On Taylor's hypothesis and advection velocities

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Abstract

In the current paper it is shown that, in flow cases when the acceleration terms of the Navier–Stokes equations are negligible, a drastic increase in the complexity of the solutions to the Navier–Stokes equations for the fluctuating velocity occurs when the quantity $U_j \partial U_i / \partial x_j$ changes from being a linear function in space to being a non-linear one. For instance, it is seen that in case it is a linear function in space, the centroid of the trajectory positions for a certain delay τ coincides with the position along the trajectory of the mean flow for the same delay, which is not true in the non-linear case. In addition, for uniform mean flow with negligible acceleration terms, sufficient conditions for the coincidence of the advection and local mean velocity are given, along with a formal proof of their sufficiency. The possibility of extending the theorem to some other cases is also discussed. A method to take into account the effect of mean velocity gradients on the velocity along a trajectory and on the trajectory position is used in an attempt to improve the accuracy of Taylor's hypothesis for shear flows. This method should allow a substantial relaxation of Lin's criterion in many cases.

1 Introduction

Taylor's hypothesis of frozen turbulence (Taylor, 1938) suggests that a turbulent pattern is changed fairly slowly as it is advected past a point. Since this allows the temporal variation of velocity obtained at one point to be interpreted as the spatial variation of the velocity at a fixed time, this hypothesis has been widely used in experimental fluid dynamics. For homogeneous, isotropic turbulence with low turbulence intensity its validity has been established thoroughly experimentally, see e.g. (Favre *et al.*, 1952; Comte-Bellot & Corrsin, 1971). However, it is also common knowledge (Lin, 1953) that the validity of the hypothesis for shear flows is doubtful, and the measurements of space-time correlation coefficients[1] in a boundary layer by Favre

[1] Some theoretical papers in the field discuss correlations, whereas almost all experimental papers consider correlation coefficients. For homogeneous and stationary turbulence the two concepts are equivalent, but we shall distinguish the two by using the notation $R_{ij}(\mathbf{x_0}, t_0, \mathbf{x_1}, t_1) \equiv \overline{u_i(\mathbf{x_0}, t_0)\, u_j(\mathbf{x_1}, t_1)}$ (the overline denotes ensemble average) for correlations and $\rho_{ij}(\mathbf{x_0}, t_0, \mathbf{x_1}, t_1) \equiv R_{ij}(\mathbf{x_0}, t_0, \mathbf{x_1}, t_1) / \left(\overline{u_i^2(\mathbf{x_0}, t_0)}\ \overline{u_j^2(\mathbf{x_1}, t_1)}\right)^{1/2}$ for correlation coefficients. For stationary and homogeneous turbulence these quantities depend only on the difference between $\mathbf{x_0}$ and $\mathbf{x_1}$ and between t_0 and t_1.

et al. (1957, 1958) confirmed this suspicion. Indeed, it was found that in the innermost region of the boundary-layer, structures were not advected by the local mean velocity, and worse, the locus of the points of maximum correlation coefficients, when the time delay was adjusted to optimise the correlation coefficient, did not coincide with the mean streamlines. In Sternberg (1962) it was concluded that "in any shear flow, the disturbance velocity at a point P is in general different from the local mean velocity". On the other hand, in Champagne *et al.* (1970); Harris *et al.* (1977) it was found that for homogeneous shear flow, the advection velocity[2] and the local mean velocity coincided to within the accuracy of the measurements, even in case of strong shear. This surprisingly simple behaviour was attributed to the absence of curvature of the mean velocity profile, but curvature of the mean velocity profile does not necessarily rule out the possibility of this simple behaviour. For example, in the core region of channel flow the validity of the Taylor hypothesis has been confirmed both experimentally (Romano, 1995) and numerically (Piomelli *et al.*, 1989; Kim & Hussain, 1994; Wilhelm *et al.*, 1998) with good agreement between the advection and local mean velocities despite the presence of a non-negligible curvature of the mean velocity profile. So far a precise criterion of when to expect this simple behaviour has not been presented, however, in this paper it is argued that, in cases when the acceleration terms of the Navier–Stokes equations are small, it is reasonable to expect that the crucial factor determining whether the advection and local mean velocities coincide, is the linearity in space of $U_j \partial U_i / \partial x_j$ (where $\mathbf{U}$ denotes the mean velocity).

Most of the research analysing Taylor's hypothesis has focused on the error due to non-negligible turbulence intensity, and on the properties of the different terms in the Navier-Stokes equations. Lin (1953) and Heskestad (1965) estimated the size of the terms of the Navier–Stokes equations in an attempt to analyse the error induced by the use of Taylor's hypothesis. Lumley (1965) made a rather complete analysis of the effects which generate the error, and found that if certain conditions are satisfied the dominant contribution to the error for large frequencies is due to fluctuating advection velocity, which clearly is a problem of increasing severity as the turbulence intensity is increased. It should be stressed that Lumley's analysis only applies for frequencies so high that the Lin (1953) criterion is satisfied, and for some shear flows this is a considerable restriction. Lumley also gave correction formulae for the advection velocity and the spectrum, applicable for the high frequencies in the case of non-negligible turbulence intensity. Lumley's correction equation was solved analytically by Champagne (1978), but since the solution over-estimated the high-frequency part of the spectrum

[2]In older literature "disturbance velocity " or "convection velocity" are typically used, whereas in more recent literature both "advection velocity" and "convection velocity" are used frequently. We will use "advection velocity" throughout, to emphasize that active turbulent quantities are advected rather than passive scalars.

considerably, the validity of the approach has not been established beyond reasonable doubt. Alternative formulae based on different assumptions have been derived by Wyngaard & Clifford (1977) whose results agree well with Lumley's, however the measurements of Antonia *et al.* (1980) did not agree well with the assumption used in Wyngaard & Clifford (1977). Recently, Lumley's analysis was used to derive correction formulae for general tensors by Hill (1996), whose article also contains an explanation of the assumptions made. In all the papers mentioned thus far it is assumed that the contribution of the acceleration terms is negligible. However, the first attempt to try to estimate the size of the error induced by the acceleration terms was made in Gledzer (1997). In the current paper the approach of Hill is extended to the case of homogeneous shear flow, and the method is used to modify Taylor's hypothesis to allow a significant relaxation of Lin's criterion for shear flows.

The introduction of the concept of an advection velocity different from the local mean velocity has extended the range of validity of Taylor's hypothesis to certain shear flows, for instance the flow in the impeller stream of a Rushton turbine (Michelet *et al.*, 1997) and the turbulent von Kármán swirling flows (Pinton & Labbé, 1994). More generally, Zaman & Hussain (1981) concluded that provided that the correct choice of the advection velocity is made, "the application of the Taylor hypothesis can be acceptable when applied to single large-scale structures that are not undergoing rapid evolution or interaction with neighbouring structures". In the same spirit, we will in this paper establish theoretically that in cases when the acceleration terms of the Navier–Stokes equations are negligible, and the mean velocity and mean vorticity fields are parallel, Taylor's hypothesis is approximately valid along individual trajectories, but, in general, we have little a priori knowledge of where these trajectories are displaced as the time progresses. In fact, the major difficulty with introducing an advection velocity different from the local mean velocity is that there is no known method to predict this advection velocity from the Navier–Stokes equations, and very little research has been done in this area. As a first step to enable the calculation of the advection velocity from simpler properties of the turbulent flow, we will in this paper give sufficient conditions which establish that the advection velocity, defined using the space-time correlation coefficients in a manner described in the next subsection, coincides with the local mean velocity.

A question which, by contrast, has received considerable attention in literature, is how to define the advection velocity, and a number of different definitions are in widespread use. Zaman & Hussain (1981) found that a good choice was the "structure passage velocity", *i.e.* the velocity of the structure centre. A difficulty with this type of definition is that it requires some structure identification procedure. For this reason, most definitions use the space-time correlation coefficients, but there are several different ways of defining the advection velocity from space-time correlation coefficients, and since this issue is of great importance to us we will devote the next subsection to it.

In section 2, we define mathematically the concepts to be used in the subsequent sections and we use these concepts to state some of the results that are be established later. In section 3 the case when $U_j\partial U_i/\partial x_j$ is a linear function in space is analysed. For most of this analysis it is assumed that the acceleration terms of the Navier–Stokes equations (pressure, Reynolds stresses and viscosity) are negligible, but towards the end of the section, the stability of the analysis to perturbations linear in time and space is investigated. In section 4 it is shown that the picture is changed significantly if $U_j\partial U_i/\partial x_j$ is a non-linear function in space.

1.1 On various definitions of advection velocity

Advection velocities defined from space-time correlation coefficients are used widely, but can be defined in at least three different ways, as shown in Fig. 1. Suppose that the direction of the spatial separation is fixed. For a given spatial separation Δx_f there is a temporal separation τ_m which maximises the value of the correlation coefficient, and it is possible to define the advection velocity $u_{c1} = \Delta x_f/\tau_m$. Alternatively, we can follow Wills (1964) and seek the spatial displacement Δx_m which maximises the correlation coefficient for a given temporal delay τ_f and define the advection velocity as $u_{c2} = \Delta x_m/\tau_f$. An alternative and more practical definition of this advection velocity was given by Fisher & Davies (1964) who realised that Δx_m was given by the point where the envelope of the temporal correlation coefficients curves for all fixed spatial separations touched the temporal correlation coefficient curve for a given spatial separation (see Fig. 2). The equivalence is clear, since for a fixed temporal separation the envelope is nothing but the maximum value obtained when varying the spatial separation. From Fig. 1 it is seen that $u_{c1} > u_{c2}$. In fact, an interesting estimate due to Comte-Bellot & Corrsin (1971) suggests that for homogeneous, isotropic turbulence

$$\frac{u_{c1}}{u_{c2}} = 1 + \frac{\overline{u^2}}{U^2}\ , \tag{1}$$

where $\overline{u^2}^{1/2}$ is the rms of the fluctuating velocity in the direction of Δx_f and U is the mean velocity, which is assumed to be in the same direction. If this formula is valid the difference between the various definitions of the advection velocity can be expected to be small unless the turbulence intensity is large. For the near wall domain of a channel, the data in Wilhelm *et al.* (1998) shows that u_{c2} agrees with a definition based on tracing high-shear layer structures, which indicates that a definition based on the space-time correlation coefficient can be consistent with the suggestions of Zaman & Hussain (1981).

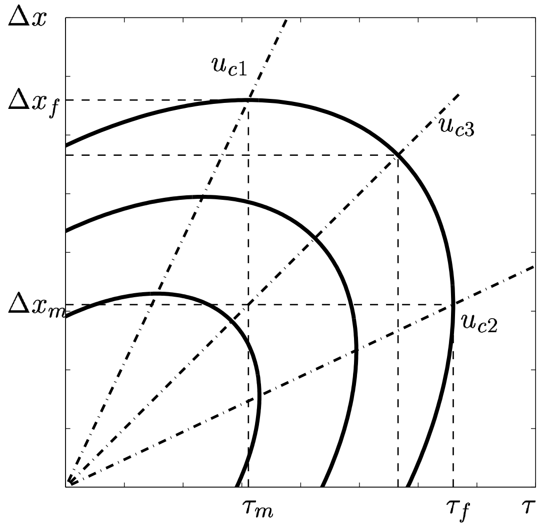

Figure 1: The solid curves show typical isocontours of the space-time correlation coefficient. Δx is the spatial separation in a certain direction, and τ is the temporal separation. The dash-dotted lines correspond to the three different definitions of advection velocity.

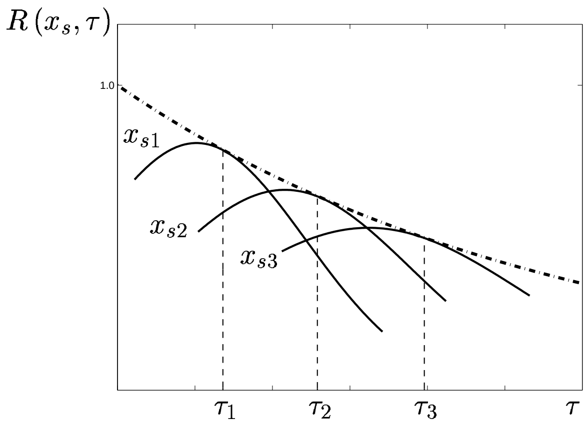

Figure 2: Space-time correlation coefficient curves $\rho(x_s, \tau)$, for three different spatial separations along one direction. The dot-dashed curve shows the envelope of the curves for all spatial separations. Consequently, τ_1 is the temporal separation then x_{s1} is the spatial separation along the chosen direction that maximises the space-time correlation coefficient. Analogous properties hold for x_{s2} and x_{s3}.

Another possible definition of an advection velocity from space-time correlation coefficients was suggested by Goldschmidt *et al.* (1981) who chose as advection velocity the quotient $u_{c3} \equiv x/\tau$ such that $\partial\rho\left(x,\tau\right)/\partial\tau = -\partial\rho\left(x,\tau\right)/\partial x$, which is well-defined if the isocontours of the space-rime correlation coefficients ressemble those in Fig. 1. The reason for this choice was to relate temporal and spatial derivatives just like in Taylor's hypothesis. An issue, which has not attracted much attention in the literature, is that all advection velocities defined thus far only contain the velocity component parallel with the fixed direction of the point separation. One of the conclusions that can be drawn from the measurements in Favre *et al.* (1957, 1958) is that in many complicated flows, such as boundary layers and jets, there is no *a priori* reason to assume that the advection velocity will be in any one particular direction. To remedy this possible shortcoming, the definition of u_{c2} can readily be modified to $\mathbf{u_{c2}} = \Delta\mathbf{x_m}/\tau_f$, where $\Delta\mathbf{x_m}$ is the spatial separation which maximises the space-time correlation coefficient for the temporal separation τ_f, taking all three dimensions into account. On the other hand, the situation for u_{c1} and u_{c3} is more complicated. For u_{c1}, the problem is that there is likely to be many points where $\partial\rho\left(x,\tau\right)/\partial\tau = 0$, and for u_{c3} it is difficult to make a geometrically meaningful definition considering the vast number of possible shapes of the four-dimensional isocontours. *For this reason* $\mathbf{u}_{c2}$ *will be the definition primarily used in this paper, and unless otherwise stated this definition will be used henceforth.*

It must be emphasised that in general the advection velocity depends on the size of the separation in time or space, or equivalently it can be seen to depend on the frequency (Fisher & Davies, 1964; Lumley, 1965). The effect is most pronounced for shear flows though it has also been observed for the high-frequencies in uniform flows (Cenedese *et al.*, 1991). The dependence of the advection velocity for shear flows on the size of the separation in time or space is not surprising, since the advection velocity, for all the definitions discussed, is really an average of the advection velocities at all points along a suitable path linking the two points at which the correlation coefficient is taken. For the near-wall region of a boundary-layer where the outgoing motion is more correlated than the ingoing, it can be expected that the advection velocity increases with the point separation, since the structures move into regions with increasing mean velocity. For u_{c2} and u_{c3}, but not for u_{c1}, this picture is consistent with the measurements in Favre *et al.* (1967). Noteworthy is that both Michelet *et al.* (1997) and Pinton & Labbé (1994) found that good agreement may be obtained for Taylor's hypothesis if the average velocity over the paths is chosen as the advection velocity. Consequently, it seems that when discussing the advection velocity at a point P the most relevant quantity is the advection velocity for an infinitesimal separation in time or in space.

Advection velocities has long been considered as an average of the advection velocities over trajectories. For example, Grant (1958) suggested that the line of maximum correlation coefficient in time and space is "something like and average streamline for particles within jets". A priori it is not clear what average to take though, since there may be a weight making certain trajectories more important than others. One interesting suggestion was given by Sternberg (1962, 1967) who assumed that vorticity is advected by the mean velocity and based on this he concluded that the advection velocity of a velocity structure is given by an average of the advection velocities of the vorticity generating it. This theory is able to give a qualitative explanation of the advection velocity distributions in the near-wall regions in a boundary-layer or a channel, but the effect on more general flows has not been calculated. In fact, the question of how to use say the Navier–Stokes equations to predict the effect of mean shear on the advection velocity has been largely ignored, as most research effort in the field seems to have been focused on the question of how to define the advection velocity and on the relationship between the various definitions. In particular, no explanation has been given as to why the advection velocity appear to coincide with the local mean velocity in case of homogeneous shear flow and in the core region of channel flow, but not for other shear flows.

In this paper we will formulate sufficient conditions which ensure that the advection velocity $\mathbf{u}_{c2}$, as the separation approaches zero, almost certainly[3] coincides with the local mean velocity (or equivalently that for a fixed temporal separation the spatial displacement of maximum space-time correlation coefficient $\Delta\mathbf{x}_m$ coincides with the location of the trajectory of the mean flow at time τ.)

2 Theoretical preliminaries

We start from the instantaneous Navier–Stokes equations

$$\frac{\partial \widetilde{u_i}}{\partial t} + \widetilde{u_j}\frac{\partial \widetilde{u_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial x_i} + \nu\frac{\partial^2 \widetilde{u_i}}{\partial x_j^2}, \qquad (2)$$

where $\widetilde{p}$ is the pressure, ρ the density and $\widetilde{u_i}$ denotes the velocity in the x_i direction of a cartesian co-ordinate system. Furthermore, it is assumed that the velocity field is solenoidal, i.e. that it satisfies the continuity equation:

$$\frac{\partial \widetilde{u_i}}{\partial x_i} = 0 \ . \qquad (3)$$

In addition, assume that there exists a stationary, ergodic [4], mean flow. (The ergodicity need not be imposed in all arguments below.) Furthermore,

[3]The "almost certainly" refers to the fact that we will only prove that under certain condition $\Delta\mathbf{x}_m$ is a *local* maximum of the space-time correlation coefficient for the temporal separation τ.

[4]defined to mean that temporal averages coincide with ensemble ones. In this paper we assume that there exists a probability measure over the ensemble of all possible flow states. For a more detailed discussion of this issue see e.g. (Frisch, 1995)

we will without additional comments assume that all moments used exist and that the correlations and the correlation coefficients are C^3 functions of space and time.

Perform a Reynolds decomposition of the velocity and pressure fields

$$\widetilde{u}_i(\mathbf{x},t) = U_i(\mathbf{x}) + u_i(\mathbf{x},t) \tag{4}$$
$$\widetilde{p}(\mathbf{x},t) = P(\mathbf{x}) + p(\mathbf{x},t) \tag{5}$$

into mean and fluctuating parts. It is evident that the continuity equation holds for each of the components U_i and u_i. In addition, the Navier–Stokes equations can be split into a mean and fluctuating part. Here, we will only be concerned with the latter:

$$\frac{\partial u_i}{\partial t} + U_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial U_i}{\partial x_j}u_j = \overline{u_j\frac{\partial u_i}{\partial x_j}} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2}\ , \tag{6}$$

where the overbar denotes an ensemble average. A central part of Taylor's hypothesis is that the advective terms in the Navier-Stokes equations dominate the dynamics on the scales studied. Hence, the terms on the right hand side: the Reynolds stresses, the pressure gradients and the viscous terms must be comparatively small. In the core region of channel flow the numerical calculations presented in Piomelli *et al.* (1989) supports this assumption. In case of high Reynolds number flows with uniform mean flow, estimates have been made by Lin (1953); Heskestad (1965) which show that the acceleration terms are comparatively small in a mean sense, though for the analysis of Taylor's hypothesis in these and most other studies, including the present one, to be strictly valid they must be uniformly small in time, space and ensemble. This condition is not likely to be met in real flows, and therefore we will investigate the sensitivity of our results to perturbations, which can be thought of as non-negligible values of one or several acceleration terms. However, for most of the article we will assume that the acceleration terms indeed are negligible, and therefore we will analyse the properties of the system of equations

$$\frac{\partial u_i}{\partial t} + (U_j + u_j)\frac{\partial u_i}{\partial x_j} + \frac{\partial U_i}{\partial x_j}u_j = 0\ . \tag{7}$$

The system of equations (7) is hyperbolic, and such systems are commonly analysed by the method of characteristics. A characteristic is nothing but a particle trajectory, and the method of characteristics is thus Lagrangian particle dynamics. Hence if we let $\mathbf{v}(t|\mathbf{x_0},t_0)$ and $\mathbf{X}(t|\mathbf{x_0},t_0)$ denote the fluctuating velocity and the coordinate at time t of a liquid particle, which was at $\mathbf{x_0}$ at time t_0, we have that (7) is equivalent to the equations (Note that both $\mathbf{v}$ and $\mathbf{X}$ depend on $\mathbf{u}(\mathbf{x_0},t)$, but we do not indicate this when we do not use it, in order to simplify the notation somewhat.)

$$\frac{dX_i}{dt} = U_i\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right) + v_i\left(t|\mathbf{x_0},t_0\right) \tag{8}$$
$$\frac{dv_i}{dt} = -\frac{\partial U_i}{\partial x_j}\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right) v_j\left(t|\mathbf{x_0},t_0\right) \ . \tag{9}$$

There is a technical difficulty with the method of characteristics that calls for a comment. There is a possibility that two characteristics cross which requires that the velocity at that point attains two different values, which is clearly unphysical. For Burger's equation (the 1-D version of (7)) this problem is typically solved by adding viscosity to the problem. It is very likely that adding viscosity will remove this difficulty in the present case as well, and thus when this difficulty arises, viscosity, and hence the acceleration terms in general cannot be negligible, in which case our model is inapplicable. We will henceforth ignore this difficulty, by saying that we assume that the velocity field is uniquely extendible in the space-time region under consideration (except on sets of probability zero in ensemble).

We may gain some insight into the system of equations (8)-(9) just by eliminating $\mathbf{v}\left(t|\mathbf{x_0},t_0\right)$ from the system:

$$\frac{d^2X_i}{dt^2} = \frac{\partial U_i}{\partial x_j}\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right)\frac{dX_j}{dt} + \frac{dv_i}{dt} \tag{10}$$
$$= \frac{\partial U_i}{\partial x_j}\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right)\frac{dX_j}{dt} - \frac{\partial U_i}{\partial x_j}\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right) v_j\left(t|\mathbf{x_0},t_0\right) \tag{11}$$
$$= \frac{\partial U_i}{\partial x_j}\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right) U_j\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right) \ . \tag{12}$$

This system has the initial conditions

$$\mathbf{X}\left(t_0|\mathbf{x_0},t_0\right) = \mathbf{x_0}, \quad \frac{d\mathbf{X}}{dt}\left(t_0|\mathbf{x_0},t_0\right) = \mathbf{U}\left(\mathbf{x_0}\right) + \mathbf{u_0} \ , \tag{13}$$

where we have used $\mathbf{u_0}$ to denote $\mathbf{u}\left(\mathbf{x_0},t_0\right)$.

The theory of systems of ordinary differential equations tells us that the properties of (12) are far more simple if $\frac{\partial U_i}{\partial x_j}\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right) U_j\left(\mathbf{X}\left(t|\mathbf{x_0},t_0\right)\right)$ is a linear function of $\mathbf{X}\left(t|\mathbf{x_0},t_0\right)$ than in the general case. This distinction is apparently new and it separates flow cases for which linearity holds, which include flow with uniform mean flow, the homogeneous shear flow and channel flow, from jets, wakes and boundary layers, which are examples of flows for which the right hand side of (12) is a non-linear function of $\mathbf{X}\left(t|\mathbf{x_0},t_0\right)$. Hence, linearity of this quantity separates rather accurately flows for which lines of maximum space-time correlation coefficients coincide with the mean streamlines and flows with more complicated behaviour. The only exception from this classification known to the author is the near-wall region of channel flow where the advection velocity is distinct from the local mean velocity, yet the right hand side of (12) vanishes. However, one must recall that we have assumed acceleration terms to be negligible, which is clearly not the case in the near-wall region of channel flow.

In the linear case we can perform a much more detailed mathematical analysis of our equations than in the general case, and for that reason we will devote the next section to the study of the linear case, and in section 4 we will discuss some properties of non-linear flows. Throughout the article we will when discussing linear (non-linear) flows without any additional modifiers refer to flows such that the acceleration terms are negligible and the right-hand side in (12) is linear (non-linear).

One striking difference between the linear and the non-linear case concerns the motion of the centroid of the ensemble of trajectories as the temporal separation increases, *i.e.* the properties of $\overline{\mathbf{X}(t|\mathbf{x_0}, t_0)}$. In the next section we will see that in the linear case $\overline{\mathbf{X}(t|\mathbf{x_0}, t_0)}$ coincides with the mean flow trajectory $\mathbf{X_m}(t|\mathbf{x_0}, t_0)$, which is given by the solution to (12)-(13) with $\mathbf{u_0} = 0$. By contrast, these quantities are normally not identical in the non-linear case as will be seen in section 4. This property, that the operations of taking average and propagating along trajectories do not commute, makes it unlikely that the locus of the space-time correlation coefficient maxima, which contain information carried by several trajectories, should coincide with the mean streamlines in the non-linear case.

For the linear case, however, the situation is more hopeful since, as will be seen below in Section 3.1,

$$\Delta\mathbf{X}(t|\mathbf{x_0}, \mathbf{u_0}, t_0) \equiv \mathbf{X}(t|\mathbf{x_0}, t_0) - \overline{\mathbf{X}(t|\mathbf{0}, t_0)} \tag{14}$$

is linear with respect to the vector $\left[\mathbf{x_0}^T \mathbf{u_0}^T\right]^T$, i.e. there are matrices A and B which are functions of t only, such that

$$\Delta\mathbf{X}(t|\mathbf{x_0}, \mathbf{u_0}, t_0) = A(t)\,\mathbf{x_0}^T + B(t)\,\mathbf{u_0}^T. \tag{15}$$

We will now give sufficient conditions which imply that the mean streamlines and the locus of the space-time correlation coefficient maxima coincide for uniform flow. To this end let us make the following definition

Definition 1 *The probability distribution function of the simultaneous occurrence of* $\mathbf{u_1}$ *at* $(\mathbf{0}, t)$ *and of* $\mathbf{u_2}$ *at* $(\mathbf{x}, t)$*, denoted by* $P(\mathbf{u_1}(\mathbf{0}, t), \mathbf{u_2}(\mathbf{x}, t))$*, is said to have parity symmetry around the origin if*

$$P(\mathbf{u_1}(\mathbf{0}, t), \mathbf{u_2}(\mathbf{x}, t)) = P(-\mathbf{u_1}(\mathbf{0}, t), -\mathbf{u_2}(-\mathbf{x}, t)) \ . \tag{16}$$

Clearly, parity symmetry can be defined around any point. A typical case where parity symmetry does not hold is helical turbulence (Moffatt & Tsinober, 1992). In most flows with uniform mean velocity it seems unlikely that the turbulence would be helical, but for example in the homogeneous shear flow case in Champagne *et al.* (1970); Harris *et al.* (1977) the situation is less clear.

We are now ready to state the following theorem

Theorem 1 *Suppose that* $\Delta\mathbf{X}\left(t|\mathbf{x_0},\mathbf{u_0},t_0\right)$ *is linear with respect to the vector* $\left[\mathbf{x_0}^T\mathbf{u_0}^T\right]^T$ *and that* $\mathbf{v}\left(t|\mathbf{x_0},\mathbf{u_0},t_0\right)$ *is independent of* $\mathbf{x_0}$ *and that its dependence with respect to* $\mathbf{u_0}$ *is linear. In addition, suppose that the fluctuating velocity field is stationary, homogeneous and uniquely extendible for* $t \leq t \leq t_1$*. Furthermore, suppose that the probability distribution* P *has parity symmetry around* $\mathbf{x_0}$ *for* $t \leq t \leq t_1$*, then for* $t \leq t \leq t_1$*, and any* i,j*, the space-time correlation coefficient* $\rho_{ij}\left(\mathbf{x},t\right)$ *has antipodal symmetry around* $\overline{\mathbf{X}\left(t|\mathbf{x_0},t_0\right)}$*, i.e. that for any vector* $\mathbf{h}$

$$\rho_{ij}\left(\overline{\mathbf{X}\left(t|\mathbf{x_0},t_0\right)}+\mathbf{h},t\right)=\rho_{ij}\left(\overline{\mathbf{X}\left(t|\mathbf{x_0},t_0\right)}-\mathbf{h},t\right). \tag{17}$$

Remark 1: A consequence of the theorem is that unless $\overline{\mathbf{X}\left(t|\mathbf{x_0},t_0\right)}$ is the maximum of the space-time correlation coefficient functions ρ_{ij} for a given temporal separation, the maximum is attained at at least two different points.

Remark 2: The conditions on $\Delta\mathbf{X}\left(t|\mathbf{x_0},\mathbf{u_0},t_0\right)$ are satisfied for all linear flows for which the acceleration terms can be neglected. The conditions on $\mathbf{v}\left(t|\mathbf{x_0},\mathbf{u_0},t_0\right)$ are met for homogeneous shear flows where the acceleration terms can be neglected, but not for other linear flows such as channel flow.

Remark 3: Since the turbulence is assumed to be homogeneous the theorem applies both to correlations R_{ij} and correlation coefficients ρ_{ij}.

Theorem 1 excludes some possibilities, but we would like to establish that $\overline{\mathbf{X}\left(t|\mathbf{x_0},t_0\right)}$ is a local maximum of the space-time correlation coefficient with temporal separation $t-t_0$. If the maximum is, in fact, a global maximum, such a result would, by virtue of the coincidence of $\overline{\mathbf{X}\left(t|\mathbf{x_0},t_0\right)}$ and $\mathbf{X_m}\left(t|\mathbf{x_0},t_0\right)$ for linear flows, shown in the next section, imply that the advection velocity defined using correlation coefficients coincides with the local mean velocity. However, to achieve this goal, we must study more detailed concepts. Suppose that we know the value of the velocity component $u_i\left(\mathbf{x_0},t\right) \equiv u_{i0}$, then what is the expectation value (over the ensemble) of the velocity component $u_i\left(\mathbf{x},t\right)$? This quantity will be denoted by

$$\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{i0},t\right)}. \tag{18}$$

We know that it is u_{i0} if $\mathbf{x}=\mathbf{x_0}$ and that it approaches zero as $|\mathbf{x}-\mathbf{x_0}|$ exceeds the integral scale. A quantity which will be of great importance to us is

$$g_{\mathbf{u_0},i}\left(\mathbf{x}-\mathbf{x_0}\right)=\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{i0},t\right)}P\left(\mathbf{u_0}\right)\ , \tag{19}$$

where $P(\mathbf{u_0})$ denotes the probability of having the velocity vector $\mathbf{u_0}$ at the point $\mathbf{x_0}$. Here, by virtue of the homogeneity, this probability is independent of $\mathbf{x_0}$ and $g_{\mathbf{u_0},i}$ is a function of $\mathbf{x}-\mathbf{x_0}$ only.

Definition 2 *We say that $g_{\mathbf{u_0},i}$ is predominantly concave if*

$$\int_{\mathbf{R}^3} u_{i0}\,\frac{\partial^2}{\partial \mathbf{l}^2}g_{\mathbf{u_0},i}\bigg|_{\mathbf{x}=\mathbf{x_0}+c\mathbf{u_0}} d\mathbf{u_0}<0, \tag{20}$$

where there is no summation over i, $\mathbf{l}$ is any unit vector and c is a constant.

Before we treat the question of when we can expect $g_{\mathbf{u_0},i}$ to be predominantly concave, we will discuss its relevance to the topic at hand. In fact, we are now ready to state a theorem, which ensures that the trajectory along the mean streamlines is a local maximum of the space-time correlation coefficient.

Theorem 2 *Suppose that all the conditions in Theorem 1 hold and that in addition $\mathbf{v}(t|\mathbf{x_0},\mathbf{u_0},t_0)$ is constant and equal to $\mathbf{u_0}$, and that $\Delta\mathbf{X}(t|\mathbf{x_0},\mathbf{u_0},t_0)$ satisfies (15) with A non-singular for all time, and $B=f(t)\,Id$, where f is a scalar function of time and Id is the identity matrix. Suppose furthermore that $g_{\mathbf{u_0},i}$ is predominantly concave, then $\overline{\mathbf{X}(t|\mathbf{x_0},t_0)}$ is a local maximum of the space-time correlation coefficient functions ρ_{ii} (no summation over i) for a given temporal separation.*

Remark 1: The conditions on $\Delta\mathbf{X}(t|\mathbf{x_0},\mathbf{u_0},t_0)$ are met for flows with uniform mean flow, for channel flow as well as for the homogeneous shear flow studied by Champagne *et al.* (1970); Harris *et al.* (1977) (but not for general linear flow) provided that the acceleration terms can be neglected. The conditions for $\mathbf{v}(t|\mathbf{x_0},\mathbf{u_0},t_0)$ are met for flows with uniform mean flow in case the acceleration terms are negligible. After the proof of the theorem in Subsection 3.3 we will discuss possible extensions to channel flow and to the homogeneous shear flow studied by Champagne *et al.* (1970); Harris *et al.* (1977).

Remark 2: Suppose we retain all the conditions in the theorem, except that we assume that there exists a unit vector $\mathbf{l}$ for which the left hand side in (20) is positive then it can be seen from the proof of the theorem that $\overline{\mathbf{X}(t|\mathbf{x_0},t_0)}$ is *not* a local maximum of the space-time correlation coefficient functions ρ_{ii}. (If there exists a unit vector $\mathbf{l}$ for which the left hand side in (20) is zero the situation is unresolved.) This shows the crucial importance of the concept of predominant concavity to the problem at hand.

Remark 3: By virtue of the assumed homogeneity the theorem holds equally well for correlations as for correlation coefficients.

We will now conclude this section with a subsection analysing the property of predominant concavity.

2.1 Analysis of the concept of predominant concavity

In general, it appears to be a difficult problem to determine whether a certain flow has the property of $g_{\mathbf{u_0},i}$ being predominantly concave. Accordingly, we will only show that this almost certainly is the case if certain conditions apply. Subsequently, we will discuss the possible dangers if these conditions are not satisfied. Note that throughout this section the summation convention does not apply for any i indices.

Assumption 1: Let us assume that $\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{i0},t\right)}/u_{i0}$ is independent of the value of u_{i0}, and thus given by some function $\mathbf{v}\left(\mathbf{x}-\mathbf{x_0}\right)$ of space. In that case we have that

$$\begin{aligned} R_{ii}\left(\mathbf{x}-\mathbf{x_0}\right) &= \int_{-\infty}^{\infty} u_{i0}\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{i0},t\right)}P\left(u_{i0}\right)du_{i0} && (21)\\ &= \overline{u_{i0}^2}\mathbf{v}\left(\mathbf{x}-\mathbf{x_0}\right)\ , && (22)\end{aligned}$$

where $R_{ii}\left(\mathbf{x}\right)$ is the spatial autocorrelation of the u_i velocity with separation vector $\mathbf{x}-\mathbf{x_0}$. Consequently,

$$\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{i0},t\right)} = \frac{u_{i0}R_{ii}\left(\mathbf{x}-\mathbf{x_0}\right)}{\overline{u_{i0}^2}}. \qquad (23)$$

With this assumption, the integral in (20) becomes

$$\int_{\mathbf{R}^3} \frac{u_{i0}^2}{\overline{u_{i0}^2}} \frac{\partial^2}{\partial x_j \partial x_k} R_{ii}\left(\mathbf{x}-\mathbf{x_0}\right)\bigg|_{\mathbf{x}-\mathbf{x_0}=c\mathbf{u_0}} l_j l_k P\left(\mathbf{u_0}\right)d\mathbf{u_0}, \qquad (24)$$

where the l_j's are components of $\mathbf{l}$. Let us change the integration variable to $\mathbf{y}\equiv c\mathbf{u_0}$, for which the integral becomes

$$\begin{aligned} & \frac{1}{|c|^3\,\overline{u_{i0}^2}}\int_{\mathbf{R}^3} y_i^2\frac{\partial^2}{\partial y_j\partial y_k}R_{ii}\left(\mathbf{y}\right)l_j l_k P\left(\frac{\mathbf{y}}{c}\right)d\mathbf{y} && (25)\\ \equiv\ & -\frac{1}{|c|^3\,\overline{u_{i0}^2}}\int_{\mathbf{R}^3} f\left(\mathbf{y}\right)\overline{h\left(\mathbf{y}\right)}d\mathbf{y} && (26)\end{aligned}$$

where

$$\begin{aligned} f\left(\mathbf{y}\right) &= -iy_i\left(-i\right)^2\frac{\partial^2}{\partial y_j\partial y_k}R_{ii}\left(\mathbf{y}\right)l_j l_k && (27)\\ h\left(\mathbf{y}\right) &= -iy_i P\left(\frac{\mathbf{y}}{c}\right)\ . && (28)\end{aligned}$$

If we use Plancherel's (or Parseval's) formula the integral in (20) becomes

$$-\frac{1}{|c|^3\,\overline{u_{i0}^2}}\int_{\mathbf{R}^3}\hat{f}\left(\mathbf{k}\right)\overline{\hat{h}\left(\mathbf{k}\right)}d\mathbf{k}, \qquad (29)$$

where $\hat{f}$ and $\hat{h}$ are the Fourier transforms of f and h respectively, which are given by

$$\hat{f}(\mathbf{k}) = \frac{\partial}{\partial k_i}\left((\mathbf{k}\cdot\mathbf{l})^2 \widehat{R_{ii}}(\mathbf{k})\right) \tag{30}$$

$$\hat{h}(\mathbf{k}) = c\frac{\partial}{\partial k_i}\widehat{P}(c\mathbf{k}). \tag{31}$$

Here $\widehat{R_{ii}}$ is the three-dimensional spectrum of the u_i velocity component, and $\widehat{P}$ is the characteristic functional of the velocity. Our next assumption restricts the choice of these functions.

Assumption 2: Assume that $\widehat{R_{ii}}$ and $\widehat{P}$ are isotropic, i.e. that they depend only on $|\mathbf{k}|$, and assume also that they are real, bounded, positive and decreasing as functions of $|\mathbf{k}|$. We furthermore assume that the logarithm of $\widehat{R_{ii}}$ is a concave function of $\ln|\mathbf{k}|$ and that $\widehat{R_{ii}}$ decays exponentially for sufficiently large values of $|\mathbf{k}|$.

If we let $\mathbf{i}$ denote the unit vector in the x_i-direction, and use this assumption it can be seen that

$$\hat{f}(\mathbf{k}) = \frac{\mathbf{k}\cdot\mathbf{i}}{|\mathbf{k}|}(\mathbf{k}\cdot\mathbf{l})^2\widehat{R_{ii}}'(|\mathbf{k}|) + 2(\mathbf{k}\cdot\mathbf{l})(\mathbf{l}\cdot\mathbf{i})\widehat{R_{ii}}(|\mathbf{k}|) \tag{32}$$

$$\hat{h}(\mathbf{k}) = c^2\frac{\mathbf{k}\cdot\mathbf{i}}{|\mathbf{k}|}\widehat{P}'(|c\mathbf{k}|), \tag{33}$$

where a prime denotes differentiation. Consequently,

$$\begin{aligned}\hat{f}(\mathbf{k})\overline{\hat{h}(\mathbf{k})} &= c^2\frac{(\mathbf{k}\cdot\mathbf{i})^2(\mathbf{k}\cdot\mathbf{l})^2}{|\mathbf{k}|^2}\widehat{R_{ii}}'(|\mathbf{k}|)\widehat{P}'(|c\mathbf{k}|)\\ &+ c^2\frac{(\mathbf{k}\cdot\mathbf{l})(\mathbf{l}\cdot\mathbf{i})(\mathbf{i}\cdot\mathbf{k})}{|\mathbf{k}|}\widehat{R_{ii}}(|\mathbf{k}|)\widehat{P}'(|c\mathbf{k}|)\end{aligned} \tag{34}$$

With assumption 2 it is clear that the first term on the right hand side of this expression is positive. The sign of the second term is typically negative, but can vary with the choice of $\mathbf{l}$ and $\mathbf{i}$. However, if we combine this with (29) we find that it is solely the second term which threatens the validity of (20). Hence, it is readily seen that the critical case is if $\mathbf{l}$ and $\mathbf{i}$ coincide, which we will assume henceforth.

Let us now introduce a polar coordinate system with the z-axis directed along $\mathbf{i}$. Hence, if (34) is integrated over the entire space it becomes

$$\begin{aligned}\int_{\mathbf{R}^3}\hat{f}(\mathbf{k})\overline{\hat{h}(\mathbf{k})}d\mathbf{k} &= 2\pi c^2\int_0^\infty\Big[|\mathbf{k}|\widehat{R_{ii}}'(|\mathbf{k}|)\int_0^\pi\cos^4\theta\sin\theta d\theta\\ &+ 2\widehat{R_{ii}}(|\mathbf{k}|)\int_0^\pi\cos^2\theta\sin\theta d\theta\Big]|\mathbf{k}|^3\widehat{P}'(|c\mathbf{k}|)\,d|\mathbf{k}| \qquad (35)\\ &= \frac{4}{5}\pi c^2\int_0^\infty\left[|\mathbf{k}|^{10/3}\widehat{R_{ii}}'(|\mathbf{k}|) + \frac{10}{3}|\mathbf{k}|^{7/3}\widehat{R_{ii}}(|\mathbf{k}|)\right]\end{aligned}$$

$$\begin{aligned}
&\times\ |\mathbf{k}|^{2/3}\,\widehat{P}'\left(|c\mathbf{k}|\right)d\,|\mathbf{k}| &(36)\\
&= \frac{4}{5}\pi c^2\int_0^\infty \left[|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)\right]'|\mathbf{k}|^{2/3}\,\widehat{P}'\left(|c\mathbf{k}|\right)d\,|\mathbf{k}|\,. &(37)
\end{aligned}$$

To satisfy predominant concavity we must have that this integral is positive due to (29).

According to assumption 2 we have that $|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)$ is zero at zero as well as at infinity. We thus have that

$$\int_0^\infty \left[|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)\right]' d\,|\mathbf{k}| = 0. \tag{38}$$

In addition, we have that $\left[|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)\right]' < 0$ if and only if $|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)$ decays. Due to the assumed concavity of the logarithm of $\widehat{R_{ii}}\left(|\mathbf{k}|\right)$ as a function of $\ln|\mathbf{k}|$, we find that there exists a $|\mathbf{k}|_{crit}$ such that

$$\begin{aligned}
\left[|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)\right]' &> 0,\ \text{if}\ |\mathbf{k}| < |\mathbf{k}|_{crit} &(39)\\
\left[|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)\right]' &< 0,\ \text{if}\ |\mathbf{k}| > |\mathbf{k}|_{crit}\ . &(40)
\end{aligned}$$

We may estimate the size of $|\mathbf{k}|_{crit}$ by noting that in the inertial range of a turbulent flow the three-dimensional autocorrelation spectrum decays as $|\mathbf{k}|^{-11/3}$ (see e.g. (Frisch, 1995)), which shows that

$$|\mathbf{k}|_{crit} \sim \frac{1}{\mathcal{L}} \tag{41}$$

where $\mathcal{L}$ denotes the integral scale. Since $\widehat{P}'\left(|c\mathbf{k}|\right) < 0$, we have thus found that predominant concavity holds if the average of the absolute value of $|\mathbf{k}|^{2/3}\,\widehat{P}'\left(|c\mathbf{k}|\right)$, weighted with the size of $\left[|\mathbf{k}|^{10/3}\,\widehat{R_{ii}}\left(|\mathbf{k}|\right)\right]'$, is less for $|\mathbf{k}| < \frac{1}{\mathcal{L}}$ than for $|\mathbf{k}| > \frac{1}{\mathcal{L}}$.

Assumption 3: Suppose that $\widehat{P}$ is Gaussian.

With this assumption we find that the absolute value of $|\mathbf{k}|^{2/3}\,\widehat{P}'\left(|c\mathbf{k}|\right)$ attains its maximum when $|\mathbf{k}| = \sqrt{\frac{5}{3}}\frac{1}{c\overline{|\mathbf{u}|^2}^{1/2}}$ where $\overline{|\mathbf{u}|^2}^{1/2}$ is the rms value of the size of the fluctuating velocity. Hence we can be almost certain that predominant concavity holds whenever

$$c \ll \sqrt{\frac{5}{3}}\frac{\mathcal{L}}{\overline{|\mathbf{u}|^2}^{1/2}} \tag{42}$$

Below we will see that c is typically the time it takes for a particle travelling with the mean flow to travel between the two points which the correlation is taken at. This indicates that unless the separation between the points is comparable to the integral scale divided by the turbulent intensity

$(\overline{|\mathbf{u}|^2}^{1/2}/U)$ predominant concavity should be a consequence of the three assumptions above. However, at such a large separation, any residual correlation would clearly be almost impossible to measure.

Before we conclude this section we should discuss briefly the possible effects of violations of the three assumptions above. Let us begin with easiest case: Assumption 3. If the characteristic functional is not Gaussian, but satisfies the requirements in Assumption 2, then we expect the results above to hold, except that the, fairly irrelevant, coefficient $\sqrt{\frac{5}{3}}$ changes.

If assumption 1 is violated it is very difficult to analyse predominant concavity. However, it is reasonable to believe that the correlation length for a large value of u_{i0} is shorter than that for a small value of u_{i0}. This is most likely going to reduce the impact of the integral scale effects, and thus the situation will be improved rather than made worse.

The most critical assumption appears to be assumption 2. The assumptions that the logarithm of $\widehat{R_{ii}}$ is concave function of $\ln|\mathbf{k}|$ and that $\widehat{R_{ii}}$ decays exponentially for sufficiently large separations are valid for most turbulent flows. The assumptions on $\widehat{P}$ are also met for characteristic functionals of near Gaussian behaviour. The problem occurs for $\widehat{R_{ii}}$ for small values of $|\mathbf{k}|$ since the spectrum should be zero at zero (due to the absence of energy at the zero wave number). Hence, $\widehat{R_{ii}}$ must be increasing for small values of $|\mathbf{k}|$. This means that for such small values of $|\mathbf{k}|$ the first term in (34) is negative. This makes it less obvious that $\mathbf{l} = \mathbf{i}$, yet this still seems very likely. If this remains so, there should be no other difficulties with this deviation from the assumption. The remaining assumption is the isotropy of $\widehat{P}$ and $\widehat{R_{ii}}$, and if this assumption fails severely it may well jeopardize predominant concavity. The exact sensitivity to anisotropy is in general difficult to estimate, but if the angular dependence of $\widehat{P}$ and $\widehat{R_{ii}}$ are known the procedure presented above can perhaps be used to determine whether predominant concavity holds.

3 Flows with uniform mean flow, channel flow and homogeneous shear flow

In this section we will study the case when $\frac{\partial U_i}{\partial x_j}(\mathbf{X}(t|\mathbf{x_0},t_0))\,U_j(\mathbf{X}(t|\mathbf{x_0},t_0))$ is a linear function of $\mathbf{X}(t|\mathbf{x_0},t_0)$. In the first subsection we will solve (7), i.e. the Navier–Stokes equations for the velocity fluctuations without the acceleration terms. Some of the implications of this analysis for the application of Taylor's hypothesis will be described briefly in the following subsection. The third subsection will largely be devoted to the proofs of Theorems 1 and 2, but a rough estimate of the size of the moving-frame correlation coefficient will also be presented. Finally we will study the sensitivity of the model to linear disturbances in space and time.

3.1 Calculation of the trajectories and the velocity fluctuations along these when the acceleration terms are neglected

First we will consider the case when $\frac{\partial U_i}{\partial x_j}(\mathbf{X}(t|\mathbf{x_0}, t_0))\, U_j(\mathbf{X}(t|\mathbf{x_0}, t_0))$ vanishes identically, which is the case for channel flow and in case of uniform mean flow, provided that the acceleration terms can be neglected. For these flows it is evident that

$$\mathbf{X}(t_0, \mathbf{x_0}, t) = \mathbf{x_0} + (\mathbf{U}(\mathbf{x_0}) + \mathbf{u_0})(t - t_0) \ . \tag{43}$$

Knowledge of the mean velocity profile will hence enable us to calculate $\mathbf{v}$ by solving the following system of ODEs

$$\frac{dv_i}{dt} = -\frac{\partial U_i}{\partial x_j}(\mathbf{x_0} + (\mathbf{U}(\mathbf{x_0}) + \mathbf{u_0})(t - t_0))\, v_j(t|\mathbf{x_0}, t_0) \ . \tag{44}$$

Due to the time dependent coefficients there will, in general, be no solution on closed form. However, in the case of uniform mean flow we trivially find that $\mathbf{v}(t) = \mathbf{u_0}$, and for channel flow the same formula holds for the v_2 and v_3 components, whereas the v_1 component is given by

$$v_1(t) = u_{01} - \left(\int_{t_0}^{t} \frac{\partial U_1}{\partial x_2}(x_{02} + u_{02}(t - t_0))\, dt\right) u_{02} \ . \tag{45}$$

The special case of constant U_i was considered by Hill (1996) in a similar fashion. However, Hill considered an advection velocity separate from both the mean and the instantaneous velocity, which was assumed to be constant both in space and in time for each advection realisation and thus it was only allowed to vary in an ensemble sense. On the other hand, for the turbulent quantities that were advected no such restrictions applied. This approach seems natural if the turbulent quantity is a passive scalar, but if it is a velocity it seems somewhat artificial, and since the success of the approach relies upon the negligibility of the right hand side of (6) it is perhaps more natural to assume that this right hand side is zero and then the need to separate between the advection velocity and the turbulent quantities vanishes.

We will now extend the previous results to the case when

$$\frac{\partial U_i}{\partial x_j}(\mathbf{X}(t|\mathbf{x_0}, t_0))\, U_j(\mathbf{X}(t|\mathbf{x_0}, t_0)) = A_{ij} X_j(t|\mathbf{x_0}, t_0) + B_i.$$

In this case we may solve (12) by rewriting it as a sixth order system of first order linear ordinary differential equations with constant coefficients, which can be solved readily. This technique will be used in subsection 3.4. However, for the most typical of linear cases, homogeneous shear flow, there is a more direct approach. In this case we assume that $U_i = U_{i0} + U_{ij}x_j$. Hence, the system (8)-(9) becomes

$$\frac{dX_i}{dt} = U_{i0} + U_{ij}X_j\,(t|\mathbf{x_0}, t_0) + v_i\,(t|\mathbf{x_0}, t_0) \tag{46}$$

$$\frac{dv_i}{dt} = -U_{ij}v_j\,(t|\mathbf{x_0}, t_0)\ . \tag{47}$$

However, (47) can be solved using standard methods (see e.g. (Coddington & Levinson, 1955))

$$v_i\,(t|\mathbf{x_0}, t_0) = \exp\left(-U_{ij}\,(t - t_0)\right)u_{j0}\ . \tag{48}$$

Since we now know the behaviour of $\mathbf{v}$ along a trajectory we can use this information to calculate these trajectories. Indeed, standard ODE theory (see e.g. (Coddington & Levinson, 1955)) tells us that the solution to (46) is given by

$$\begin{aligned} X_i\,(t|\mathbf{x_0}, t_0) &= \mathbf{x_0} + \int_{t_0}^{t} [\exp\left(U_{ij}\,(t - t_0 - \tau)\right)U_{j0} \\ &+ \exp\left(U_{ij}\,(t - t_0 - 2\tau)\right)u_{j0}]\,d\tau. \end{aligned} \tag{49}$$

To simplify this expression, let us assume that the 3×3 matrix U_{ij} is invertible, and let us denote the inverse by U^{jk}, *i.e.* we have that $U_{ij}U^{jk} = U^{ij}U_{jk} = \delta_{ik}$. Hence (49) becomes

$$\begin{aligned} X_i\,(t|\mathbf{x_0}, t_0) &= \mathbf{x_0} + [\exp(U_{ij}\,(t - t_0)) - \delta_{ij}]\,U^{jk}U_{k0} \\ &+ \frac{1}{2}[\exp\left(U_{ij}(t - t_0)\right) - \exp(-U_{ij}(t - t_0))]U^{jk}u_k(t_0, 0)\,. \end{aligned} \tag{50}$$

We should now briefly discuss the case when U_{ij} is not invertible, in which case (50) must be modified slightly. We will demonstrate this modification under the condition that U_{ij} is diagonisable. In this case there is an invertible tensor E_{ij} and numbers λ_i, at least one of which is zero, such that

$$U_{ij} = E_{ik}\lambda_k\delta_{kl}E^{lj}\ , \tag{51}$$

with summation on k even though it occurs three times. This implies that the integral causing difficulties becomes

$$\int_{t_0}^{t} \exp\left(-U_{ij}\tau\right)d\tau = E_{ik}\int_{t_0}^{t} \exp\left(-\lambda_k\tau\right)d\tau E^{kj} \tag{52}$$

This last integral is

$$\int_{t_0}^{t} \exp\left(-\lambda_k\tau\right)d\tau = \begin{cases} \dfrac{1}{\lambda}\left(1 - \exp\left(-\lambda_k\,(t - t_0)\right)\right), & \text{if } \lambda_k \neq 0 \\ t - t_0, & \text{if } \lambda_k = 0 \end{cases} \tag{53}$$

Similar techniques should work in the non-diagonisable case as well, however, if U_{ij} is nilpotent it may be easier to express the exponential function as a polynomial and perform the integration directly.

To summarize this analysis let us note that two of the main features of the results obtained in the linear case, when the acceleration terms are neglected, are that Lagrangian velocity the $\mathbf{v}$ for all times is a linear function of its initial value $\mathbf{u_0}$ and independent of $\mathbf{x_0}$ and that the difference between a trajectory passing through $\mathbf{x_0}$ at the time t_0 and the mean flow trajectory (the solution with $\mathbf{u_0} \equiv \mathbf{0}$) passing through the origin at the time t_0 is a linear function of the vector $\left[\mathbf{x_0}^T \mathbf{u_0}^T\right]^T$. This shows that the conditions of Theorem 1 are satisfied in this case. If U_i is constant in space, we have found that the Lagrangian velocity is constant and that the difference between a trajectory passing through $\mathbf{x_0}$ at the time t_0 and the mean flow trajectory (the solution with $\mathbf{u_0} \equiv \mathbf{0}$) passing through the same point at that time is just the time multiplied by $\mathbf{u_0}$, which shows that in this case the conditions of Theorem 2 are satisfied.

3.2 Implications for the use of Taylor's hypothesis

Taylor's frozen turbulence hypothesis (Taylor, 1938) is the assumption that a turbulent velocity field changes relatively slowly as it is advected past a given point. Using this hypothesis it is possible to interpret the temporal variation of the velocity at a point as the spatial variation of the velocity at a fixed time along some curve. Consequently, typical applications of Taylor's hypothesis are for the measurement of structure functions, correlations and spectra. If we limit ourselves to linear flows where the acceleration terms are of little significance, the hypothesis seems to work fairly well. There is, of course, an error due to the hypothesis, and this error can be divided into two different parts. Some of it is due to variation of the velocity along a trajectory, and the rest of the error stems from the fact that the advection velocity varies and hence all trajectories do not coincide. The error of the second kind has been considered by several authors (see e.g. (Lumley, 1965; Wyngaard & Clifford, 1977; Hill, 1996)), and in the two first of these articles correction formulae for the calculation of the spectrum were proposed. In Hill (1996) the two methods were compared and extended to give a correction for a wide range of tensors. The error due to the variation of the velocity along a trajectory has, on the other hand, received less attention. Indeed, as we have already seen the velocity along a trajectory is constant if the mean flow is constant and the acceleration terms are neglected. Whereas we have been unable to weaken the second requirement, our calculations above allow us to take into account the variation of the velocity along a trajectory due to mean shear.

Even though, as we shall see in the next section, our methods cannot be extended easily to cover more general mean velocity profiles, we can, by approximating the general mean velocity profile with a locally linear rather than a locally constant profile, expect significantly improved accuracy for Taylor's hypothesis applied to general shear flow, when we incorporate (48) and (50) into our formulae for the derivation of spatial correlations from temporal ones.

Traditionally, the Lin (1953) criterion

$$|U| \gg \frac{1}{\kappa}\left|\frac{\partial U}{\partial y}\right| \tag{54}$$

has been used to determine for which wavenumbers κ we can expect the effects of mean velocity gradient to be negligible. However, if we can take into account the local value of the mean velocity gradient it seems reasonable that this criterion should be replaced by a criterion where the mean velocity gradient is replaced by the difference between the local mean velocity gradient at a given point along a trajectory and that at the point of measurement. This suggests that Lin's criterion can be relaxed to

$$|U| \gg \frac{1}{\kappa^2}\left\|\frac{\partial^2 U}{\partial y_i \partial y_j}\right\| \ , \tag{55}$$

which in most situations is a considerable relaxation.

To illustrate how one could use (48) and (50) when invoking Taylor's hypothesis we will briefly discuss one way of estimating the spatial correlation between two points along a streamline of the mean flow, from the time series of all three velocity components obtained at a single point, which we assume to be the origin. What we would like to calculate is

$$R_{ij}\left(\mathbf{X}_m\left(t|\mathbf{0},t-\tau\right)\right) = \overline{u_i\left(\mathbf{0},t\right) u_j\left(\mathbf{X}_m\left(t|\mathbf{0},t-\tau\right),t\right)}, \tag{56}$$

where τ is the temporal separation of the two points along a streamline of the mean flow and $\mathbf{u_0}$ is the instantaneous velocity at the origin at time $t-\tau$. However, the only instantaneous information we have access to is $u_i\left(\mathbf{0},t\right)$ for all three values of i (in addition it is assumed that we have access to sufficient information about the mean flow to enable us to calculate the local mean velocity gradients).

If we neglect the acceleration terms and assume that the mean velocity profile is close to linear, our previous analysis shows that

$$u_j\left(\mathbf{X}\left(t|\mathbf{0},t-\tau\right),t\right) = F_{jk}\left(\tau\right) u_k\left(\mathbf{0},t-\tau\right) \ , \tag{57}$$

where F_{ij} is an invertible matrix valued function of the temporal separation given in (48). In this flow case we also have that $\mathbf{X}_m\left(t|\mathbf{0},t-\tau\right) = \overline{\mathbf{X}\left(t|\mathbf{0},t-\tau\right)}$, and therefore we have at each time access to the value of u_j at some point $\mathbf{X}\left(t|\mathbf{0},t-\tau\right)$ separated from the point where we would like to know u_j by $\Delta\mathbf{X}(t|\mathbf{0},\mathbf{u_0},t_0)$. The essential approximation used in Taylor's hypothesis is to let the unknown value of $u_j(\mathbf{X}_m(t|\mathbf{0},t-\tau),t)$ be approximated by the value of $u_j(\mathbf{X}(t|\mathbf{0},t-\tau),t)$, which we have access to. Of course, we may use Taylor's formula to describe the difference between the measured correlation and the real correlation as

$$\begin{aligned}
&\Delta R_{ij}(\mathbf{X}_m(t|\mathbf{0},t-\tau)) = \\
= \;& \overline{u_i(\mathbf{0},t)\frac{\partial u_j}{\partial x_k}(\mathbf{X}(t|\mathbf{0},t-\tau),t)\,\Delta X_k(t|\mathbf{0},\mathbf{u_0},t-\tau)} + \\
+ \;& \overline{\frac{1}{2}u_i(\mathbf{0},t)\frac{\partial^2 u_j}{\partial x_k\partial x_l}(\xi(t),t)\Delta X_k(t|\mathbf{0},\mathbf{u_0},t-\tau)\Delta X_l(t|\mathbf{0},\mathbf{u_0},t-\tau)},
\end{aligned} \tag{58}$$

where $\xi(t)$ is a point between $\mathbf{X}_m(t|\mathbf{0},t-\tau)$ and $\mathbf{X}(t|\mathbf{0},t-\tau)$ at each time. From (50) we have that

$$\Delta X_k(t|\mathbf{0},\mathbf{u_0},t_0) = G_{km}(t-t_0)\,u_m(\mathbf{0},t_0) \tag{59}$$

where G_{km} is a matrix valued function of the temporal separation, which is $\mathcal{O}(t)$ for small t. We can hence express the first of the error terms in (58) as

$$G_{km}(\tau)\,F^{-1}_{mn}(\tau)\,\overline{u_i(\mathbf{0},t)\frac{\partial u_j}{\partial x_k}(\mathbf{X}(t|\mathbf{0},t-\tau),t)\,u_n(\mathbf{X}(t|\mathbf{0},t-\tau),t)}\;. \tag{60}$$

In the special case of uniform mean flow we have that $G_{km}(\tau)\,F^{-1}_{mn}(\tau)$ is only a linear function of τ multiplied by the identity matrix, but for general linear flow there are higher order terms in τ with off diagonal components. If we, however, assume that τ is small enough to enable us to approximate $G_{km}(\tau)\,F^{-1}_{mn}$ by a function of time multiplied by the identity matrix the expression becomes

$$f(\tau)\,\overline{u_i(\mathbf{0},t)\frac{\partial u_j}{\partial x_k}(\mathbf{X}(t|\mathbf{0},t-\tau),t)\,u_k(\mathbf{X}(t|\mathbf{0},t-\tau),t)}\;. \tag{61}$$

In this case, we may apply the continuity equation to find that this is

$$f(\tau)\,\overline{\frac{\partial}{\partial x_k}\left(u_i(\mathbf{0},t)\,u_j(\mathbf{X}(t|\mathbf{0},t-\tau),t)\,u_k(\mathbf{X}(t|\mathbf{0},t-\tau),t)\right)}\;, \tag{62}$$

where the derivative is applied to the point $\mathbf{X}(t|\mathbf{0},t-\tau)$. Now, if we consider the average as an ensemble average and assume homogeneity we may replace the derivative with one at the origin instead provided that we insert a minus sign. Consequently, the term becomes

$$-f(\tau)\,F_{jl}(\tau)\,F_{km}(\tau)\,\overline{\frac{\partial u_i}{\partial x_k}(\mathbf{0},t)\,u_l(\mathbf{0},t-\tau)\,u_m(\mathbf{0},t-\tau)}\;. \tag{63}$$

It seems likely that a velocity gradient at a point will be essentially uncorrelated to the velocities at the same point but at a different time (this is a slight variation of the assumption of the independence of the small and the large scales used by Heskestad (1965) and others), in which case this term will vanish. In any case, all the factors in this expression except for the derivative is available to us, and estimating the derivative with Taylor's hypothesis should be much safer than estimating spatial correlations, and therefore we could calculate the value of this expression from our measurement data.

To summarize, we can say that if we have uniform flow with negligible acceleration terms, which is homogeneous and for which the small and large scales are essentially independent of each other then the first term in (58) should vanish. For general linear flows satisfying all the other assumptions we can perform a power series expansion in τ of the term to find that for small τ it should be of order $\mathcal{O}\left(\tau^2\right)$. Hence we may conclude that if we use the method outlined in this subsection the error due to Taylor's hypothesis for the spatial separation $\mathbf{X}_m\left(t|\mathbf{0},t-\tau\right)$ along a mean streamline is of the order of magnitude $\mathcal{O}\left(\tau^2\right)$ for small τ for any linear flow. This error is one order smaller than that expected from Lin's criterion so, at least for linear flows, our method seems to offer some improvement. Clearly, correction formulae of the Lumley or the Wyngaard & Clifford type can be implemented in the present case as well, to increase the accuracy of the measurement of the spectrum even further.

3.3 On the properties of moving-frame correlation coefficients

The primary aim of this subsection is to prove Theorems 1 and 2. After the proofs we will discuss possible extensions of the theorems.

In the previous two subsections the aim was to understand the variation of the velocity along a trajectory and how this knowledge could be used to improve the accuracy of Taylor's hypothesis. Now consider the situation we studied in the last subsection, *i.e* calculation of the correlation coefficient between u_i at the origin and u_j at a point on the mean trajectory given by $\mathbf{X}_m\left(t|\mathbf{0},t-\tau\right)$, for some τ. In the case of a linear flow with negligible acceleration terms, we found that what we had access to was the value of u_j in a cluster of points, the exact locations of which depend on the u_j. Since we had a linear flow with negligible acceleration terms the centroid of the cluster was $\mathbf{X}_m\left(t|\mathbf{0},t-\tau\right)$. Our aim in the last subsection was to try to extract some information about the desired correlation coefficient from the data we had access to.

Suppose now that there were a point $\overline{\mathbf{x}}$ such that $u_j\left(\overline{\mathbf{x}},t\right)$ was more correlated with the velocity at our cluster of points than $u_j\left(\mathbf{X}_m\left(t|\mathbf{0},t-\tau\right),t\right)$. In this case our application of Taylor's hypothesis would yield more information about the correlation coefficient between u_i at the origin and u_j at $\overline{\mathbf{x}}$ than that between u_i at the origin and u_j at $\mathbf{X}_m(t|\mathbf{0}, t-\tau)$. Of course, this is the situation where an advection velocity different from the local mean velocity is typically introduced. However, such an amendment to Taylor's hypothesis has not been justified with direct use of the Navier–Stokes equations. Fortunately, for linear flows with negligible acceleration terms, the experimental and numerical evidence suggests that the advection velocity and local mean velocities coincide. On the other hand, for non-linear flows this does not seem to be the case. In the next section we will see that for non-linear flows the centroid of the trajectories for some temporal separation will no longer coincide with the position along the trajectory of the mean flow for the same temporal separation. Clearly, the information exists along the individual trajectories and hence it is more likely that the location of the point where u_j attains its maximum correlation coefficient with the u_js in the cluster, is determined by the statistical properties of the individual trajectories rather than by properties of the mean flow, which is not really involved in the problem. Indeed, one is tempted to conjecture that the location of the point where u_j attains its maximum correlation coefficient with the u_js in the cluster, is given by the centroid of the cluster for all flows where the acceleration terms can be neglected, yet it should be stressed that there is no need for the proper averaging process to be this simple. By proving Theorems 1 and 2 we will obtain sufficient conditions for the validity of this conjecture in the case of uniform mean flow.

Proof of Theorem 1

Since the turbulence is assumed to be stationary and homogeneous, we may, for convenience of notation, choose $t_0 = 0$ and $\mathbf{x_0} = \mathbf{0}$.

To prove the theorem we will establish that for any $\mathbf{h}$ we have that

$$\overline{u_i(\mathbf{0},0)\, u_j\left(\overline{\mathbf{X}(t|\mathbf{0},0)} + \mathbf{h}, t\right)} = \overline{u_i(\mathbf{0},0)\, u_j\left(\overline{\mathbf{X}(t|\mathbf{0},0)} - \mathbf{h}, t\right)} \tag{64}$$

Since we assume that $R_{ij}(\mathbf{x}, t)$ is a differentiable function, the relation (64) implies that the gradient of $R_{ij}(\mathbf{x}, t)$ vanishes in $\overline{\mathbf{X}(t|\mathbf{0},0)}$, which is hence a stationary point for $R_{ij}(\mathbf{x}, t)$. Because of homogeneity the same property holds for $\rho_{ij}(\mathbf{x}, t)$.

We have that

$$\begin{aligned} &\overline{u_i(\mathbf{0},0)\, u_j\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)} = \\ &= \int_{-\infty}^{\infty} u_i^a \int_{\mathbf{R}^3} u_j^b P\left(u_i^a(\mathbf{0},0), \mathbf{u}^b\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)\right) d\mathbf{u}^b du_i^a\ , \end{aligned} \tag{65}$$

where the superscripts a and b are used to identify the variables, and $P\left(u_i^a(\mathbf{0},0), \mathbf{u}^b\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)\right)$ denotes the probability distribution of the simultaneous occurrence of u_i at $(\mathbf{0},0)$ and of $\mathbf{u}$ at $\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)$.

Now let us introduce the notation $\mathbf{q}(\mathbf{h}) = \overline{\mathbf{X}(t|\mathbf{0},0)} + \mathbf{h}$, $\mathbf{u_t}(\mathbf{q}(\mathbf{h}),t) \equiv \mathbf{u}(\mathbf{q}(\mathbf{h}),t)$. Of course, in a given realisation $\mathbf{u_t}$ depends on the position, but when we use it as a stochastic variable we will suppress this dependence when it does not matter, and only indicate it when it matters, *i.e* when we discuss the probability density of it occurring.

By the unique extendability, and the assumptions on $\Delta\mathbf{X}$ and $\mathbf{v}$ we have that (see Fig. 3 for an illustrations of these quantities)

$$\begin{aligned} \mathbf{v}(0|\mathbf{q}(\mathbf{h}),t) &= C(t)\,\mathbf{u_t} \qquad (66)\\ \mathbf{X}(0|\mathbf{q}(\mathbf{h}),t) &= A(t)\,\mathbf{u_t} + B(t)\,\mathbf{h} \equiv \mathbf{X_0}(t,\mathbf{u_t},\mathbf{h}) \qquad (67)\end{aligned}$$

where $A(t)$, $B(t)$ and $C(t)$ are matrix valued functions of t. Consequently,

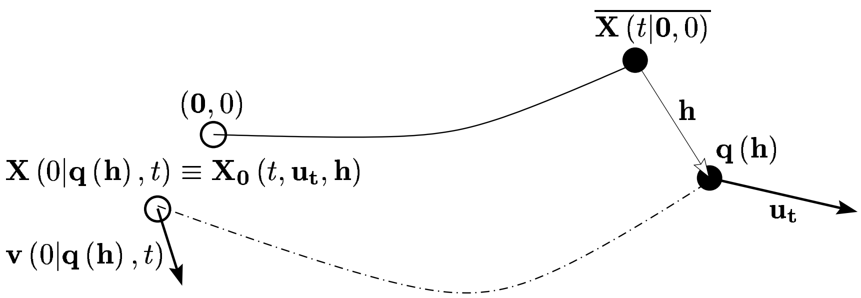

Figure 3: An illustration of the trajectory quantities used in the proofs of Theorems 1 and 2. Filled circles illustrate positions at time t and unfilled circles positions at time 0. The solid line illustrates the motion of the centroid of the ensemble at time t of the trajectories emanating from the origin at time 0. The dashed-dotted line illustrates the trajectory backwards in time of a particle at position $\mathbf{q}(\mathbf{h})$ at time t with velocity $\mathbf{u_t}$.

we have that

$$P\Big(u_i^a(\mathbf{0},0)\,,\mathbf{u_t}^b(\mathbf{q}(\mathbf{h}),t)\Big) = P\Big(u_i^a(\mathbf{0},0)\,,\Big(C(t)\,\mathbf{u_t}^b\Big)(\mathbf{X_0}(t,\mathbf{u_t},\mathbf{h}),0)\Big)\,. \qquad (68)$$

From (67) we have that $\mathbf{X_0}(t,-\mathbf{u_t},-\mathbf{h}) = -\mathbf{X_0}(t,\mathbf{u_t},\mathbf{h})$ for any $\mathbf{h}$ and $\mathbf{u_t}$. If we use this, (66) and the parity assumption we have that

$$\begin{aligned} & P\Big(u_i^a(\mathbf{0},0)\,,\mathbf{u_t}^b(\mathbf{q}(\mathbf{h}),t)\Big) = \qquad (69)\\ =\; & P\Big(u_i^a(\mathbf{0},0)\,,\Big(C(t)\,\mathbf{u_t}^b\Big)(\mathbf{X_0}(t,\mathbf{u_t},\mathbf{h}),0)\Big) \qquad (70)\\ =\; & P\Big(-u_i^a(\mathbf{0},0)\,,-\Big(C(t)\,\mathbf{u_t}^b\Big)(-\mathbf{X_0}(t,\mathbf{u_t},\mathbf{h}),0)\Big) \qquad (71)\end{aligned}$$

$$= P\left(-u_i^a(\mathbf{0},0), \left(C(t)\left(-\mathbf{u_t}^b\right)\right)(\mathbf{X_0}(t,-\mathbf{u_t},-\mathbf{h}),0)\right) \quad (72)$$
$$= P\left(-u_i^a(\mathbf{0},0), -\mathbf{u_t}^b(\mathbf{q}(-\mathbf{h}),t)\right) \quad (73)$$

Now consider,

$$\overline{u_i(\mathbf{0},0)\, u_j(\mathbf{q}(\mathbf{h}),t)} - \overline{u_i(\mathbf{0},0)\, u_j(\mathbf{q}(-\mathbf{h}),t)} = \quad (74)$$
$$= \int_{-\infty}^{\infty} u_i^a \int_{\mathbf{R}^3} u_j^b P\left(u_i^a(\mathbf{0},0), \mathbf{u}^b(\mathbf{q}(\mathbf{h}),t)\right) d\mathbf{u}^b du_i^a -$$
$$- \int_{-\infty}^{\infty} u_i^a \int_{\mathbf{R}^3} u_j^b P\left(u_i^a(\mathbf{0},0), \mathbf{u}^b(\mathbf{q}(-\mathbf{h}),t)\right) d\mathbf{u}^b du_i^a\ , \quad (75)$$

if the inner integrations are performed, it is seen from the relation between (69) and (73) and the simple fact that $u_i^a u_j^b = (-u_i^a)\left(-u_j^b\right)$ that the outer integral over $u_i^a \geq 0$ for the first term in (75) equals the outer integral over $u_i^a \leq 0$ for the second term and vice versa. Hence (75) vanishes and the condition (64) has been established. **qed**

Proof of Theorem 2

Once again, the turbulence is assumed to be stationary and homogeneous, and thus we may choose $t_0 = 0$ and $\mathbf{x_0} = \mathbf{0}$. We must prove that for any $\mathbf{h}$, and for any sufficiently small $\delta > 0$ we have that

$$\overline{u_i(\mathbf{0},0)\, u_i\left(\overline{\mathbf{X}(t|\mathbf{0},0)} + \delta\mathbf{h}, t\right)} < \overline{u_i(\mathbf{0},0)\, u_i\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)}\ . \quad (76)$$

With $\mathbf{q}$ and $\mathbf{u}_t$ defined as in proof of Theorem 1 we have, by assumption, for all $\mathbf{h}$ and for any sufficiently small $\delta > 0$, (see Fig. 3 for an illustration of these quantities)

$$\mathbf{v}(0|(\mathbf{q}(\delta\mathbf{h}),t)) = \mathbf{u_t} \quad (77)$$
$$\mathbf{X}(0|(\mathbf{q}(\delta\mathbf{h}),t)) = f(t)\, I\mathbf{u_t} + G(t)\,\delta\mathbf{h} \equiv \mathbf{X_0}(t,\mathbf{u_t},\delta\mathbf{h}) \quad (78)$$

where $f(t)$ is a scalar function of time, $G(t)$ a matrix valued function of time with non-zero eigenvalues and I the identity matrix. Consequently, we have that

$$P\left(u_i^a(\mathbf{0},0), \mathbf{u_t}^b(\mathbf{q}(\delta\mathbf{h}),t)\right) = P\left(u_i^a(\mathbf{0},0), \mathbf{u_t}^b(\mathbf{X_0}(t,\mathbf{u_t},\delta\mathbf{h}),0)\right)\ . \quad (79)$$

Hence, in order to calculate the space-time correlation coefficients in (76) we can equivalently calculate the single-time correlation coefficient of u_i at the origin and at a cluster of points for which the positions depend linearly on $\mathbf{u_t}$. For the right hand side we have $\mathbf{h} = 0$ and hence the cluster has the centroid at the origin, whereas for the left-hand side when $\delta\mathbf{h} \neq 0$ we have that the centroid is located at $G(t)\,\delta\mathbf{h} \neq 0$ (G is assumed to be non-singular).

In general, we have that the probability density of both a and b occuring, $P(a,b)$, is equal to the conditional probability density of a occuring provided that b has occured, $P(a|b)$, multiplied by the probability density of b occuring, $P(b)$. If this notation for the probability densities is used, and the notation introduced in (18) and (19) is recalled, then the discussion in the previous paragraph implies that

$$\begin{aligned}
&\overline{u_i(\mathbf{0},0)\,u_i\left(\overline{\mathbf{X}(t|\mathbf{0},0)},t\right)} = \\
&= \int_{\mathbf{R}^3} u_{ti}^b \int_{-\infty}^{\infty} u_i^a P\left(u_i^a(\mathbf{0},0), \mathbf{u_t}^b\left(\mathbf{X_0}\left(t,\mathbf{u_t}^b,\mathbf{0}\right),0\right)\right) du_i^a d\mathbf{u_t}^b \quad (80)\\
&= \int_{\mathbf{R}^3} u_{ti}^b \overline{u_i(\mathbf{0},0|\mathbf{X_0}(t,\mathbf{u_t}^b,\mathbf{0}),u_{ti}^b,0)} P\left(\mathbf{u_t}^b\left(\mathbf{X_0}\left(t,\mathbf{u_t}^b,\mathbf{0}\right),0\right)\right) d\mathbf{u_t}^b (81)\\
&= \int_{\mathbf{R}^3} u_{ti}^b g_{\mathbf{u_t}^b,i}\left(-\mathbf{X_0}\left(t,\mathbf{u_t}^b,\mathbf{0}\right)\right) d\mathbf{u_t}^b, \quad (82)
\end{aligned}$$

where we have used the fact that the inner integral in (80) is the expectation velocity of u_i at the origin provided that it is known in $\mathbf{X_0}(t,\mathbf{u_t},\mathbf{0})$ at the same time multiplied by the probability of having the velocity $\mathbf{u_t}$ at $\mathbf{X_0}(t,\mathbf{u_t},\mathbf{0})$.

Similarly, we find that

$$\overline{u_i(\mathbf{0},0)\,u_i\left(\overline{\mathbf{X}(t|\mathbf{q}(\delta\mathbf{h}),0)},t\right)} = \int_{\mathbf{R}^3} u_{ti}^b g_{\mathbf{u_t}^b,i}\left(-\mathbf{X_0}\left(t,\mathbf{u_t}^b,\delta\mathbf{h}\right)\right) d\mathbf{u_t}^b. \quad (83)$$

According to Theorem 1 we have that

$$\begin{aligned}
&\overline{u_i(\mathbf{0},0)\,u_i\left(\overline{\mathbf{X}(t|\mathbf{q}(\delta\mathbf{h}),0)},t\right)} = \\
&= \frac{1}{2}\left(\overline{u_i(\mathbf{0},0)\,u_i\left(\overline{\mathbf{X}(t|\mathbf{q}(\delta\mathbf{h}),0)},t\right)} + \overline{u_i(\mathbf{0},0)\,u_i\left(\overline{\mathbf{X}(t|\mathbf{q}(-\delta\mathbf{h}),0)},t\right)}\right)\\
&= \int_{\mathbf{R}^3} u_{ti}^b \frac{1}{2}\left[g_{\mathbf{u_t}^b}\left(-\mathbf{X_0}\left(t,\mathbf{u_t}^b,\delta\mathbf{h}\right)\right) + g_{\mathbf{u_t}^b}\left(-\mathbf{X_0}\left(t,\mathbf{u_t}^b,-\delta\mathbf{h}\right)\right)\right] d\mathbf{u_t}^b, (84)
\end{aligned}$$

where we have used the fact that $\mathbf{X_0}$ depends linearly on $\delta\mathbf{h}$. A one-dimensional Taylor expansion of $g_{\mathbf{u_t}^b}$ around $-\mathbf{X_0}\left(t,\mathbf{u_t}^b,\mathbf{0}\right)$ along the direction of $-G(t)\,\mathbf{h}$ shows that

$$\begin{aligned}
&\overline{u_i(\mathbf{0},0)\,u_i\left(\overline{\mathbf{X}(t|\mathbf{q}(\delta\mathbf{h}),0)},t\right)} - \overline{u_i(\mathbf{0},0)\,u_i\left(\overline{\mathbf{X}(t|\mathbf{0},0)},t\right)} = \\
&= \frac{1}{2}\int_{\mathbf{R}^3} u_{ti}\, \frac{\partial^2}{\partial \mathbf{l}^2} g_{\mathbf{u_t},i}\Big|_{\mathbf{x}=-\mathbf{X_0}(t,\mathbf{u_t},\mathbf{0})} d\mathbf{u_t}\delta^2\,|G(t)\,\mathbf{h}|^2 + \mathcal{O}\left(\delta^3\,|G(t)\,\mathbf{h}|^3\right), (85)
\end{aligned}$$

where $\mathbf{l}$ is a unit vector in the direction of $G(t)\,\mathbf{h}$. For any fixed time t we have that $\mathbf{X_0}(t,\mathbf{u_t},\mathbf{0}) = k\mathbf{u_t}$ for some k, and hence by Definition 2 we have that the first term on the right hand side of (85) is strictly negative. Furthermore, it is the term of lowest order in δ and hence for sufficiently small $\delta > 0$ we have that the right-hand side in (85) is strictly negative. This establishes (76) and hence the theorem. **qed**

From this proof we note that if $\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{j0},t\right)P}\left(\mathbf{u_0},\mathbf{x_0}\right)$ is predominantly concave then Theorem 2 would also hold for the cross-correlation coefficients ρ_{ij}. However, the Taylor stress $\overline{u_iu_j}$ is often negative and in that case we would be more interested in when the space-time correlation coefficient function has a (negative) minimum, and in this case the theorem holds if $\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{j0},t\right)P}\left(\mathbf{u_0},\mathbf{x_0}\right)$ is predominantly convex (instead of < 0 use > 0 in the definition). The major reason for not including these correlation coefficients in the theorem is that it is less evident that the maximum of the correlation coefficient is obtained for zero separation, and if not, we cannot expect them to be predominantly concave (convex) when the correlation coefficient at zero separation is positive (negative).

For channel flow, Theorems 1 and 2 do not apply. All conditions on $\mathbf{X}$ are satisfied, but the conditions on $\mathbf{v}$ are not. In fact, only v_1 fails to meet the required criteria, and hence the theorems should apply for ρ_{22} and ρ_{33}. For Theorem 1 the problem with ρ_{11} is that $\mathbf{v}\left(t|\mathbf{x_0},0\right)$ does not depend linearly $u_2\left(\mathbf{x_0},0\right)$ in this case. In fact, linearity is an unnecessarily strict requirement, since all we require is, in fact, that $\mathbf{v}\left(t|\mathbf{x_0},0\right)$ is an odd function of $\mathbf{u}\left(\mathbf{x_0},0\right)$, but for channel flow this relaxation does not help us. Indeed, if Theorem 1 is to hold for the ρ_{11} correlation coefficient in channel flow, the parity symmetry must be replaced by some other far more complicated relation. For Theorem 2 the situation is worse since the very dependence of $v_1\left(t|\mathbf{x_0},0\right)$ on $u_2\left(\mathbf{x_0},0\right)$ violates the conditions.

The primary reason why we did not allow $v_1\left(t|\mathbf{x_0},t\right)$ to depend on $u_2\left(\mathbf{x_0},t\right)$ in Theorem 2 is best illustrated by what would happen in the case of homogeneous shear flow considered by Champagne *et al.* (1970); Harris *et al.* (1977). In this case we have that

$$v_1\left(t|\mathbf{x_0},0\right) = u_1\left(\mathbf{x_0},0\right) + \frac{\partial U_1}{\partial x_2}u_2\left(\mathbf{x_0},0\right)t\ , \tag{86}$$

with $\partial U_1/\partial x_2 > 0$, and with this relation the only way that we can pursue the argument in the proof of Theorem 2 easily is if we make the additional assumption that $\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{j0},t\right)P}\left(u_{j0},\mathbf{x_0}\right)$ is predominantly concave. However, ordinarily we must in this flow case have that $\overline{u_iu_j}$ is negative, and even if $\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{j0},t\right)P}\left(\mathbf{u_0},\mathbf{x_0}\right)$ is predominantly convex we cannot extend Theorem 2 without difficulty. Indeed, the dependence of $u_1\left(\mathbf{x},t\right)$ on $u_2\left(\mathbf{X}\left(0|\mathbf{x},t\right),0\right)$, which is growing linearly with t, will give a negative contribution to ρ_{11} the relative importance of which (compared to the positive contribution from the dependence of $u_1\left(\mathbf{X}\left(0|\mathbf{x},t\right),0\right)$) will be growing linearly with time. Similarly, the overall concavity of the function that will replace $g_{\mathbf{u_t},i}$ in this case will be successively eroded by the convex contribution from $\overline{u_i\left(\mathbf{x},t|\mathbf{x_0},u_{j0},t\right)P}\left(\mathbf{u}_0,\mathbf{x_0}\right)$. Hence, whereas it is clear that this violation of the conditions required for Theorem 2 has little impact on validity of the theorem for small separations, its implications for large separations remain somewhat unclear. The same conclusion is largely valid in the more complicated cases, such as channel flow or general linear flow. There are, however, other difficulties with the homogeneous shear flow considered by Champagne *et al.* (1970); Harris *et al.* (1977), since it is not homogeneous in the streamwise direction, and since its parity symmetry is quite questionable. In fact, if Theorem 1 is to hold, which is consistent with all presented data, then the departure from parity symmetry must cancel the effects of inhomogeneity in some sense.

We will conclude this section by an estimate of the shape of the moving frame correlation of a frame moving with the local mean velocity in case all the assumptions in Theorem 2 are satisfied. This moving frame correlation is given by $\overline{u_i(\mathbf{0},0)\, u_i\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)}$ which according to (82) is given by

$$\int_{\mathbf{R}^3} u_i g_{\mathbf{u},i}\left(-\mathbf{X_0}(t,\mathbf{u},\mathbf{0})\right) d\mathbf{u} \; . \tag{87}$$

If we now, as a rough assumption, assume that the functional form of $\overline{u_i(\mathbf{x},t|\mathbf{x_0},u_{i0},t)}$ is independent of the value of u_{i0}, (Assumption 1 in section 2.1) we find from (23) that

$$\overline{u_i(\mathbf{0},0)\, u_i\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)} \approx \int_{\mathbf{R}^3} \frac{u_i^2}{\overline{u_i^2}} R_{ii}\left(\mathbf{X_0}(t,\mathbf{u},\mathbf{0})\right) P(\mathbf{u})\, d\mathbf{u} \tag{88}$$

with no summation on i. Here we have used homogeneity to avoid having to specify the point in which we are to have $\mathbf{u}$. If we assume that P is Gaussian we have that $u_i^2 P(\mathbf{u})$ attains its maximum value at

$$\mathbf{u}_{max}^{\pm} \equiv \left(u_{i,max} = \pm\sqrt{2\overline{u_i^2}}, u_{j,max} = 0, u_{k,max} = 0\right), \tag{89}$$

and as $\overline{u_i^2}$ decreases the distribution approaches a delta function multiplied by $\overline{u_i^2}/2$. If P is not Gaussian but still reasonably close to it, these properties are still valid with the exception that we might get a coefficient different from $\sqrt{2}$ in (89). If we as a first approximation replace $u_i^2 P(\mathbf{u})$ in (88) by a delta function multiplied by $\overline{u_i^2}/2$ at $\mathbf{u}_{max}^{\pm}$ given by (89) we find that

$$\begin{aligned}
&\overline{u_i(\mathbf{0},0)\, u_i\left(\overline{\mathbf{X}(t|\mathbf{0},0)}, t\right)} \\
\approx\; & \frac{1}{2}\left[R_{ii}\left(\mathbf{X_0}(t,\mathbf{u}_{max}^{+},\mathbf{0})\right) + R_{ii}\left(\mathbf{X_0}(t,\mathbf{u}_{max}^{-},\mathbf{0})\right)\right] && (90)\\
=\; & \frac{1}{2}\left[R_{ii}\left(f(t)\sqrt{2\overline{u_i^2}},0,0\right) + R_{ii}\left(-f(t)\sqrt{2\overline{u_i^2}},0,0\right)\right] && (91)
\end{aligned}$$

where we have used (78) and where $R_{ii}(a,b,c)$ denotes the autocorrelation of u_i with separation $x_i = a,\ x_j = b,\ x_k = c$, where $i,\ j,\ k$ all are different, but the order is arbitrary. In case of uniform mean flow, channel flow or the homogeneous shear flow studied by Champagne *et al.* (1970); Harris *et al.* (1977) we have that $f(t) = t$.

Hence if we consider the moving frame correlation

$$R_{11}^{MF}(t) \equiv \overline{u_1(0,\mathbf{0})\, u_1\left(t, \overline{\mathbf{X}(t|\mathbf{0},0)}\right)},$$

and use the fact that the autocorrelation is typically symmetric in the x_1-direction, (91) suggests that $R_{11}^{MF} \approx R_{11}\left(\sqrt{2\overline{u_i^2}}t, 0, 0\right)$, *i.e.* the moving frame correlation is given by the R_{11} autocorrelation in the x_1 direction. Hence, the integral time scale of the moving frame correlation, T_{11}^{MF} is related to the integral length scale of the autocorrelation L_{11} according to $L_{11} = \sqrt{2}\overline{u_i^2}^{1/2} T_{11}^{MF}$. Apart from the constant, this result was obtained in Corrsin (1963) by making assumptions about the shape of the spectrum, which is a completely different method.

To test the validity of the formula (91) it could be compared with experimentally measured auto- and moving frame correlation coefficients. Such measurements have been made for example by Comte-Bellot & Corrsin (1971) for roughly isotropic turbulence, by Champagne *et al.* (1970); Harris *et al.* (1977) for homogeneous shear flow with two different values of the mean shear, and by Romano (1995) for channel flow. According to (91), the separation for which the autocorrelation coefficient attains a certain value should be $\sqrt{2}\overline{u_i^2}^{1/2}/U$ times the separation when the moving frame correlation coefficient attains the same value. Unfortunately, for most of these measurements the value of $\sqrt{2}\overline{u_i^2}^{1/2}/U$ is fairly small, and therefore testing (91) involves measuring very small distances in the published figures, which introduces an error too large for a satisfactory comparison with theory.

Interestingly, for $R_{22}^{MF}(t)$ (91) gives us an approximation in terms of the R_{22} autocorrelation along the x_2 axis, and for R_{33}^{MF} we have an approximation in terms of the R_{33} autocorrelation along the x_3 axis. If we were to calculate $R_{12}^{MF}(t)$ using the same technique we would find that it primarily depends on the autocorrelation R_{12} along the lines $x_2 = \pm x_1$.

3.4 Sensitivity of the model to linear disturbances in space and time

The purpose of the subsection is to study the sensitivity of the solutions for linear U_i obtained in subsection 3.1 to small acceleration terms, which are sufficiently small so that they can be linearised in time and space. Since these terms are, in general, unknown this is merely a sensitivity analysis, which is meant to indicate roughly the magnitude of the error caused by neglecting them. However, the Reynolds stress terms are stationary in time, and can thus, at least in principle, be measured. The formulae developed here can in fact be used to correct for a linear approximation of the Reynolds stress term.

In this subsection we will add a linear right-hand side to (7) in the case when U is linear. In the next section we will consider general disturbances, but then we are only able to analyse the system for small times. In the case under consideration here we may proceed in the same manner as in Section 2 to obtain the system of equations

$$\frac{dX_i}{dt} = U_{i0} + U_{ij}X_j\left(t|\mathbf{x_0},t_0\right) + v_i\left(t|\mathbf{x_0},t_0\right) \tag{92}$$

$$\frac{dv_i}{dt} + U_{ij}v_j\left(t|\mathbf{x_0},t_0\right) = A_i + B_{i0}\left(t-t_0\right) + B_{ij}X_j\left(t|\mathbf{x_0},t_0\right) \ . \tag{93}$$

At this point the argument becomes easier if we employ vector/matrix notation.Thererfore we will write $\mathbf{v}(t)$ for $v_i\left(t|\mathbf{x_0},t_0\right)$, $\mathbf{X}(t)$ for $X_i(t)$, $\mathbf{U_o}$ for U_{i0}, $\mathcal{U}$ for U_{ij}, $\mathbf{A}$ for A_i, $\mathbf{B}$ for B_{i0} and $\mathcal{B}$ for B_{ij}. The system above can thus be written

$$\frac{d}{dt}\begin{bmatrix}\mathbf{v}\\ \mathbf{X}\end{bmatrix} = \begin{bmatrix}-\mathcal{U} & \mathcal{B}\\ \mathcal{I} & \mathcal{U}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{X}\end{bmatrix} + \begin{bmatrix}\mathbf{U_0}\\ \mathbf{A}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{B}\end{bmatrix}(t-t_0) \tag{94}$$

where $\mathcal{I}$ denotes the identity matrix.

We now let $\mathbf{w} = [\mathbf{v}\ \ \mathbf{X}]^T$ and let $\mathcal{L}$ denote the 2×2-matrix in (94). If we assume that $\det\mathcal{L}\neq 0$ we have from standard ODE theory that

$$\mathbf{w} = Ce^{\mathcal{L}(t-t_0)} - \mathcal{L}^{-1}\left[\mathbf{u_0}\ \ \mathbf{A}\right]^T - \mathcal{L}^{-1}\left[\mathbf{0}\ \ \mathbf{B}\right]^T(t-t_0) - \mathcal{L}^{-2}\left[\mathbf{0}\ \ \mathbf{B}\right]^T \tag{95}$$

where C is a constant to be chosen to satisfy the initial values. However, to calculate $e^{\mathcal{L}(t-t_0)}$ which dominates the dynamics of the system, we must diagonalise a 6×6-matrix, which involves a fair degree of work, and obscures the insight into the relation with the problem without acceleration terms. These drawbacks are remedied, however, by the following proposition:

Proposition 1 *With $\mathcal{L}$ defined as above, we have that*
(a) If λ is an eigenvalue to $\mathcal{L}$ and $\mathbf{w}_e = [\mathbf{v}_e\ \ \mathbf{X}_e]^T$ a corresponding eigenvector, then λ^2 is an eigenvalue to $\mathcal{U}^2+\mathcal{B}$ and $\mathbf{X}_e$ is a corresponding eigenvector.
(b) Conversely, if μ^2 is an eigenvalue to $\mathcal{U}^2+\mathcal{B}$ and $\mathbf{s}_e$ a corresponding eigenvector, then $\pm\mu$ are eigenvalues to $\mathcal{L}$ with corresponding eigenvectors $\left[-\left(\mathcal{U}\mp\mu\mathcal{I}\right)\mathbf{s}_e\ \ \mathbf{s}_e\right]^T$.

Proof (a) We have assumed that

$$\begin{bmatrix}-\mathcal{U} & \mathcal{B}\\ \mathcal{I} & \mathcal{U}\end{bmatrix}\begin{bmatrix}\mathbf{v}_e\\ \mathbf{X}_e\end{bmatrix} = \lambda\begin{bmatrix}\mathbf{v}_e\\ \mathbf{X}_e\end{bmatrix} \ . \tag{96}$$

The equation for the second row yields that

$$\mathbf{v}_e = \lambda\mathbf{X}_e - \mathcal{U}\mathbf{X}_e \ . \tag{97}$$

When this relation is substituted into the equation for the first row we get

$$\left(\mathcal{U}^2 + \mathcal{B}\right) \mathbf{X}_e = \lambda^2 \mathbf{X}_e \tag{98}$$

which proves (a).
(b) We now assume that

$$\left(\mathcal{U}^2 + \mathcal{B}\right) \mathbf{s}_e = \mu^2 \mathbf{s}_e \ . \tag{99}$$

To begin with, we consider

$$\begin{bmatrix} -\mathcal{U} & \mathcal{B} \\ \mathcal{I} & \mathcal{U} \end{bmatrix} \begin{bmatrix} -\left(\mathcal{U} \mp \mu\mathcal{I}\right) \mathbf{s}_e \\ \mathbf{s}_e \end{bmatrix} = \begin{bmatrix} \left(\mathcal{U}^2 \mp \mu\mathcal{U} + \mathcal{B}\right) \mathbf{s}_e \\ \pm\mu\mathbf{s}_e \end{bmatrix} \tag{100}$$

If we now use (99) this expression becomes

$$\begin{bmatrix} \left(\mathcal{U}^2 \mp \mu\mathcal{U} + \mathcal{B}\right) \mathbf{s}_e \\ \pm\mu\mathbf{s}_e \end{bmatrix} = \begin{bmatrix} \left(\mu^2\mathcal{I} \mp \mu\mathcal{U}\right) \mathbf{s}_e \\ \pm\mu\mathbf{s}_e \end{bmatrix} \tag{101}$$

$$= \pm\mu \begin{bmatrix} -\left(\mathcal{U} \mp \mu\mathcal{I}\right) \mathbf{s}_e \\ \mathbf{s}_e \end{bmatrix} \tag{102}$$

which proves (b). **qed**

Not only does Proposition 1 show that it suffices to linearise a 3×3-matrix to diagonalise $\mathcal{L}$, but in addition it shows how a small $\mathcal{B}$ affects the eigenvalues of $\mathcal{U}$. In fact the problem is on a form where the standard asymptotic techniques can be applied directly.

It is also worthwhile to note that in the case of constant U_i treated in the previous section we have that $\mathcal{U} = 0$. Hence, the short term behaviour is essentially determined by the eigenvalues of $\sqrt{\mathcal{B}}$.

4 General shear flow

The moment we step from linear U_i to quadratic U_i the system of equations (12) becomes non-linear, and thus finding solutions to the system becomes a very tough task. The complexity of the solutions also increases considerably in a way that has strong physical implications. To illustrate this increased complexity we will study the short time behaviour of the solutions to the full system, including the acceleration terms, by expressing it as a power series in $(t - t_0)$. Needless to say, once $\mathbf{X}$ has been found $\mathbf{v}$ can be obtained as an infinite series. First, however, we will make a simplification of (12) valid in case of irrotational mean flow, which reveals some information about the change of the fluctuating velocity along a trajectory in this case.

4.1 The case when U_i is irrotational

When the mean vorticity is exactly zero (note that no assumption is made about the fluctuating vorticity), then we may find a quantity conserved by the system (12) (In fact, it is really sufficient that $\mathbf{U} \times \mathbf{\Omega} = 0$, which is a somewhat weaker condition than $\mathbf{\Omega} = 0$.) This can be used to derive a Bernoulli type equation, which is useful when discussing Taylor's hypothesis. For the remainder of this subsection we assume that

$$\Omega_k = \epsilon_{kij} \frac{\partial U_i}{\partial x_j} = 0 \ . \tag{103}$$

Now let us multiply (12) with $d\mathbf{X}/dt$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\frac{d\mathbf{X}}{dt} \cdot \frac{d\mathbf{X}}{dt} \right) = \frac{d\mathbf{X}}{dt} \cdot \frac{d^2\mathbf{X}}{dt^2} &= U_j\left(\mathbf{X}\right) \frac{\partial U_i}{\partial x_j} \left(\mathbf{X}\right) \frac{dX_i}{dt} && (104) \\ &= U_j\left(\mathbf{X}\right) \frac{\partial U_j}{\partial x_i} \left(\mathbf{X}\right) \frac{dX_i}{dt} && (105) \\ &= \frac{1}{2} \frac{D\left(\mathbf{U}\cdot\mathbf{U}\right)}{Dt} \left(\mathbf{X}\right) \ . && (106) \end{aligned}$$

Using the initial conditions we have established that

$$\left| \frac{d\mathbf{X}}{dt} \right|^2 - \left|\mathbf{U}\left(\mathbf{X}\right)\right|^2 = 2\mathbf{U}\left(\mathbf{x_0}\right) \cdot \mathbf{u}\left(\mathbf{x_0}, t_0\right) + \left|\mathbf{u}\left(\mathbf{x_0}, t_0\right)\right|^2 , \tag{107}$$

for all t. Firstly, this equation tells us that the speed of the particle trajectory in this model only depends on the fluctuating velocity at the initial position and on the variation of the mean velocity along the trajectory. Secondly, we may substitute (8) for dX_i/dt to obtain

$$\begin{aligned} & 2\mathbf{U}\left(\mathbf{X}\left(t|\mathbf{x_0}, t_0\right)\right) \cdot \mathbf{u}\left(\mathbf{X}\left(t|\mathbf{x_0}, t_0\right), t\right) + \left|\mathbf{u}\left(\mathbf{X}\left(t|\mathbf{x_0}, t_0\right), t\right)\right|^2 = \\ = \ & 2\mathbf{U}\left(\mathbf{x_0}\right) \cdot \mathbf{u}\left(\mathbf{x_0}, t_0\right) + \left|\mathbf{u}\left(\mathbf{x_0}, t_0\right)\right|^2 , && (108) \end{aligned}$$

for all t. This implies that the quantity $2\mathbf{U}\left(\mathbf{x_0}\right) \cdot \mathbf{u}\left(\mathbf{x_0}, t_0\right) + \left|\mathbf{u}\left(\mathbf{x_0}, t_0\right)\right|^2$ is conserved along trajectories provided of course that the acceleration terms are negligible. Assuming that the turbulence is not too anisotropic and that $|\mathbf{U}|^2 >> |\mathbf{u}|^2$ the first term should dominate the second by a factor in the order of $|\mathbf{u}| / |\mathbf{U}|$. Consequently, this model satisfies some variant of Taylor hypothesis along each trajectory for the fluctuating velocity component which is parallell to the mean flow at the instantaneous location. However, we do not know the location of a given trajectory at a certain point in time, so we have no a priori way of knowing the location of the point of maximum correlation coefficient in this case (which may or may not be at the centroid of the cluster of trajectory locations at each point in time). Note that this picture is quite consistent with the conclusions of Zaman & Hussain (1981), that unless the large-scale structures are not undergoing rapid evolution (which they often would if subjected to strong mean vorticity) then Taylor's hypothesis work reasonably well provided that we adjust the advection velocity.

The only case when $U_j \partial U_i / \partial x_j$ is non-linear for which we have been able to find an exact solution is the rather non-physical situation for which U_i only depends on x_i and does so quadratically. The solution can then be expressed in terms of Jacobian elliptic functions (see e.g. (Abramowitz & Stegun, 1970)). The full solution is presented in the appendix, but we mention that for all solution cases the trajectory position depends on the fluctuating velocity at the initial point in a highly non-linear way, and that for almost all the solution cases the trajectory approaches infinity in finite time, which clearly illustrates the limitations of the model as well as the complexity of the situation.

4.2 The small time behaviour of the system including acceleration terms

We would now like to consider how the situation is affected by including the acceleration terms, which we have neglected so far. These terms include the Reynolds stress terms, the pressure terms and the viscous terms. If we derive an equation analogous to (12) from (6) rather than from (7) we obtain

$$\begin{aligned} \frac{d^2 X_i}{dt^2} &= U_j \frac{\partial U_i}{\partial x_j} + \overline{u_j \frac{\partial u_i}{\partial x_j}} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} && (109) \\ &\equiv f_i (\mathbf{X}) + r_i (\mathbf{X}) + p_{,i} (t, \mathbf{X}) + v_i (t, \mathbf{X}) && (110) \end{aligned}$$

Recently, Gledzer (1997) computed corrections to Taylor's hypothesis for the acceleration terms. A crucial approximation in his analysis was, however, to approximate the value of the acceleration terms along the trajectory by their value in one particular point. In order to solve (110) we will generalise this approach and replace the value of the acceleration terms along the trajectory by their Taylor series in one particular point, which we choose to be the origin. This will allow us to find a solution for small values of $t - t_0$ to (110) on the form

$$\begin{aligned} X_i &= X_i^{(1)} (t - t_0) + \frac{1}{2} X_i^{(2)} (t - t_0)^2 + \frac{1}{6} X_i^{(3)} (t - t_0)^3 + \\ &+ \frac{1}{24} X_i^{(4)} (t - t_0)^4 + \mathcal{O}\left((t - t_0)^5 \right) . && (111) \end{aligned}$$

where $X_i^{(j)}$ denotes the value of the jth derivative of X_i at time t_0. This solution can be used to indicate the speed at which the centroid of the locations of the trajectories at any point in time will depart from the location of the trajectory of the mean flow at the same point in time. Furthermore, the solution could easily be used to calculate $\mathbf{v} (t | \mathbf{0}, t_0)$ along each trajectory in terms of a power series in $(t - t_0)$. Hence the accuracy of Gledzer's approximation, mentioned above, could be estimated for small temporal separations.

For brevity of notation we will henceforth assume that $t_0 = 0$. To this end we will use the Taylor expansions of f_i, r_i and $p_{,i}$, and thus we must assume that these terms are sufficiently regular to render this approach valid. For the remainder of this subsection we will use

$$\begin{aligned}
f_i(\mathbf{x}) &= f_i^0 + f_{i,j}x_j + \frac{1}{2}f_{i,jk}x_jx_k + \mathcal{O}\left(|\mathbf{x}|^3\right) && (112)\\
r_i(\mathbf{x}) &= r_i^0 + r_{i,j}x_j + \frac{1}{2}r_{i,jk}x_jx_k + \mathcal{O}\left(|\mathbf{x}|^3\right) && (113)\\
p_{,i}(\mathbf{x}) &= p_{,i}^0 + p_{,i0}t + p_{,ij}x_j + \frac{1}{2}\left(p_{,i00}t^2 + (p_{,ij0} + p_{,i0j})\,tx_j + p_{,ijk}x_jx_k\right) + \\
&+ \mathcal{O}\left(|(t,\mathbf{x})|^3\right) && (114)\\
v_i(\mathbf{x}) &= v_i^0 + v_{i,0}t + v_{i,j}x_j + \frac{1}{2}\left(v_{i,00}t^2 + (v_{i,j0} + v_{i,0j})\,tx_j + v_{i,jk}x_jx_k\right) + \\
&+ \mathcal{O}\left(|(t,\mathbf{x})|^3\right) && (115)
\end{aligned}$$

where a zero after the comma in the subscript indicates derivation with respect to time, a letter after the comma in the subscript indicates derivation with respect to a spatial direction and a zero in the superscript indicates the value of the function at time zero. After substitution of these expressions into (110) we obtain upon equating terms the following equations.

$$\begin{aligned}
\mathcal{O}(1): \quad X_i^{(2)} &= f_i^0 + r_i^0 + p_{,i}^0 + v_i^0 && (116)\\
\mathcal{O}(t): \quad X_i^{(3)} &= p_{,i0} + v_{i,0} + (f_{i,j} + r_{i,j} + p_{,ij} + v_{i,j})\,X_j^{(1)} && (117)\\
&= p_{,i0} + v_{i,0} + (f_{i,j} + r_{i,j} + p_{,ij} + v_{i,j})\left(U_j^0 + u_j^0\right) && (118)\\
\mathcal{O}\left(t^2\right): \quad X_i^{(4)} &= p_{,i00} + v_{i,00} + (f_{i,j} + r_{i,j} + p_{,ij} + v_{i,j})\,X_j^{(2)} + \\
&+ (p_{,ij0} + p_{,i0j} + v_{i,j0} + v_{i,0j})\,X_j^{(1)} + \\
&+ (f_{i,jk} + r_{i,jk} + p_{,ijk} + v_{i,jk})\,X_j^{(1)}X_k^{(1)} && (119)\\
&= p_{,i00} + v_{i,00} + (f_{i,j} + r_{i,j} + p_{,ij} + v_{i,j}) \times \\
&\times \left(f_j^0 + r_j^0 + p_{,j}^0 + v_j^0\right) + (p_{,ij0} + p_{,i0j} + v_{i,j0} + v_{i,0j}) \times \\
&\times \left(U_j^0 + u_j^0\right) + (f_{i,jk} + r_{i,jk} + p_{,ijk} + v_{i,jk}) \times \\
&\times \left(U_j^0 + u_j^0\right)\left(U_k^0 + u_k^0\right) && (120)
\end{aligned}$$

This gives the position of a trajectory for small times. For most applications we cannot trace individual trajectories so we would like to relate this information to information about trajectories of the mean flow, which we could calculate beforehand. If we repeat the above procedure we obtain for the short time behaviour of $\mathbf{X_m}$ defined in Section 2 (remember that the f_is and the r_is are already averaged, whereas the $p_{,i}$s and the v_is have vanishing means).

$$\begin{aligned} X_{mi}^{(2)} &= f_i^0 + r_i^0 && (121)\\ X_{mi}^{(3)} &= (f_{i,j} + r_{i,j})\, X_{mj}^{(1)} && (122)\\ &= (f_{i,j} + r_{i,j})\, U_j^0 && (123)\\ X_{mi}^{(4)} &= (f_{i,j} + r_{i,j})\, X_{mj}^{(2)} + (f_{i,jk} + r_{i,jk})\, X_{mj}^{(1)} X_{mk}^{(1)} && (124)\\ &= (f_{i,j} + r_{i,j}) \left(f_j^0 + r_j^0\right) + (f_{i,jk} + r_{i,jk})\, U_j^0 U_k^0 && (125) \end{aligned}$$

We thus have that

$$\begin{aligned} X_i(t) - X_{mi}(t) &= u_i^0 t + \frac{1}{2}\left(p_{,i}^0 + v_i^0\right) t^2 + \frac{1}{6}\Big[p_{,i0} + v_{i,0}\\ &+ (p_{,ij} + v_{i,j})\, U_j^0 + (f_{i,j} + r_{i,j} + p_{,ij} + v_{i,j})\, u_j^0\Big] t^3\\ &+ \frac{1}{24}\Big[p_{,i00} + v_{i,00} + (p_{,ij} + v_{i,j})\left(f_j^0 + r_j^0\right)\\ &+ (f_{i,j} + r_{i,j} + p_{,ij} + v_{i,j})\left(p_{,j}^0 + v_j^0\right)\\ &+ (p_{,ij0} + p_{,i0j} + v_{i,j0} + v_{i,0j})\left(U_j^0 + u_j^0\right)\\ &+ (f_{i,jk} + r_{i,jk})\left(u_j^0 U_k^0 + u_k^0 U_j^0 + u_j^0 u_k^0\right)\\ &+ (p_{,ijk} + v_{i,jk})\left(U_j^0 + u_j^0\right)\left(U_k^0 + u_k^0\right)\Big] t^4 + \mathcal{O}\left(t^5\right) \quad (126) \end{aligned}$$

Fortunately, the quantity of primary interest is the ensemble average of this expression. Upon avaraging several terms vanish and what is left is

$$\begin{aligned} \overline{X_i - X_{mi}} &= \frac{1}{6}\overline{(p_{,ij} + v_{i,j})\, u_j^0}\, t^3 + \frac{1}{24}\Big[\overline{(p_{,ij} + v_{i,j})\left(p_{,j}^0 + v_j^0\right)}\\ &+ \overline{(p_{,ij0} + p_{,i0j} + v_{i,j0} + v_{i,0j})\, u_j^0} + \overline{(f_{i,jk} + r_{i,jk})\, u_j^0 u_k^0}\\ &+ \overline{(p_{,ijk} + v_{i,jk})\left(U_j^0 u_k^0 + u_j^0 U_k^0 + u_j^0 u_k^0\right)}\Big]\, t^4 + \mathcal{O}\left(t^5\right) \quad (127) \end{aligned}$$

From this we see that the centroid of the trajectories now departs from the trajectory of the averaged flow as $\mathcal{O}\left(t^3\right)$ for small t. However, for homogeneous turbulence we find that the coefficient in front of the t^3-term vanishes as is seen from

$$\begin{aligned} \overline{p_{,ij} u_j^0} &= -\frac{1}{\rho}\overline{\frac{\partial^2 p}{\partial x_i \partial x_j}(0, \mathbf{0})\, u_j(0, \mathbf{0})} && (128)\\ &= -\frac{1}{\rho}\overline{\frac{\partial}{\partial x_j}\left(\frac{\partial p}{\partial x_i}(0, \mathbf{0})\, u_j(0, \mathbf{0})\right)} && (129)\\ &= -\frac{1}{\rho}\frac{\partial}{\partial x_j}\overline{\frac{\partial p}{\partial x_i}(0, \mathbf{0})\, u_j(0, \mathbf{0})} && (130) \end{aligned}$$

$$
\begin{aligned}
\overline{v_{i,j}u_j^0} &= \nu \overline{\frac{\partial^3 u_i}{\partial x_k^2 \partial x_j}\left(0,\mathbf{0}\right) u_j\left(0,\mathbf{0}\right)} && (131)\\
&= \nu \overline{\frac{\partial}{\partial x_j}\left(\frac{\partial^2 u_i}{\partial x_k^2}\left(0,\mathbf{0}\right) u_j\left(0,\mathbf{0}\right)\right)} && (132)\\
&= \nu \frac{\partial}{\partial x_j}\overline{\frac{\partial^2 u_i}{\partial x_k^2}\left(0,\mathbf{0}\right) u_j\left(0,\mathbf{0}\right)}, && (133)
\end{aligned}
$$

where we have used the continuity equation $\partial u_i/\partial x_i = 0$. Similarly, several of the coefficients for the t^4-terms will vanish for homogeneous flows, however, one term that will not vanish, in general, is

$$
\frac{1}{48} f_{i,jk}\overline{u_j u_k} t^4 \tag{134}
$$

where it is easily checked that

$$
f_{i,jk} = \frac{\partial^3 U_i}{\partial x_j \partial x_k \partial x_l} U_l + 2\frac{\partial^2 U_i}{\partial x_j \partial x_l}\frac{\partial U_l}{\partial x_k} + \frac{\partial U_i}{\partial x_l}\frac{\partial^2 U_l}{\partial x_j \partial x_k}\,. \tag{135}
$$

Hence for general shear flows, when this quantity does not vanish, we find that the centroid of the trajectories departs from the trajecory of the averaged flow at least as fast as $\mathcal{O}\left(t^4\right)$ for small t, which underlines the importance of separating the two concepts.

5 Conclusion

In this paper we have demonstrated that, in the case when the acceleration terms of the Navier–Stokes equations are negligible, a drastic increase in the complexity of the solutions to the Navier–Stokes equations for the fluctuating velocity occurs when the quantity $U_j \partial U_i/\partial x_j$ changes from being a linear function of space to being a non-linear one. For instance, in case it is a linear function in space, we have demonstrated that the centroid of the trajectory positions for a certain delay τ coincides with the position of the trajectory for the mean flow for the same delay, and, by contrast, this does not hold in the non-linear case, when the behaviour is far more complex. This may explain why the loci of the maxima of the space-time correlation coefficient coincide with the mean streamlines for homogeneous, isotropic turbulence, for the homogeneous shear flow and for the core region of channel flow, but not for the wall region of channel flow, for boundary layers, for jets and for wakes. Indeed, in the former cases acceleration terms are small and $U_j \partial U_i/\partial x_j$ is a linear (or constant) function of space, whereas in the latter cases at least one of these conditions is violated.

In the case of uniform mean flow, with negligible acceleration terms sufficient conditions for the coincidence of the advection and local mean velocities have been given, along with a formal proof of their sufficiency. It has also been argued that these conditions can be modified to extend the above property to certain other cases including the core region in channel flow and homogeneous shear flow, at least when the temporal separations, which enter into the definition of the advection velocity, are small. It should be noted, however, that to establish these results we had to introduce the concept of predominant concavity, and at least the first assumption we made when analysing this concept seems rather difficult to test experimentally, but possibly DNS could used to achieve this.

We have also shown how our method to take into account the effect of mean velocity gradients on the velocity along a trajectory and on the trajectory position can be used in an attempt to improve the accuracy of Taylor's hypothesis for shear flows. At least in principle our method can allow a substantial relaxation of Lin's criterion in many cases. By using non-intrusive experimental techniques it should be quite possible to examine experimentally the importance of taking into account the effect of the mean velocity gradients when applying Taylor's hypothesis.

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A Exact solution in case of a quadratic U in one dimension

Suppose that U_i is a quadratic function of x_1 only (*i.e.* independent of x_2 and x_3). Then (12) becomes for X_1

$$\begin{aligned} \frac{d^2X_1}{dt^2} &= U_1(X_1)\frac{dU_1}{dX_1}(X_1) && (136)\\ &= \left(\alpha_0+\alpha_1X_1+\alpha_2X_1^2\right)(\alpha_1+2\alpha_2X_1) && (137)\end{aligned}$$

Now this equation contains only dU_1/dx_1, and thus we can assume that U_i is irrotational. Thus we may use (107) to obtain

$$\left(\frac{dX_1}{dt}\right)^2 = K+U_1^2, \tag{138}$$

where

$$K = u(t_0,0)\,(2\alpha_0+u(t_0,0))\ . \tag{139}$$

This implies that

$$\frac{dX_1}{dt} = S\sqrt{K+U_1(X_1)^2}\ , \tag{140}$$

where $S = sign(\alpha_0+u(t_0,0))$ if this expression is non-zero, otherwise $S = sign(\alpha_0\alpha_1)$. If this expression is also zero then $X_1\equiv 0$ so S is immaterial. The equation (140) is separable, and hence the solution is given by

$$\int_0^y \frac{dz}{\sqrt{K+(\alpha_0+\alpha_1z+\alpha_2z^2)^2}} = S(t-t_0)\ . \tag{141}$$

This integral can be expressed in terms of elliptic integrals, and hence after some manipulations, the solution can be expressed using Jacobian elliptic functions (see e.g. (Abramowitz & Stegun, 1970)). We refrain from presenting the detailed calculations and just give the results. Let

$$K_1=\frac{K}{\alpha_2^2},\quad \Delta=\frac{\alpha_0}{\alpha_2}-\frac{\alpha_1^2}{4\alpha_2^2},\quad Y_1=X_1+\frac{\alpha_1}{\alpha_2} \tag{142}$$

In all the solutions it is assumed that $\alpha_2\neq 0$.

1. The case when $K>0$. Let

$$R=\sqrt{\Delta^2+K_1},\ \beta=\sqrt{\frac{1}{2}\left(\sqrt{\Delta^2+K_1}-\Delta\right)},\ A=\sqrt{\frac{\sqrt{R}+\beta}{\sqrt{R}-\beta}}\ . \tag{143}$$

Then we have that

$$Y_1=-\sqrt{R}\frac{1+\frac{1}{A}\mathrm{sc}\left(\left[-\alpha_2A\sqrt{R-\beta^2}S(t-t_0)+ph\right]|\frac{2\beta\sqrt{R}}{(\beta+\sqrt{R})}\right)}{1-\frac{1}{A}\mathrm{sc}\left(\left[-\alpha_2A\sqrt{R-\beta^2}S(t-t_0)+ph\right]|\frac{2\beta\sqrt{R}}{(\beta+\sqrt{R})}\right)}, \tag{144}$$

where

$$ph = \mathrm{sc}^{-1}\left(A\frac{\alpha_1/(2\alpha_2)+\sqrt{R}}{\alpha_1/(2\alpha_2)-\sqrt{R}}|\frac{2\beta\sqrt{R}}{\left(\beta+\sqrt{R}\right)}\right) \tag{145}$$

and where $sc\,(u|m) = \frac{sn(u|m)}{cn(u|m)}$ is a Jacobian elliptic function, and where the notation $y = sc^{-1}\,(u|m)$ means that $u = sc\,(y|m)$.

2. The case when $K < 0$, $\Delta > \sqrt{-K_1}$. Let

$$A = \sqrt{\Delta + \sqrt{-K_1}}, \quad B = \sqrt{\Delta - \sqrt{-K_1}}\,. \tag{146}$$

Then we have that

$$Y_1 = B\mathrm{sc}\left([\alpha_2 AS\,(t-t_0)+ph]\,|\frac{2\sqrt{-K_1}}{\Delta+\sqrt{-K_1}}\right)\,, \tag{147}$$

where

$$ph = \mathrm{sc}^{-1}\left(\frac{\alpha_1}{2\alpha_2 B}|\frac{2\sqrt{-K_1}}{\Delta+\sqrt{-K_1}}\right)\,, \tag{148}$$

and where the notation is the same as in the previous case.

3. The case when $K < 0$, $|\Delta| < \sqrt{-K_1}$. Let

$$A = \sqrt{\Delta + \sqrt{-K_1}}, \quad B = \sqrt{\sqrt{-K_1} - \Delta}\,. \tag{149}$$

Then we have that

$$Y_1 = B\mathrm{nc}\left(\left[\alpha_2\sqrt{A^2+B^2}S\,(t-t_0)+ph\right]|\frac{\Delta+\sqrt{-K_1}}{2\sqrt{-K_1}}\right)\,, \tag{150}$$

where

$$ph = \mathrm{nc}^{-1}\left(\frac{\alpha_1}{2\alpha_2 B}|\frac{\Delta+\sqrt{-K_1}}{2\sqrt{-K_1}}\right)\,, \tag{151}$$

and where $nc\,(u|m) = \frac{1}{cn(u|m)}$ is a Jacobian elliptic function, and where the notation $y = nc^{-1}\,(u|m)$ means that $u = nc\,(y|m)$.

4. $K < 0$, $\Delta < -\sqrt{-K_1}$. Let

$$A = \sqrt{-\Delta + \sqrt{-K_1}}, \quad B = \sqrt{-\Delta - \sqrt{-K_1}}\,. \tag{152}$$

Then we have that

$$Y_1 = A\mathrm{dc}\left([\alpha_2 AS\,(t-t_0)+ph]\,|\frac{-\Delta-\sqrt{-K_1}}{-\Delta+\sqrt{-K_1}}\right)\,, \tag{153}$$

where

$$ph = \mathrm{dc}^{-1}\left(\frac{\alpha_1}{2\alpha_2 A}|\frac{-\Delta-\sqrt{-K_1}}{-\Delta+\sqrt{-K_1}}\right)\ , \tag{154}$$

and where $dc\,(u|m) = \frac{dn(u|m)}{cn(u|m)}$ is a Jacobian elliptic function, and where the notation $y = dc^{-1}\,(u|m)$ means that $u = dc\,(y|m)$.

5. $K = 0$, $\Delta > 0$. In this case the solution is

$$Y_1 = \sqrt{\Delta}\frac{\frac{\alpha_1}{2\alpha_2\sqrt{\Delta}} + \tan\left(\alpha_2\sqrt{\Delta}S\,(t-t_0)\right)}{1 - \frac{\alpha_1}{2\alpha_2\sqrt{\Delta}}\tan\left(\alpha_2\sqrt{\Delta}S\,(t-t_0)\right)}\ . \tag{155}$$

6. $K = 0$, $\Delta < 0$. In this case the solution is

$$Y_1 = \sqrt{-\Delta}\frac{\frac{\alpha_1}{2\alpha_2} + \sqrt{-\Delta} + \left(\frac{\alpha_1}{2\alpha_2} - \sqrt{-\Delta}\right)\exp\left(2\alpha_2\sqrt{-\Delta}S\,(t-t_0)\right)}{\frac{\alpha_1}{2\alpha_2} + \sqrt{-\Delta} - \left(\frac{\alpha_1}{2\alpha_2} - \sqrt{-\Delta}\right)\exp\left(2\alpha_2\sqrt{-\Delta}S\,(t-t_0)\right)}\ . \tag{156}$$

7. $K = \Delta = 0$. In this case the solution is

$$Y_1 = \frac{\alpha_1}{\alpha_2}\frac{1}{2-\alpha_1 S\,(t-t_0)}\ . \tag{157}$$

8. $K < 0$, $\Delta = \sqrt{-K}$, $\alpha_1 \neq 0$. In this case the solution is

$$Y_1 = -\frac{\sqrt{2\Delta}\frac{\alpha_1}{\alpha_2}\left(\sqrt{\left(\frac{\alpha_1}{2\alpha_2}\right)^2 + 2\Delta} - \sqrt{2\Delta}\right)\exp\left(\alpha_2\sqrt{2\Delta}S\,(t-t_0)\right)}{\left(\sqrt{\left(\frac{\alpha_1}{2\alpha_2}\right)^2 + 2\Delta} - \sqrt{2\Delta}\right)^2\exp\left(2\alpha_2\sqrt{2\Delta}S\,(t-t_0)\right) - \frac{\alpha_1^2}{4\alpha_2^2}}\ . \tag{158}$$

9. $K < 0$, $\Delta = -\sqrt{-K}$, $\alpha_1 \neq 0$. In this case the solution is

$$Y_1 = -\sqrt{-2\Delta}\sqrt{1 + \left(\frac{\sqrt{-1-\frac{1}{2\Delta}\left(\frac{\alpha_1}{2\alpha_2}\right)^2} + \tan\left(\alpha_2\sqrt{-2\Delta}S\,(t-t_0)\right)}{1 - \sqrt{-1-\frac{1}{2\Delta}\left(\frac{\alpha_1}{2\alpha_2}\right)^2}\tan\left(\alpha_2\sqrt{-2\Delta}S\,(t-t_0)\right)}\right)^2}\ . \tag{159}$$

It can be seen that the conditions $\Delta^2 + K = 0$, $\alpha_1 \neq 0$ imply that

$$\frac{1}{-2\Delta}\left(\frac{\alpha_1}{2\alpha_2}\right)^2 \geq 1\ . \tag{160}$$

10. $K < 0$, $\Delta^2 + K = 0$, $\alpha_1 = 0$. These conditions imply that $u_0 = -\alpha_0$. Hence, it can be seen that X_1 and all its derivatives vanish at $t = t_0$. Thus

$$X_1 \equiv 0 \tag{161}$$

Some comments should be made about these solutions. Firstly, except for some trivial cases all of the above solutions blow-up in finite time. This only manifests the local character of the validity of the quadratic approximation. Moreover, it is far from unlikely that the method breaks down earlier when the characteristics cross each other. Secondly, all solutions have a strongly non-linear dependence on $u_1(t_0, 0)$.