NOTE ON MIYASHITA-ULBRICH ACTION AND H-SEPARABLE EXTENSION

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ABSTRACT. The Miyashita-Ulbrich action is an action of a Hopf algebra A on the centralizer E^C associated to an A-Galois extension B/C with algebra homomorphism $\alpha : B \to E$. Doi and Takeuchi [DT] ask when the action of a Hopf algebra A on the centralizer E^C of a ring extension E/C comes from such an A-Galois extension B/C. They provide an affirmative answer for Azumaya algebra E with subalgebra C such that E_C is a progenerator. In this note we observe how their proof extends to an H-separable extension E/C with the same condition on E_C. Similarly, we establish the converse: if E/C is an H-separable, right A-Galois extension, then E^C is a left A^*-Galois extension over the center $Z(E)$.

1. INTRODUCTION

We let R be a commutative ground ring, A be a Hopf algebra over R which is finite projective as an R-module, and A^* its dual Hopf algebra. If B is an associative unital algebra and M is a unitary B-bimodule, we let M^B denote the central elements of M: $M^B = \{m \in M | \ mb = bm, \forall b \in B\}$. Ulbrich [U] defines an action of A on the centralizer $V := V_B(C) = B^C$ of an A-Galois extension B/C. If $\beta : B \otimes_C B \to B \otimes A$ denotes the Galois isomorphism given by $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$, the action of an $a \in A$ on x in the centralizer V is given by

$$x \triangleleft a = \sum_i b_i x b'_i \tag{1}$$

where $\sum_i b_i \otimes b'_i = \beta^{-1}(1 \otimes a)$. It is easy to compute that this is a module action with invariant subalgebra $Z(B)$, the center of B. Indeed, it is a measuring action with V becoming a right A-module algebra [U, II].

Doi and Takeuchi [DT] extend this action to the centralizer E^C of an algebra extension E/C with algebra homomorphism $\alpha : B \to E$ by means of a π-method. The mapping π is the map induced from the isomorphism β by applying the functor $\mathrm{Hom}^{\ell}_B(-, E)$, where E is a B-bimodule via α. The C-bimodule isomorphism obtained induces

$$\pi : \mathrm{Hom}(A, E^C) \xrightarrow{\cong} \mathrm{Hom}_{C-C}(B, E), \quad \pi(f)(b) = b_{(0)} f(b_{(1)}). \tag{2}$$

For every $x \in E^C$, there is a map $x^{(\ )}$ in $\mathrm{Hom}(A, E^C)$ defined as $x^{(\ )} = \pi^{-1}(\lambda_x)$ where λ_x is left multiplication by x on B. This obtains a right measuring action of A on E^C by

$$x \triangleleft a := x^a, \tag{3}$$

2000 *Mathematics Subject Classification.* 12F10,16W30.

Key words and phrases. Hopf-Galois extension, centralizer, Miyashita-Ulbrich action, H-separable extension, Azumaya algebra.

the value of $x^{(\ )}$ on a. This action extends Ulbrich's action, is characterized by

$$xb = b_{(0)}(x \triangleleft b_{(1)}) \tag{4}$$

for every $b \in B, x \in E^C$, and is called the Miyashita-Ulbrich action [DT].

The authors of [DT] ask whether a measuring action of a Hopf algebra A on the centralizer E^C of a ring extension E/C is the Miyashita-Ulbrich action of some A-Galois extension B/C with algebra homomorphism $\alpha : B \to E$ [DT, p. 489]. They prove that this is the case for Azumaya algebra E with subalgebra C such that E_C is a progenerator. Their method of proof is general and careful enough to invite an extension of their answer to the more general H-separable extensions.

An H-separable extension is a certain type of separable extension which generalizes Azumaya algebra [H1]. Given an Azumaya algebra D and algebra C, the tensor algebra $C \otimes D$ is an H-separable extension of C [H1, Prop. 3.1]. Another example is the algebra extension B/C where B is an Azumaya algebra with subalgebra C such that B_C is projective and C is a direct summand of B as C-bimodules (cf. [S1, Prop. 1.4] and [K, 2.9]). H-separable extensions have several well-known properties in common with Azumaya algebras which we review in Section 2. Certain H-separable extension are automatically Frobenius extensions [S1, S2]. If B/C is an H-separable extension, then B_C a projective module implies B_C is finitely generated [T].

In this paper we note that the problem of Doi and Takeuchi has an affirmative answer as well for an H-separable extension E/C where E_C is a projective generator (Theorem 3.2). We moreover show that an H-separable right A-Galois extension B/C induces a left A^*-Galois extension $V/Z(B)$ (Theorem 3.1). The proofs of these two theorems follows [DT] except for a modification required by the fact that the center Z no longer is $R1_E$.

2. H-separable extension

Let B be a ring and M a B-bimodule. M is *centrally projective* if M is isomorphic to a B-bimodule direct summand of a finite direct sum of B with itself:

$$M \oplus * \cong {}_B(B \oplus \cdots \oplus B)_B.$$

Note that either the left or right B-dual $\mathrm{Hom}_B(M, B)$ is then centrally projective. A ring extension B/C is *H-separable* if the natural B-bimodule $B \otimes_C B$ is centrally projective [H1].

It follows from Hirata's extension of Morita theory [H1] that the centralizer $V = V_B(C)$ is a finitely generated projective module over the center Z of B, and there is a B-bimodule isomorphism,

$$B \otimes_C B \xrightarrow{\cong} \mathrm{Hom}_Z(V, B), \quad b \otimes b' \mapsto (v \mapsto bvb'). \tag{5}$$

From this it follows that B/C is a separable extension [H1, HS], as the name should indicate. Moreover, we obtain the characterization:

Proposition 2.1 ([S1]). The ring extension B/C is H-separable if and only if for every B-bimodule M, the multiplication mapping $V \otimes_Z M^B \to M^C$ is an isomorphism.

By letting $M = \mathrm{End}_{\mathcal{Z}} B$ and $\mathcal{E} := \mathrm{End}(B_C)$ we obtain the next corollorary, which is important to this note. Let $\rho_v \in \mathcal{E}$ denote right multiplication by $v \in V$.

Corollary 2.2 ([H2]). If B/C is H-separable, there is a ring isomorphism

$$B \otimes_Z V^{\mathrm{op}} \xrightarrow{\cong} \mathcal{E}, \quad b \otimes v \longmapsto \lambda_b \circ \rho_v \tag{6}$$

and a similarly defined ring isomorphism

$$V \otimes_Z V^{\mathrm{op}} \cong \mathrm{End}_{C-C}(B). \tag{7}$$

Now we call a ring extension B/C a (right) *HS-separable extension* if 1) it is an H-separable extension; and 2) B_C is a projective generator (therefore B_C is a progenerator [T]). [S2] has conditions where right and left extensions of this type are equivalent.

The characterization below of HS-separable extensions is key to this note. For any ring extension B/C, the functor (of induction) $B \otimes_C -$ from the category $\mathcal{C}$ of left C-modules into the category $\mathcal{D}$ of B-V-bimodules M such that $M^Z = M$, has right adjoint $G : \mathcal{D} \to \mathcal{C}$ given by $G(M) = M^V$. The associated counit of adjunction is given by (8), though not necessarily an isomorphism.

Lemma 2.3. *A ring extension B/C is HS-separable if and only if there is a ring isomorphism (6), V is finitely generated projective over Z, and G is an equivalence where multiplication leads to the isomorphism,*

$$B \otimes_C P^V \xrightarrow{\cong} P, \tag{8}$$

for every B-V-bimodule $P \in \mathcal{D}$.

Proof. ($\Rightarrow$) We have already noted that the isomorphism (6) and the condition on V hold for H-separable extensions. Since B_C is a progenerator, $B \otimes_C -$ is a (Morita) equivalence of $\mathcal{C}$ with the category of $\mathrm{End}(B_C) \cong B \otimes V^{\mathrm{op}}$-modules, i.e. the category $\mathcal{D}$. By uniqueness of right adjoint, G is an inverse equivalence of induction and (8) is a natural isomorphism.

($\Leftarrow$) Induction is clearly an equivalence of $\mathcal{C}$ and $\mathcal{D}$, so B_C is a progenerator by (6) and the Morita theorems. Since V is finitely generated projective over Z, the assumption that $\mathcal{E} \cong B \otimes_Z V^{\mathrm{op}}$ implies that $\mathcal{E}$ is centrally projective as a B-bimodule [S2, Lemma 3]. Then B/C is H-separable, since $B \otimes_C B$ is the left B-dual of $\mathcal{E}$ [S3, Cor. 2.3]. □

3. Hopf-Galois actions on the centralizer

We continue our notation V for the centralizer and Z for the center of the overalgebra; as before, A is a Hopf algebra, which is finitely generated projective over the commutative ground ring R, with comultiplication denoted by Δ and counit ε. The next theorem generalizes [DT, 5.2, (i) $\Rightarrow$ (ii)] and part of [U, Satz 2.7].

Theorem 3.1. *Suppose B/C is an H-separable right A-Galois extension. Then V/Z is a left A^*-Galois extension.*

Proof. The Miyashita-Ulbrich action $V \otimes A \to V$ induces via duality a left A^*-comodule algebra structure on V. The values of the coaction are denoted by $x \mapsto x_{(-1)} \otimes x_{(0)} \in A^* \otimes V$, whence the dual action is given by $x_{(-1)}(a)x_{(0)} = x \triangleleft a$ for every $x \in V, a \in A$. The coinvariants are the same as the invariant subalgebra, namely $Z = Z(B)$.

Now consider the commutative diagram (Fig. 1). The bottom horizontal arrow is a standard isomorphism due to the assumption that A is finite projective over

$$\begin{array}{ccc} V \otimes_Z V & \cong & \mathrm{End}_{C-C}(B) \\ {\scriptstyle \beta'} \downarrow & & \uparrow {\scriptstyle \pi} \\ A^* \otimes V & \cong & \mathrm{Hom}(A, V) \end{array}$$

FIGURE 1

R. The top horizontal arrow is the isomorphism (7) as a left V-isomorphism. β' is the left Galois map given by

$$x \otimes y \mapsto x_{(-1)} \otimes x_{(0)} y$$

for every $x, y \in V$. π is the Doi-Takeuchi map π discussed above in the case $E = B$. The diagram commutes since

$$x \otimes y \mapsto x_{(-1)} \otimes x_{(0)} y \mapsto x^{(\,)} y \stackrel{\pi}{\longmapsto} \lambda_x \circ \rho_y,$$

which is the image of $x \otimes y \in V \otimes V$ under the isomorphism (7). It follows that β' is an isomorphism and V/Z a left A^*-Galois extension. □

The next result is a type of converse of the theorem above. Recall the category $\mathbf{G}_C^{A,E}$ in [DT, Section 6] consisting of all triples (B, i, α) such that B is a right A-Galois extension with invariants isomorphic to C via i and algebra homomorphism $\alpha : B \to E$. The morphisms in this category satisfy two obvious commuting triangles and are then isomorphisms [DT, p. 509].

In the isomorphism class of (B, i, α) is a canonical representative $\hat{B} \subseteq E \otimes A$ with right A-comodule structure given by the restriction of $\mathrm{id}_E \otimes \Delta$ to $\hat{B}$, which is the image of $\hat{\alpha} : B \to E \otimes A$ given by $\hat{\alpha}(b) = \alpha(b_{(0)}) \otimes b_{(1)}$ [DT, Theorem (6.12)]. This map is injective if E_C is faithfully flat. That B/C is a right A-Galois extension translates to the conditions [DT, (6.6)-(6.8)] for $\hat{B}$ being right Galois over $C = C \otimes 1_A$, including the condition

$$\eta : \; E \otimes_C \hat{B} \stackrel{\cong}{\longrightarrow} E \otimes A, \quad \eta(x \otimes \hat{b}) = (x \otimes 1)\hat{b} \tag{9}$$

Each isomorphism class (B, i, α) in $\mathbf{G}_C^{A,E}$ determines a measuring (Miyashita-Ulbrich) action $E^C \otimes A \to E^C$ denoted by $\triangleleft_B$ and characterized by Eq. (4). Denote the set of right measuring actions of A on E^C, where Z is contained in the invariants $V^{(A)}$, by $\mathrm{Meas}(E^C/Z, A)$.

Theorem 3.2. *Suppose E/C is an HS-separable extension with right measuring action $\triangleleft$ of A on its centralizer V such that $V^{(A)} \supseteq Z$. Then there is a right A-Galois extension B/C with algebra homomorphism $\alpha : B \to E$ such that $\triangleleft_B = \triangleleft$. Moreover, $\mathrm{Meas}(E^C/Z, A)$ is in 1-1 correspondence with the isomorphism classes of $\mathbf{G}_C^{A,E}$ via $\hat{B} \mapsto \triangleleft_{\hat{B}}$.*

Proof. The main steps of the proof, following [DT, (6.13)-(6.20)], are to define an E-V-bimodule structure on $E \otimes A$ from the given action $\triangleleft$, define a right Galois subalgebra

$$\hat{B}_{\triangleleft} = (E \otimes A)^V$$

over C, and prove that $\triangleleft_{\hat{B}_{\triangleleft}} = \triangleleft$. The E-V-bimodule structure on $E \otimes A$ is indicated by

$$e(e' \otimes a)v = ee'(v \triangleleft a_{(1)}) \otimes a_{(2)} \tag{10}$$

Note that $(E \otimes A)^Z = E \otimes A$ since $Z \subset V^{(A)}$. That $\eta : E \otimes_C \hat{B}_{\triangleleft} \to E \otimes A$ is an isomorphism follows from our assumption that E/C is HS-separable and Lemma 2.3. That $\hat{B}_{\triangleleft}$ is a subalgebra which is right Galois over C follows from the proof of [DT, Lemma (6.17)]. The map $\alpha_{\triangleleft} : \hat{B}_{\triangleleft} \to E$ is $\mathrm{id}_E \otimes \varepsilon$. That the measuring action on E^C associated with $\hat{B}_{\triangleleft}$ is $\triangleleft$ itself follows from the proof of [DT, Lemma (6.18)].

To finish the proof that $\mathbf{G}_C^{A,E} \leftrightarrow \mathrm{Meas}(E^C/Z, A)$ we must show that given $\hat{B} \in \mathbf{G}_C^{A,E}$, the construction applied to the associated measuring action $\triangleleft_{\hat{B}}$ leads back to $\hat{B}$: i.e., $\hat{B}_{\triangleleft_{\hat{B}}} = \hat{B}$. Now $\hat{B} \subseteq \hat{B}_{\triangleleft_{\hat{B}}}$ by the computation in the proof of [DT, Lemma (6.19)]. A proper inclusion is impossible since $E \otimes_C \hat{B} = E \otimes_C \hat{B}_{\triangleleft_{\hat{B}}} = E \otimes A$ and E_C a generator implies the existence of a right C-module projection $E_C \to C_C$. □

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