RESOLVENT ESTIMATES IN l_p FOR DISCRETE LAPLACIANS ON IRREGULAR MESHES AND MAXIMUM-NORM STABILITY OF PARABOLIC FINITE DIFFERENCE SCHEMES

Michel Crouzeix and Vidar Thomée

Abstract. In an attempt to show maximum-norm stability and smoothing estimates for finite element discretizations of parabolic problems on nonquasi-uniform triangulations we consider the lumped mass method with piecewise linear finite elements in one and two space dimensions. By an energy argument we derive resolvent estimate for the associated discrete Laplacian, which is then a finite difference operator on an irregular mesh, which show that this generates an analytic semigroup in l_p for $p < \infty$, uniformly in the mesh, assuming in the two-dimensional case that the triangulations are of Delaunay type, and with a logarithmic bound for $p = \infty$. By a different argument based on a weighted norm estimate for a discrete Green's function this is improved to hold without a logarithmic factor for $p = \infty$ in one dimension under a weak mesh-ratio condition. Our estimates are applied to show stability also for time stepping methods.

1. Introduction.

Recently several papers have appeared concerning stability and smoothing properties with respect to the maximum-norm of finite element discretizations of parabolic problems, see, e.g., Palencia [4], [5], Schatz, Thomée, and Wahlbin [8], [9], and Thomée and Wahlbin [10] and [11]. In contrast to the corresponding investigations in L_2-norm these results require so called inverse properties of the families of finite element spaces, which restricts the associated triangulations to quasi-uniform ones. In an attempt to remove such restrictive assumptions we consider the very special cases of spatially one and two-dimensional problems, piecewise linear approximating functions, and with numerical quadrature in the inner product containing the time derivative. The discretization method then reduces to the lumped mass method and may be considered as a finite difference method with variable mesh-width. The proofs of our stability results are carried out by deriving resolvent estimates for discrete analogues of the Laplacian.

We consider the model initial boundary value problem

$$u_t = \Delta u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \quad \text{with } u(\cdot, 0) = v \quad \text{in } \Omega, \tag{1.1}$$

where Ω is either the one-dimensional interval $(0, 1)$ or a convex plane domain with smooth boundary $\partial\Omega$. Introducing the solution operator $E(t)$ by $u(t) = E(t)v$, the

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maximum-principle for (1.1) shows that this operator, the semigroup $E(t) = e^{t\Delta}$ generated by the Laplacian, is a contraction semigroup in the Banach space $\mathcal{C}_0(\Omega)$,

$$(1.2)\qquad \|E(t)v\|_\infty \le \|v\|_\infty, \quad \text{where} \quad \|v\|_\infty = \sup_\Omega |v(x)|,$$

and this immediately implies the corresponding result with respect to the norm in $L_p = L_p(\Omega)$, where $1 \le p < \infty$. We also have the smoothing property

$$\|E'(t)v\|_p \le Ct^{-1}\|v\|_p, \quad \text{for } t > 0,\ 1 \le p \le \infty, \quad \text{where} \quad \|v\|_p = \|v\|_{L_p(\Omega)}.$$

This property shows that $E(t)$ is an analytic semigroup in L_p, which is equivalent to an estimate for the resolvent $R(\lambda;\Delta) = (\lambda I - \Delta)^{-1}$: For some $\theta \in (\frac{1}{2}\pi, \pi)$,

$$(1.3)\qquad \|R(\lambda;\Delta)\|_p \le \frac{M}{|\lambda|}, \quad \text{for } |\arg\lambda| \le \theta, \quad 1 \le p \le \infty.$$

This in turn is the same as saying that the solution of the elliptic problem

$$(1.4)\qquad \lambda u - \Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega,$$

satisfies the inequality

$$(1.5)\qquad \|u\|_p \le \frac{M}{|\lambda|}\|f\|_p, \quad \text{for } |\arg\lambda| \le \theta.$$

For $p = \infty$ this is not as trivial as the proof of (1.2), and the domain of $-\Delta$ in $\mathcal{C}_0(\Omega)$ has to be restricted to u with $\Delta u = 0$ on $\partial\Omega$, see, e.g., Pazy [7].

For later reference we sketch a proof in the one-dimensional case for $1 < p < \infty$ which yields a bound that is not uniform in p. For this, we multiply (1.4) by $\varphi = \bar{u}|u|^{p-2}$ and integrate over Ω to obtain, after integration by parts in the second term,

$$(1.6)\qquad \lambda\|u\|_p^p + \int_0^1 u'(\bar{u}|u|^{p-2})'\,dx = (f, u|u|^{p-2}), \quad \text{where } (v,w) = \int_0^1 v\bar{w}\,dx.$$

To proceed we think of (1.6) as a relation of the form

$$(1.7)\qquad ae^{i\varphi} + be^{i\psi} = c, \quad \text{with } a, b > 0,\ \varphi,\ \psi \in R.$$

and note that, by multiplication by $e^{-i\varphi}$ and taking real parts, since $\cos(\psi - \varphi) \ge 0$, this implies

$$(1.8)\qquad a \le |c|, \quad \text{if} \quad |\varphi - \psi| \le \frac{\pi}{2}.$$

To apply this to (1.6), with $\varphi = \arg\lambda$, we show that the argument ψ of the integral term satisfies $|\psi| \le \arcsin|1 - 2/p|$. For this we study the integrand

$$K_p = u'(\bar{u}|u|^{p-2})' = \frac{p}{2}|u'|^2|u|^{p-2} + \frac{p-2}{2}(u')^2\bar{u}^2|u|^{p-4}.$$

Setting $u'^2\bar{u}^2 = re^{i\omega}$, we have $K_p = (1/2)r^2|u|^{p-4}(p+(p-2)e^{2i\omega})$, so that we easily find $|\arg K_p| = |\arg(p+(p-2)e^{2\mathrm{i}\omega})| \le \arcsin(|p-2|/p)$. Application of (1.8) now gives

$$|\lambda|\,\|u\|_p^p \le |(f, u|u|^{p-2})| \le \|f\|_p \|u\|_p^{p-1}, \quad \text{for } |\arg\lambda| \le \frac{\pi}{2} - \arcsin|1-\frac{2}{p}|,$$

which shows

$$\|R(\lambda;\Delta)\|_p \le \frac{1}{|\lambda|}, \qquad \text{for } |\arg\lambda| \le \theta_p = \arccos|1-\frac{2}{p}| \le \frac{\pi}{2}. \tag{1.9}$$

We now want to derive a bound for the resolvent in a wider sector which extends to the left halfplane. For this we use (1.9) (with λ replaced by μ) to obtain

$$\begin{aligned}\|R(\lambda;\Delta)\|_p &\le \|R(\mu;\Delta)\|_p/(1-|\lambda-\mu|\|R(\mu;\Delta)\|_p)\\ &\le \frac{1}{|\mu|-|\lambda-\mu|}, \quad \text{if } |\arg\mu| = \theta_p, \quad |\lambda-\mu|/|\mu| < 1.\end{aligned}$$

Letting $|\mu| \to \infty$ we find $|\mu| - |\lambda-\mu| \to |\lambda|\cos(|\arg\lambda| - \theta_p)$ and hence, with $M_p(\varphi) = 1/(\cos(|\varphi| - \theta_p)$,

$$\|R(\lambda;\Delta)\|_p \le \frac{M_p(\arg\lambda)}{|\lambda|}, \quad \text{for } \theta_p \le |\arg\lambda| < \theta_p + \frac{\pi}{2}. \tag{1.10}$$

In particular, for $p \ge 2$, if we assume that $|\arg\lambda| \le \frac{\pi}{2} + \arcsin(1/\sqrt{p})$, then $\cos(|\arg\lambda| - \theta_p) \ge \cos(\arcsin(1/\sqrt{p}) + \arcsin(1-2/p)) = 1/\sqrt{p}$, and hence

$$\|R(\lambda;\Delta)\|_p \le \frac{\sqrt{p}}{|\lambda|}, \qquad \text{for } |\arg\lambda| \le \frac{\pi}{2} + \arcsin\frac{1}{\sqrt{p}}; \tag{1.11}$$

for $p < 2$ the corresponding inequalities hold with p replaced by the conjugate exponent $p' = p/(p-1)$ in the bounds; note that $|1-2/p'| = |1-2/p|$.

The estimate (1.9) shows that $E(t)$ can be extended into a contraction semigroup in L_p for t in the same sector in the complex plane, i.e.,

$$\|E(t)\|_p \le 1, \qquad \text{for } |\arg t| \le \theta_p, \quad 1 < p < \infty. \tag{1.12}$$

The inequality (1.10) (or (1.11)) shows that the semigroup $E(t)$ is analytic in L_p, and by the theory of analytic semigroups one may conclude that in addition to stability $E(t)$ has the smoothing property $\|E'(t)\|_p \le Cp/t$ for $p \ge 2$ (cf. Pazy [7]). Using (1.12) one may demonstrate directly the sharper smoothing estimate

$$\|E'(t)\|_p \le \frac{1}{2t}(\sqrt{p-1} + \frac{1}{\sqrt{p-1}}), \qquad \text{for real } t > 0. \tag{1.13}$$

In fact, using an eigenfunction expansion we note that $E(t) = e^{t\Delta}$ may be thought of as analytic in t in L_2. Using the Cauchy formula in the circle $\gamma_r(t)$ with center at $t \in R_+$ and radius $r = t\sin\theta_p$ we may write

$$E'(t) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_r(t)} \frac{E(\tau)}{(\tau-t)^2}\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} \frac{E(t+re^{\mathrm{i}\varphi})}{(re^{\mathrm{i}\varphi})^2}\,r\,e^{\mathrm{i}\varphi}d\varphi,$$

and hence (1.12) yields $\|E'(t)\|_p \le 1/r$. Since $\sin^2\theta_p = 1-\cos^2\theta_p = 1-(1-2/p)^2 = 4(p-1)^2/p^2$, this shows (1.13).

We owe to Cesar Palencia the observation that a slightly weaker version of (1.12) may be obtained by complex interpolation (see, e.g., Davies [4], Theorem 1.4.2) between $\|E(t)\|_2 \le 1$ for $\mathrm{Re}\, t \ge 0$, which is obvious by spectral representation, and $\|E(t)\|_\infty \le 1$ for $t \ge 0$, which follows by the maximum principle. For $p > 2$ we then obtain $\|E(t)\|_p \le 1$ for $|\arg t| \le \pi/p$ (which angle is smaller than $\theta_p \approx 2/\sqrt{p}$ as $p \to \infty$). This method also yields somewhat weaker resolvent estimates than (1.10) and (1.11).

The inequality (1.12) appears indeed to be optimal in the sense that $E(t)$ cannot be a contraction in a wider sector than $|\arg t| \le \theta_p$. To elucidate this we consider the case that $\Omega = R$ and $p > 2$. We note that if $\|E(\rho e^{\mathrm{i}\theta})v\|_p = \|u(\rho e^{\mathrm{i}\theta})\|_p$ is a contraction for $\rho \ge 0$, then, for appropriate initial values v,

$$\frac{d}{d\rho}\int_R |u(\rho e^{\mathrm{i}\theta})|^p d\rho\Big|_{\rho=0} = \mathrm{Re}\int_R |v|^{p-2}\bar{v}v'' e^{\mathrm{i}\theta}dx$$
$$= -\,\mathrm{Re}\int_R \big(\frac{p}{2}|v|^{p-2}|v'|^2 + \frac{p-2}{2}|v|^{p-4}\bar{v}^2(v')^2\big)e^{\mathrm{i}\theta}dx \le 0.$$

Choosing $v(x) = e^{-e^{i\omega}x^2/2}$, with ω arbitrary in $(-\pi/2, \pi/2)$, this shows

$$\mathrm{Re}\int_R |v|^p x^2(p + (p-2)e^{2\mathrm{i}\omega})e^{\mathrm{i}\theta}dx \ge 0,$$

or $|\theta + \arg(1 + (1-2/p)e^{2\mathrm{i}\omega})| \le \pi/2$ for $|\omega| \le \pi/2$. But since we easily find $\max_{|\omega|\le\pi/2} |\arg(p + (p-2)e^{2\mathrm{i}\omega})| = \arcsin(1-2/p) = \pi/2 - \theta_p$ we conclude that $|\theta| \le \theta_p$.

We now consider semidiscretization in space of (1.1) by piecewise linear finite elements. In the one-dimensional case ($d = 1$), let $0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary partition of $\bar\Omega$ and let S_h be the continuous piecewise linear functions on $\bar\Omega$ which vanish at $x = 0$ and $x = 1$. In the two-dimensional case ($d = 2$), let $\mathcal{T}_h$ denote a regular triangulation of Ω and let S_h be the continuous piecewise linear functions on $\mathcal{T}_h$ which vanish on $\partial\Omega$. In each case the corresponding semidiscrete problem is then to find $U = U(t) \in S_h$ such that

$$(1.14) \qquad (U_t, \chi) + (\nabla U, \nabla\chi) = 0, \quad \forall \chi \in S_h, \quad \text{for } t \ge 0, \quad \text{with } U(0) = V.$$

The solution operator $E_h(t)$ defined by $U(t) = E_h(t)V$ is then the semigroup generated by the discrete Laplacian defined by $(\Delta_h\psi, \chi) = -(\nabla\psi, \nabla\chi)$ for $\psi, \chi \in S_h$. Under the assumption of quasi-uniformity of the partitions it was shown in Crouzeix, Larsson, and Thomée [3] for $d = 1$ and for general dimension in Thomée and Wahlbin [11] that the analogue of (1.3) holds for Δ_h and that thus $E_h(t)$ is an analytic semigroup in S_h, equipped with the L_p-norm, uniformly in p and h.

The problem we want to address here is thus the possibility of removing the restrictive and undesirable assumption of quasi-uniformity of the partition. For $d = 1$ this assumption means that with $h_i = x_i - x_{i-1}$ the ratios h_i/h_j are bounded for all i, j, and for $d = 2$ that each triangle of $\mathcal{T}_h$ contains a circle of radius ch with $c > 0$ independent of the maximal diameter h of the triangles of $\mathcal{T}_h$.

In order to show stability results for nonquasi-uniform partitions we have had to modify our semidiscrete method (1.14) by applying a quadrature formula to the first term. We begin to do this for $d = 1$. Noting that $U \in S_h$ is defined by the values $U_j = U(x_j)$ we introduce the discrete inner product

$$(V,W)_h = \sum_{j=1}^{N-1} \kappa_j V_j \bar{W}_j, \quad \text{where } \kappa_j = \tfrac{1}{2}(h_j + h_{j+1}).$$

The semidiscrete problem now reads

$$(U_t, \chi)_h + (U', \chi') = 0, \quad \forall \chi \in S_h, \quad \text{for } t \ge 0, \quad \text{with } U(0) = V. \tag{1.15}$$

The definition of the discrete Laplacian $\Delta_h : S_h \to S_h$ takes the form

$$(\Delta_h \psi, \chi)_h = -(\psi', \chi'), \quad \forall \psi, \chi \in S_h, \tag{1.16}$$

and we find easily

$$(\Delta_h U)_j = -\Big(\frac{U_{j+1} - U_j}{h_{j+1}} - \frac{U_j - U_{j-1}}{h_j}\Big)/\kappa_j, \quad \text{for } j = 1, \dots, N-1. \tag{1.17}$$

The parabolic problem (1.15) may also be written as the semidiscrete finite difference equation

$$U_t - \Delta_h U = 0, \quad \text{for } t \ge 0, \quad \text{with } U(0) = V. \tag{1.18}$$

Denoting by $E_h(t) = e^{t\Delta_h}$ the solution operator of (1.15) (or (1.18)), it is easy to see that this is a contraction semigroup on S_h with respect to the maximum-norm

$$\|E_h(t)V\|_\infty \le \|V\|_\infty, \quad \text{where } \|V\|_\infty = \|V\|_{h,\infty} = \max_j |V_j|.$$

This follows easily by a discrete maximum-principle but may also be expressed by saying that the resolvent $R(\lambda; \Delta_h) = (\lambda I - \Delta_h)^{-1}$ satisfies

$$\|R(\lambda; \Delta_h)\|_\infty \le \frac{1}{\lambda}, \quad \text{for } \lambda > 0; \tag{1.19}$$

this is a special case of the discrete analogue of (1.9) which is contained in Theorem 2.1 below.

We are also able to show that the analogues of (1.10) - (1.13) hold in this case with respect to the natural discrete L_p-norm,

$$\|V\|_{h,p} = \Big(\sum_{j=1}^{N} \kappa_j |V_j|^p\Big)^{1/p}. \tag{1.20}$$

For $p = \infty$ one may use the analogue of (1.11) to derive a corresponding maximum-norm bound with the factor $\sqrt{p}$ replaced by $|\log \underline{\kappa}|^{1/2}$ where $\underline{\kappa} = \min_j \kappa_j$; this will result in a smoothing estimate in maximum-norm with a factor $|\log \underline{\kappa}|^{1/2}$.

Under the very weak condition on the partition that $h_j/h_{j\pm 1}$ is bounded we are also able to show in Theorem 3.1 that the resolvent is bounded in maximum-norm, and hence uniformly in L_p for $1 \le p \le \infty$. The proof is based on that in Crouzeix, Larsson, and Thomée [3] and uses a weighted norm estimate for a discrete Green's function, with a weight depending on the partition.

The approach taken above for $1 < p < \infty$ carries over to two space dimensions and we thus show in Theorem 4.1 analogues of the resolvent estimates (1.9) - (1.11), for piecewise linear finite elements on not necessarily quasi-uniform triangulations, but assuming these to be of Delaunay type. Again this will show a maximum-norm estimate with a constant depending this time on the logarithm of the minimum area of a mesh neighborhood of a vertex.

In Section 5 we shall apply our resolvent estimates to show some stability estimates for fully discrete finite difference schemes for our parabolic equations in one and two space dimensions.

We begin to consider the stability of the fully discrete backward Euler method for $d = 1$: Let $0 = t_0 < t_1 < \cdots < t_n < \dots$ be a partition of the positive time axis, set $k_n = t_n - t_{n-1}$, and let $U^n = U(t_n)$. With $\bar\partial_k U^n = (U^n - U^{n-1})/k$ we may then pose the problem

$$\bar\partial_{k_n} U^n - \Delta_h U^n = 0, \quad \text{for } n \ge 1, \quad \text{with } U^0 = V.$$

Setting $E_{kh} = (I - k\Delta_h)^{-1}$ this may also be expressed as

$$U^n = E_{k_n h} U^{n-1} = E_{kh}(t_n)V, \quad \text{where } E_{kh}(t_n) = \prod_{j=1}^{n} E_{k_j h}. \tag{1.21}$$

Since $E_{kh} = k^{-1} R(k^{-1}; \Delta_h)$, (1.19) shows that $\|E_{kh}\|_{h,p} \le 1$ and hence

$$\|E_{kh}(t_n)\|_{h,p} \le \prod_{j=1}^{n} \|E_{k_j h}\|_{h,p} \le 1, \quad \text{for } 1 \le p \le \infty.$$

A corresponding result for $d = 2$ will follow from Theorem 4.1 below.

For the purpose of treating more general time-stepping methods of the form (1.21) where now $E_{kh} = r(k\Delta_h)$, with $r(\lambda)$ is a rational function, we show in Section 5 a slight modification of a Banach space result of Bakaev [2] which permits application of our resolvent estimates in the complex plane. In particular, if $r(\lambda)$ is A-acceptable, so that $|r(\lambda)| \le 1$ for $\operatorname{Re}\lambda \le 0$, and if $|r(\infty)| < 1$ then the resolvent estimate (1.10) implies a stability bound of the form

$$\|E_{kh}(t_n)V\|_{h,p} \le C_p \|V\|_{h,p}, \quad \text{for } 2 \le p < \infty,$$

with a certain modification for $p = \infty$. If $|r(\infty)| = 1$ a logarithmic factor may have to be added to the stability bound unless the time stepping is quasi-uniform, see Section 5 for details.

2. Resolvent estimates in one dimension.

We begin by showing a resolvent estimate in the discrete L_p-norm which is valid for any choice of the partitions of $\Omega = (0, 1)$.

Theorem 2.1. *With Δ_h defined by (1.16) we have*

$$\|R(\lambda;\Delta_h)\|_{h,p} \le \frac{1}{|\lambda|}, \quad \text{for } |\arg\lambda| \le \theta_p = \arccos|1-\frac{2}{p}|, \quad 1 \le p \le \infty, \tag{2.1}$$

and, with $M_p(\varphi) = 1/\cos(|\varphi| - \theta_p)$,

$$\|R(\lambda;\Delta)\|_p \le \frac{M_p(\arg\lambda)}{|\lambda|}, \quad \text{for } \theta_p \le |\arg\lambda| < \theta_p + \frac{\pi}{2}, \quad 1 \le p \le \infty. \tag{2.2}$$

Further, with $\underline{\kappa} = \min_i \kappa_i$

$$\|R(\lambda;\Delta_h)\|_\infty \le e\frac{|\log\underline{\kappa}|^{1/2}}{|\lambda|}, \quad \text{for } |\arg\lambda| \le \frac{\pi}{2} + \arcsin\frac{1}{|\log\kappa|^{1/2}}. \tag{2.3}$$

For the proof we need the following lemma:

Lemma 2.1. *Let z and w be two complex numbers and set*

$$H_p = (w-z)(\bar{w}|w|^{p-2} - \bar{z}|z|^{p-2}), \quad \text{where } 1 < p < \infty.$$

Then

$$|\arg H_p| \le \arcsin|1 - \frac{2}{p}|.$$

Proof. Setting $d = w - z$ and $\varphi(t) = d\overline{(z+td)}\,|z+td|^{p-2}$ we may write

$$H_p = d\,\overline{(z+d)}\,|z+d|^{p-2} - d\,\bar{z}|z|^{p-2} = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\,dt,$$

and it hence suffices to show $|\arg\varphi'(t)| \le \arcsin|1-2/p|$. With $d^2\,\overline{(z+td)}^2 = re^{\mathrm{i}\omega}$ we have

$$\varphi'(t) = \frac{p}{2}|d|^2|z+td|^{p-2} + \frac{p-2}{2}d^2\,\overline{(z+td)}^2|z+td|^{p-4} = \tfrac{1}{2}|z+td|^{p-4}r^2(p+(p-2)e^{2\mathrm{i}\omega}),$$

and the desired result then follows as for the continuous case in the introduction. □

Proof of Theorem 2.1. Introducing the discrete elliptic problem to find $U \in S_h$ from

$$\lambda U - \Delta_h U = F, \tag{2.4}$$

we have $U = R(\lambda;\Delta_h)F$ so that the statement (2.1) will follow from

$$\|U\|_{h,p} \le \frac{1}{|\lambda|}\|F\|_{h,p}, \quad \text{for } \lambda > 0. \tag{2.5}$$

We obtain from (2.4)

$$\lambda(U,\chi)_h + (U',\chi') = (F,\chi)_h, \quad \forall\chi \in S_h, \tag{2.6}$$

and choosing $\chi = I_h(U|U|^{p-2})$ where I_h is the interpolant into S_h we have

$$(U', \chi') = \sum_{j=1}^{N} h_j^{-1} H_{p,j}, \quad \text{where } H_{p,j} = (U_j - U_{j-1})(\bar{U}_j|U_j|^{p-2} - \bar{U}_{j-1}|U_{j-1}|^{p-2}).$$

Note that each $H_{p,j}$ is of the form of H_p in Lemma 2.1, and this lemma therefore shows $|\arg(U', \chi')| \leq \arcsin|1 - 2/p|$. We may now think of (2.6) as a relation of the form (1.7), and the argument in the proof of (1.9) may again be applied to show (2.1). We then deduce (2.2) from (2.1) in the same way as (1.10) follows from (1.9) (for $p = \infty$, let $p \to \infty$). In particular (1.11) holds in $\|\cdot\|_{h,p}$ with Δ replaced by Δ_h.

To show (2.3) we note that, with k suitable,

$$\|U\|_{h,p} \leq \|U\|_\infty = \kappa_k^{-1/p}(\kappa_k|U_k|^p)^{1/p} \leq \underline{\kappa}^{-1/p}\|U\|_{h,p}, \quad \text{for } U \in S_h.$$

Therefore for $p = |\log \underline{\kappa}|$, we have $\|R(\lambda; \Delta_h)\|_\infty \leq e\,\|R(\lambda; \Delta_h)\|_{h,p}$, and the desired result now follows from the analogue of (1.11). □

In the same way as in the introduction, Theorem 2.1 can be translated into properties for the semigroup $E_h(t) = e^{t\Delta_h}$.

Corollary 2.1. *We have*

$$\|E_h(t)\|_{h,p} \leq 1, \quad \textit{for } |\arg t| \leq \theta_p = \arccos|1 - \frac{2}{p}|, \quad 1 \leq p \leq \infty, \tag{2.7}$$

and

$$t\|E_h'(t)\|_{h,p} \leq \frac{1}{2}(\sqrt{p-1} + \frac{1}{\sqrt{p-1}}), \quad \textit{for } t > 0, \quad 1 < p < \infty. \tag{2.8}$$

3. A logarithm free maximum-norm estimate for the resolvent for $d = 1$.

In this section we show a maximum-norm bound for the resolvent which is valid in any sector in the complex plane not containing the negative real axis, under a mild condition on the partition.

Theorem 3.1. *Assume that $h_{j\pm1}/h_j \leq \mu$. Then, for any $\theta < \pi$, we have, with $C = C_{\theta,\mu}$,*

$$\|R(\lambda; \Delta_h)\|_{h,p} \leq \frac{C}{|\lambda|}, \quad \textit{for } |\arg \lambda| \leq \theta, \quad 1 \leq p \leq \infty.$$

Proof. It suffices to consider the case $p = \infty$. The proof is a modification of that in [3]. We remark that the problem (2.4) can be written, with α_k, β_k positive, cf. (1.17),

$$-\alpha_k U_{k+1} + (\alpha_k + \beta_k + \lambda)U_k - \beta_k U_{k-1} = F_k,$$

Assuming that k is such that $\|U\|_\infty = |U_k|$, and noting that $\alpha_k + \beta_k = 2/(h_k h_{k+1})$ we deduce

$$|\lambda|\,|U_k| \leq \|F\|_\infty + \frac{4}{h_k h_{k+1}}|U_k|. \tag{3.1}$$

Assume first $|\lambda| \geq 6/(h_k h_{k+1})$ so that

$$\frac{4}{h_k h_{k+1}} \leq \frac{2|\lambda|}{3}.$$

Then (3.1) implies

$$\|U\|_\infty = |U_k| \leq \frac{3}{|\lambda|}\|F\|_\infty,$$

which is the desired result for these λ.

The main part of the proof will concern the remaining case $|\lambda| \leq 6/(h_k h_{k+1})$. We introduce the discrete Green's function $G_j = G_{j,k}(\lambda) \in S_h$ by

$$(\chi', G') + \lambda(\chi, G)_h = \chi_k, \quad \forall \chi \in S_h,$$

and use it in the representation

$$U_k = (U', G') + \lambda(U, G)_h = (F, G)_h.$$

Since thus

$$|U_k| \leq \|F\|_{h,\infty}\|G\|_{h,1},$$

it suffices to show

$$\|G\|_{h,1} \leq \frac{C}{|\lambda|}, \quad \text{for } |\lambda| \leq 6/(h_k h_{k+1}). \tag{3.2}$$

This will be done by a weighted norm technique.

In [3], where quasi-uniformity of the partition was assumed, the weight function used was of the form $\omega(x) = e^{\gamma\xi|x-x_k|}$, with γ appropriately small. Note that then, for $x = x_j > x_k$, $\omega(x_j) = \prod_{i=k+1}^{j} e^{\gamma\xi h_i}$, and similarly for $x_j < x_k$. Here we set, with $\xi = |\lambda|^{1/2}$ and γ to be chosen later,

$$\omega_j = \begin{cases} 1, & \text{for } j = k, \\ \prod_{i=k+1}^{j}(1 + \gamma\xi h_i), & \text{for } j > k, \\ \prod_{i=j+1}^{k}(1 + \gamma\xi h_i), & \text{for } j < k. \end{cases}$$

We shall show that

$$\|G\omega\|_h \leq C\xi^{-3/2} \quad \text{and} \quad \|\omega^{-1}\|_h \leq C\xi^{-1/2}.$$

Together these inequalities imply

$$\|G\|_{h,1} \leq \|G\omega\|_h\|\omega^{-1}\|_h \leq C\xi^{-2} = C|\lambda|^{-1},$$

which is our claim (3.2). We begin with the bound for ω^{-1}. For this we note that $\omega_j^{-1} \leq (1 + |x_j - x_k|\gamma\xi)^{-1}$, and hence

$$\begin{aligned} \|\omega^{-1}\|_h^2 = \sum_{j=1}^{N-1} \kappa_j \omega_j^{-2} &\leq \tfrac{1}{2}(h_k + h_{k+1}) + \frac{1+\mu}{2}\int_0^1 (1 + |y - x_k|\gamma\xi)^{-2} dy \\ &\leq \kappa_k + (1+\mu)\int_0^\infty (1 + y\gamma\xi)^{-2} dy = \frac{C}{\gamma\xi} + Ch_k \leq C\xi^{-1}. \end{aligned}$$

For the desired bound for $G\omega$, let $V \in S_h$ be defined by $V_j = G_j\omega_j$ for $j = 0, \dots, M$. We shall then show that, for γ suitably chosen,

$$\|V'\|^2 + \xi^2\|V\|_h^2 \le C\xi^{-1}. \tag{3.3}$$

We note that with $W \in S_h$ defined by $W_j = G_j\omega_j^2 = V_j\omega_j$, for $j = 0, \dots, M$, we have $\|V\|_h^2 = (W, G)_h$ and $(W', G') + \lambda(W, G)_h = W_k = V_k$. Hence

$$\|V'\|^2 + \lambda\|V\|_h^2 = \|V'\|^2 - (W', G') + (W', G') + \lambda(W, G)_h = \|V'\|^2 - (W', G') + V_k.$$

Since $|\arg\lambda| \le \theta < \pi$ and $|a + b\,e^{\mathrm{i}\theta}| \ge \cos(\theta/2)(a+b)$ when $a,\, b > 0$, this shows

$$(\|V'\|^2 + \xi^2\|V\|_h^2)\cos\frac{\theta}{2} \le \big|\|V'\|^2 + \lambda\|V\|_h^2\big| \le |V_k| + \|V'\|^2 - (W', G'). \tag{3.4}$$

We can bound the first term on the right by

$$\begin{aligned}|V_k| &\le \|V\|_\infty \le \sqrt{\|V'\|\,\|V\|_h} \\ &\le \frac{1}{8\epsilon\xi} + 2\epsilon\xi\,\|V'\|\,\|V\|_h \le \frac{1}{8\epsilon\xi} + \epsilon\,(\|V'\|^2 + \xi^2\|V\|_h^2).\end{aligned} \tag{3.5}$$

For the last term on the right in (3.4) we write

$$\|V'\|^2 - (W', G') = \sum_{j=1}^{N} h_j^{-1}\big(|V_j - V_{j-1}|^2 - (W_j - W_{j-1})(G_j - G_{j-1})\big) = \sum_{j=1}^{N} h_j^{-1}B_j.$$

Here, since $W_j = V_j\omega_j,\ G_j = V_j\omega_j^{-1}$,

$$\begin{aligned}B_j &= V_j\bar V_{j-1}(\omega_j\omega_{j-1}^{-1} - 1) + \bar V_j V_{j-1}(\omega_{j-1}\omega_j^{-1} - 1) \\ &= V_j\bar V_j(\omega_j\omega_{j-1}^{-1} + \omega_{j-1}\omega_j^{-1} - 2) - V_j\,\overline{\partial_j V}(\omega_j\omega_{j-1}^{-1} - 1) - \bar V_j\,\partial_j V(\omega_{j-1}\omega_j^{-1} - 1).\end{aligned}$$

Here

$$\omega_j\omega_{j-1}^{-1} - 1 = \begin{cases} h_j\gamma\xi, & \text{if } j > k, \\ -h_j\gamma\xi(1 + h_j\gamma\xi)^{-1}, & \text{if } j \le k, \end{cases}$$

and similarly

$$\omega_{j-1}\omega_j^{-1} - 1 = \begin{cases} -h_j\gamma\xi(1 + h_j\gamma\xi)^{-1}, & \text{if } j > k, \\ h_j\gamma\xi, & \text{if } j \le k, \end{cases}$$

and hence

$$\omega_j\omega_{j-1}^{-1} + \omega_{j-1}\omega_j^{-1} - 2 = (h_j\gamma\xi)^2(1 + h_j\gamma\xi)^{-1}.$$

Thus, using also $h_j \le \mu h_{j-1}$,

$$h_j^{-1}|B_j| \le h_j\gamma^2\xi^2|V_j|^2 + 2\gamma\xi|V_j|\,|\partial V_j| \le \epsilon h_j^{-1}|\partial V_j|^2 + C_\epsilon\gamma^2\xi^2(h_j + h_{j-1})|V_j|^2,$$

from which we deduce

$$\|V'\|^2 - (W', G') \le \epsilon\|V'\| + 2C_\epsilon\gamma^2\xi^2\|V\|_h^2.$$

Choosing $\epsilon = (\cos(\theta/2))/4,\ \gamma^2 = \epsilon/(2C_\epsilon)$, and using (3.5) we finally obtain the estimate (3.3) from (3.4), with the constant $C = 1/(\cos(\theta/2))^2$. The proof is now complete. □

4. Resolvent estimates in l_p in two dimensions.

In this section we consider the initial boundary value problem (1.1) where Ω is a two-dimensional convex domain and $\partial\Omega$ is smooth. Let $\mathcal{T}_h = \{\tau\}$ be triangulations of $\Omega \subset R^2$ without any particular restrictions, and let S_h denote continuous piecewise linear functions on $\mathcal{T}_h$, which vanish on $\partial\Omega$. Let $\{P_j\}_1^{N_h}$ denote the inner vertices of S_h, and $\{P_j\}_{N_h+1}^{N_h+M_h}$ the ones on $\partial\Omega$. Further, let $\{\Phi_j\}_1^{N_h+M_h} \subset S_h$ be the corresponding basis functions, with $\Phi_j(P_i) = \delta_{ij}$, and set $U_j = U(P_j)$ for $U \in S_h$, so that $U_j = 0$ for $j = N_h+1, \ldots, N_h+M_h$.

With $(\cdot,\cdot)$ the inner product in $L_2(\Omega)$ the semidiscrete problem is now to find $U(t) \in S_h$ such that

$$(U_t, \chi)_h + (\nabla U, \nabla \chi) = 0 \quad \forall \chi \in S_h, \quad \text{for } t \geq 0, \tag{4.1}$$

where the discrete inner product is defined using quadrature by

$$(V, W)_h = \sum_{\tau \in \mathcal{T}_h} Q_\tau(V\bar{W}), \quad \text{where } Q_\tau(f) = \tfrac{1}{3}\text{area}(\tau) \sum_{P_j \in \bar{\tau}} f(P_j),$$

or

$$(V, W)_h = \sum_{j=1}^{N_h} \kappa_j V_j \bar{W}_j, \quad \text{where } \kappa_j = \frac{1}{3} \sum_{P_j \in \bar{\tau}} \text{area}(\tau).$$

The semidiscrete problem (4.1) may now be written

$$U_t - \Delta_h U = 0, \quad \text{for } t \geq 0, \quad \text{with } U(0) = V,$$

where $\Delta_h : S_h \to S_h$ is defined by

$$-(\Delta_h \psi, \chi)_h = (\nabla \psi, \nabla \chi), \quad \psi, \chi \in S_h.$$

We find

$$(\Delta_h U)_j = -\sum_{i=1}^{N_h} \kappa_j^{-1} \alpha_{ji} U_i, \quad \text{where } \alpha_{ji} = (\nabla \Phi_j, \nabla \Phi_i), \quad \text{for } j = 1, \cdots, N_h.$$

We note that $\sum_{i=1}^{N_h+M_h} \alpha_{ji} = 0$ since $\sum_{i=1}^{N_h+M_h} \Phi_i = 1$.

The basis of our l_p analysis is the following lemma where for an edge e_j of $\mathcal{T}_h$ defined by two neighbors P_{j_1} and P_{j_2}, $\partial_j U = U_{j_1} - U_{j_2}$.

Lemma 4.1. *For every edge of the triangulation there is a real-valued constant* $\gamma_j = -\alpha_{j_1 j_2}$ *such that*

$$(\nabla U, \nabla \chi) = \sum_j \gamma_j \partial_j U \cdot \overline{\partial_j \chi}, \quad \forall U, \chi \in S_h.$$

If $\mathcal{T}_h$ *satisfies the Delaunay condition the constants* γ_j *are nonnegative.*

Proof. It suffices to remark that, noting that $U_j = \chi_j = 0$ for $N_h+1 \leq j \leq N_h+M_h$,

$$(\nabla U, \nabla \chi) = \sum_{i,j=1}^{N_h+M_h} \alpha_{ij} U_i \bar{\chi}_j = \sum_{i \neq j} \alpha_{ij}(U_i - U_j)(\bar{\chi}_j - \bar{\chi}_i).$$

Now, the Delaunay condition means that for all edges of $\mathcal{T}_h$, the sum of the two opposite angles is $\leq \pi$, which is equivalent to saying that $\alpha_{ij} \leq 0$ for $i \neq j$. □

We are now ready to state the two-dimensional analogue of Theorem 2.1.

Theorem 4.1. *Assume that the triangulations $\mathcal{T}_h$ satisfy the Delaunay condition. Then the resolvent estimate (2,1) and (2.2) hold with respect to the discrete norm defined in (1.20).*

Further, with $\underline{\kappa} = \min_i \kappa_i$,

$$\|R(\lambda; \Delta_h)\|_\infty \le e\frac{|\log(\underline{\kappa}/|\Omega|)|^{1/2}}{|\lambda|}, \quad \textit{for } |\arg\lambda| \le \frac{\pi}{2} + \arcsin\frac{1}{|\log(\underline{\kappa}/|\Omega|)|^{1/2}}.$$

Proof. We set $\chi = I_h(U|U|^{p-2})$ in (4.1) and note that by Lemma 4.1

$$(\nabla U, \nabla\chi) = \sum_j \gamma_j \partial_j U \partial_j (\bar{U}|U|^{p-2}) = \sum_j \gamma_j A_j.$$

By Lemma 2.1 we find $|\arg A_j| \le \arcsin|1-2/p|$ and hence $|\arg(\nabla U, \nabla\chi)| \le \arcsin|1-2/p|$. As in the proof of Theorem 2.1 this implies the l_p-estimates (2.1) and (2.2) as in the proof of Theorem 2.1. This time, with k appropriate,

$$\|U\|_\infty = |U_k| \le \kappa_k^{-1/p}(\kappa_k|U_k|^p)^{1/p} \le \underline{\kappa}^{-1/p}\|U\|_{h,p}, \quad \text{for } U \in S_h.$$

and the maximum-norm estimate follows as in (2.3), with $p = |\log(\underline{\kappa}/|\Omega|)|$. □

The key points in our proof are the lumped mass discretization of the inner product in $L^2(\Omega)$ and the positivity of the γ_j in Lemma 4.1. Therefore our result is still valid in three dimensions when the γ_j are nonnegative (this is true for some meshes but we do not know if it is compatible with efficient mesh refinement procedures. For instance, the Delaunay procedure does not imply this property in three dimensions). Similarly, Theorem 4.1 is valid in the plane for the linear non conforming element under the asumption that the γ_j are nonnegative, which is the case if and only if in each triangle the angles are at most $\pi/2$.

5. Stability of time stepping schemes.

For the purpose of treating fully discrete methods of the form (1.21) with $E_{kh} = r(k\Delta_h)$ and $r(\lambda)$ a rational function we now show the following abstract result based on Bakaev [1], [2], which is geared to application of our above resolvent estimates in the complex plane.

Theorem 5.1. *Assume that A generates an analytic semigroup in a Banach space $\mathcal{B}$, so that*

$$\|R(\lambda; A)\| \le \frac{M}{|\lambda|}, \quad \textit{for } |\arg\lambda| \le \tfrac{1}{2}\pi + \delta, \quad \textit{with } \delta \in (0, \tfrac{1}{2}\pi).$$

Let $r(\lambda)$ be a nonconstant A-acceptable rational function, so that $|r(\lambda)| \le 1$ for $\operatorname{Re}\lambda \le 0$. Let $\{k_j\}_{j=1}^n$ be any sequence of positive numbers. Then there exists a positive constant $C = C(r)$ such that, with $\bar{k}_n = \max_{j\le n}$, $\underline{k}_n = \min_{j\le n} k_j$ and $r_\infty = r(\infty)$,

$$\text{(5.1)} \qquad \|\prod_{j=1}^n r(k_j A)\| \le \begin{cases} CM(1+|\log\delta|), & \textit{if } |r_\infty| < 1, \\ CM(1+|\log\delta| + \log(\bar{k}_n/\underline{k}_n)), & \textit{if } |r_\infty| = 1. \end{cases}$$

Proof. Since the $r(k_jA)$ commute we may assume without loss of generality that $k_j \le k_{j+1}$ for $j \ge 1$ so that $\bar{k}_n = k_n$, $\underline{k}_n = k_1$. Assume first that $|r_\infty| < 1$. We may write (with $\prod_{l=1}^0 r(k_lA) = I$)

$$\prod_{j=1}^n r(k_jA) = r_\infty^n I + \sum_{j=0}^{n-1} r_\infty^{n-j} B_j, \quad \text{where } B_j = (r(k_{j+1}A) - r_\infty I)\prod_{l=1}^j r(k_lA),$$

and hence

$$\|\prod_{j=1}^n r(k_jA)\| \le |r_\infty|^n + \sum_{j=0}^\infty |r_\infty|^j \max_{j \le n-1} \|B_j\|.$$

Since $|r_\infty| < 1$ it thus suffices to bound B_j by the right hand side of (5.1). For this we shall first show that for some constants $a,\ b > 0$, and m,

$$(5.2) \qquad |r(\lambda)| \le \begin{cases} e^{a|\lambda|}, & \text{for } |\lambda| \le \epsilon, \\ e^{-b\delta|\lambda|}, & \text{for } |\lambda| \le \epsilon,\ |\arg\lambda| = \tfrac{1}{2}\pi + \delta, \end{cases}$$

and

$$(5.3) \qquad |r(\lambda) - r_\infty| \le m/|\lambda|, \quad \text{for } |\lambda| \ge \epsilon,\ \operatorname{Re}\lambda \le 0.$$

If $|r(0)| < 1$ it is clear that (5.2) holds for ϵ small and a, b appropriate, so we may assume now that $|r(0)| = 1$. The first inequality then follows at once for ϵ small enough since $\log|r(\lambda)| = O(|\lambda|)$ for $|\lambda|$ small. The second then holds with $b = (\sin\delta/\delta)\inf_{|\lambda|\le\epsilon,\, \operatorname{Re}\lambda<0}(\log|r(\lambda)|/\operatorname{Re}\lambda|)$. If the infimum is attained for some λ with $\operatorname{Re}\lambda < 0$, then clearly $b > 0$. Otherwise, by compactness, the infimum is attained as a limit $\lambda_n \to \mathrm{i}y$ with $y \in R$, and we have $|r(\mathrm{i}y)| = 1$. We can write $r(\mathrm{i}y+z)/r(\mathrm{i}y) = 1 + c_k z^k + O(z^{k+1})$ with $c_k \ne 0$ and then $\log|r(\mathrm{i}y+z)| \sim \operatorname{Re}(c_k z^k)$. Since $|r(\mathrm{i}y+z)| \le 1$ for $\operatorname{Re} z \le 0$, we conclude $k = 1$ and $c_1 > 0$. Therefore $\log|r(\lambda_n)|/\operatorname{Re}\lambda_n \sim c_1$ and $b = c_1 \sin\delta/\delta > 0$. The estimate for $r(\lambda) - r_\infty$ of (5.3) also follows since this rational function has no pole for $\operatorname{Re}\lambda \le 0$ and vanishes at $\lambda = \infty$.

We may write, with Γ_j a suitable contour surrounding the spectrum of A, by the Dunford-Taylor functional calculus,

$$B_j = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_j} (r(k_{j+1}\lambda) - r_\infty) \prod_{l=1}^j r(k_l\lambda) R(\lambda; A)\, d\lambda,$$

Recalling that $k_j \le k_{j+1}$ for $j \ge 1$, we shall choose $\Gamma_j = \Gamma_j^0 \cup \Gamma_j^1 \cup \Gamma_j^2$ where, with $t_j = \sum_{l=1}^j k_l$,

$$\begin{aligned} \Gamma_j^0 &= \{\lambda;\ |\lambda| = \epsilon/t_j,\ |\arg\lambda| \le \tfrac{1}{2}(\pi+\delta)\}, \\ \Gamma_j^1 &= \{\lambda;\ |\arg\lambda| = \tfrac{1}{2}(\pi+\delta),\ \epsilon/t_j \le |\lambda| \le \epsilon/k_j\}, \\ \Gamma_j^2 &= \{\lambda;\ |\arg\lambda| = \tfrac{1}{2}(\pi+\delta),\ |\lambda| \ge \epsilon/k_j\}. \end{aligned}$$

Starting with Γ_j^0 we have, since $|r(\lambda) - r_\infty| \le 2$ there,

$$\begin{aligned} \Big|\int_{\Gamma_j^0}\Big| &\le 2M \int_{|\lambda|=\epsilon/t_j} \prod_{l=1}^j e^{ak_l|\lambda|} \frac{|d\lambda|}{|\lambda|} \\ &= 2M(\epsilon/t_j)^{-1} \int_{|\lambda|=\epsilon/t_j} e^{at_j|\lambda|}|d\lambda| = 2Me^{a\epsilon}\, 2\pi = CM. \end{aligned}$$

For Γ_j^2 we note that $|r(k_l\lambda)| \le 1$ for $\lambda \in \Gamma_j^2$, $l = 1, \dots, j$, and that $k_{j+1}|\lambda| \ge k_j|\lambda| \ge \epsilon$ and hence

$$\Big|\int_{\Gamma_j^2}\Big| \le mM\int_{\epsilon/k_j}^{\infty}\frac{1}{k_{j+1}|\lambda|}\frac{|d\lambda|}{|\lambda|} \le mMk_j^{-1}\int_{\epsilon/k_j}^{\infty}\frac{dx}{x^2} = CM.$$

Finally, since $k_l|\lambda| \le k_j|\lambda| \le \epsilon$ for $l \le j$ on Γ_j^1,

$$\begin{aligned}\Big|\int_{\Gamma_j^1}\Big| \le & 2M\int_{\epsilon/t_j}^{\epsilon/k_j}\prod_{l=1}^{j} e^{-b\delta k_l|\lambda|}\frac{|d\lambda|}{|\lambda|} \le 2M\int_{\epsilon/t_j}^{\infty} e^{-b\delta t_j x}\frac{dx}{x} \\ = & CM\int_{b\epsilon\delta}^{\infty} e^{-x}\frac{dx}{x} \le CM(1+|\log\delta|).\end{aligned}$$

This completes the proof when $|r_\infty| < 1$.

Let now $|r_\infty| = 1$. In this case we have, for $\epsilon < 1$ small enough,

$$|r(\lambda)| \le \begin{cases} e^{a|\lambda|}, & \text{for } |\lambda| \le \epsilon, \\ e^{a/|\lambda|}, & \text{for } |\lambda| \ge 1/\epsilon, \\ e^{-b\delta|\lambda|}, & \text{for } |\lambda| \le \epsilon,\ |\arg\lambda| = \frac12(\pi+\delta), \\ e^{-b\delta/|\lambda|}, & \text{for } |\lambda| \ge 1/\epsilon,\ |\arg\lambda| = \frac12(\pi+\delta), \end{cases} \tag{5.4}$$

This time we write

$$\prod_{j=1}^{n} r(k_jA) = r_\infty^n I + \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_n}\prod_{j=1}^{n} r(k_j\lambda)R(\lambda;A)\,d\lambda.$$

where Γ_n divides the complex plane into two parts, one that contains the spectrum of A and one that contains the poles of $r(\lambda)$. We choose $\Gamma_n = \cup_{l=0}^{4}\Gamma_n^l$ where with $t_n = \sum_{j=1}^{n} k_j$ and $\tilde t_n = (\sum_{j=1}^{n} k_j^{-1})^{-1}$,

$$\begin{aligned}\Gamma_n^0 &= \{\lambda;\ |\lambda| = \epsilon/t_n,\ |\arg\lambda| \le \tfrac12(\pi+\delta)\}, \\ \Gamma_n^1 &= \{\lambda;\ |\arg\lambda| = \tfrac12(\pi+\delta),\ \epsilon/t_n \le |\lambda| \le \epsilon/k_n\}, \\ \Gamma_n^2 &= \{\lambda;\ |\arg\lambda| = \tfrac12(\pi+\delta),\ \epsilon/k_n \le |\lambda| \le 1/(\epsilon k_1)\}, \\ \Gamma_n^3 &= \{\lambda;\ |\arg\lambda| = \tfrac12(\pi+\delta),\ 1/(\epsilon k_1) \le |\lambda| \le 1/(\epsilon\tilde t_n)\}, \\ \Gamma_n^4 &= \{\lambda;\ |\lambda| = 1/(\epsilon\tilde t_n),\ |\arg\lambda| \le \tfrac12(\pi+\delta)\},\end{aligned}$$

Here, using the first two bounds of (5.2),

$$\Big|\int_{\Gamma_n^0\cup\Gamma_n^4}\Big| \le M\int_{|\lambda|=\epsilon/t_n} e^{at_n|\lambda|}\frac{|d\lambda|}{|\lambda|} + M\int_{|\lambda|=1/(\epsilon/\tilde t_n)} e^{a/(\tilde t_n|\lambda|)}\frac{|d\lambda|}{|\lambda|} = CM.$$

Further,

$$\begin{aligned}\Big|\int_{\Gamma_n^1\cup\Gamma_n^3}\Big| \le & CM\int_{\epsilon/t_n}^{\epsilon/k_n} e^{-b\delta t_n x}\frac{dx}{x} + CM\int_{1/(\epsilon k_1)}^{1/(\epsilon\tilde t_n)} e^{-b\delta/(\tilde t_n x)}\frac{dx}{x} \\ = & CM\int_{b\epsilon\delta}^{\infty} e^{-x}\frac{dx}{x} \le CM(1+|\log\delta|).\end{aligned}$$

Finally,

$$\Big|\int_{\Gamma_n^2}\Big| \le CM\int_{\epsilon/k_n}^{1/(\epsilon k_1)}\frac{dx}{x} \le CM(1+\log(k_n/k_1).$$

Together these estimates complete the proof when $|r_\infty| = 1$. □

Application of this result to $E_{kh}(t_n) = \prod_{j=1}^n E_{k_j h}$, where $E_{kh} = r(k\Delta_h)$, with $|r_\infty| < 1$, then shows by Theorem 2.1, in one space dimension ($d = 1$), without any restriction on the partitions in space and time (for simplicity we assume $p \ge 2$)

$$\|E_{kh}(t_n)V\|_{h,p} \le \begin{cases} Cp^{1/2}\log p\|V\|_{h,p}, & \text{for } 2 \le p < \infty,\\ C|\log \underline{\kappa}|^{1/2}(1+\log|\log\underline{\kappa}|)\|V\|_{h,\infty}, & \text{for } p = \infty. \end{cases} \tag{5.5}$$

and

$$\|E_{kh}(t_n)V\|_{h,\infty} \le C_\mu\|V\|_{h,\infty}, \quad \text{if } h_j/h_{j\pm1} \le \mu \quad \text{for all } j \ge 0.$$

In two space dimensions ($d = 2$) the estimates in (5.5) remain valid, provided the triangulations are of Delaunay type.

In the case $|r_\infty| = 1$ stability factor in Theorem 5.1 contains the additional term $\log(\bar{k}_n/\underline{k}_n)$ which is bounded when the time stepping is quasi-uniform. As may be seen from the proof, this term arises in the estimate along Γ_n^2, and one may show that this term may be bounded even for some nonquasi-uniform partitions of the time axis. Consider for example the Crank-Nicolson method, with $r(\lambda) = (1+\lambda/2)/(1-\lambda/2)$. Here one finds easily that (5.4) holds with $\epsilon = 1$. Using for instance $k_j = k_1 j^q$, with $q > 1$, we have that $k_j|\lambda| \le 1$ for $j \le l$ when $|\lambda| \in (1/k_{l+1}, 1/k_l)$ and hence on this interval, since $t_l \ge cl^{q+1}$,

$$\Big|\prod_{j=1}^n r(k_j\lambda)\Big| \le e^{-b\delta\sum_{j=1}^l k_j|\lambda|} \le e^{-b\delta t_l/k_{l+1}} \le e^{-c\delta l}.$$

Hence

$$\Big|\int_{\Gamma_n^2}\Big| \le CM\sum_{l=1}^{n-1}\int_{1/k_{l+1}}^{1/k_l} e^{-c\delta l}\frac{dx}{x} \le CM\sum_{l=1}^{n-1} e^{-c\delta l}\log(k_{l+1}/k_l) \le \frac{CM}{1-e^{-c\delta}} \le \frac{CM}{\delta}.$$

In particular then, the term $\log(\bar{k}_n/\underline{k}_n)$ may be replaced by δ^{-1}, and stability holds for $\delta > 0$ fixed. We shall not pursue this investigation further here.

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M. Crouzeix, IRMAR, Université de Rennes I, Campus de Beaulieu, F-35042 Rennes Cedex, France

E-mail address: crouzeix@maths.univ-rennes1.fr

V. Thomée, Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

E-mail address: thomee@math.chalmers.se