General n-dimensional Tauberian problems with application to the Laplace- and Stieltjes transforms

Abstract: A general theorem on the closure of translates in certain weighted spaces in R^n is proved and as a consequence a general n-dimensional Tauberian theorem. This is applied to the n-dimensional Laplace transform and to the one-dimensional Stieltjes- and Weierstrass transforms. AMS subject classification is primary 40E05 and secondary 44A30.

0 Introduction

Let K, ϕ and ψ be functions from R^n to R which belong to some specific classes of functions which will be defined later in the text.

Suppose that

(0.1) $$K * \phi(x) \sim K * \psi(x), \quad x \to +\infty.$$

Under certain restrictions on ψ and with a Tauberian condition on ϕ we will show that

(0.2) $$\phi(x) \sim \psi(x), \quad x \to +\infty.$$

Here

$$K * \phi(x) = \int K(x-u)\phi(u)du,$$

where we let an unspecified region of integration be R^n throughout the text.

By $x = (x_1, x_2, \ldots, x_n) \to +\infty$ we mean that $x_k \to +\infty, k = 1, 2, \ldots, n$, and by (0.2) we mean that

$$\phi(x) = \psi(x) + o(\psi(x)) \quad \text{when} \quad x \to +\infty,$$

with a corresponding interpretation for (0.1).

The class of kernels considered here will be chosen so that it includes a wide variety of well-known transformation kernels. As specific examples we apply the general results to the n-dimensional Laplace transform and the one-dimensional Stieltjes- and Weierstrass transforms.

Problems of this kind for the one dimensional Laplace transform has been treated earlier by the author in [7].

The method used depends on an n-dimensional analogue of a theorem on the closure of the span of translates of the transformation kernels in a certain weighted space. One-dimensional closure theorems of this kind was first proved by Nyman [14] and Korenblum [11]. The methods used in this paper are developments of ideas used by the author in ([5], [6], [7]).

1 Preliminaries

We will use the following notations beside the ones used in the introduction.

If

$$x = (x_1, x_2, \ldots, x_n) \in R^n, \quad y = (y_1, y_2, \ldots, y_n) \in R^n$$

then

$$x \leq y \quad \text{if} \quad x_k \leq y_k \quad \text{for all} \quad k = 1, 2, \ldots, n,$$

with a corresponding meaning for $x < y$.

We also let R^n_+ be all $x \in R^n$ such that $x \geq \mathbf{0} = (0, 0, \ldots, 0)$.

Furthermore, we let

$$x \cdot y = \sum_{k=1}^{n} x_k y_k \, , \; |x| = \sqrt{x \cdot x} \, , \; x \otimes y = (x_1 y_1, x_2 y_2, \ldots, x_n y_n),$$
$$\delta x = (\delta x_1, \delta x_2, \ldots, \delta x_n) \text{ if } \delta \text{ is a real number},$$
$$\|x\| = \sum_{k=1}^{n} \max(0, x_k), \text{ (which is a pseudonorm in } R^n \text{ but a norm in } R^n_+),$$
$$\exp x = (\exp x_1, \exp x_2, \ldots, \exp x_n), \quad \mathbf{1} = (1, 1, \ldots, 1).$$

If $x > \mathbf{0}$ then

$$\ln x = (\ln x_1, \ln x_2, \ldots, \ln x_n), \; x^y = \exp(y \cdot \ln x),$$
$$\frac{x}{y} = \Big(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n}\Big).$$

We use standard notations for the Fouriertransform, thus

$$\hat{K}(x) = \int \exp(-ix \cdot t) K(t) dt.$$

We also introduce weight-functions p defined in R^n such that

$$\begin{aligned} &p(x) \geq p(0) = 1 \\ &p(x+y) \leq p(x)p(y) \\ &p(rx) \geq p(x) \quad \text{if } r \text{ real and } r \geq 1 \end{aligned}$$

In this paper we use weight-functions p of the form

$$p(x) = (1+|x|)^{\mu} \exp(\|m \otimes x\|), \ \text{ where } \mu \in R_+ \text{ and } m \in R^n_+.$$

We call a function p from R^n to R non-decreasing if

$$p(x) \leq p(y) \quad \text{when } x \leq y.$$

DEFINITION. For any weight-function p we let $L^1(p)$ consist of all measurable functions H such that

$$\|H\|_p^1 = \int |H(-x)|p(x)dx < \infty$$

We also let $L^{\infty}(p)$ denote the space of all measurable functions ϕ such that

$$\|\phi\|_p^{\infty} = ess \sup_{-\infty<x<\infty} \frac{|\phi(x)|}{p(x)} < \infty.$$

We see that $L^1(p)$ is a Banachspace under this norm with $L^{\infty}(p)$ as its dual space, which means that any bounded linear functional on $L^1(p)$ is of the form

$$K \to \int K(-x)\phi(x)dx$$

for some function $\phi \in L^{\infty}(p)$. (Cf. e.g. [16] p. 136)

For convenience we let C stand for positive constants not necessarily the same each time.

2 Some general Tauberian Theorems

We first introduce the class of kernels considered.

DEFINITION. By $E(\alpha, \beta, M)$ we denote all integrable functions K defined in R^n such that:

1^0 $\hat{K}(t) \neq 0$ *for all* $t \in R^n$

2^0 *The function* g *defined by* $g(t) = \hat{K}(t)^{-1}$ *can be analytically continued in a region* $-\alpha < \text{Im } t < \beta$ *for some* $\alpha, \beta > 0$

3^0 *This function* g *satisfies an inequality*

$$|g(t)| \leq C \exp(M(x))$$

for all $t = x + iy$ *such that* $-\alpha < y < \beta$ *and for some function* M *from* R^n *to* R_+.

THEOREM 1. *Let* p *be a weight-function of the form*

$$p(x) = (1 + |x|)^\mu \exp(\|m \otimes x\|), x \in R^n, \text{ for some } \mu \text{ in } R_+ \text{ and } m \text{ in } R_+^n.$$

If $K \in L^1(p) \cap E(\alpha, \beta, M)$ *with* $\alpha > m$ *and if* $M(x) = o\Big(\sum_{k=1}^n \exp(\pi \frac{|x_k|}{\alpha_k})\Big)$ *when* $|x| \to \infty$, *then the span of translates of* K *is dense in* $L^1(p)$.

PROOF. We will prove that for any $\phi \in L^\infty(p)$ and any K which satisfies the condition above then

(2.1) $$K * \phi(x) = 0 \quad \text{for all} \quad x \in R^n$$

implies that

$$\phi(x) = 0 \quad \text{a.e. in } R^n.$$

For any real $\varepsilon > 0$ such that $\alpha > m + \frac{3\varepsilon}{\pi}\alpha$ and $\beta - \frac{2\varepsilon}{\pi}\alpha > 0$, and for any $\omega \in R^n$ consider the function

(2.2) $$Q(x) = \frac{1}{(2\pi)^n} \int \exp(ix \cdot u) h(u - \omega) g(u) du$$

where

(2.3) $$h(u) = \exp\Big[- A \cdot \exp\Big(\pi\frac{u}{\alpha} + i\Big(\frac{\pi}{2} - 2\varepsilon\Big)\mathbf{1}\Big) - A \cdot \exp\Big(- \pi\frac{u}{\alpha} - i\Big(\frac{\pi}{2} - 2\varepsilon\Big)\mathbf{1}\Big)\Big]$$

with an $A \in R_+^n$ such that $\cos\Big(\frac{\pi}{2} - \varepsilon\Big)A = \mathbf{1}$.

In (2.2) we now make the substitution

$$u = t + is \quad \text{where} \quad - \alpha + \frac{2\epsilon}{\pi}\alpha < s < \frac{2\varepsilon}{\pi}\alpha,$$

and after a translation of the region of integration we obtain that

(2.4) $$Q(-x) = \frac{1}{(2\pi)^n} \exp(s \cdot x) \int \exp(-ix \cdot t) h(t - \omega + is) g(t + is) dt$$

If now $s = (s_1, s_2, \ldots, s_n)$ is chosen so that

$$s_k = -\alpha_k + \frac{3\varepsilon}{\pi}\alpha_k \quad \text{when} \quad x_k > 0$$

and

$$s_k = +\varepsilon\frac{\alpha_k}{\pi} \quad \text{when } x_k \leq 0$$

then we can see that there exists a positive number δ such that

$$|Q(-x)p(x)| \leq C \exp(-\delta|x|) \quad \text{for all} \quad x \in R^n. \tag{2.5}$$

Hence $Q \in L^1(p)$.

If

$$K * \phi = \psi$$

then clearly $\psi \in L^\infty(p)$ and

$$Q * (K * \phi) = Q * \psi. \tag{2.6}$$

The conditions above are enough to prove that

$$Q * (K * \phi) = (Q * K) * \phi. \tag{2.7}$$

The method above to prove formulas (2.4) and (2.5) can also be used to prove that if h is defined as in (2.3) then $h = \hat{H}$ for some function $H \in L^1(p)$. Now

$$Q * K(u) = H(u)\exp(i\omega \cdot u), \tag{2.8}$$

which follows from the fact that

$$\hat{Q}(u) = h(u - \omega)g(u)$$

and hence

$$(Q * K)\hat{}(u) = \hat{Q}(u)\hat{K}(u) = h(u - \omega).$$

By use of (2.1), (2.7) and (2.8) we have that

$$\int \exp(-i\omega \cdot u)H(-u)\phi(u + x)du = 0 \quad \text{for all} \quad \omega, x \in R^n \tag{2.9}$$

The uniqueness of the Fouriertransform now implies that

$$H(-u)\phi(u + x) = 0 \text{ a.e. in } u \text{ for any } x \in R^n,$$

and since H is non-trivial, we have finally proved that

$$\phi(u) = 0 \text{ a.e. in } R^n.$$

It now follows from the Hahn-Banach theorem that the span of translates of K is dense in $L^1(p)$. (Cf. [16] p.114).

REMARK. In thesis 1950 Nyman [14] gave, in the one-dimensional case, the necessary and sufficient conditions for the span of translates of a kernel $K \in L^1(p)$ to be dense in $L^1(p)$, if $p(x) = \exp(\alpha x)$ for $x \geq 0$ and $p(x) = \exp(-\beta x)$ for $x < 0$. He proved that $\hat{K}(t) \neq 0$ in the closed strip $-\alpha \leq \operatorname{Im} t \leq \beta$ and $K \in E(\alpha, \beta, M)$ with $M(x) = o\Big(\exp \frac{\pi|x|}{\alpha+\beta}\Big)$ when $x \to \pm\infty$, are the appropriate conditions.

Essentially the same result was proved by Korenblum [11] in 1958.

In this connection it may also be noted that a well-known result by Levinson [12], also in the one-dimensional case, shows that if $m > \alpha$ then

$$H(x) = O\Big(\exp(-mx)\Big) \quad \text{and} \quad \hat{H}(x) = O\Big(\exp(-\exp(\frac{\pi}{\alpha}x))\Big), \quad x \to +\infty$$

implies that $H(x) = 0$ a.e.

THEOREM 2. *Let K, ϕ and ψ be functions from R^n to R such that K is non-negative and fulfills the conditions of Theorem 1 with $p(x) = (1+|x|)\exp(\|m \otimes x\|)$. Furthermore, let ψ be positive and non-decreasing and let $\phi(x)$ and $\psi(x)$ be bounded when $\|x\|$ is bounded.*

If now

(2.10) $$K * \phi(x) \sim K * \psi(x), \quad x \to +\infty$$

and also

(2.11) $$K * \phi(x) = O(K * \psi(x)) \quad \text{when} \quad \|x\| \to +\infty$$

and if for any real $\delta > 0$ there exist an X in R such that

(2.12) $$\psi(x+y) \leq (1+\delta)\exp(m \cdot y)\psi(x) \text{ when } \|x\| \geq X,\ y \geq 0.$$

and if

(2.13) $$\lim_{h \to 0+} \liminf_{\|x\| \to \infty} \inf_{x \leq y \leq x+h} \Big(\frac{\phi(y) - \phi(x)}{\psi(x)}\Big) = 0$$

then

(2.14) $$\phi(x) \sim \psi(x), \quad x \to +\infty.$$

PROOF. As a first step we will prove that

$$|\phi(x)| \leq C(1 + \psi(x)) \quad \text{for all} \quad x \in R^n.$$

In proving this we use that

(2.15) $$\psi(x+u) \leq C\exp(\|m \otimes u\|)(1 + \psi(x)) \quad \text{for any } u, x \in R^n.$$

This follows since ψ is non-decreasing and hence

$$\psi(x+u) \le \psi(x+y) \quad \text{if} \quad y = (\|u_1\|, \|u_2\|, \dots, \|u_n\|) \quad \text{for any } x, u \in R^n$$

and then by (2.12)

$$\psi(x+u) \le C \exp(\|m \otimes u\|)\psi(x) \text{ for any } u \in R^n \text{ if } \|x\| \text{ is large enough.}$$

Since $\psi(x)$ is bounded when $\|x\|$ is bounded, we easily get (2.15).

We now start using (2.13) and the fact that $\phi(x)$ is bounded when $\|x\|$ is bounded to see that

$$\phi(y) - \phi(x) \ge -C(1+\psi(x)), \quad x \le y \le x + 2\,\mathbf{1} \quad \text{for all } x \in R^n. \tag{2.16}$$

If $x \le y \le x + \mathbf{1}$ then

$$K * \phi(y+\mathbf{1}) - K * \phi(x) = \int K(-u)(\phi(u+y+\mathbf{1}) - \phi(u+x))du = I_1 + I_2$$

where I_1 is the integral taken over all $S = \{u \in R^n : -\mathbf{1} \le u \le \mathbf{0}\}$ and I_2 is the integral taken over the rest of R^n.

By (2.15) and (2.16) we see that

$$I_2 \ge -C(1+\psi(x))$$

and hence by (2.11) and (2.15) we have that

$$I_1 \le K * \phi(y+\mathbf{1}) - K * \phi(x) + C(1+\psi(x)) \le C(1+\psi(x)) \tag{2.17}$$

if $\|x\|$ is large enough.

When $u \in S$ we write

$$\begin{aligned} &\phi(u+y+\mathbf{1}) - \phi(u+x) = \phi(u+y+\mathbf{1}) - \phi(y) + \phi(x) - \phi(u+x) + \phi(y) - \phi(x) \ge \\ &\ge -C(1+\psi(x)) + \phi(y) - \phi(x) \end{aligned}$$

and see that

$$I_1 \ge (-C(1+\psi(x)) + \phi(y) - \phi(x)) \int_S K(-u)du.$$

Since $\phi(x)$ is bounded if $\|x\|$ is bounded we have from (2.17) that

$$\phi(y) - \phi(x) \le C(1+\psi(x)) \text{ for all } x, y \in R^n \text{ such that } x \le y \le x + \mathbf{1}.$$

We combine this inequality with (2.16) and see that

$$|\phi(u+x) - \phi(x)| \le C(1+\psi(x)), \quad \mathbf{0} \le u \le \mathbf{1} \text{ for any } x \in R^n. \tag{2.18}$$

Hence for any $u \geq 0$ and any $u \leq 0$ we have by (2.15) that

$$|\phi(u+x) - \phi(x)| \leq C(1+|u|)\exp(\|m \otimes u\|)(1+\psi(x)) \leq Cp(u)(1+\psi(x)).$$

For any other value of u we let

$$x_0 = x,\ x_1 = x + (u_1, 0, 0, \ldots),\ x_2 = x_1 + (0, u_2, 0, \ldots),\ x_3 = x_2 + (0, 0, u_3, 0, \ldots), \ldots$$

and in this case we see after some calculations using (2.18) and (2.15) that

$$|\phi(u+x) - \phi(x)| = |\sum_{q=1}^{n}(\phi(x_q) - \phi(x_{q-1}))| \leq$$
$$\leq C\sum_{q=1}^{n}(1+|u_q|)\exp(\|m_q u_q\|)(1+\psi(x_{q-1})) \leq Cp(u)(1+\psi(x))$$

Hence

(2.19) $\qquad |\phi(u+x) - \phi(x)| \leq Cp(u)(1+\psi(x))$ for all u and x in R^n.

Now since $K \in L^1(p)$

$$|K * \phi(x) - \phi(x)\int K(-u)du| = |\int K(-u)(\phi(u+x) - \phi(x))du| \leq$$
$$\leq C(1+\psi(x))\int K(-u)p(u)du \leq C(1+\psi(x)).$$

Finally we use (2.11) and the fact that $\phi(x)$ is bounded when $\|x\|$ is bounded to see that

(2.20) $\qquad |\phi(x)| \leq C(1+\psi(x)) \quad$ for all $x \in R^n$.

Hence the first step of the proof is completed.

For any function $H \in L^1(p)$ and for any $\varepsilon > 0$ we now can, using *Theorem 1*, find a finite linear combination K_ε of translates of K,

$$K_\varepsilon(x) = \sum_{k=1}^{m} a_k K(x - \lambda_k),$$

such that

$$\|K_\varepsilon - H\|_p^1 < \varepsilon.$$

We write

$$H * \phi = H * \psi + (H - K_\varepsilon) * (\phi - \psi) + K_\varepsilon * (\phi - \psi).$$

By (2.10) and (2.12) we see that

$$K_\varepsilon * (\phi - \psi) = o(1)K_\varepsilon * \psi(x) = o(\psi(x)), \quad x \to +\infty.$$

By (2.20) we have that

$$(H - K_\varepsilon) * (\phi - \psi)(x) = O(1)(|H - K_\varepsilon| * (1 + \psi(x)), \quad x \to +\infty,$$

and hence by (2.15) we have that

$$(H - K_\varepsilon) * (\phi - \psi)(x) = O(1) \int |H(-u) - K_\varepsilon(-u)|(1 + \psi(u + x))du =$$
$$= O(1)(1 + \psi(x)) \int |H(-u) - K_\varepsilon(-u)|p(u)du = O(1)(1 + \psi(x))\|H - K_\varepsilon\|_p^1, \quad x \to +\infty.$$

Since ε is arbitrary, it follows that

$$H * \phi(x) = H * \psi(x) + o(\psi(x)), \quad x \to +\infty. \tag{2.21}$$

For any positive real number h we now let H be the characteristic function on the set $S_h = \{u \in R^n : -h\ \mathbf{1} \le u \le \mathbf{0}\}$ multiplied by h^{-n}. Then by (2.21)

$$h^{-n} \int_{S_h} \phi(x - u)du = h^{-n} \int_{S_h} \psi(x - u)du + o(\psi(x)), \quad x \to +\infty.$$

We divide this expression with $\psi(x)$ and see that

$$\frac{\phi(x)}{\psi(x)} = h^{-n} \int_{S_h} \frac{\phi(x)}{\psi(x)} du = h^{-n} \int_{S_h} \frac{\phi(x) - \phi(x - u)}{\psi(x)} du + h^{-n} \int_{s_h} \frac{\phi(x - u)}{\psi(x)} du =$$
$$= h^{-n} \int_{S_h} \frac{\phi(x) - \phi(x - u)}{\psi(x)} du + h^{-n} \int_{S_h} \frac{\psi(x - u)}{\psi(x)} du + o(1), \quad x \to \infty.$$

By use of (2.12) and (2.13) we see that if h is small enough then to any real $\varepsilon > 0$ there exists an $x_1 \in R^n$ such that

$$\frac{\phi(x)}{\psi(x)} < 1 + \varepsilon \quad \text{if} \quad x \ge x_1.$$

If on the other hand, H is the characteristic function on $D_h = \{u \in R^n : 0 \le u \le h\ \mathbf{1}\}$ multiplied by h^{-n} we can in an analogous way prove that there exists an $x_2 \in R^n$ so that

$$\frac{\phi(x)}{\psi(x)} > 1 - \varepsilon \quad \text{if} \quad x \ge x_2.$$

Hence

$$\phi(x) \sim \psi(x), \quad x \to +\infty,$$

and we have proved *Theorem 2.*

3 Results for the n-dimensional Laplace transform

We suppose that α and β are real functions of bounded variation defined in R^n_+. We also suppose that

$$\alpha(t) = \beta(t) = 0 \quad \text{if any} \quad t_k = 0,\ k = 1, 2, \ldots, n.$$

This means that α and β belong to an n-dimensional analogue of class V_0 in [1].

We use the following notations for the corresponding Laplace transforms:

$$F(s) = \int_{R^n_+} \exp(-s \cdot t) d\alpha(t), \quad s > 0, \tag{3.1}$$

$$G(s) = \int_{R^n_+} \exp(-s \cdot t) d\beta(t), \quad s > 0, \tag{3.2}$$

where we suppose that the integrals are boundedly convergent for any $s > 0$ (cf. [1]).

THEOREM 3. *Let β be positive and non-decreasing and suppose that*

$$F(s) \sim G(s), \quad s \to \mathbf{0}+, \tag{3.3}$$

and that

$$F(s) = O(G(s)) \quad \textit{when} \quad \min_{k=1,2,\ldots,n} s_k \to 0+. \tag{3.4}$$

If for any real number $\delta > 0$ there exist an $m \in R^n_+$ and a positive real number T such that

$$\beta(r \otimes t) \leq (1+\delta) r^m \beta(t),\ r \geq \mathbf{1},\ \|t\| \geq T \tag{3.5}$$

and if

$$\lim_{\lambda \to 1+} \liminf_{\|x\| \to \infty} \inf_{x \leq t \leq \lambda x} \left(\frac{\alpha(t) - \alpha(x)}{\beta(x)} \right) = 0 \tag{3.6}$$

then

$$\alpha(t) \sim \beta(t), \quad t \to \infty. \tag{3.7}$$

PROOF. We make a partial integration in (3.1) and obtain that

$$F(s) = s^{\mathbf{1}} \int_{R^n_+} \exp(-s \cdot t) \alpha(t) dt, \quad s > 0,$$

where the integral is absolutely convergent (cf. [1]). We do the same in (3.2) and get a corresponding result for $G(s)$.

We now make the substitutions

$$s = \exp(-x) \quad \text{and} \quad t = \exp u$$

and let

$$\phi(x) = \alpha(\exp x) \quad \text{and} \quad \psi(x) = \beta(\exp x).$$

In this way (3.3) is transformed into (2.10), that is

$$K * \phi(x) \sim K * \psi(x), \quad x \to +\infty,$$

where

$$K(x) = \exp(-\mathbf{1} \cdot \exp(-x) - \mathbf{1} \cdot x)$$

and

$$\hat{K}(t) = \prod_{k=1}^{n} \Gamma(1 + it_k), \quad t = (t_1, t_2, \ldots, t_n).$$

Hence K fulfills the conditions required in *Theorem 2*. That $K \in E(\alpha, \beta, M)$ for any $\alpha > 0$ and properly chosen β and M follows from Stirlings formula (cf. e.g. [6] p. 231). It is also easy to see that the other conditions of *Theorem 2* are fulfilled since (2.11), (2.12) and (2.13) are consequences of (3.4), (3.5) and (3.6) respectively. Now the conclusion (3.7) follows from (2.14) and hence *Theorem 3* is a consequence of *Theorem 2*.

We also have the following corollary to *Theorem 3*:

COROLLARY 1. *Let $m \in R_+^n$ and let A be a positive real number. Suppose that*

(3.8) $$F(s) \sim As^{-m}, \quad s \to \mathbf{0}+$$

and that

(3.9) $$F(s) = O(s^{-m}) \quad \textit{when} \quad \min_{k=1,2,\ldots,n} s_k \to 0+.$$

Furthermore, suppose that

(3.10) $$\lim_{\lambda \to 1+} \liminf_{||x|| \to \infty} \inf_{x \le t \le \lambda x} \left(\frac{\alpha(t) - \alpha(x)}{x^m} \right) = 0,$$

then

(3.11) $$\alpha(t) \sim \frac{At^m}{\prod_{k=1}^{m} \Gamma(1 + m_k)}, \quad t \to \infty.$$

PROOF. In *Theorem 3* we let

$$\beta(t) = \frac{At^m}{\prod_{k=1}^{n} \Gamma(1 + m_k)}, \quad \text{if} \quad t > \mathbf{0},$$

and

$$\beta(t) = 0 \quad \text{if any} \quad t_k = 0,\ k = 1, 2, \ldots, n.$$

In this case,

$$G(s) = As^{-m}, \quad s > \mathbf{0},$$

and the corollary follows from this.

REMARK. From this corollary it follows that Theorem 3 *is an n-dimensional generalisation of the classical Hardy-Littlewood-Karamata theorem for the Laplace transform (cf. [8], [10], [13]). If* $n = 1$ *condition (3.4) is included in (3.3) and hence the classical one-dimensional results are included in* Theorem 3.

The two-dimensional case of this corollary was first treated by Delange [4] in the special case when $m = \mathbf{0}$*. He then used Tauberian conditions strong enough to imply bounded convergence of the Laplace transform. More recently the multidimensional case of* Corollary 1 *has been treated among others by Celidze ([2], [3]), Stadtmüller and Trautner [17] and Omey and Willekens [15]. They also use stronger conditions than ours which imply both bounded convergence and condition (3.9).*

In Theorem 3 *we could equally well let* α *and* β *be measures on* R_+^n*. We just have to replace* $\alpha(t)$ *and* $\beta(t)$ *in (3.5), (3.6) and (3.7) with* $\int_{0\le s\le t} d\alpha(s)$ *and* $\int_{0\le s\le t} d\beta(s)$ *respectively. (Cf. e.g. [5] p. 48).*

4 Consequences for the one-dimensional Stieltjes- and Weierstrass transforms

Suppose that α and β are real-valued functions of bounded variation defined in R_+ with $\alpha(0) = \beta(0) = 0$. For the Stieltjes transform we give the following example as a consequence of *Theorem 2.*

THEOREM 4. *Suppose that* β *is positive and non-decreasing and that*

(4.1) $$F(s) = \int_0^\infty \frac{d\alpha(t)}{s+t} \sim G(s) = \int_0^\infty \frac{d\beta(t)}{s+t}, \quad s \to \infty$$

and also that for any real number δ *there exist a real number* m, $0 \le m < 1$, *and a real number* T *such that*

(4.2) $$\beta(rt) \le (1+\delta) r^m \beta(t), \quad r \ge 1,\, t \ge T.$$

Furthermore, suppose that

(4.3) $$\lim_{\lambda\to 1+} \liminf_{x\to\infty} \inf_{x\le t\le \lambda x} \Big(\frac{\alpha(t)-\alpha(x)}{\beta(x)}\Big) = 0,$$

then

(4.4) $$\alpha(t) \sim \beta(t), \quad t \to \infty.$$

PROOF. After a partial integration in (4.1), we have that

$$F(s) = \int_0^\infty \frac{\alpha(t)dt}{(s+t)^2}$$

and with a corresponding result for $G(s)$.

We now make the substitutions

$$s = \exp x \quad \text{and} \quad t = \exp y$$

and thus transform the problem into *Theorem 2* with

$$K(x) = \Big(\exp\Big(\frac{x}{2}\Big) + \exp\Big(-\frac{x}{2}\Big)\Big)^{-2}, \ \phi(x) = \alpha(\exp x) \text{ and } \psi(x) = \beta(\exp x).$$

In this case

$$sF(s) = K * \phi(x) \quad \text{and} \quad \hat{K}(x) = 2\pi x(\exp(\pi x) - \exp(-\pi x))^{-1}.$$

We omit the details of the proof.

Since

$$\int_0^\infty \frac{t^\alpha}{(s+t)^2} dt = \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{s^{1-\alpha}}, \quad -1 < \alpha < 1, \text{(cf. e.g. [18], p.184)},$$

we have the following corollary if we let

$$\beta(t) = A\frac{t^{1-\gamma}}{\Gamma(\gamma)\Gamma(2-\gamma)}, \quad 0 < \gamma \leq 1,$$

in *Theorem 4*.

COROLLARY 2. *If* $0 < \gamma \leq 1$ *and if for any positive number* A

$$F(s) \sim As^{-\gamma}, \quad s \to \infty$$

and if also

$$\lim_{\lambda \to 1+} \liminf_{x \to \infty} \inf_{x \leq t \leq \lambda x} \Big(\frac{\alpha(t) - \alpha(x)}{x^{1-\gamma}}\Big) = 0,$$

then

$$\alpha(t) \sim A\frac{t^{1-\gamma}}{\Gamma(\gamma)\Gamma(2-\gamma)}, \quad t \to \infty.$$

This is a classical Tauberian theorem for the Stieltjes transform in a general version.

We finally give an example for the Weierstrass transform. The Weierstrass transform is the convolution transform

$$f(x) = \int K(x-t)\phi(t)dt$$

where

$$K(x) = (4\pi)^{-1/2} \exp\Big(-\frac{x^2}{4}\Big), \quad \text{(cf. [9] p. 174)},$$

and we just give the following corollary to Theorem 2.

COROLLARY 3. *Suppose that for any positive number A and positive natural number n*

$$f(x) \sim Ax^n, \quad x \to +\infty,$$

and that $\phi(x)$ is bounded when x is bounded above.
If also

$$\lim_{h\to 0+} \liminf_{x\to+\infty} \inf_{x\le y\le x+h} \left(\frac{\phi(y)-\phi(x)}{x^n}\right) = 0$$

then

$$\phi(x) \sim Ax^n, \quad x \to +\infty.$$

PROOF. Suppose that

$$H_n(x) = (-1)^n \exp(x^2) D^n \exp(-x^2), \quad n = 0, 1, 2, \ldots$$

are the Hermite polynomials.

We use the one-dimensional analogue of *Theorem 2* with the Weierstrass kernel K and with

$$\psi(x) = H_n(\frac{x}{2}).$$

The result now follows from the fact that

$$H_n(\frac{x}{2}) \sim x^n, \quad x \to +\infty,$$

and that

$$\int K(x-t) H_n(\frac{t}{2}) dt = x^n, \quad \text{(cf. [9] p. 178).}$$

The other conditions of *Theorem 2* are clearly fulfilled.

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