qquad \alpha = \rho_r = \frac{1}{2}(1 + b + a(r-1)), \ \beta = \frac{1}{2}(1 + b), \ \gamma = \frac{a}{2} - \frac{1}{2}(1 + b).$$

Theorem 6.1. *With the above notation the map $e_{\mathbf{m}}$ to ε_m extends to a unitary operator from $(H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}})_0$ onto $L^2(\Sigma, d\mu)$, and the r-tuple of operators $(\mathcal{L}_{1,\frac{a}{2}}, \mathcal{L}_{2,\frac{a}{2}}, \ldots, \mathcal{L}_{r,\frac{a}{2}})$ is unitarily equivalent to the the r-tuple of multiplication operators $(\mathcal{L}_1(\underline{\lambda}), 0, \ldots, 0)$; the vector $e_0 = 1 \otimes 1$ and the function $\varepsilon_0(\underline{\lambda}) = 1$ are cyclic vectors in the corresponding spaces. The spectrum of the r-tuple $(\mathcal{L}_{1,\frac{a}{2}}, \mathcal{L}_{2,\frac{a}{2}}, \ldots, \mathcal{L}_{r,\frac{a}{2}})$ of generators of invariant differential operators on $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$ is*

$$\{(-(\lambda^2 + \rho_r^2), 0, \ldots, 0); \lambda \in \Sigma\}.$$

7. Quantization of the minimal complementary series representations

Let $\mathcal{C}_{\frac{a}{2} - \frac{1+b}{2} + k}$ be the discrete part appearing in the decomposition in Theorem 6.1 and we refer it as a complementary series representation. In this section we construct directly an intertwining operator from the complementary series $\mathcal{C}_{\frac{a}{2} - \frac{1+b}{2} + k}$ into the space $S_2(H^{\frac{a}{2}})$ of Hilbert-Schmidt operators on $H^{\frac{a}{2}}$ using the expansion (5.13). Here k are nonnegative integers and $\frac{a}{2} - \frac{1+b}{2} + k < 0$. The intertwining operator can formally be defined on all spherical functions $\phi_{\underline{\lambda}}$ for all k. However we prove that it maps the $\phi_{\underline{\lambda}}$ into a Hilbert-Schmidt operator if and only if k satisfies the above condition. This gives an alternative proof that those spherical representations appear discretely in the decomposition of the tensor product $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$, and a quantization of the spherical representations as operators on the minimal representation $H^{\frac{a}{2}}$.

Theorem 7.1. *Let $\underline{\lambda}$ be as (5.3) with $\lambda = -i(\frac{a-1-b}{2} + k)$, and $0 \leq k < \frac{a}{2}(r-2) + 1 + b$ being integers. If*

$$q > \frac{1 + b + a(r-1)}{(1 + b + \frac{a}{2}(r-2) - k)}$$

then the map

$$\phi_{\underline{\lambda}}(z) \mapsto h^{-\frac{a}{2}}(z,w)\phi_{\underline{\lambda}}(z,w) = \sum_{\mathbf{m}} d_{\mathbf{m}}^{-\frac{1}{2}} \frac{1}{(1)_m(\frac{a}{2})_m} S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a - (1+b)}{2}) e_{\mathbf{m}}(z,w)$$

extends to a linear operator from the linear span of $\{\phi_\lambda(gz); g \in G\}$ (of all translations of ϕ_λ under G) into the Schatten von Neumann class $\mathfrak{S}_q$ of bounded operators on $H^{\frac{a}{2}}$. Define the norm of an element in the linear span to be the Schatten - von Neumann norm of the corresponding operator. The closure of the linear span of $\{\phi_\lambda(gz); g \in G\}$ is then an G-invariant Banach space.

The rest of this section is devoted to the proof.

Recall the expansion (5.13). From which we get immediately an expansion for its polarization,

$$h^{-\frac{a}{2}}(z,w)\phi_\lambda(z,w) = \sum_{\mathbf{m}} d_{\underline{\mathbf{m}}}^{-\frac{1}{2}} \frac{1}{(1)_m(\frac{a}{2})_m} S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2}) e_{\underline{\mathbf{m}}}(z,w).$$

Here we use the S_m to denote the continuous dual Hahn polynomials; see Section 9.

The operators $(\frac{a}{2})_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}$ are pairwise orthogonal projections, their $\mathfrak{S}_q$ norm satisfy

$$\|(\frac{a}{2})_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}\|_q^q = \dim(\mathcal{P}_{\underline{\mathbf{m}}}) = d_{\underline{\mathbf{m}}},$$

and

$$\|e_{\underline{\mathbf{m}}}\|_q^q = d_{\underline{\mathbf{m}}}^{1-\frac{q}{2}};$$

thus

$$(7.1) \quad \|h^{-\frac{a}{2}}\phi_\lambda\|_q^q = \sum_{\mathbf{m}} \left(d_{\underline{\mathbf{m}}}^{-\frac{1}{2}} \frac{1}{(1)_m(\frac{a}{2})_m} \right)^q \left(S_m(\lambda^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2}) \right)^q d_{\underline{\mathbf{m}}}^{1-\frac{q}{2}}$$

Let us take $\lambda = -i(\frac{a}{2} - \frac{1}{2}(1+b) + k)$. Then

$$
\begin{aligned}
& S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2}) \\
(7.2) \quad & = (\frac{a}{2}r)_m (1+b+\frac{a}{2}(r-1))_m \\
& \times {}_3F_2\left(-m, \frac{a}{2}r+k, 1+b+\frac{a}{2}(r-2)-k; \frac{a}{2}r, 1+b+\frac{a}{2}(r-1); 1\right).
\end{aligned}
$$

To estimate it for large m we use the Thomae's transformation formula (see [6], p. 59),

$$
\begin{aligned}
& {}_3F_2\left(-m, \frac{a}{2}r+k, 1+b+\frac{a}{2}(r-2)-k; \frac{a}{2}r, 1+b+\frac{a}{2}(r-1); 1\right) \\
& = \frac{(\frac{a}{2}+k)_m}{(1+b+\frac{a}{2}(r-1))_m} {}_3F_2\left(-m, -k, 1+b+\frac{a}{2}(r-2)-k; \frac{a}{2}r, 1-m-\frac{a}{2}-k; 1\right)
\end{aligned}
$$

The term ${}_3F_2$ in r.h.s. is now bounded for all m, due to the appearance of the parameter $-k$ so that it is a finite sum of k-terms, each of which is bounded. Thus

we find

$$|S_m(\lambda_1^2, \rho_r, \frac{1}{2}(1+b), \frac{a-(1+b)}{2})| \approx (\frac{a}{2}r)_m(1+b+\frac{a}{2}(r-1))_m\frac{(\frac{a}{2}+k)_m}{(1+b+\frac{a}{2}(r-1))_m}$$

$$= (\frac{a}{2}r)_m(\frac{a}{2}+k)_m$$

Also, the dimension $d_{\mathbf{m}}$ of the space $\mathcal{P}_{\mathbf{m}}$ has been calculate by Upmeier [28]:

$$d_{\mathbf{m}} = \frac{(\frac{a}{2}r)_m(1+b+\frac{a}{2}(r-1))_m}{(1)_m(\frac{a}{2})_m}.$$

We can find the estimate of each term in (7.1):

$$\left(d_{\mathbf{m}}^{-\frac{1}{2}}\frac{1}{(1)_m(\frac{a}{2})_m}\right)^q((\frac{a}{2}r)_m(\frac{a}{2}+k)_m)^q d_{\mathbf{m}}^{1-\frac{q}{2}}$$

(7.3)
$$= \left(\frac{(\frac{a}{2}r)_m(\frac{a}{2}+k)_m}{(1)_m(\frac{a}{2})_m}\right)^q d_{\mathbf{m}}^{1-q}$$

$$\approx m^{q(\frac{a}{2}r+\frac{a}{2}+k-1-\frac{a}{2})}m^{(1-q)(\frac{a}{2}r+1+b+\frac{a}{2}(r-1)-\frac{a}{2}-1)}$$

$$\approx m^{a(r-1)+b+q(k-1-b-\frac{a}{2}(r-2))}.$$

Thus (7.1) is convergent if and only if

$$a(r-1)+b+q(k-1-b-\frac{a}{2}(r-2)) < -1$$

or

$$q(1+b+\frac{a}{2}(r-2)-k) > 1+b+a(r-1)$$

Since $q > 0$, $k = 0, 1\ldots$, and $r \geq 2$, the above condition is equivalent to

$$0 \leq k < \frac{a}{2}(r-2)+1+b$$

and

$$q > \frac{1+b+a(r-1)}{(1+b+\frac{a}{2}(r-2)-k)}.$$

This proves the first part of the theorem, and the remaining part is then follows by abstract arguments.

Remark 7.2. Observe that since $k \geq 0$ the cut-off in the theorem is

$$\frac{1+b+a(r-1)}{(1+b+\frac{a}{2}(r-2)-k)} > 1.$$

Namely those operators are never trace class. When $q = 2$, the condition amounts to

$$0 \leq k < \frac{1+b}{2}-\frac{a}{2}$$

which is our previous condition in Section 6. Notice further that when

$$\frac{1+b}{2}-\frac{a}{2} \leq k < \frac{a}{2}(r-2)+1+b$$

the operator $h^{-\frac{a}{2}}(z,w)\phi_{\underline{\lambda}}(z,w)$ is in $\mathfrak{S}_q$ with $q > 2$, since $q-2 > 0$ in this case.

Remark 7.3. The unitary complementary series $\mathcal{C}_{\frac{a}{2}-\frac{1+b}{2}+k}$ for the group $SU(N,2)$ have also been discovered previously; see [13] Theorem 2.1 (a)(iii). (In that paper Knapp and Speh classified the unitary principal series representations $Ind_{MAN}(\tau \times \underline{\lambda})$ induced from a minimal parabolic subgroup MAN with τ being an one-dimensional representation of M. In our case τ is the trivial representation; our series $\mathcal{C}_{\frac{a}{2}-\frac{1+b}{2}+k}$ are exactly the all those classified in Theorem 2.1(a)(iii) there. However our result also gives the K-type structures of the representations.)

8. POSITIVE DEFINITENESS OF SPHERICAL FUNCTIONS

Proposition 8.1. *The spherical function $\phi_{\underline{\lambda}}(z)$ is positive definite for all $\underline{\lambda}$ in (5.3) and $\lambda \in \Sigma$.*

Proof. Fix $\lambda_0 \in \mathbb{R}^+$. For any $0 < \delta < \lambda_0$ take nonnegative C^∞-function f on $(0, \infty)$ with compact support $(\lambda_0 - \delta, \lambda_0 - \delta)$ so that

$$\int_{\lambda_0 - \delta}^{\lambda_0 - \delta} |f(\lambda)|^2 d\mu(\lambda) = 1.$$

Consider the element

$$F_\delta = \int_{\lambda_0 - \delta}^{\lambda_0 - \delta} f(\lambda) h^{-\frac{a}{2}}(z, w) \phi_{\underline{\lambda}}(z, w) d\mu(\lambda)$$

By Theorem 6.1 F_δ is a unit vector in $H^{\frac{a}{2}} \otimes \overline{H^{\frac{a}{2}}}$. But

$$(\pi(\frac{a}{2}) \otimes \overline{\pi(\frac{a}{2})}(g) F_\delta, F_\delta) = \int_{\lambda_0 - \delta}^{\lambda_0 - \delta} f(\lambda) \phi_{\underline{\lambda}}(g \cdot 0) d\mu(\lambda)$$

with is positive definite function of $g \in G$. Let $\delta \to 0$. We claim that

$$(\pi(\frac{a}{2}) \otimes \overline{\pi(\frac{a}{2})}(g) F_\delta, F_\delta) \to \phi_{\underline{\lambda}}(g \cdot 0),$$

therefore $\phi_{\underline{\lambda}}(g \cdot 0)$ is a point-wise limit of positive definite functions, and consequently is positive definite. The proof of the above claim is a routine method. Indeed for any $\epsilon > 0$ let δ be chosen so that $|\phi_\lambda(g \cdot 0) - \phi_{\lambda_0}(g \cdot 0)| < \epsilon$ when $|\lambda - \lambda_0| \leq \delta$. Then

$$(\pi(\frac{a}{2}) \otimes \overline{\pi(\frac{a}{2})}(g) F_\delta, F_\delta) - \phi_{\underline{\lambda}}(g \cdot 0) = (\int_{\lambda_0 - \delta}^{\lambda_0 - \delta} f(\lambda)(\phi_{\underline{\lambda}}(g \cdot 0) - \phi_{\lambda_0}(g \cdot 0)) d\mu(\lambda)$$

and its absolute value is dominated, using Cauchy-Schwarz inequality, by

$$\epsilon \int_{\lambda_0 - \delta}^{\lambda_0 - \delta} |f(\lambda)| d\mu(\lambda) \leq \epsilon \left(\int_{\lambda_0 - \delta}^{\lambda_0 - \delta} |f(\lambda)|^2 d\mu(\lambda) \right)^{\frac{1}{2}} \mu(\lambda_0 - \delta, \lambda_0 - \delta)^{\frac{1}{2}} \leq \epsilon,$$

since μ is a probability measure, $\mu(\lambda_0 - \delta, \lambda_0 - \delta) \leq 1$. $\square$

9. Appendix: Orthogonality relation of Continuous Dual Hahn polynomials

We summarize here some formulas that we used in this paper; see [30] and [14]. The continuous dual Hahn polynomials are defined by

$$(9.1) \qquad S_m(x^2, \alpha, \beta, \gamma) = (\alpha + \beta)_m (\alpha + \gamma)_m \widetilde{S}_m(x^2, \alpha, \beta, \gamma)$$

where

$$(9.2) \qquad \widetilde{S}_m(x^2) = \widetilde{S}_m(x^2, \alpha, \beta, \gamma) = {}_3F_2\left(-m, \alpha + ix, \alpha - ix; \alpha + \beta, \alpha + \gamma; 1\right).$$

Then the functions $\widetilde{S}_m(x^2, \alpha, \beta, \gamma)$ satisfy the recurrence relation (see [14])

$$(9.3)$$
$$-(\alpha^2 + x^2)\widetilde{S}_m(x^2) = A_{m+1}\widetilde{S}_{m+1}(x^2) - (A_{m+1} + C_{m-1})\widetilde{S}_m(x^2) + C_{m-1}\widetilde{S}_{m-1}(x^2),$$

where

$$A_m = (m + \alpha + \beta)(m + \alpha + \gamma), \quad C_m = (m+1)(m + \beta + \gamma).$$

Their orthogonality relation is given in the following

Theorem 9.1. *Let*

$$(9.4) \qquad s_m(x^2, \alpha, \beta, \gamma) = \left(\frac{(\alpha + \beta)_m (\alpha + \gamma)_m}{(1)_m (\beta + \gamma)_m}\right)^{\frac{1}{2}} \widetilde{S}_m(x^2, \alpha, \beta, \gamma)$$

If α, β, γ are positive then

$$\int_0^\infty s_m(x^2, \alpha, \beta, \gamma) s_l(x^2, \alpha, \beta, \gamma) d\mu(x) = \delta_{ml}.$$

where $d\mu(x)$ on $(0, \infty)$ is the measure

$$d\mu(x) = d\mu_{\alpha,\beta,\gamma}(x) = \frac{1}{2\pi} \frac{1}{\Gamma(\alpha + \beta)\Gamma(\alpha + \gamma)\Gamma(\beta + \gamma)} \left|\frac{\Gamma(\alpha + ix)\Gamma(\beta - ix)\Gamma(\gamma + ix)}{\Gamma(2ix)}\right|^2.$$

If $\gamma < 0$, α and β are positive then

$$(9.5) \qquad \int_{(0,\infty)\cup\{i(\gamma+k); k=0,1,\ldots,\gamma+k<0\}} s_m(x^2, \alpha, \beta, \gamma) s_l(x^2, \alpha, \beta, \gamma) d\mu(x) = \delta_{ml},$$

where $d\mu(x)$ is the sum of the above measure on $(0, \infty)$ and the discrete measure on $\{i(\gamma + k); 0 \le k < -\gamma\}$:

$$(9.6)$$
$$\sum_{0 \le k < \gamma} \frac{\Gamma(\gamma + \beta)\Gamma(\gamma + \alpha)\Gamma(\beta - \gamma)\Gamma(\alpha - \gamma)}{\Gamma(-2\gamma)}$$
$$\times \frac{(2\gamma)_k (\gamma + 1)_k (\gamma + \beta)_k (\gamma + \alpha)_k}{(\gamma)_k (\gamma - \beta + 1)_k (\gamma - \alpha + 1)_k k!} (-1)^k \delta_{i(\gamma+k)}.$$

Here $\delta_{i(\gamma+k)}$ stands for the Dirac measure at the given point.

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