MODIFIED INFINITE ELEMENT METHODS

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This work is devoted to the study of infinite element techniques for low frequency scattering problems, which play an important role in the simulation of acoustical radiation problems. The standard infinite element method is modified for the low wave number regime. This special modification is motivated and described in this work. From the presented theoretical analysis it is clearly visible that this adaption of the method allows for an efficient treatment of scattering problems at the low frequency end.

1. Introduction

The infinite element method (IEM) has been introduced by Bettess[3] to solve exterior problems. The original idea has been successively refined in the past two decades and more recently a large body of improvements of the original method has appeared in the literature. A patent[27] has even been awarded to a specific infinite element. Recently the IEM is also being applied in the time domain by Cipolla and Butler[8], and is even extended to electromagnetics by Demkowicz et. al.[7]. For a complete overview over the early developments we refer to Bettess[4]. The improvements and applications of the IEM during the 90ies are summerized in the papers by Astley[1,2], Geers[13], Gerdes[18], and Ihlenburg[22,23]. We refer the reader to these articles for a complete overview of the existing literature on infinite elements.

We note here that most of the exisiting work on the IEM has been applied to solve a variety of scattering problems in the medium frequency range. In terms of the non dimensional wave number k, in general problems with $k > 1$ were addressed (assuming that the characteristic length scale of the problem is unity). In particular, the complications in the solution of radiation problems are assumed to depend strongly on large values of k, due to the oscillatory character of the solution and the resulting pollution effects[16] that are present. Although this is certainly the case, it seems to have been overseen that the efficient solution of the low frequency regime is equally important for the detection and classification of submerged objects. As was recently pointed out in a theoretical study by Astley[1], the various infinite elements perform poorly for small k if the so called approximability condition is not met. That is, if the solution is significantly depending on higher order modes, which are not included in the infinite element, then the error in the infinite element is significant in the low frequency range. The analysis by Astley relies on a modal decomposition, which was first used by Demkowicz and Gerdes[10] and this analysis is also applied in this work to the modified infinite elements.

We emphasize that in principle other techniques exist that can be an alternative to the infinite element methodology. Among these is for example the technique to truncate the computational domain at some distance away from the scatterer and to impose a boundary condition at the artificial boundary. In general, a global DtN condition is applied, as is done in the work of Pinsky[26]. This exact condition is often approximated by a local absorbing boundary condition and the locality of the boundary operator and the distance at which it is applied significantly influence the computational efficiency of such a strategy. Additionally, it was pointed out by Astley[1] that such local techniques also have performance problems in the low frequence range, whereas the global DtN is in principle exact, but truncations of summations need to be applied for computational purposes. For more details on absorbing boundary conditions we refer to the books by Givoli[19] and Ihlenburg[22] and the references therein. We point out that the IEM can nevertheless outperform such local and global techniques, depending on which physical quantities are of interest in the problem.

Another alternative is the boundary element method[9] but the non local character of the computations are CPU time intensive, as was pointed out by Burnett[5]. The doubly asymptotic expansions of Geers[14] are another alternative for transient computations.

A theoretically consistant variational formulation for the exterior acoustical radiation problem has to be carefully derived and the corresponding space setting uses weighted Sobolev spaces that were introduced by Leis[24]. This variational formulation, for which a complete mathematical theory exists, or a different approach derived by Burnett[5], for which a similar mathematical theory does not exist, are used in all modern publications on infinite elements. The accuracy of the infinite elements depends then on the choice of the finite dimensional subspace, i.e. the accuracy depends on the particular choice of the radial shape functions in the infinite element. The choice of shape functions is motivated by the Atkinson-Wilcox expansion theorem[29] in spherical coordinates, which has been extended to general ellipsoidal coordinate systems by Burnett and Holford[6]. These radial shape functions are up to linear combinations of the form $\Psi_j(r) = \exp(ikr)/r^j$, $1 \le j \le N$. The number N of such radial shape functions that need to be included in order to solve a given radiation problem has to be determined and this issue has been analyzed by Astley[1], Demkowicz and Gerdes[10], Babuska and Shirron[28], and recently in a more general context by Demkowicz and Ihlenburg[12]. As mentioned above, these works were mainly concerned with the case $k > 1$, but pointed out the importance of using shape functions that are consistent with the radial expansion[29]. In the present work we apply shape functions that fit into the framework of the mathematically correct variational formulations but are not compeletely consistent with the radial expansion theorem. The shape functions being chosen are of the form $\Psi_j(r) = \exp(ikr)/r^j$, $1 \leq j \leq L$, which are identical to the above shape functions, and additionally we also use $\Psi_{j+L}(r) = r^{-j-L}$, $1 \leq j \leq M$. These shape functions are similar to those that are being used to solve the exterior Laplace problem[15]. The anlaysis below shows that excellent results can be achieved with this choice for small k, although it is not strictly consistent with the radial expansion theorems. The motiviation for this choice is that the Helmholtz equation reduces to the Laplace equation as k approaches zero. Therefore we introduce terms in the definition of the shape functions that are stemming from the solution of the limiting Laplace problem.

The content of this paper is outlined as follows. In section 2 we present the rigid scattering problem and the standard infinite element. The modified infinite element is presented in section 3, and section 4 presents a modal error analysis. Numerical results for the modal analysis are discussed in section 5 and we finish the presentation with conclusions in section 6.

2. Infinite Element Formulations

2.1. *The exterior Helmholtz problem*

Let Ω be a rigid obstacle and the domain exterior to the obstacle is the exterior domain $\Omega^e = \mathbb{R}^3\backslash\Omega$. The classical formulation of the scattering problem is to find a function $u = u(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega^e$, which satisfies:

- the Helmholtz equation in the exterior domain

$$-\Delta u - k^2 u = 0 \qquad \text{in} \quad \Omega^e, \tag{2.1}$$

  where k is the wave number,

- a Neumann boundary condition on the scatterer

$$\nabla_n u = g \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega, \tag{2.2}$$

- and the Sommerfeld radiation condition at infinity

$$\left| \frac{\partial u}{\partial n} - iku \right| = O\left(\frac{1}{r^2}\right). \tag{2.3}$$

2.2. *Variational Formulations*

The unconjugated Astley-Leis formulation reads as

$$\begin{cases} \text{Find } u \in H_{1,w}(\Omega^e) \quad \text{such that} \\ \displaystyle\int_{\Omega^e} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega^e - k^2 \int_{\Omega^e} u \, v \, d\Omega^e = \int_{\partial\Omega} g \, v \, dS \quad \forall \, v \in H_{1,w^*}(\Omega^e), \end{cases} \tag{2.4}$$

The conjugated Astley-Leis formulation is stated similarly, except for the complex conjugate over the test function v. The weights are defined as $w = r^{-2}$, and $w^* = r^2$. The weighted Sobolev spaces are then defined as

$$H_{1,w}(\Omega^e) = \{u : \|u\|_{1,w} < \infty\} \tag{2.5}$$

with the norm $\|u\|_{1,w}$ corresponding to the inner product

$$(u,v)_{1,w} = \int_{\Omega^e} w\, u\, \overline{v} + w\, \nabla u \cdot \nabla \overline{v}\; d\Omega^e + \int_{\Omega^e} \left(\frac{\partial u}{\partial r} - iku\right) \overline{\left(\frac{\partial v}{\partial r} - ikv\right)}\; d\Omega^e. \tag{2.6}$$

The unconjugated Burnett formulation can be written as

$$\left\{ \begin{array}{l} \text{Find } u \in H_{1,w}(\Omega^e) \text{ such that } \forall\, v \in H_{1,w}(\Omega^e) \\ \lim\limits_{\gamma \to \infty} \left(\displaystyle\int_{\Omega^e_\gamma} \nabla u \cdot \nabla v \; - k^2\, u\, v\; d\Omega^e_\gamma - ik \int_{S_\gamma} uv\; dS_\gamma \right) = \displaystyle\int_{\partial\Omega} g\, v\, dS, \end{array} \right. \tag{2.7}$$

The conjugated Burnett formulation is obtained similarly if the complex conjugate is applied to test function v. For a more detailed derivation of these variational formulations we refer to Gerdes[17].

2.3. *The infinite element*

In Gerdes and Demkowicz[10] we described in detail how an exact solution to (2.1) can be derived for a spherical scatterer by the separation of variables procedure. The exact solution is then of the form

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} h_n(kr) P_n^m(\cos\theta)\, (A_{nm}\cos(m\phi) + B_{nm}\sin(m\phi)), \tag{2.8}$$

where h_n are spherical Hankel functions of the first kind, P_n^m are the Legendre functions, and A_{nm}, B_{nm} are coefficients that can be determined if a Neumann boundary condition is known on the circumscribing sphere[15].

The radial expansion (2.8) converges outside the smallest sphere that circumscribes the scatterer to the solution[29] and similar results in general spheroidal coordinates were recently established by Holford[21]. We note that u in (2.8) depends radialy on the spherical Hankel functions[25] which are defined as

$$h_n(kr) = \sum_{m=0}^{n} \frac{\exp(ikr)}{r^{m+1}} \frac{\exp\left(-i\frac{\pi}{2}(n+1)\right)}{k(2k)^m} i^m \left(n + \frac{1}{2}, m\right) \tag{2.9}$$

with

$$\left(n + \frac{1}{2}, m\right) = \left\{ \begin{array}{ll} 1 & m = 0 \\ \displaystyle\prod_{k=1}^{m} (n+k) \cdot \prod_{k=1}^{m} \frac{(n-m+k)}{k} & m \geq 1. \end{array} \right.$$

The radial dependence of the solution motivates the definition of the radial shape functions in the infinite element and by tensor product structure we define an infinite element shape function N_m as

$$N_m(r, \boldsymbol{\xi}) = \psi_j(r) \cdot \varphi_i(\boldsymbol{\xi}). \tag{2.10}$$

Here φ_i are standard polynomial finite element shape functions defined on a reference finite element $\hat{K}$ and ψ_j are radial shape functions, which are usually defined by

$$\psi_j(r) = \frac{\exp{(ikr)}}{r^{j+l}}, \qquad j \geq 1. \tag{2.11}$$

This is consistent with the radial expansion (2.8). In (2.11) l is determined by the particular variational formulation. We have $l = 2$ for the test functions in the Astley-Leis formulation and otherwise always $l = 0$.

3. Modified Infinite Element

The modified infinite element for low frequencies uses the standard variational formulations (2.4) and (2.7) and is defined by different radial shape functions. These modified radial shape functions $\Psi_j(r)$ with

$$\Psi_j(r) = r^{-j}\exp(ikr),\ 1 \leq j \leq L, \qquad \Psi_{j+L}(r) = r^{-j-L},\ 1 \leq j \leq M \tag{3.12}$$

are satisfying the radial decay rates but are not all oscillatory.

We emphasize that in the variational formulations (2.4) and (2.7) the solution u belongs to the space $H_{1,w}$ in which the Sommerfeld radiation condition is directly incorporated. This implies that the modified radial shape functions have to satisfy the Sommerfeld condition in order to be within the theoretical framework. We immediately see that r^{-1} does not satisfy the radiation condition but that r^{-j} for $j > 1$ satisfies this condition. Therefore, we must always have $L \geq 1$ and then the modified radial shape functions can be used to create a finite dimensional subspace of $H_{1,w}$. It is evident, that a modified infinite element shape function $N_m(r, \boldsymbol{\xi})$ is defined similar to (2.10) but incorporates the modified radial shape functions (3.12).

The radial test functions are defined analogously to (3.12), except for the r^{-2} weighting in the Astley-Leis formulation and the application of the complex conjugate in the conjugated formulations.

4. Modal Error Analysis

We assume that the scatterer Ω has the form of the unit sphere, and by separation of variables the solution u of (2.1) is of the form (2.8). A given Neumann boundary condition on $\partial\Omega$ allows to uniquely determine u. In this context we normalize the Hankel functions h_n so that they satisfy the condition

$$-\frac{\partial}{\partial r}(h_n(kr))(1) = 1. \tag{4.13}$$

For the modal error analysis we are searching for an approximate solution of the form

$$u^N(r,\theta,\phi) = \sum_{n=0}^{N}\sum_{m=0}^{n} h_n(kr)P_n^m(\cos\theta)\,(A_{nm}\cos(m\phi) + B_{nm}\sin(m\phi)). \tag{4.14}$$

This truncated solution u^N assumes an exact approximation in angular direction, and possible errors result only from the radial approximation. We emphasize here, that such an analysis has first been used in the work of Demkowicz and Gerdes[10], in Gerdes[17], and more recently in the work of Astley[1]. In the infinite elements we are not directly using the Hankel functions but replace them with the radial infinite element shape functions, i.e. we are seeking a truncated solution u^N of the form

$$u^N(r,\theta,\phi) = \sum_{n=0}^{N}\sum_{m=0}^{n} \Psi_n(r)P_n^m(\cos\theta)\,(A_{nm}\cos(m\phi) + B_{nm}\sin(m\phi)). \tag{4.15}$$

The coefficients A and B in (4.15) are different from the coefficients in (4.14), nevertheless the key to the modal analysis is to substitute (4.15) into the weak formulation. Then we integrate in angular direction and make use of the L^2 orthogonality of the Legendre and trigonometric functions. For the details of the derivation we refer to Demkowicz and Gerdes[10]. The resulting one dimensional weak form describes the approximability of the j-th order radial component of the solution by linear combinations of functions Ψ_n, and for the conjugated Astley-Leis formulation the radial form is given by

$$\begin{aligned}&\sum_{n=0}^{N}\left\{ j(j+1)\int_1^\infty \frac{1}{r^2}\Psi_n\overline{X_m}dr + \int_1^\infty \Psi_n'\overline{\left(\frac{X_m}{r^2}\right)'}dr - k^2\int_1^\infty \Psi_n\overline{X_m}dr\right\} u_n^j \\ &= \overline{X_m(1)}, \quad m = 0,\ldots,N\end{aligned} \tag{4.16}$$

The radial unconjugated Astley-Leis and the (un)conjugated Burnett formulations are obtained similarly, compare Gerdes[17]. We note that the Hankel functions can be used in principal for the test functions X_m in (4.16). For practical purposes we use the functions $\exp(ikr)/r^{n+1}$ instead of $h_n(kr)$. The trial functions Ψ_n are the infinite element radial shape functions, and in the case of the modified infinite element that we want to study in this work, we apply the functions defined in (3.12). System (4.16) is then solved for the coefficients u_n^j, which are used in the spectral approximation of the j-th radial component of u, i.e.

$$u_j(r) \approx u_j^N = \sum_{n=0}^{N} u_n^j\Psi_n(r). \tag{4.17}$$

We note that u_j is basically h_j. The error of this approximation can be measured in the component energy norm $\|u_j - u_j^N\|_{j,E}$, which was derived in Demkowicz and Gerdes[10], and is defined by

$$\|u\|_{j,E}^2 = \int_1^\infty |u'|^2dr + j(j+1)\int_1^\infty \frac{1}{r^2}|u|^2dr + \int_1^\infty |u|^2dr. \tag{4.18}$$

We emphasize that the definition of (4.18) is motivated by theoretical considerations from the derivation of (4.16). Popular error measures that are often being considered by practitioners and that are motivated from physical considerations are the surface inertia and resistance. The surface inertia is defined as the real part of the acoustic response on the surface of the scatterer and the surface resistance is the scaled imaginary part of the acoustic response, compare e.g. Astley[1] and Geers[14].

5. Numerical Results

We present the component errors for the solution of (4.16) for various wave numbers. We focus in particular on the performance of the modified infinite elements for the conjugated Astley-Leis formulation which is most suitable for the solution of transient problems, as was pointed out by Astley[1]. The performance of the modified infinite elements for the unconjugated Astley-Leis formulation and the Burnett formulations can be assessed similarly.

Figure 1 shows the error in the component energy norm for $k = 0.1$ for the conjugated Astley-Leis formulation with the standard and the modified radial shape functions with $L = 1$. The graph shows the error that results from the approximation of the j-th solution component with N radial shape functions. The resulting errors are basically identical for both sets of radial shape functions. The only slight difference results from the minimal error that occurs in the approximation of the $j = 1$ component with the modified element.

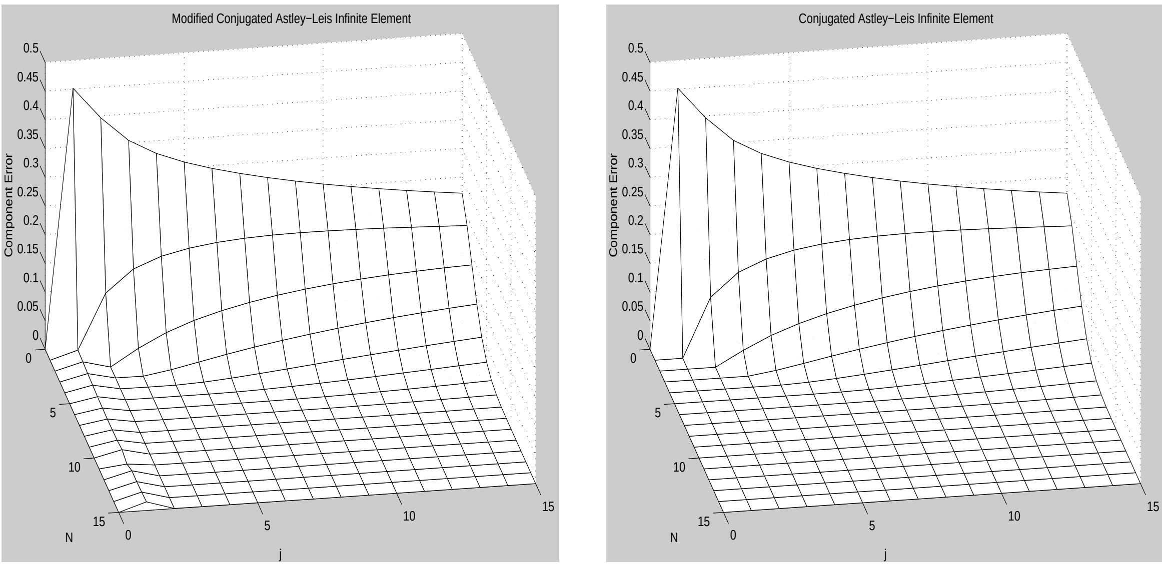

Fig. 1. Component energy error for wave number $k = 0.1$.

Figure 2 shows results for a very small wave number of $k = 0.001$, and we clearly see again an identical performance in both cases, and even the case $j = 1$ is resolved by the modified radial shape functions. It is also evident that the error is zero if the approximability condition[1] is satisfied, i.e. if $N \geq j$.

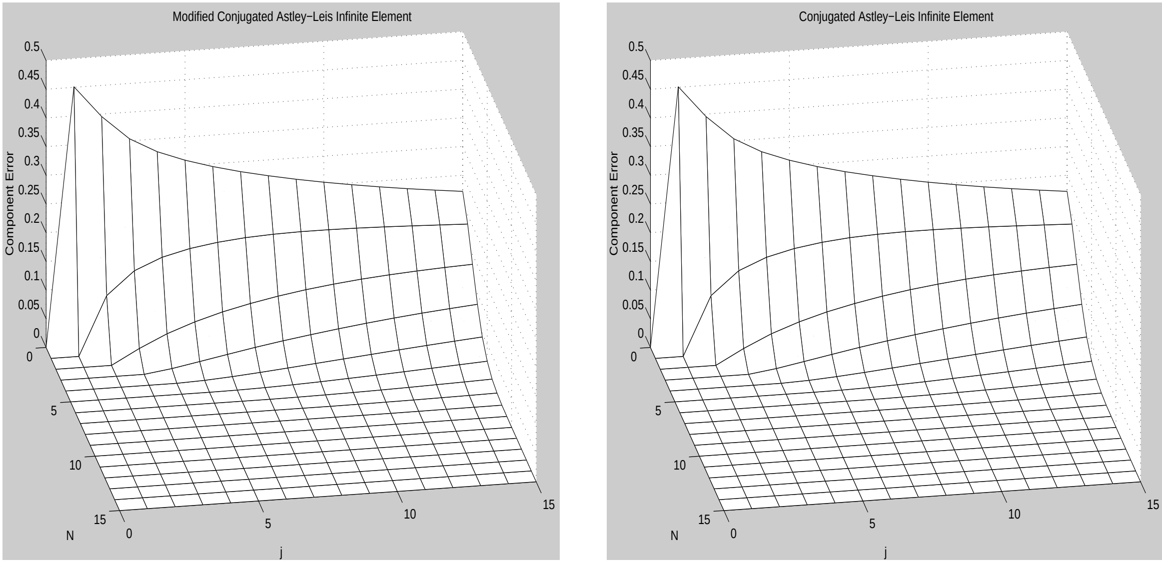

Fig. 2. Component energy error for wave number $k = 0.001$.

From Figures 1 and 2 we can conclude that both sets of radial shape functions perform equally well if the error is measured in the component energy norm. This conclusion is in contrast to the results of Astley[1], but we emphasize that the component errors depend on the particular norm that is used to measure the error.

Figure 3 shows the component error for a larger wave number $k = 2$, and it is obvious that the modified infinite element does not lead to a satisfactory performance when the error is measured in the component energy norm.

The definition of the component energy norm is motivated by the theoretical considerations that lead to (4.16). Practitioners are often interested in quantities that have a physical meaning, as for example the inertia. In Figure 4 we show the relative error in inertia for an exact solution with $j = 15$ components over a range of wave numbers. We see that the modified infinite element with $N = 2, 4, 6$ clearly outperforms the standard infinite element by two orders of magnitude over a small range of wave numbers. These results for the physical quantity are opposite to the results for the component energy norm.

6. Conclusions

The new feature that we introduce to infinite elements in this work is the use of radial shape functions that are not consistent with the radial expansion developed by Atkinson and Wilcox but that are consistent with the mathematical framework on infinite elements. We emphasize that these modified infinite element radial shape functions (3.12) satisfy the Sommerfeld radiation condition and therefore allow to define a finite dimensional subspace

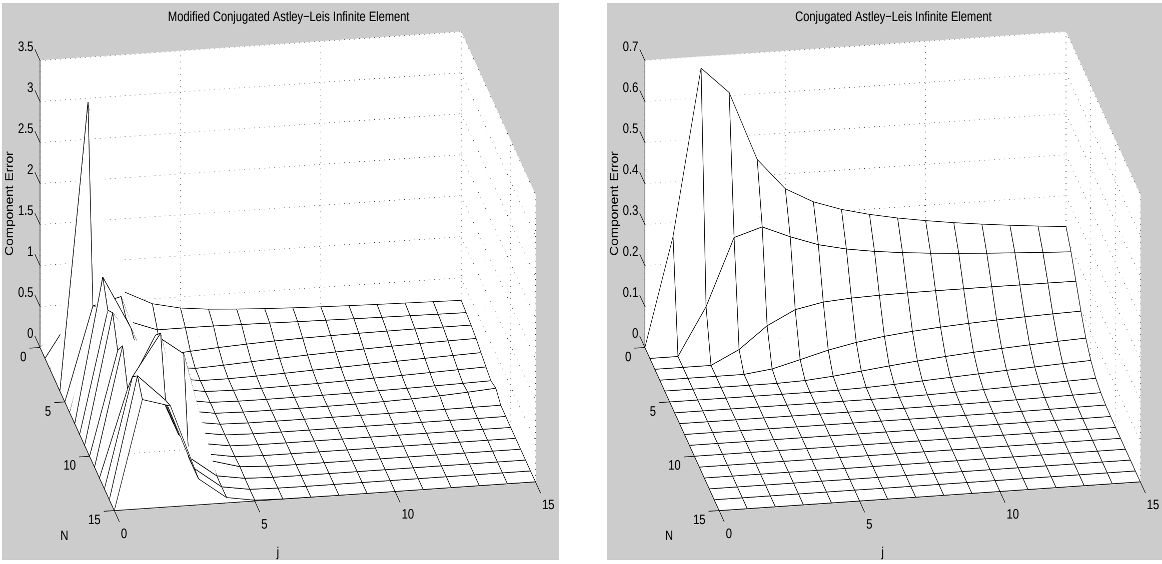

Fig. 3. Component energy error for wave number $k = 2$.

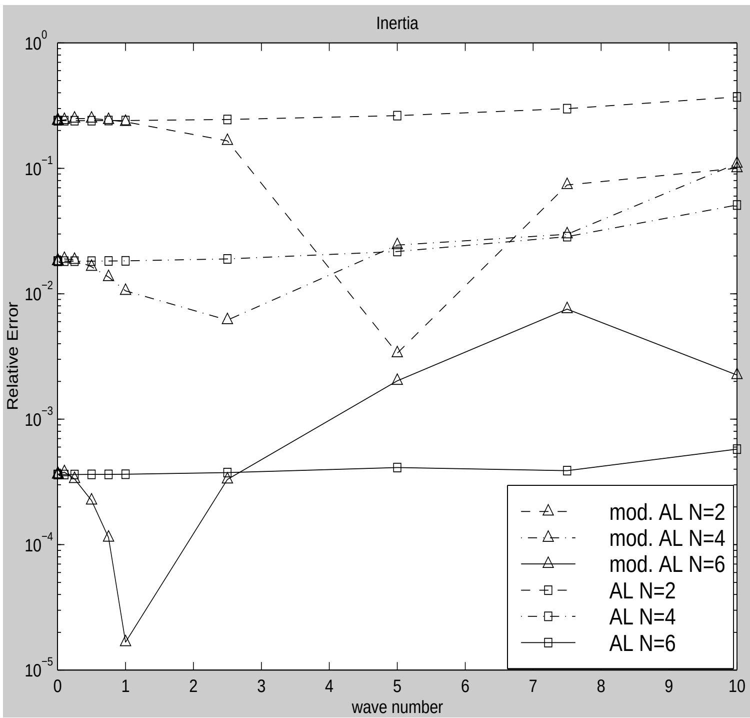

Fig. 4. Relative error in inertia for exact solution with j=15 modal components.

of $H_{1,w}$. The framework for the previously developed convergence analysis is satisfied and a modal analysis can be performed.

The numerically obtained results for the modal analysis allow for two conclusions. First, the standard and the modified infinite element deliver similar results for small wave numbers if the modal error is measured in the component energy norm. Practically the results are identical for wave numbers smaller than unity. Second, when the inertia is used as the error measure, then the results are orders of magnitude different in a wave number window which depends on the number of terms N that are used in the approximation.

Evidently, this modified infinite element has the potential for an effective approximation of low frequence scattering problems but further theoretical analysis is needed to define finite dimensional subspaces of $H_{1,w}$ that allow for optimal component approximability.

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