The spectra of the self – adjoint extensions of a symmetric operator S inside a gap of S

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Abstract

For a self – adjoint operator A with spectral family $E_A(\cdot)$ and a Borel set B in $\mathbb{R}$ let A_B and $A_{\mathbb{R}\setminus B}$ be the unique self – adjoint operator in $\operatorname{ran} E_A(B)$ and $\operatorname{ran} E_A(\mathbb{R}\setminus B)$, respectively, such that

$$A = A_B \oplus A_{\mathbb{R}\setminus B}.$$

Let S be a symmetric operator with deficiency indices (n, n) and gap J. Then for every self – adjoint operator A^{aux} such that

$$\dim \operatorname{ran} E_{A^{aux}}(J) \leq n$$

there exists a self – adjoint extension A of S such that

$$A_J \simeq A_J^{aux}.$$

Key words: Spectral family, self – adjoint extension, inverse problem, Schrödinger operator, δ – interaction, moments

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1 Introduction

Let S be a symmetric operator. Suppose that the open interval J is a gap of S. In the year 1947 M. G. Krein [13] has given the complete answer to the question about which kinds of spectra the self – adjoint extensions of S can have inside J in the special case when the deficiency indices $n_\pm(S) := \dim\ker(S^* \mp i)$ of S are finite. In the present paper I shall give the complete answer in the general case.

Above question is part of a great research program. During the last seventy years one has tried to find methods which make it possible to determine the spectral families $E_A(\cdot)$ of self – adjoint operators A occurring in mathematical physics, e.g. Schrödinger operators. This was, of course, partly motivated by the fact that the measure

$$\mu_{f,A}(\cdot) :=\| E_A(\cdot)f \|^2$$

equals the probability distribution for the energy if A is the Hamiltonian of a quantum system and the system is in the (normalized) state f.

Usually one is not able to give explicitly the spectral family $E_A(\cdot)$ but one derives certain useful partial results. E.g., it might be possible to determine the eigenvalues E and corresponding eigenvectors f of A, i.e. the real numbers E and vectors $f \neq 0$ such that $\mu_{f,A} =\| f \|^2 \delta_E$, δ_E being the Dirac measure with mass in E. It might also be possible to determine the spectrum $\sigma(A)$ of A, i.e. the smallest closed subset C of $\mathbb{R}$ such that

$$\mu_{f,A}(\mathbb{R} \setminus C) = 0$$

for all f in the underlying Hilbert space. Moreover every self – adjoint operator A can be uniquely represented as

$$A = A^{ac} \oplus A^{pp} \oplus A^{sc}$$

where A^{ac}, A^{pp} and A^{sc} is a self – adjoint operator in the closed subspace of all f such that the measure $\mu_{f,A}$ is absolutely continuous with respect to the Lebesgue measure, a pure point measure and singular with respect to the Lebesgue measure and continuous, respectively. One is also strongly interested in the sets

$$\sigma_{ac}(A) := \sigma(A^{ac}), \quad \sigma_{pp}(A) := \sigma(A^{pp}), \quad \sigma_{sc}(A) := \sigma(A^{sc}).$$

Note that $\sigma_{pp}(A)$ equals the closure $\overline{\sigma_p(A)}$ of the set $\sigma_p(A)$ of eigenvalues of A.

While for a long time one had concentrated on the absolutely continuous spectrum $\sigma_{ac}(\cdot)$ and the point spectrum $\sigma_{pp}(\cdot)$ it became clear during the last decade that "singular continuous spectrum $\sigma_{sc}(\cdot)$ is generic" (cf. the article [15] by B. Simon and references given therein) and one has investigated this kind of spectrum more detailed. In particular, it has been shown that in a wide variety of applications it is useful to know whether the measure $\mu_{f,A}$ is α – dimensional, i.e. vanishes at every set with Hausdorff – dimension less than α and there exists a Borel set B with Hausdorff – dimension α such that $\mu_{f,A}(\mathbb{R} \setminus B) = 0$, cf. the recent work [12] by A. Kiselev and Y. Last and references given therein. For arbitrary $\alpha \in [0, 1]$ let $\sigma_\alpha(A)$ be the smallest closed set C such that $\mu_{f,A}(\mathbb{R} \setminus C) = 0$ for every f such that $\mu_{f,A}$ is α – dimensional.

In a wide variety of models in quantum physics the information about the Hamiltonian of a given system is incomplete in the sense that one is given a symmetric operator S which has infinitely many self – adjoint extensions and one only knows that the Hamiltonian is one of these extensions. We refer to the books [2], [3] and [4] by S. Albeverio et al. and references given therein for a discussion of many of these models, namely the so called "zero – range interaction" or "δ – interaction models".

If only incomplete information is available then one often studies an inverse problem. The starting point are given properties and one wants to find out whether there exists a self – adjoint extension with the preassigned properties. One of the first well known results of this kind is due to K. O. Friedrichs [11] and M. Stone [16]. They have shown that the open interval $(-\infty, b)$ is a gap of the symmetric operator S, i.e. there exists at least one self – adjoint extension A_∞ of S such that $\sigma(A_\infty) \cap (-\infty, b) = \emptyset$, if and only if

$$(Sf, f) \geq b \parallel f \parallel^2, \quad f \in D(S).$$

M.G. Krein [13] has shown that the bounded interval (a, b) is a gap of the symmetric operator S if and only if

$$\parallel (S - \frac{b+a}{2})f \parallel \geq \frac{b-a}{2} \parallel f \parallel, \quad f \in D(S).$$

If the closure $\bar{S}$ of S is not self – adjoint then in addition to the self – adjoint extensions of S preserving the gap J of S there exist other self – adjoint extensions A of S such that $\sigma(A) \cap J \neq \emptyset$. One might wonder which kinds of spectra these other self – adjoint extensions can have inside the gap J. As mentioned M.G. Krein has given the complete answer to this question in the special case when the deficiency indices of S are finite and in this paper I shall give the complete answer in the case of infinite deficiency indices.

Thus we know which kinds of spectra the self – adjoint extensions of a symmetric operator S can have inside a gap of S. We do not know which kinds of spectra are possible on the whole real axis. In order to treat this problem it might be useful to combine the Weyl function approach by V. A. Derkach and M. M. Malamud ([9], [10]) with the methods described in the present paper.

In the following ideas and results from the articles [1], [5], [6], [7] and [8] will play an important role and I shall scetch what is really new in the present paper. Let T be an operator, $f \in D(T^m)$ for all $m \in \mathbb{N}$ and μ a measure with finite moments of arbitrary order such that

$$(T^m f, T^l f) = \int \lambda^{m+l} \mu(d\lambda), \quad l, m = 0, 1, \ldots$$

and μ is uniquely determined by the sequence of its moments (this holds, e.g., if the support of μ is compact). Then μ is uniquely determined by f and T and one might define $\mu_{f,T} := \mu$. If the operator T is self – adjoint then this definition is in accordance with the above one, i.e. $\mu_{f,T}(\cdot) = \| E_T(\cdot) f \|^2$, $E_T(\cdot)$ being the spectral family of T.

If T is self – adjoint, $\mu_{g,T}$ is a pure point measure and $\mu_{f,T}$ a continuous measure then $f \perp g$. In particular, if $f \neq 0$ and $\mu_{f,T}$ a continuous measure then f cannot be approximated by a sequence $\{f_n\}_{n \in \mathbb{N}}$ such that μ_{f_n} is a pure point measure for infinitely many n.

Let S be a symmetric operator. Suppose that the open interval J is a gap of S and the deficiency indices of S are infinite. Let μ be a finite measure with compact support in J. Then, in contradistinction to the self – adjoint case, there exist f and f_n, $n \in \mathbb{N}$, such that the measures μ_{f,S^*} and μ_{f_n,S^*} are defined, μ_{f_n,S^*} is a pure point measure for every $n \in \mathbb{N}$,

$$\mu_{f,S^*} = \mu, \quad \text{and} \quad S^{*m} f_n \longrightarrow S^{*m} f, \quad n \to \infty, \quad m = 0, 1, \ldots .$$

An explicit construction of such f and f_n, $n \in \mathbb{N}$, is given in the proof of the Lemma 10 below (under mild additional conditions on μ). If one combines this new result, which might be of interest in itself, with ideas in the mentioned articles then one gets a proof of the main theorem of the present paper.

2 Results and Proofs

Let A be a self – adjoint operator in a Hilbert space $\mathcal{H}$. By the spectral theorem, A has a unique representation of the form

$$A = \int \lambda dE_A(\lambda).$$

Here E_A denotes the unique mapping from the Borel algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ into the set of orthogonal projections in $\mathcal{H}$ with the following properties:

(i) For every $f \in \mathcal{H}$ the mapping $B \mapsto \| E_A(B)f \|^2$ from $\mathcal{B}(\mathbb{R})$ to the interval $[0, \| f \|^2]$ is a measure with total mass $\| f \|^2$. This measure will be denoted by $\mu_{f,A}$.

(ii)

$$\begin{aligned} D(A) &= \{f \in \mathcal{H} : \int t^2 \mu_{f,A}(dt) < \infty\}, \\ (Af, f) &= \int t \mu_{f,A}(dt), \quad f \in D(A). \end{aligned}$$

For every Borel set $B \subset \mathbb{R}$ the operator $E_A(B)$ will be called the spectral projector corresponding to A and B.

It easily follows from the spectral theorem, that for every Borel set $B \subset \mathbb{R}$ we have

$$\mathcal{H} = \operatorname{ran} E_A(B) \oplus \operatorname{ran} E_A(\mathbb{R} \setminus B)$$

and there exist unique self – adjoint operators A_B in $\operatorname{ran}(E_A(B))$ and $A_{\mathbb{R}\setminus B}$ in $\operatorname{ran}(E_A(\mathbb{R} \setminus B))$ such that

$$A = A_B \oplus A_{\mathbb{R}\setminus B}.$$

Inside B the operators A and A_B have the same eigenvalues and for every eigenvalue $E \in B$ of A the multiplicity $\operatorname{mult}(E, A)$ of E as an eigenvalue of A equals $\operatorname{mult}(E, A_B)$.

For open sets J we have in addition that

$$\mu_{f,A}(B) = \mu_{f,A_J}(B)$$

for every Borel set $B \subset J$ and every $f \in \mathcal{H}$. In particular, we have

$$\sigma(A) \cap J = \sigma(A_J) \cap J, \quad \sigma_{ac}(A) \cap J = \sigma_{ac}(A_J) \cap J, \quad \sigma_{sc}(A) \cap J = \sigma_{sc}(A_J) \cap J,$$

and for every $\alpha \in [0, 1]$

$$\sigma_\alpha(A) \cap J = \sigma_\alpha(A_J) \cap J.$$

Let S be a symmetric operator with deficiency indices (n, n). Suppose that the open interval J is a gap of S. It easily follows from von Neumann's extension theory that

$$\dim \operatorname{ran} E_A(J) \leq n$$

for every self – adjoint extension A of S; "dim" means dimension in the sense of Hilbert space theory, i.e. the cardinality of any orthonormal base. Up to unitary equivalence this is the only restriction for the operators A_J, A being a self – adjoint extension of S:

Theorem 1 *Let S be a symmetric operator in the Hilbert space $\mathcal{H}$. Suppose that the open interval J is a gap of S and the deficiency indices of S equal (n, n). Let A^{aux} be any self – adjoint operator such that*

$$dim\, ran\, E_{A^{aux}}(J) \leq n.$$

Then there exists a self – adjoint extension A of S such that

$$A_J \simeq A_J^{aux}.$$

Remark 2 In particular, A and A^{aux} have the same eigenvalues inside J and for every eigenvalue $E \in J$ of A we have

$$\operatorname{mult}(E, A) = \operatorname{mult}(E, A^{aux}).$$

Moreover

$$\sigma(A)\cap J = \sigma(A^{aux})\cap J, \quad \sigma_{ac}(A)\cap J = \sigma(A^{aux}_{ac})\cap J, \quad \sigma(A_{sc})\cap J = \sigma(A^{aux}_{sc})\cap J,$$

and for every $\alpha \in [0,1]$

$$\sigma_\alpha(A) \cap J = \sigma_\alpha(A^{aux}) \cap J.$$

Remark 3 The theorem had been formulated as a conjecture in [1].

Remark 4 In the special case when the deficiency index n is finite the theorem has already been proved by M.G.Krein ([13]).

The first step in the proof of the theorem is to show that the problem can be reduced considerably. In fact one needs only to prove the following:

There exists a closed linear subspace $\mathcal{H}_0$ of $\mathcal{H}$ and a self – adjoint operator M in the Hilbert space $\mathcal{H}_0$ such that

$$M \subset S^* \quad \text{and} \quad M \simeq A^{aux}_J.$$

This is a consequence of the following

Lemma 5 *([1], Lemma 2.1 and Lemma 2.2)*

(i) Let S be a symmetric operator in the Hilbert space $\mathcal{H}$. Let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}$ and M a self–adjoint operator in the Hilbert space $\mathcal{H}_0$. Suppose that M is a restriction of the adjoint S^* of S. Then the operator

$$S_M := S^*_{| D(S)+D(M)}, \tag{1}$$

i. e. the restriction S_M of S^* to the space

$$D(S) + D(M) := \{f + g : f \in D(S), g \in D(M)\},$$

can be represented in the form

$$S_M = M \oplus G_0 \tag{2}$$

for a unique symmetric operator G_0 in the Hilbert space $\mathcal{H}_0^\perp$.

(ii) If in addition the open interval J is a gap of S and

$$\sigma(M) \subset \bar{J}$$

then J is also a gap of G_0 and S has a self – adjoint extension A such that

$$A_J = M_J.$$

Proof: Let $f, \tilde{f} \in D(S)$ and $g, \tilde{g} \in D(M)$. We have

$$\begin{aligned}(S_M(f+g), \tilde{f}+\tilde{g}) &= (Sf, \tilde{f}+\tilde{g}) + (S^*g, \tilde{f}) + (Mg, \tilde{g}) \\ &= (f, S^*(\tilde{f}+\tilde{g})) + (g, S\tilde{f}) + (g, M\tilde{g}) \\ &= (f+g, S_M(\tilde{f}+\tilde{g})).\end{aligned}$$

Thus S_M is a symmetric operator in the Hilbert space $\mathcal{H}$.

Let $f \in D(S_M)$. For every $n \in \mathbb{Z}$ let

$$P_n := E_M([n, n+1))P_{\mathcal{H}_0},$$

$P_{\mathcal{H}_0} : \mathcal{H} \longrightarrow \mathcal{H}$ being the orthogonal projection onto $\mathcal{H}_0$. Since M is a self–adjoint operator in the Hilbert space $\mathcal{H}_0$ it follows from the spectral theorem that for every $n \in \mathbb{Z}$ the operator P_n is an orthogonal projection in $\mathcal{H}$ onto the closed subspace $\mathrm{ran}(P_n)$ of $\mathcal{H}$,

$$\mathrm{ran}(P_n) \subset D(M), \quad n \in \mathbb{Z}, \tag{3}$$

$$\mathrm{ran}(P_n) \perp \mathrm{ran}(P_m), \quad n \neq m, \tag{4}$$

$$\sum_{n \in \mathbb{Z}} P_n = P_{\mathcal{H}_0}, \tag{5}$$

$$P_n M g = M P_n g, \quad g \in D(M), n \in \mathbb{Z}. \tag{6}$$

Thus we have

$$\begin{aligned}(P_n S_M f, g) &= (S_M f, g) = (f, Mg) \\ &= (P_n f, Mg) = (M P_n f, g)\end{aligned}$$

for every $g \in \operatorname{ran}(P_n)$. In the second step we have used (3) and the facts that S_M is symmetric and M a restriction of S_M. In the third step we have used (6) and in the last step again (3). Thus we have

$$P_n S_M f = M P_n f, \quad n \in \mathbb{Z}. \tag{7}$$

Since, by (7), (4) and (5),

$$(MP_n f, MP_k f) = (P_n S_M f, P_k S_M f) = 0, \quad k \neq n,$$

and

$$\sum_{n\in\mathbb{Z}} \| MP_n f \|^2 = \sum_{n\in\mathbb{Z}} \| P_n S_M f \|^2 = \| P_{\mathcal{H}_0} f \|^2 < \infty,$$

the sequence $\{M \sum_{n=-N}^{N} P_n f\}_{N\in\mathbb{N}}$ converges in $\mathcal{H}_0$. Since M is closed and, by (5), $\lim_{N\to\infty} \sum_{n=-N}^{N} P_n f = P_{\mathcal{H}_0} f$ it follows that

$$P_{\mathcal{H}_0} f \in D(M) \tag{8}$$

and

$$MP_{\mathcal{H}_0} f = \lim_{N\to\infty} \sum_{n=-N}^{N} MP_n f = \sum_{n\in\mathbb{Z}} P_n S_M f = P_{\mathcal{H}_0} S_M f. \tag{9}$$

Here again we have used (7) and (5). By (8) and (9),

$$S_M = M \oplus G_0,$$

where

$$G_0 := S_{M|\, D(S_M)\cap\mathcal{H}_0^\perp}.$$

G_0 is a symmetric operator in the Hilbert space $\mathcal{H}_0^\perp$ since S_M is a symmetric operator in $\mathcal{H}$. Obviously the above decomposition of S_M is unique.

(ii) We shall consider the case when $J = (-\infty, b)$. The proof in the other case is virtually the same.

Assume that

$$(G_0 f, f) < b \| f \|^2 \tag{10}$$

for some $f \in D(G_0)$. We choose $g \in D(S)$ and $h \in D(M)$ such that

$$f = g + h.$$

Then we have

$$\begin{array}{rcl} (Sg, g) & = & (S_M(f-h), f-h) \\ & = & (G_0 f, f) + (Mh, h) \\ & < & b\,(\,(f,f) + (h,h)\,) \\ & = & b \,\|\, f - h \,\|^2 = b \,\|\, g \,\|^2 . \end{array} \tag{11}$$

In the second and the fourth step we have used that $S_M = M \oplus G_0$ and in the third step the assumption (10) and the hypothesis that $\sigma(M) \subset \bar{J}$ and consequently $(Mh, h) \leq b \,\|\, h \,\|^2$ for every $h \in D(M)$. (11) is a contradiction to the hypothesis that $(-\infty, b)$ is a gap of S. Thus we have shown that $(G_0 f, f) \geq b \,\|\, f \,\|^2$ for every $f \in D(G_0)$, i.e. that $(-\infty, b)$ is a gap of G_0, provided $(-\infty, b)$ is a gap of S.

Since J is a gap of G_0 we can choose a self – adjoint operator G in the Hilbert space $\mathcal{H}_0^\perp$ such that $G_0 \subset G$ and $\sigma(G) \cap J = \emptyset$. Then the operator $A := M \oplus G$ is a self – adjoint extension of S and

$$A_J = M_J \oplus G_J = M_J .$$

□

In the special case when the deficiency indices (n, n) of the symmetric operator S are finite Theorem 1 easily follows from the above lemma. In fact let A^{aux} be any self – adjoint operator such that

$$\dim \operatorname{ran} E_{A^{aux}}(J) \leq n < \infty. \tag{12}$$

Let $\{E_i\}_{i=1}^k$ be the family of eigenvalues of A^{aux} inside J where every eigenvalue $E \in J$ of A^{aux} occurs exactly $\operatorname{mult}(E, A^{aux})$ many times in $\{E_i\}_{i=1}^k$. By (12), $k \leq n$. Since the kernel $\mathcal{N}_E$ of $S^* - E$ is n – dimensional for every regular point E of S we can choose, by induction, an orthonormal system $\{e_i\}_{i=1}^k$ such that $S^* e_i = E_i e_i$ for $i = 1, \dots, k$. Obviously the restriction M of S^* to the space $\mathcal{H}_0 := \operatorname{span}\{e_i : i = 1, \dots, k\}$ is a self – adjoint operator in $\mathcal{H}_0$, $M = M_J$ and $M \simeq A_J^{aux}$. Thus it follows from the above lemma that there exists a self – adjoint extension A of S such that $A_J \simeq A_J^{aux}$.

In what follows we shall concentrate on the case when the deficiency indices of S are infinite. Then it is often difficult to give directly a self – adjoint operator M in some closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $M \subset S^*$ and $M \simeq A_J^{aux}$. However, since A_J^{aux} is a self – adjoint operator, it can be represented as

$$A_J^{aux} \simeq \bigoplus_{i\in I} Q_{\mu_i}$$

for some suitably chosen family of finite measures μ_i on the Borel algebra of $\mathbb{R}$ (cf. [17], Theorem 7.18). Here Q_μ denotes the operator in $L^2(\mathbb{R},\mu)$ which is defined by

$$\begin{aligned} D(Q_\mu) &:= \{f \in L^2(\mathbb{R},\mu) : \int t^2 \mu(dt) < \infty\}, \\ Q_\mu f(t) &:= t\, f(t) \quad \mu\text{– a.e,} \quad f \in D(Q_\mu). \end{aligned}$$

Thus first we shall concentrate on the simpler problem how to determine a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ and a self – adjoint operator M in $\mathcal{H}_0$ such that

$$M \subset S^* \quad \text{and} \quad M \simeq Q_\mu$$

if a measure μ is given. Once this problem is solved it is easy to complete the proof of the theorem. Several ideas will be used for the solution of this simpler problem. One of these ideas is virtually contained in the proof of the spectral theorem as given in E. Nelson's book [14], cf. the following lemma and its proof.

Lemma 6 *Let S be a symmetric operator, μ a finite measure on the Borel algebra of $\mathbb{R}$ and $f \in \cap_{m\in\mathbb{N}} D(S^{*m})$ such that the support of μ is compact and*

$$(S^{*m}f, S^{*l}f) = \int t^{m+l}\mu(dt), \quad m,l = 0,1,2,\ldots \tag{13}$$

Then there exists a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ and a self – adjoint operator M in $\mathcal{H}_0$ such that

$$M \subset S^* \quad \textit{and} \quad M \simeq Q_\mu.$$

Proof: Let $\mathcal{H}_0^0$ be the span of the vectors $f, S^*f, S^{*2}f, \ldots$ and $\mathcal{H}_0$ the closure of $\mathcal{H}_0^0$. It follows from (13) that for every finite family $\{\alpha_i\}_{i=0}^m$ in $\mathbb{C}$

$$\| \sum_{i=0}^m \alpha_i S^{*i} f \|^2 = \int |\sum_{i=0}^m \alpha_i t^i|^2 \mu(dt).$$

Since $\mathcal{H}_0^0$ is dense in $\mathcal{H}_0$ and the set of μ – equivalence classes of polynomials is dense in $L^2(\mathbb{R}, \mu)$ it follows that there exists a unique unitary transformation

$$U : \mathcal{H}_0 \longrightarrow L^2(\mathbb{R}, \mu)$$

satisfying

$$(U \sum_{i=0}^{m} \alpha_i S^{*i} f)(t) = \sum_{i=0}^{m} \alpha_i t^i$$

for μ – a.e. $t \in \mathbb{R}$ for every finite family $\{\alpha_i\}_{i=0}^{m}$ in $\mathbb{C}$.

Let

$$M := U^{-1} Q_\mu U.$$

Obviously M is a bounded self – adjoint operator in $\mathcal{H}_0$ and $Mg = S^* g$ for every $g \in \mathcal{H}_0^0$. Since $\mathcal{H}_0^0$ is dense in $\mathcal{H}_0$, M is bounded and S^* is closed it follows that $M \subset S^*$. □

It is easy to give f with the required properties provided μ is a pure point measure:

Example 7 Let S be a symmetric operator. Suppose that the open interval J is a gap of J and the deficiency indices of S are infinite. Let $\mu = \sum_{n=1}^{\infty} \beta_n \delta_{E_n}$ for some summable family $\{\beta_n\}_{n \in \mathbb{N}}$ and some bounded family $\{E_n\}_{n \in \mathbb{N}}$ in J. Since for every $E \in J$ the space $\mathcal{N}_E$ (recall that $\mathcal{N}_E$ denotes the kernel of $S^* - E$) is infinite dimensional we can choose, by induction, an orthonormal family $\{e_n\}_{n \in \mathbb{N}}$ in $D(S^*)$ such that

$$S^* e_n = E_n e_n$$

for all $n \in \mathbb{N}$.

Obviously

$$f := \sum_{n=1}^{\infty} \sqrt{\beta_n} e_n \in \bigcap_{m \in \mathbb{N}} D(S^{*m}),$$

$$S^{*m} f = \sum_{n=1}^{\infty} E_n^m \sqrt{\beta_n} e_n, \quad m = 1, 2, \dots,$$

and consequently

$$(S^{*m} f, S^{*l} f) = \int t^{m+l} \mu(dt), \quad m, l = 0, 1, 2, \dots$$

□

Now let μ be any finite measure. Due the fact that the set of finite pure point measures on the Borel algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ is dense in the set of all finite measures on $\mathcal{B}(\mathbb{R})$ it is natural that one tries to use the following strategy for the construction of a vector f satisfying

$$(S^{*m}f, S^{*l}f) = \int t^{m+l}\mu(dt), \quad m,l = 0,1,2,\ldots$$

One chooses pure point measures μ_n such that

$$\mu_n \longrightarrow \mu \quad \text{weakly.}$$

Then one tries to find vectors f_n such that

$$(S^{*m}f_n, S^{*l}f_n) = \int t^{m+l}\mu_n(dt), \quad m,l = 0,1,2,\ldots,$$

$$f_n \longrightarrow f$$

for some vector f and $\{S^{*m}f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $n \in \mathbb{N}$. If one has shown that there exists such approximating vectors f_n then it is easily seen that f has the required property.

In order to find the approximating vectors we shall apply the following elementary geometrical result:

Lemma 8 *Let P and Q be orthogonal projections. Suppose that there exists an $c > 0$ such that*

$$\| Ph \| \geq c \| h \|, \quad h \in ran(Q).$$

Then for every $h \in ran(Q)$ there exists an $f \in ran(P)$ such that $Qf = h$ and $\| Qf \| \geq c \| f \|$.

Proof: For every $h \in \text{ran}(Q)$ we have

$$(QPh, h) = (Ph, Ph) \geq c^2 \| h \|^2 .$$

For $h \in \text{ran}(Q)$, $h \perp \text{ran}(QPQ)$, it follows that

$$0 = (QPh, h) \geq c^2 \| h \|^2,$$

and therefore $h = 0$. Thus ran (QPQ) is dense in ran (Q).

Moreover we have for $h \in$ ran (Q) that

$$\begin{aligned} \| Ph \|^2 = (QPh, h) &\leq \| QPh \| \| h \| \\ &\leq \| QPh \| \frac{1}{c} \| Ph \| . \end{aligned}$$

Thus

$$\| QPh \| \geq c \| Ph \| .$$

Thus ran $(QPQ) = \text{ran}(Q)$. $\square$

The following lemma gives estimates for the distance of various vectors to their orthogonal projection onto $\mathcal{N}_E$ where E is any point in any gap J of the symmetric operator S. It will be important that we get upper bounds which do not depend on the special choice of the symmetric operator S with gap J.

Lemma 9 *Let S be a symmetric operator in the Hilbert space $\mathcal{H}$. Suppose that the open interval J is a gap of S. Let $E \in J$, $\{E_i\}_{i \in I}$ a family in J and $\{e_i\}_{i \in I}$ an orthonormal system such that $e_i \in \mathcal{N}_{E_i}$ for every $i \in I$. Let P be the orthogonal projection onto $\mathcal{N}_E$ and Q the orthogonal projection onto the space*

$$\mathcal{H}_0 := \overline{span\{e_i : i \in I\}}.$$

Then the following holds.

(i) For all $h \in \mathcal{H}_0$ we have

$$\| h - Ph \| \leq \frac{1}{dist\,(E, \partial J)} \sup_{i \in I} |E - E_i| \ \| h \| .$$

(ii) If

$$\sup_{i \in I} |E - E_i| < dist\,(E, \partial J)$$

then for every $h \in \mathcal{H}_0$ there exists an $f \in \mathcal{N}_E$ such that

$$Qf = h, \quad \| f \|^2 \leq \frac{1}{1 - \frac{\sup_{i \in I} |E - E_i|^2}{dist(E, \partial J)^2}} \ \| h \|^2 .$$

Proof: (i) Without loss of generality we may assume that the operator S is closed. Then $\mathrm{ran}(S-E)$ is a closed subspace of $\mathcal{H}$ and we have

$$(\mathrm{ran}\, P)^{\perp} = \mathcal{N}_E^{\perp} = \mathrm{ran}(S-E).$$

Let $h \in \mathcal{H}_0$. We choose normalized vectors $\tilde{e}_1 \in (\mathrm{ran}\, P)^{\perp}$ and $\tilde{e}_2 \in \mathrm{ran}\, P$ such that

$$h = (\tilde{e}_1, h)\tilde{e}_1 + (\tilde{e}_2, h)\tilde{e}_2.$$

Without loss of generality we may also assume that the supremum of the numbers $|E-E_i|$ where the supremum is taken over all $i \in I$ is finite. Since the operator S^* is closed and $S^* e_i = E_i e_i$ for every $i \in I$ and $\{e_i\}_{i\in I}$ is an orthonormal family we get that $h \in D(S^*)$ and

$$\| (S^* - E)h \| \leq \sup_{i\in I} |E-E_i| \, \| h \| .$$

We choose $g \in D(S)$ such that

$$\tilde{e}_1 = (S-E)g.$$

We have

$$\| g \| \leq \| (S-E)^{-1} \| \, \| \tilde{e}_1 \| \leq \frac{1}{\mathrm{dist}(E, \partial J)}.$$

Thus

$$\begin{aligned} |(\tilde{e}_1, h)| &= |((S-E)g, h)| \\ &= |(g, (S^*-E)h)| \\ &\leq \frac{1}{\mathrm{dist}\,(E,\partial J)} \, \| (S^*-E)h \| \\ &\leq \frac{1}{\mathrm{dist}\,(E,\partial J)} \sup_{i\in I} |E-E_i| \, \| h \| . \end{aligned}$$

Since $h - Ph = (\tilde{e}_1, h)\tilde{e}_1$ the assertion (i) is proved.

(ii) is an immediate consequence of (i) and the Lemma 8. □

For every self – adjoint operator A the pure point spectral space of A and its continuous spectral space are orthogonal. Thus, in particular, non – zero elements in the continuous spectral subspace of A cannot be approximated by vectors in its pure point spectral subspace. The proof of the following lemma will show that in a sense the opposite is true for the adjoint of a symmetric operator with infinite deficieny indices.

Lemma 10 *Let S be a symmetric operator. Suppose that the open interval J is a gap of S and the deficiency indices of S are infinite. Let $E \in J$ and*

$$0 < \varepsilon < \frac{1}{96}\, dist(E, \partial J).$$

Then for every finite measure μ on the Borel algebra of $\mathbb{R}$ such that $\mu(\mathbb{R} \setminus (E-\varepsilon, E+\varepsilon)) = 0$ there exists an f with the following properties:

(i)

$$f \in \bigcap_{m \in \mathbb{N}} D(S^{*m}).$$

(ii) For all $l, m = 0, 1, 2, \ldots$ we have

$$(S^{*l}f, S^{*m}f) = \int t^{l+m} \mu(dt).$$

Remark: The reader might be surprised about the constant 1/96. Actually the claim of the lemma holds under much weaker conditions on the measure μ. It suffices to require that $\mu(\mathbb{R} \setminus J) = 0$ and μ has finite moments of arbitrary order, cf. Corollary 12 below. It is not even necessary that μ is uniquely determined by the sequence of its moments and, in particular, it is not necessary that the support of the measure μ is a compact subset of J. However, the above hypothesis is convenient and, in particular, one can check easily that all constants $c_1, \ldots, c_8$ in the proof below are well defined.

Proof: We define, by induction, a sequence $\{f_n\}$ converging to a limit f with the required properties.

$\underline{n = 0}$

We choose any point $E' \in J$ and any normalized vector $e \in \mathcal{N}_{E'}$ and put

$$f_0 := \mu(\mathbb{R})^{1/2} e.$$

$\underline{n \Rightarrow n+1}$

Suppose that 2^n pairwise disjoint subintervals $I_1, I_2, \dots, I_{2^n}$ of $(E-\varepsilon, E+\varepsilon)$, points $E_j \in I_j$, $j = 1, \dots, 2^n$, and an orthonormal system $\{e_j\}_{j=1}^{2^n}$ have been chosen such that

(i) the length of I_j equals $2^{-n+1}\,\varepsilon$ for every $j = 1, \dots, 2^n$ and $(E-\varepsilon, E+\varepsilon)$ equals the union of these intervals and

(ii) $e_j \in \mathcal{N}_{E_j}$ for every $j = 1, \dots, 2^n$.

Let

$$f_n := \sum_{j=1}^{2^n} \sqrt{\mu(I_j)}e_j.$$

For $j = 1, \dots, 2^n$ we choose pairwise disjoint intervals I_{jk}, $k = 1, 2$, such that $I_j = I_{j1} \cup I_{j2}$ and the length of I_{jk} equals $2^{-n}\,\varepsilon$ for $k = 1, 2$.

We choose points $E_{jk} \in I_{jk}$ and put, as an abbreviation, for $j = 1, \dots, 2^n$ and $k = 1, 2$

$$\alpha_j := \sqrt{\mu(I_j)}, \quad \alpha_{jk} := \sqrt{\mu(I_{jk})}$$

and denote by P_{jk} the orthogonal projection onto $\mathcal{N}_{E_{jk}}$.

We shall choose, by induction over j, normalized vectors $e_{jk} \in \mathcal{N}_{E_{jk}}$ such that with

$$f_{n+1} := \sum_{j=1}^{2^n} \sum_{k=1}^{2} \alpha_{jk} e_{jk} \tag{14}$$

we have for all $m = 0, 1, 2, \dots,$

$$\| S^{*m} f_{n+1} - S^{*m} f_n \| \leq c(m) 2^{-n/4} \tag{15}$$

where $c(m)$ is a finite constant which does not depend on n.

In order to find such vectors e_{jk} we note that, by the Lemma 9, the linear mappings P_{jk} are injective. By hypothesis, the spaces $\mathcal{N}_{E'}$ are infinite dimensional for all $E' \in J$ and therefore $\dim \mathcal{N}_{E'} \cap V^{\perp} = \infty$ for every finite dimensional space V. Thus, by induction over j, we can choose an orthonormal system $\{g_j\}_{j=1,\dots,2^n}$ with the following properties:

(i) $g_j \in \mathcal{N}_{E_j}$ for all $j = 1, \dots, 2^n$.

(ii) $(g_j, e_i) = 0$ for all $i, j = 1, \dots, 2^n$.

(iii) $P_{jk} g_j \perp \{e_1, \dots, e_{2^n}\}$ for all $j = 1, \dots, 2^n$, $k = 1, 2$.

(iv) $P_{jk} g_j \perp g_i$ for all $1 \leq i, j \leq 2^n$, $i \neq j$, $k = 1, 2$.

Now we shall define the e_{jk} by induction.

$\underline{j = 1}$

We choose an angle φ such that, with

$$\tilde{e}_{11} := \sin\varphi \cdot e_1 + \cos\varphi \cdot g_1, \tilde{e}_{12} := \cos\varphi \cdot e_1 - \sin\varphi \cdot g_1,$$

we have

$$\alpha_1 e_1 = \alpha_{11} \tilde{e}_{11} + \alpha_{12} \tilde{e}_{12}.$$

Let $k = 1, 2$. Since $|E_{1k} - E_1| \leq 2\varepsilon 2^{-n}$ it follows from the Lemma 9 that

$$\| e_1 - P_{1k} e_1 \| \leq c_1 2^{-n} \tag{16}$$

for the constant $c_1 := 2\varepsilon / \text{dist}(E, \partial J)$. Let Q be the orthogonal projection onto

$$\mathcal{H}_1 := \text{span}\{e_2, \dots, e_{2^n}, g_2, \dots, g_{2^n}\}.$$

Since $e_1 \perp \mathcal{H}_1$ it follows from (16) that

$$\| QP_{1k} e_1 \| \leq c_1 2^{-n}.$$

By Lemma 9, this implies that there exists an $s_k \in \mathcal{N}_{E_{1k}}$ such that

$$s_k \in \mathcal{N}_{E_{1k}}, \quad Qs_k = QP_{1k} e_1$$

and

$$\| s_k \| \leq c_2 2^{-n}$$

for the constant $c_2 := (1 - (2\varepsilon)^2)/(\varepsilon + \text{dist}(E, \partial J))^2)^{-1/2} \cdot c_1$.

We put

$$d_{1k} := P_{1k} e_1 - s_k.$$

Then we have

$$d_{1k} \in \mathcal{N}_{E_{1k}}, \quad d_{1k} \perp \mathcal{H}_1$$

and

$$\| e_1 - d_{1k} \| \leq c_3 2^{-n}$$

for the constant $c_3 := c_1 + c_2$.

By Lemma 9, we have

$$\| g_1 - P_{1k} g_1 \| \leq c_1 2^{-n}$$

and, by the choice of the g_j, we have in addition that

$$P_{1k} g_1 \perp \mathcal{H}_1.$$

By choosing suitable linear combinations of the vectors d_{1k} and $P_{1k} g_1$ one can find e_{1k} with the following properties:

$$e_{1k} \in \mathcal{N}_{E_{1k}}, \quad k = 1, 2, \tag{17}$$

$$\| e_{1k} \| = 1, \quad e_{11} \perp e_{12}, \tag{18}$$

$$e_{1k} \perp \mathcal{H}_1 \tag{19}$$

and

$$\| e_{1k} - \tilde{e}_{1k} \| \leq c_8 2^{-n} \tag{20}$$

for the constant c_8 defined below. In fact, we have

$$\| \alpha e_1 + \beta_1 g_1 - (\alpha d_{1k} + \beta P_{1k} g_1) \|^2 \leq 2 c_3^2 2^{-2n} (|\alpha|^2 + |\beta|^2) \tag{21}$$

for all α and β. In particular, with

$$\begin{aligned} f_{11} &:= \sin\varphi \cdot d_{11} + \cos\varphi \cdot P_{11} g_1, \\ f_{12} &:= \cos\varphi \cdot d_{12} - \sin\varphi \cdot P_{12} g_1, \end{aligned}$$

we have

$$\| \tilde{e}_{11} - f_{11} \| \leq c_4 2^{-n}, \tag{22}$$

and

$$\| \tilde{e}_{12} - f_{12} \| \leq c_4 2^{-n} \tag{23}$$

for the constant $c_4 := 2c_3$.

Let P be the orthogonal projection onto the space spanned by $\{d_{12}, P_{12}g_1\}$. By (21), we have

$$\| Ph \|^2 \geq (1 - c_5^2 \cdot 2^{-2n}) \| h \|^2 \tag{24}$$

for all h in the space spanned by $\{d_{11}, P_{11}g_1\}$ for the constant $c_5 := 2c_4/(1 - c_4)$.

Since $(\tilde{e}_{11}, \tilde{e}_{12}) = 0$ we have

$$(f_{11}, f_{12}) = (f_{11} - \tilde{e}_{11}, f_{12}) + (\tilde{e}_{11}, f_{12} - \tilde{e}_{12}).$$

By (21) – (23), this implies that

$$|(f_{11}, f_{12})| \leq c_6 \cdot 2^{-n} \tag{25}$$

for the constant $c_6 := c_4(2 + c_4)$.

It easily follows from (21) – (25) that

$$\| f_{11} \| \geq 1 - c_4 \cdot 2^{-n} > 0$$

and

$$\| f_{12} - (f_{11}, f_{12}) \frac{Pf_{11}}{\| Pf_{11} \|^2} \| \geq 1 - c_7 \cdot 2^{-n} > 0$$

for the constant $c_7 := c_4 + c_6(1 - c_4)^{-1}(1 - c_5^2)^{-1/2}$. Thus the vectors

$$e_{11} := \frac{f_{11}}{\| f_{11} \|}$$

and

$$e_{12} := \frac{f_{12} - (f_{11}, f_{12}) \frac{Pf_{11}}{\|Pf_{11}\|^2}}{\| f_{12} - (f_{11}, f_{12}) \frac{Pf_{11}}{\|Pf_{11}\|^2} \|}$$

are well – defined and, by (21) – (25), satisfy the conditions (17) – (20) above with $c_8 := 2c_7$.

Before we proceed with the step $j \Rightarrow j + 1$ in the induction let us note the following fact: Since for $i = 1, \ldots, 2^n$ the spaces spanned by e_i, g_i are two

– dimensional and pairwise orthogonal we can, in addition to $\tilde{e}_{11}$ and $\tilde{e}_{12}$ choose vectors $\tilde{e}_{ik}$, $i = 2, \ldots, 2^n$, $k = 1, 2$ such that

$$\tilde{e}_{ik} \in \operatorname{span}(\{e_i, g_i\}),$$

$$\alpha_i e_i = \alpha_{i1}\tilde{e}_{i1} + \alpha_{i2}\tilde{e}_{i2} \tag{26}$$

for all $i = 2, \ldots, 2^n$, $k = 1, 2$ and

$$(\tilde{e}_{ik}, \tilde{e}_{jk'}) = \delta_{ij}\delta_{kk'}.$$

$\underline{j \Rightarrow j+1}$

Suppose that for $i = 1, \ldots, j$ and $k = 1, 2$ vectors e_{ik} with the following properties are constructed:

$$e_{ik} \in \mathcal{N}_{E_{ik}}. \tag{27}$$

$$e_{i1} \perp e_{i2}, \quad \| e_{ik} \| = 1. \tag{28}$$

$$\begin{aligned} e_{ik} \quad \perp \quad & \{e_{i'k'} : 1 \le i' \le i-1, k' = 1, 2\} \cup \{e_{i'} : i+1 \le i' \le 2^n\} \\ & \cup \{g_{i'} : i+1 \le i' \le 2^n\}. \end{aligned} \tag{29}$$

$$\| e_{ik} - \tilde{e}_{ik} \| \le c_8 2^{-n} \tag{30}$$

for the constant c_8 defined above.

Then $e_{j+1\,1}$ and $e_{j+1\,2}$ are constructed exactly in the same way as e_{11} and e_{12}. Only instead of $\mathcal{H}_1$ one chooses the span of the vectors $e_{11}, e_{12}, \ldots, e_{j1}, e_{j2}, e_{j+2}, \ldots, e_{2^n}, g_{j+2}, \ldots, g_{2^n}$.

By induction, we get now vectors e_{ik}, $i = 1, \ldots, 2^n$, $k = 1, 2$, such that (27) – (30) hold for all $i = 1, \ldots, 2^n$, $k = 1, 2$.

Next we shall show that the inequalities (15) hold. Let $m = 0, 1, 2, \ldots,$ We have

$$\begin{aligned} & \| S^{*m} f_{n+1} - S^{*m} f_n \|^2 \\ = \; & \| \sum_{j=1}^{2^n} \left(E_j^m \alpha_j e_j - E_{j1}^m \alpha_{j1} e_{j1} - E_{j2}^m \alpha_{j2} e_{j2} \right) \|^2 \\ = \; & \sum_{j=1}^{2^n} t_j - 2\mathrm{Re} \sum_{j=2}^{2^n} r_j. \end{aligned}$$

Here we have put as an abbreviation

$$t_j :=\| E_j^m \alpha_j e_j - E_{j1}^m \alpha_{j1} e_{j1} - E_{j2}^m \alpha_{j2} e_{j2} \|^2$$

and

$$r_j := (\sum_{i=1}^{j-1} \sum_{k=1}^{2} E_i^m \alpha_{ik} \tilde{e}_{ik}, E_{j1}^m \alpha_{j1} e_{j1} + E_{j2}^m \alpha_{j2} e_{j2})$$

and used that $e_{ik} \perp e_{jk'}$ if $i \neq j$ or $k \neq k'$, $e_i \perp e_j$ for $i \neq j$ and for $k, k' = 1, 2$ we have $e_{ik} \perp \tilde{e}_{jk'}$ if $i < j$.

Since $\alpha_{j1}^2 + \alpha_{j2}^2 = \alpha_j^2$, $\alpha_{j1}\tilde{e}_{j1} + \alpha_{j2}\tilde{e}_{j2} = \alpha_j e_j$ and $|E_j - E_{jk}| \leq 2\varepsilon 2^{-n}$ for $j = 1, \ldots, 2^n$, $k = 1, 2$, it follows from (30) that

$$t_j \leq \alpha_j^2 2^{-2n} \cdot c^{(m)}, \quad 1 \leq j \leq 2^n, \tag{31}$$

for some finite constant $c^{(m)}$ which neither depends on j nor on n. Since

$$\sum_{j=1}^{2^n} \alpha_j^2 = \mu(\mathbb{R}) \tag{32}$$

it follows that

$$\sum_{j=1}^{2^n} t_j \leq c^{(m)} \mu(\mathbb{R}) 2^{-2n}.$$

By Schwarz' inequality and since $e_j \perp e_i$ for $i \neq j$ we have

$$\begin{aligned} |r_j| \quad \leq \quad & \| E_1^m \alpha_1 e_1 + \ldots + E_{j-1}^m \alpha_{j-1}^m e_{j-1} \| \cdot \\ & \| E_{j1}^m \alpha_{j1} e_{j1} + E_{j2}^m \alpha_{j2} e_{j2} - E_j^m \alpha_j e_j \| . \end{aligned}$$

By (32) and since $(e_i, e_j) = \delta_{ij}$ for all $1 \leq i, j \leq 2^n$ the first factor on the right hand side is bounded by a finite constant which neithers depends on j nor on n. The second factor equals $t_j^{1/2}$.

By (31), (32) and Schwarz' inequality this implies that

$$|\sum_{j=2}^{2^n} r_j| \leq c_m \mu(\mathbb{R})^{1/2} 2^{-n/2}$$

for some finite constant c_m which does not depend on n. Thus the inequality (15) is proved.

By (15), the sequence $(S^{*m}f_n)_{n\in N}$ converges for $m = 0, 1, 2, \ldots$ Let $f := \lim_{n\to\infty} f_n$. Since the operator S^* is closed we get that $f \in D(S^*)$ and $S^*f = \lim_{n\to\infty} S^*f_n$. By repeating this argument, we get that $f \in D(S^{*m})$ and

$$S^{*m}f = \lim_{n\to\infty} S^{*m}f_n \tag{33}$$

for all $m = 0, 1, 2, \ldots$.

Now note that for all $n \in \mathbb{N}$ and $l, m = 0, 1, 2, \ldots$ we have

$$(S^{*m}f_n, S^{*l}f_n) = \int t^{m+l}\mu_n(dt) \tag{34}$$

where

$$\mu_n := \sum_{j=1}^{2^n} \mu(I_j)\delta_{E_j}$$

and for the definition of the I_j and E_j we refer to the first paragraph of the step $\underline{n \Rightarrow n+1}$ in the above construction via induction. Obviously the measures μ_n weakly converge to μ and it follows from (33) and (34) that

$$(S^{*m}f, S^{*l}f) = \int t^{m+l}\mu(dt)$$

for all $l, m = 0, 1, 2, \ldots$ Thus the lemma is proved. $\square$

Lemma 11 *Let S be a symmetric operator in the Hilbert space $\mathcal{H}$. Suppose that the open interval J is a gap of S and the deficiency indices (n, n) of S are infinite. Let μ be a finite measure on the Borel algebra of $\mathbb{R}$ such that $\mu(\mathbb{R}\setminus(E-\varepsilon, E+\varepsilon)) = 0$ for some point $E \in J$ and some $\varepsilon < \frac{1}{96}\,dist(E, \partial J)$. Then there exists a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$, a self – adjoint operator M in $\mathcal{H}_0$ and a symmetric operator G_0 in $\mathcal{H}_0^\perp$ with the following properties:*

(i) $S \subset M \oplus G_0$.

(ii) $M \simeq Q_\mu$.

(iii) J is a gap of G_0.

(iv) Let (n_0, n_0) be the deficiency indices of G_0. Then $n_0 \geq n$.

Proof: First we shall consider the case when the deficiency index n is strictly larger than the cardinality $\aleph_0$ of the set of integers.

By the lemmata 10 and 6, we can choose a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ and a self – adjoint operator M in $\mathcal{H}_0$ such that

$$M \subset S^* \quad \text{and} \quad M \simeq Q_\mu.$$

By Lemma 5, the operator S_M, defined by

$$S_M := S^*_{|\, D(S)+D(M)},$$

can be represented as

$$S_M = M \oplus G_0$$

for some symmetric operator G_0 with gap J. Obviously $S \subset S_M$. Thus we need only to prove that $n_0 \geq n$ where (n_0, n_0) are the deficency indices of G_0.

Let $z \in \mathbb{C} \setminus \mathbb{R}$. Let $g \in \operatorname{ran}(S-z)^\perp \cap \mathcal{H}_0^\perp$. Then $(g, (M-z)h) = 0$ for all $h \in D(M)$ since M is an operator in $\mathcal{H}_0$. It follows that $g \in \operatorname{ran}(S_M - z)^\perp$. Since $G_0 \subset S_M$ it follows that

$$n_0 \geq \dim\left(\operatorname{ran}(S-z)^\perp \cap \mathcal{H}_0^\perp\right).$$

Since $M \simeq Q_\mu$ we have, in particular, that $\mathcal{H}_0 \simeq L^2(\mathbb{R}, \mu)$. Thus $\dim \mathcal{H}_0 \leq \aleph_0$. Since $\dim \operatorname{ran}(S-z)^\perp = n > \aleph_0$ it follows that

$$\dim\left(\operatorname{ran}(S-z)^\perp \cap \mathcal{H}_0^\perp\right) \geq n - \aleph_0 = n$$

and the lemma is proved in the case when the deficiency index n is strictly larger than $\aleph_0$.

Now let $n = \aleph_0$. Choose infinitely many pairwise different points E_n, $n \in \mathbb{N}$, satisfying

$$\varepsilon < |E - E_n| < \tilde{\varepsilon}, \quad n \in \mathbb{N},$$

for some $\tilde{\varepsilon} < \frac{1}{96}\operatorname{dist}(E, \partial J)$ and put

$$\mu' := \sum_{n=1}^{\infty} 2^{-n} \delta_{E_n}, \quad \tilde{\mu} := \mu + \mu'.$$

By the Lemmata 10 and 6, we can choose a closed subspace $\tilde{\mathcal{H}}_0$ of $\mathcal{H}$ and a self – adjoint operator $\tilde{M}$ in $\tilde{\mathcal{H}}_0$ such that

$$\tilde{M} \simeq Q_{\tilde{\mu}} \quad \text{and} \quad \tilde{M} \subset S^*.$$

Since obviously

$$Q_{\tilde{\mu}} \simeq Q_\mu \oplus Q_{\mu'}$$

there exist self – adjoint operators M and M' such that

$$\tilde{M} = M \oplus M', \quad M \simeq Q_\mu, \quad M' \simeq Q_{\mu'}.$$

By Lemma 5 and with the notation of this lemma, we have

$$S \subset S_M = M \oplus G_0 \subset S_{\tilde{M}} = M \oplus M' \oplus \tilde{G}_0 \tag{35}$$

for suitably chosen symmetric operators G_0 and $\tilde{G}_0$ and J is a spectral gap both of G_0 and $\tilde{G}_0$.

Since $\tilde{G}_0$ is symmetric and has a gap it has a self – adjoint extension $\tilde{G}$. By (35), $M' \oplus \tilde{G}$ is a self – adjoint extension of G_0 and, since $M' \simeq Q_{\mu'}$, each of the infinitely many points E_n, $n \in \mathbb{N}$, inside the gap J of G_0 is an eigenvalue of M' and therefore also of the self – adjoint extension $M' \oplus \tilde{G}$ of G_0. Thus the symmetric operator G_0 must have infinite deficiency indices and the lemma is also proved in the case when the deficiency indices of S equal $\aleph_0$. $\square$

It is now easy to complete the proof of Theorem 1. Let (n, n) be the deficiency indices of the symmetric operator S. We have already completed the proof of the theorem in the case when n is finite, cf. the considerations following the proof of Lemma 5.

Next consider the case when $n = \aleph_0$. Since the operator A_J^{aux} is self – adjoint there exists a family $\{\mu_i\}_{i \in I}$ of finite measures μ_i, $i \in I$, on the Borel algebra of $\mathbb{R}$ such that

$$A_J^{aux} \simeq \bigoplus_{i \in I} Q_{\mu_i}. \tag{36}$$

Since the spectral projector $E_{A_J^{aux}}(\mathbb{R} \setminus J)$ corresponding to A_J^{aux} and $\mathbb{R} \setminus J$ equals zero we have

$$\mu_i(\mathbb{R} \setminus J) = 0, \quad i \in I.$$

Since, by hypothesis, A_J^{aux} is a self – adjoint operator in an Hilbert space with dimension less than or equal to n and here $n = \aleph_0$ all but at most countably many of the measures μ_i equal zero.

We can choose pairwise disjoint Borel sets B_k, $k \in \mathbb{N}$, such that

$$J = \bigcup_{k \in \mathbb{N}} B_k \quad \text{and} \quad B_k \subset (E_k - \varepsilon_k, E_k + \varepsilon_k), \quad k \in \mathbb{N},$$

where for every $k \in \mathbb{N}$ both $E_k \in J$ and $\varepsilon_k < \frac{1}{96}\text{dist}\,(E_k, \partial J)$. Since obviously

$$Q_\mu = \bigoplus_{k \in \mathbb{N}} Q_{1_{B_k}\,\mu}$$

(1_B denotes the characteristic function of B) for every finite measure μ on the Borel algebra of $\mathbb{R}$ satisfying $\mu(\mathbb{R} \setminus J) = 0$ we may assume that $I = \mathbb{N}$ for the index I in the representation (36) and that for every $n \in \mathbb{N}$ there exists $E_n \in J$ and $\varepsilon_n < \frac{1}{96}\text{dist}\,(E, \partial J)$ such that

$$\mu_n(\mathbb{R} \setminus (E_n - \varepsilon_n, E_n + \varepsilon_n)) = 0.$$

By Lemma 11, there exist closed subspaces $\mathcal{H}_1$ and $\tilde{\mathcal{H}}_2$ of $\mathcal{H}$, a self – adjoint operator M_1 in $\mathcal{H}_1$ and a symmetric operator G_1 in $\tilde{\mathcal{H}}_2$ such that the following holds:

(i) $\mathcal{H} = \mathcal{H}_1 \oplus \tilde{\mathcal{H}}_2$.

(ii) $S \subset M \oplus G_1$.

(iii) $M_1 \simeq Q_{\mu_1}$.

(iv) J is a gap of G_1.

(v) The deficiency indices of G_1 are infinite.

Since

$$S \subset M_1 \oplus G_1 \subset (M_1 \oplus G_1)^* = M_1 \oplus G_1^* \subset S^*$$

the operator G_1^* in $\tilde{\mathcal{H}}_2$ is a restriction of S^*.

By replacing $\mathcal{H}$ by $\tilde{\mathcal{H}}_2$ and S by G_1 in the above considerations, we get that there exist closed subspaces $\mathcal{H}_2$ and $\tilde{\mathcal{H}}_3$ of $\tilde{\mathcal{H}}_2$ and a symmetric operator G_2 in $\tilde{\mathcal{H}}_3$ with the following properties:

(i) $\tilde{\mathcal{H}}_2 = \mathcal{H}_2 \oplus \tilde{\mathcal{H}}_3$.

(ii) $G_1 \subset M_2 \oplus G_2$.

(iii)$M_2 \simeq Q_{\mu_2}$.

(iv) J is a gap of G_2.

(v) The deficiency indices of G_2 are infinite.

Note that

$$M_2 \subset S^*$$

since

$$G_1 \subset M_2 \oplus G_2 \subset (M_2 \oplus G_2)^* = M_2 \oplus G_2^* \subset G_1^* \subset S^*.$$

By repeating the above arguments, we can prove, by induction, that there exist closed subspaces $\mathcal{H}_n$ and self – adjoint operators M_n in $\mathcal{H}_n$, $n \in \mathbb{N}$, with the following properties:

(i)

$$\mathcal{H}_n \perp \mathcal{H}_j, \quad \text{if} \quad n \neq j.$$

(ii)

$$M_n \simeq Q_{\mu_n}, \quad n \in \mathbb{N}. \tag{37}$$

(iii)

$$M_n \subset S^* \quad \text{for all} \quad n \in \mathbb{N}.$$

Obviously

$$M := \bigoplus_{n\in\mathbb{N}} M_n$$

is a self – adjoint operator in the closed subspace

$$\mathcal{H}_0 := \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$$

of $\mathcal{H}$ and, by (37) and since

$$A_J^{aux} \simeq \bigoplus_{n\in\mathbb{N}} Q_{\mu_n},$$

we have that

$$M \simeq A_J^{aux}.$$

Since $M_n \subset S^*$ for all $n \in \mathbb{N}$ and S^* is closed we have

$$M \subset S^*.$$

It follows now from Lemma 5 that S has a self – adjoint extension A such that

$$A_J \simeq A_J^{aux}.$$

Thus the theorem is also proved in the case when $n = \aleph_0$. The proof in the case when $n > \aleph_0$ is the same. One has only to work with transfinite induction instead of induction. □

Corollary 12 *Let S be a symmetric operator in the Hilbert space $\mathcal{H}$. Suppose that the open interval J is a gap of S and the deficiency indices of S are infinite. Let μ be a measure on the Borel algebra of $\mathbb{R}$ such that*

$$\mu(\mathbb{R} \setminus J) = 0 \quad \textit{and} \quad \int t^{2n} \mu(dt) < \infty, \quad n = 0, 1, 2, \ldots$$

Then there exists an f such that $f \in D(S^{*m})$ for all $m \in \mathbb{N}$ and

$$(S^{*m} f, S^{*l} f) = \int t^{m+l} \mu(dt), \quad m, l = 0, 1, 2, \ldots$$

Proof: By Theorem 1, there exist a self – adjoint extension A of S and a unitary transformation U such that

$$A_J = U^{-1} Q_\mu U.$$

Since $S \subset A = A^* \subset S^*$ and $A_J \subset A$ we have $A_J^m \subset S^{*m}$ for all $m \in \mathbb{N}$. It follows that for $f := U^{-1}1$ (1 being the function which equals 1 everywhere)

$$\begin{aligned} (S^{*m} f, S^{*l} f) &= (A_J^m f, A_J^l f) \\ &= (Q_\mu^m 1, Q_\mu^l 1) \\ &= \int t^{m+l} \mu(dt), \quad m, l = 0, 1, 2, \ldots \end{aligned}$$

□

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