On The Limiting Behaviour Of An Epidemic In Two Interacting Populations

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Abstract

In this note, we consider a stochastic model for the spread of an epidemic in a closed population consisting of two groups, in which infectives cannot move between groups, but are able to infect outside their own group. Using the matrix-geometric method, we obtain a recursive relationship for the Laplace transform of the joint distribution of some interesting quantities. We also derive the distribution of the total observed size of the epidemic in the case of a general infection mechanism.

Keywords Epidemic model; Matrix-Geometric; Final size

1. Introduction

We consider a stochastic model for an epidemic taking place in a heterogeneous population consisting of two groups. The infection can be transmitted both within and between the groups. ¿From the standpoint of the infection mechanism, our model is a special generalisation of a model considered by Gani and Yakowitz (1995) in the case of a closed population. Similar models have also been studied by Bailey (1975) (chapter 11) and O'Neill (1995) who derived a class of results for the probability of ultimate extinction. Here, we use a matrix-geometric method (cf. Neuts (1981)) similar to that of Booth (1989) to obtain the distribution of the total numbers of infections that occur in the entire population. The use of the matrix-geometric method in the study of epidemics was pioneered by Gani and Perdue (1984).

The paper is structured as follows. A formal description of the model is given in section 2. In section 3, we account for the matrix-geometric method and in section 4 we show a recursive relationship for the Laplace transform of the joint distribution of some quantities of interest. The total size distribution is discussed in section 5 while section 6 is devoted to a simple numerical example. Some of the derivations call for tedious algebraic manipulations that are presented in the appendices.

2. The model

In what follows we consider a model for the spread of an epidemic in a closed population consisting of two groups of individuals G_1 and G_2. The following notation will be used throughout the article. $X_i(t)$ and $Y_i(t)$ stand for the numbers of susceptibles and infectives at time t for the ith group with $(X_1(0), X_2(0), Y_1(0), Y_2(0)) = (n_1, n_2, a_1, a_2)$. In each group the rate of infection is related to the number of susceptibles and infectives in the two groups. Infectives in group G_i, $i = 1, 2$ are removed at rate $\mu_i \geq 0$, so the processes is completely determined by $\{(X_1(t), X_2(t), Y_1(t), Y_2(t)); t \geq 0\}$. This process is a homogeneous Markov chain on the state space

$$S = \{(x, y, u, v); 0 \leq x \leq n_1,\ 0 \leq y \leq n_2,\ 0 \leq u \leq N_1^x,\ 0 \leq v \leq N_2^y\},$$

where $N_1^x = n_1 + a_1 - x$, and $N_2^y = n_2 + a_2 - y$, with the following infinitesimal transitions:

$$\begin{array}{ll}
(X_1, X_2, Y_1, Y_2) \to (X_1 - 1, X_2, Y_1 + 1, Y_2) & \text{at rate } f_{X_1X_2Y_1Y_2, X_1-1X_2Y_1+1Y_2} \\
(X_1, X_2, Y_1, Y_2) \to (X_1, X_2 - 1, Y_1, Y_2 + 1) & \text{at rate } f_{X_1X_2Y_1Y_2, X_1X_2-1Y_1Y_2+1} \\
(X_1, X_2, Y_1, Y_2) \to (X_1, X_2, Y_1, Y_2) & \text{at rate } f_{X_1X_2Y_1Y_2} \\
(X_1, X_2, Y_1, Y_2) \to (X_1, X_2, Y_1 - 1, Y_2) & \text{at rate } \mu_1 Y_1 \\
(X_1, X_2, Y_1, Y_2) \to (X_1, X_2, Y_1, Y_2 - 1) & \text{at rate } \mu_2 Y_2.
\end{array}$$

with the conventions that $f_{ijlr,i'j'l'r'} = 0$ if $(i, j, l, r) \notin S$ or $(i', j', l', r') \notin S$, and $f_{ij00,i-1j10} = f_{ij00,ij-101} = 0$. Let

$$P_{ijlr}(t) = P(X_1(t) = i, X_2(t) = j, Y_1(t) = l, Y_2(t) = r), \quad \text{for } t \geq 0.$$

Then the forward Kolmogorov equations take the form

$$\begin{aligned}
\frac{\partial P_{ijlr}(t)}{\partial t} &= f_{ijlr} P_{ijlr}(t) + f_{ijl+1r,ijlr} P_{ijl+1r}(t) + f_{ijlr+1,ijlr} P_{ijlr+1}(t) \\
&\quad + f_{i+1jl-1r,ijlr} P_{i+1jl-1r}(t) + f_{ij+1lr-1,ijlr} P_{ij+1lr-1}(t) \qquad (1)
\end{aligned}$$

with the conventions that $P_{ijlr}(t) \equiv 0$ if $(i, j, l, r) \notin S$ and $P_{n_1n_2a_1a_2}(0) = 1$.

3. The matrix-geometric method

The standard p.g.f. methods are now ineffective, as was shown by Bailey (1975) . However, the Kolmogorov equations can be solved recursively using the matrix-geometric method. For $i = 0, ..., n_1$, $j = 0, ..., n_2$ and $l = 0, ..., N_1^i$, let A_{ij}^l, B_{ij}^{l+1}, D_{i+1j}^{l-1} and H_{ij}^l be respectively the diagonal matrices with r-th diagonal element equal to $f_{ijlr}, \mu_1(l + 1), f_{i+1jl-1r,ijlr}$ and $f_{ij+1lr-1}$, $r = 0, .., N_2^j$, and let C_{ij}^l be the matrix of the same dimension with $(r, r + 1) - th$ entries equal to $\mu_2(r + 1)$, $r = 0, .., N_2^j - 1$ and all others equal to 0. In addition, take $P_{ij}^l(t) = (P_{ijl0}(t), P_{ijl1}(t), \dots P_{ijlN_2^j-1}(t), P_{ijlN_2^j}(t))^T$. Equations (1) take now the form

$$\frac{\partial P_{ij}^l(t)}{\partial t} = A_{ij}^l P_{ij}^l(t) + C_{ij}^l P^l ij(t) + D_{i+1j}^{l-1} P_{i+1j}^{l-1}(t) + H_{ij}^l P_{ij+1}^{*l}(t) + B_{ij}^{l+1} P_{ij}^{l+1}(t), \qquad (2)$$

where $P^{*l}_{ij}(t) = (0, (P^l_{ij}(t))^T)^T$.

Furthermore, we introduce the column vectors $P_{ij}(t) = ((P^0_{ij}(t))^T, \dots, (P^l_{ij}(t))^T, \dots, (P^{N^i_1}_{ij}(t))^T)$, the block matrices $D_{i+1j} = diag(D^l_{i+1j}, 0 \le l \le N^{i+1}_1), H_{ij} = diag(H^l_{ij}, 0 \le l \le N^i_1)$ and a matrix F_{ij} the (l,l)-th block of which equals $A^l_{ij} + C^l_{ij}$, $l = 0,..,N^i_1$, its $(l, l+1)$-th block is equal to B^{l+1}_{ij}, $l = 0,..,N^{i+1}_1$, and all other blocks being equal to zero.

For each matrix A of order $(N^{i+p}_1 + 1)(N^{j+q}_2 + 1)$, for $0 \le p \le n_1 - i$ and $0 \le q \le n_2 - j$ we define an augmented matrix

$$A(p,q) = \begin{pmatrix} \Theta^{pq}_{ij} & 0 \\ 0 & A \end{pmatrix},$$

where Θ^{pq}_{ij} is the zero matrix of order $q(N^i_1 + 1) + p(N^j_2 + 1) - pq$, and for each vector $U(t)$ of dimension $(N^{i+p}_1 + 1)(N^{j+q}_2 + 1)$, we also define

$$U(t,p,q) = ((\theta^{pq}_{ij})^T, U^T(t))^T,$$

where θ^{pq}_{ij} is the $q(N^i_1 + 1) + p(N^j_2 + 1) - pq$ zero column vector, and

$$P^*_{ij+1}(t) = ((P^{*0}_{ij+1}(t))^T, \dots, (P^{*N^i_1}_{ij+1}(t))^T)^T.$$

Using the previous notation, equations (2) lead to

$$\frac{\partial P_{ij}(t)}{\partial t} = F_{ij}P_{ij}(t) + D_{i+1j}(1,0)P_{i+1j}(t,1,0) + H_{ij}P^*_{ij+1}(t). \qquad (3)$$

To obtain an appropriate form for the above equations which can help us to solve (1), we investigate the possible relationship between $P^*_{ij}(t)$ and $P_{ij}(t,0,1)$. For this, let T_{ij} be the matrix of rank $(N^i_1 + 1)(N^j_2 + 1)$ where

$$(T_{ij})_{mn} = \begin{cases} 1 & \text{if } m = r(N^j_2 + 1) + k \text{ and } n = N^i_1 + rN^j_2 + k, \\ & \text{with } 0 \le r \le N^1_j,\ 1 \le k \le N^j_2 \\ 0 & \text{otherwise.} \end{cases}$$

By rearrangement we have $P^*_{ij+1}(t) = T_{ij}P_{ij+1}(t,1,0)$ and by substitution in (3) we obtain

$$\frac{\partial P_{ij}(t)}{\partial t} = F_{ij}P_{ij}(t) + D_{i+1j}(1,0)P_{i+1j}(t,1,0) + H_{ij}T_{ij}P_{ij+1}(t,0,1), \qquad (4)$$

for $0 \le i \le n_1$ and $0 \le j \le n_2$.

The limiting distribution of the process can now be studied using Laplace transformation

$$\hat{P}_{ij}(v) = \int_0^{+\infty} e^{-vt}P_{ij}(t)\,dt.$$

Equation (4) becomes

$$\hat{P}_{n_1n_2}(v) = (vI_{n_1n_2} - F_{n_1n_2})^{-1}E, \qquad (5)$$

and

$$\hat{P}_{ij}(v) = (vI_{ij} - F_{ij})^{-1}D_{i+1j}(1,0)\hat{P}_{i+1j}(v,1,0) + (vI_{ij} - F_{ij})^{-1}H_{ij}T_{ij}\hat{P}_{ij+1}(v,0,1), \qquad (6)$$

for $0 \le i \le n_1, 0 \le j \le n_2$, and $i + j \ne n_1 + n_2$, where $E = P_{n_1n_2}(0) = (0,..,0,1)^T$ and I_{ij} denote the identity matrix of order $(N^i_1 + 1)(N^j_2 + 1)$.

4. The solution

First we determine the Laplace transforms of the probabilities $P_{ijlr}(v)$. It can be shown that $F_{ij}(v) = (vI_{ij} - F_{ij})^{-1}$ has the form

$$F_{ij}(v) = \begin{pmatrix} F_{ij}^{00}(v) & F_{ij}^{01}(v) & . & . & . & F_{ij}^{0h}(v) & . & F_{ij}^{0N_1^i}(v) \\ 0 & F_{ij}^{11}(v) & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & . \\ . & . & . & F_{ij}^{ll}(v) & . & F_{ij}^{lh}(v) & . & F_{ij}^{lN_1^i}(v) \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 0 & F_{ij}^{N_1^iN_1^i}(v) \end{pmatrix},$$

where for $0 \leq l \leq h \leq N_1^i$, $F_{ij}^{lh}(v)$ is a block of rank $N_2^j + 1$. Moreover It can be verified (cf. the Appendix) that

$$[F_{ij}^{lh}(v)]_{rs} = \begin{cases} C_{ij}(v,l,h,r,s) & \text{if } 0 \leq r \leq s \leq N_2^j \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where

$$C_{ij}(v,l,h,r,s) = \mu_1^{h-l}\mu_2^{s-r}\frac{h!s!}{l!r!}\sum_{I\in D_{rs}^{h-l}}\prod_{k=0}^{h-l}\prod_{q=i_k}^{i_{k+1}} f(v,\mu_1,\mu_2,i,j,l+k,q), \tag{8}$$

where $f(v,\mu_1,\mu_2,i,j,l,r) = (v+\mu_1 l+\mu_2 r+f_{ijlr,i-1jl+1r}+f_{ijlr,ij-1lr+1})^{-1}$, $i_0 = r$, $i_{h-l+1} = s$ and

$$D_{rs}^{h-l} = \begin{cases} \{(i_1,i_2,\ldots,i_{h-l} \leq s) \;/\; r \leq i_1 \leq i_2 \leq \ldots \leq i_{h-l} \leq s\} & \text{if } l < h \\ \emptyset & \text{if } l = h \end{cases} \tag{9}$$

with the conventions that

$$\prod_{p\in B} A_p = 1 \;\text{ and }\; \sum_B 1 = 1 \text{ if } B = \emptyset \;\text{ and }\; A_p > 0. \tag{10}$$

The quantities $C_{ij}(v,l,h,r,s)$ can be calculated (cf. the Appendix) using the following recursive relationship, for $0 \leq l \leq h \leq N_1^i$

$$C_{ij}(v,l,l,r,s) = \mu_2^{s-r}\frac{s!}{r!}\prod_{q=r}^{s} f(v,\mu_1,\mu_2,i,j,l,q), \tag{11}$$

and

$$C_{ij}(v,l,h,r,s) = \mu_1 h\sum_{p=r}^{s}\mu_2^{s-p}\frac{s!}{p!}\prod_{q=p}^{s} f(v,\mu_1,\mu_2,i,j,h,q)C_{ij}(v,l,h-1,p,s). \tag{12}$$

For $m,n = 0,...,(N_1^i+1)(N_2^j+1)-1$, let l,h and r,s be, respectively, the quotient and rest of the Euclidean division of m,n by N_2^j+1. We have

$$[F_{ij}(v)D_{i+1j}(1,0)]_{mn} = \begin{cases} C_{ij}(v,l,h,r,s)\,f_{i+1jh-1s,ijhs} & \text{if } r \leq s \text{ and } l \leq h,\; h \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Similarly, if l, h and $r, s-1$ are, respectively, the quotients and rests of the Euclidean division of m by N_2^j+1 and $n-N_1^i-1$ by N_2^j, then

$$[F_{ij}(v)H_{ij}T_{ij}]_{mn} = \begin{cases} C_{ij}(v,l,h,r,s)\, f_{ij+1hs-1,ijhs} & \text{if } r \le s,\ s \ge 1 \text{ and } l \le h \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Since $\hat{P}_{ijlr}(v)$ and $\hat{P}_{i+1jl-1s}(v)$ correspond, respectively, to the $(l(N_2^j+1)+r)$-th elements of the vectors $\hat{P}_{ij}(v)$ and $\hat{P}_{i+1j}(v,1,0)$, while $\hat{P}_{ij+1lr-1}(v)$ correspond to the $(N_1^i+lN_2^j+r)$-th element of $\hat{P}_{i+1j}(v,0,1)$, then, using (5)-(7) and the previous result, we find

$$\hat{P}_{n_1n_2lr}(v) = C_{n_1n_2}(v,l,a_1,r,a_2), \tag{15}$$

$$\begin{cases} \hat{P}_{in_2lr}(v) = \sum\limits_{\substack{l\le h\le N_1^i, h\ge 1\\ r\le s\le a_2}} C_{in_2}(v,l,h,r,s)\, f_{i+1n_2h-1s,in_2hs}\hat{P}_{i+1n_2h-1s}(v) \\ \hat{P}_{n_1jlr}(v) = \sum\limits_{\substack{l\le h\le a_1\\ r\le s\le N_2^j, s\ge 1}} C_{n_1j}(v,l,h,r,s)\, f_{n_1j+1hs-1,n_1jhs}\hat{P}_{n_1j+1h-1s}(v) \end{cases} \tag{16}$$

for $i=0,..,n_1-1$ and $j=0,..,n_2-1$, and

$$\begin{aligned} \hat{P}_{in_2-jlr}(v) = & \sum_{\substack{l\le h\le N_1^i, h\ge 1\\ r\le s\le a_2+j}} C_{in_2-j}(v,l,h,r,s)\, f_{i+1n_2-jh-1s,in_2-jhs}\hat{P}_{i+1n_2-jh-1s}(v) \\ & + \sum_{\substack{l\le h\le N_1^i\\ r\le s\le a_2+j, s\ge 1}} C_{in_2-j}(v,l,h,r,s)\, f_{in_2-j+1hs-1,in_2-jhs}\hat{P}_{in_2-j+1hs-1}(v) \end{aligned} \tag{17}$$

for $i=0,..,n_1-1$ and $j=1,..,n_2$. ¿From $(15)-(17)$ we conclude that the Laplace transforms can be solved recursively.

5. The total size

The asymptotic behaviour of the process $\{(X_1(t), X_2(t), Y_1(t), Y_2(t)); t \ge 0\}$ can be described using $(14)-(16)$, (7) and the identity $\lim_{t\to\infty} P_{ijlr}(t) = \lim_{v\to 0}(v\hat{P}_{ijlr}(v))$. The epidemic ends as soon as the numbers of infectives in both group become zero. Let π_{ij} denote the probability that exactly i and j of initially susceptible individuals ultimately escape the epidemic in G_1 and G_2, respectively. In order to determine this probability, it is necessary to calculate the limit $\lim_{v\to 0}(vC_{ij}(v,0,h,0,s)) = C_{ij}(h,s)$. We show (cf. the appendix) that such a limit exists and is, for $h=0$,

$$C_{ij}(0,s) = \mu_2^s s! \prod_{q=1}^{s} f(\mu_1,\mu_2,i,j,0,q) \tag{18}$$

and for $h>0$,

$$C_{i,j}(h,s) = \mu_1^h\mu_2^s h! s! \sum_{0\le p\le s} \left\{ \prod_{q=1}^{p} f(\mu_1,\mu_2,i,j,0,q) \sum_{I_p\in B_{ps}^{h-1}} \prod_{k=0}^{h-1} \prod_{q=i_{p_k}}^{i_{p_{k+1}}} f(\mu_1,\mu_2,i,j,l+k,q) \right\} \tag{19}$$

where

$$B_{ps}^{h-1} = \begin{cases} \{(i_{p_1}, ..., i_{p(h-1)}) \,/\, p \leq i_{p_1} \leq ... \leq i_{p(h-1)} \leq s\} & \text{if } h > 1 \\ \emptyset & \text{if } h = 1, \end{cases}$$

$f(\mu_1, \mu_2, i, j, l, r) = (\mu_1 l + \mu_2 r + f_{ijlr,i-1jl+1r} + f_{ijlr,ij-1lr+1})^{-1}$, $i_{p_0} = p$ and $i_{p_h} = s$.
Finally, (15) implies that

$$\pi_{n_1 n_2} = \lim_{v \to 0} v \hat{P}_{n_1 n_2 00}(v) = C_{n_1, n_2}(a_1, a_2)$$

Similarly, from (16) and (17) respectively, it can be shown that for $i = 0, ..., n_1 - 1$ and $j = 0, ..., n_2 - 1$.

$$\begin{aligned} \pi_{n_1 j} &= \sum_{h=1}^{a_1} \sum_{s=1}^{N_2^j} C_{n_1 j}(h, s) f_{n_1 j+1hs-1, n_1 jhs} \hat{P}_{n_1 j+1hs-1}(0) \\ &+ \sum_{s=2}^{N_2^j} C_{n_1 j}(0, s) f_{n_1 j+10s-1, n_1 j0s} \hat{P}_{n_1 j+10s-1}(0), \\ \pi_{in_2} &= \sum_{s=1}^{a_2} \sum_{h=1}^{N_1^i} C_{in_2}(h, s) f_{i+1n_2 h-1s, in_2 hs} \hat{P}_{i+1n_2 h-1s}(0) \\ &+ \sum_{h=2}^{N_1^i} C_{in_2}(h, 0) f_{i+1n_2 h-10, in_2 h0} \hat{P}_{i+1n_2 h-10}(0), \end{aligned}$$

and

$$\begin{aligned} \pi_{ij} &= \sum_{h=1}^{N_1^i} \sum_{s=1}^{N_2^j} C_{ij}(h, s)[f_{ij+1hs-1, ijhs} \hat{P}_{ij+1hs-1}(0) + f_{i+1jh-1s, ijhs} \hat{P}_{i+1jh-1s}(0)] \\ &+ \sum_{s=2}^{N_2^j} C_{ij}(0, s) f_{ij+10s-1, ij0s} \hat{P}_{ij+10s-1}(0) + \sum_{h=2}^{N_1^i} C_{ij}(h, 0) f_{i+1jh-10, ijh0} \hat{P}_{ij+1h0}(0). \end{aligned}$$

These probabilities can be determined using (18) and (19) and by means of the recursive equations (16) − (17).

6. A numerical example

If $\mu_1 = \mu_2 = 1$ and assuming a modified epidemic model (see e.g., O'Neill (1995); Ball and O'Neill (1993)):

$$\begin{aligned} f_{X_1 X_2 Y_1 Y_2, X_1 - 1 X_2 Y_1 + 1 Y_2} &= X_1(\beta_{11} Y_1 (X_1 + Y_1)^{-1} + \beta_{12} Y_2 (X_1 + Y_2)^{-1}) \\ f_{X_1 X_2 Y_1 Y_2, X_1 X_2 - 1 Y_1 Y_2 + 1} &= X_2(\beta_{21} Y_2 (X_1 + Y_2)^{-1} + \beta_{22} Y_2 (X_2 + Y_2)^{-1}). \end{aligned}$$

where for $i, j = 1, 2$ β_{ij} is the rate for a susceptible in group i to get infected by an infective from group j.

Using the methods presented in this paper it is straightforward to obtain numerical results.

Tables 1-3 illustrate some results using the initial conditions $n_1 = n_2 = 5$, $a_1 = 0$ and $a_2 = 1$. The tables have different $(\beta_{11}, \beta_{12}, \beta_{11}, \beta_{21}, \beta_{22})$ values, thus illustrating the effects upon the total size distribution. Each table displays the final size distribution for a modified stochastic epidemic.

Table.1

$\beta_{11} = 0.4,\ \beta_{12} = 0.3,\ \beta_{21} = 4,\ \beta_{22} = 2$

		$n_1 - X_2(\infty)$					
		0	1	2	3	4	5
	0	0.3429	0.0787	0.0396	0.0284	0.0296	0.0812
	1	0.0063	0.0063	0.0065	0.0078	0.0135	0.1110
	2	0.0002	0.0005	0.0008	0.0014	0.0037	0.0948
$n_1 - X_1(\infty)$	3	0.0000	0.0000	0.0001	0.0002	0.0008	0.0684
	4	0.0000	0.0000	0.0000	0.0000	0.0002	0.0459
	5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0290

For Table 1 we note that $\beta_{11} + \beta_{21} \geq \beta_{22} + \beta_{12} \geq 1$. This implies that the first group acts as an important source of infection for the population as a whole, but that susceptibles in this group have few contacts with infectives in both groups $(\beta_{12} \leq 1,\ \beta_{11} \leq 1)$, so infections transmitted to group 1, whether from 1 or 2, tend to die out quickly. This is, however, compensated for since the parameters of the second group are above the threshold.

Table.2

$\beta_{11} = 0.01,\ \beta_{12} = 0.03,\ \beta_{21} = 0.5,\ \beta_{22} = 5$

		$n_1 - X_2(\infty)$					
		0	1	2	3	4	5
	0	0.2294	0.0491	0.0247	0.0202	0.0319	0.5608
	1	0.0009	0.0007	0.0006	0.0008	0.0018	0.0732
	2	0.0000	0.0000	0.0000	0.0000	0.0001	0.0056
$n_1 - X_1(\infty)$	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003
	4	0.0000	0.0000	0.0000	0.0000	0.0002	0.0000
	5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

On the other hand, for Table 2, we have $\beta_{22} \geq 1$, $\beta_{12} \leq 1$, $\beta_{22}+\beta_{12} \geq 1$, and $\beta_{11}+\beta_{21} \leq 1$, so the parameters of the first group are below the threshold while the parameters of the second group are abobe it. In the case of Table 3 all parameters have low values so the epidemic as a whole dies out quickly with high probability.

Table.3

$\beta_{11} = \beta_{22} = 0.5,\ \beta_{12} = 0.1,\ \beta_{21} = 0.4$

		$n_1 - X_2(\infty)$					
		0	1	2	3	4	5
	0	0.6667	0.1333	0.0518	0.0243	0.0123	0.0065
	1	0.0214	0.0156	0.0109	0.0076	0.0053	0.0040
	2	0.0039	0.0043	0.0039	0.0033	0.0028	0.0027
$n_1 - X_1(\infty)$	3	0.0011	0.0016	0.0018	0.0018	0.0017	0.0020
	4	0.0004	0.0007	0.0009	0.0010	0.0011	0.0013
	5	0.0001	0.0003	0.0005	0.0006	0.0008	0.0010

7. Discussion

We have used the matrix-geometric method to study the behaviour of a stochastic model of an epidemic in a population consisting of two interacting groups. In some instances this method seems to be more tractable than direct analysis. In particular this is the case of the derivation of the results of section 3. In section 4 we were able to obtain a recursive relation for the Laplace transform.

Some comments are in order concerning the comparison between the matrix approach we have adopted and other methods. In the process of showing a simple connection between the Gontcharoff polynomials and the total size distribution Ball and O'Neill (1997) considered a simple infection mechanism. The method presented in this paper reduces this assumption to what might be called the general infection mechanism. On the other hand in a computer program, round-off errors occurring in the computation of the two-type Gontcharoff polynomials would manifest themselves in the successive steps and would be magnified. In our approach round-off errors would occur in the computation of the quantities C_{ij}, but the round-off error occurring from the computation of one C_{ij} does not affect the computation of any other $C_{i'j'}$, $(i, j \neq i', j')$ and hence round-off errors are not magnified.

Appendix

In what follows we give proofs of certain facts needed in the article.

Proof of (7) and (8)

For $i=0,...,n_1, j=0,...,n_2$ and $l=0,...,N_1^i$, let $\overline{H}_{ij}^l$, $\overline{C}_{ij}^l$, respectively, be the diagonal matrices with rth diagonal element equal to $f_{ijlr,ij-1lr+1}$ and μ_{2r}, $\quad r=0,..,N_2^j$, and let $\triangle_j$ be the matrix of rank N_2^j+1 with $(r,r+1)-$th element equal to 1 and all others equal to 0. We also define the matrix of rank $(N_1^i+1)(N_2^j+1)$

$$\triangle_{ij}=\begin{pmatrix} 0 & I_j & 0 & . & . & . & . & 0 \\ . & 0 & I_j & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & I_j & . & . & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & I_j & 0 \\ . & . & . & . & . & . & . & I_j \\ 0 & . & . & . & . & . & . & 0 \end{pmatrix},$$

where I_j denotes the identity matrix of rank N_2^j+1.
By using the matrices defined in the section 2, we take

$$B_{ij}=diag(B_{ij}^l, 0\le l\le N_1^i)$$

and

$$Z_{ij}=diag(Z_{ij}^l, 0\le l\le N_1^i),$$

where

$$Z_{ij}^l=\overline{C}_{ij}^l+D_{ij}^l-C_{ij}^l+\overline{H}_{ij}^l=(I_j-\triangle_j)\overline{C}_{ij}^l+D_{ij}^l+\overline{H}_{ij}^l$$

the last equation being true because $C_{ij}^l=\triangle_j\overline{C}_{ij}^l$.
Since $A_{ij}^l=-B_{ij}^l-\overline{H}_{ij}^l-\overline{C}_{ij}^l-D_{ij}^l$ then $vI_{ij}-F_{ij}=vI_{ij}+Z_{ij}+B_{ij}-\triangle_{ij}B_{ij}$ and it follows that

$$\text{(A.1)}\qquad \begin{aligned} F_{ij}(v) &= [vI_{ij}+Z_{ij}+B_{ij}-\triangle_{ij}B_{ij}]^{-1} \\ &= [(vI_{ij}+Z_{ij}+B_{ij})(I_{ij}-(vI_{ij}+Z_{ij}+B_{ij})^{-1})\triangle_{ij}\,B_{ij}]^{-1} \\ &= [I_{ij}-(vI_{ij}+Z_{ij}+B_{ij})^{-1}\triangle_{ij}\,B_{ij}]^{-1}(vI_{ij}+Z_{ij}+B_{ij})^{-1}. \end{aligned}$$

The off-diagonal form of $\triangle_{ij}$ and the upper triangular form of $M_{ij}(v)=(tIij+Z_{ij}+B_{ij})^{-1}$ imply that $(M_{ij}(v)\triangle_{ij}\,B_{ij})^l\equiv 0$ for all integers $l>N_1^i$. Hence

$$[I_{ij}-M_{ij}(v)\triangle_{ij}\,B_{ij}]^{-1}=\sum_{l=0}^{N_1^i}[M_{ij}(v)\triangle_{ij}\,B_{ij}]^l=R_{ij}(v).$$

Let R_{ij}^{lh} and $M_{ij}^{lh}(v)$ be, respectively, the (l,h)-th blocks of the matrices $R_{ij}(v)$ and $M_{ij}(v)$ of ranks N_2^j+1. Since for $k=0,...,N_1^i$ the (l,h)-th block of $[M_{ij}(v)\triangle_{ij}\,B_{ij}]^k$ is equal to

$$M_{ij}^{ll}(v)B_{ij}^{l+1}M_{ij}^{l+1l+1}(v)B_{ij}^{l+2}\dots M_{ij}^{l+k-1l+k-1}(v)B_{ij}^{l+k}\text{ if }h=l+k\text{ and 0 otherwise,}$$

then, for $0 \le l \le h \le N_1^i$,

$$(A.2) \qquad R_{ij}^{lh}(v) = \prod_{k=l}^{h-1} M_{ij}^{kk}(v) B_{ij}^{k+1} = \prod_{k=l}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}.$$

The diagonal form by blocks of Z_{ij} implies that $M_{ij}^{lh}(v) = 0$ if $l \neq h$, thus, for each $l, h = 0, .., N_1^i$,

$$(A.3) \qquad F_{ij}^{lh}(v) = \sum_{k=0}^{N_1^i} R_{ij}^{lk}(v) M_{ij}^{kh}(v) = \begin{cases} R_{ij}^{lh}(v) M_{ij}^{hh}(v) & \text{if } l \le h \\ 0 & \text{otherwise.} \end{cases}$$

Now, for $l = 0, ..., N_1^i$, we have

$$\begin{aligned} M_{ij}^{ll}(v) &= [(vI_{ij} + Z_{ij} + B_{ij})^{-1}]^{ll} = (tI_{ij}^{ll} + Z_{ij}^{l} + B_{ij}^{l})^{-1} \\ &= [vI_j + \overline{C}_{ij}^l + D_{ij}^l + B_{ij}^l + \overline{H}_{ij}^l - \triangle_j \overline{C}_{ij}^l]^{-1} \\ &= [I_j - Y_{ij}^l(v)]^{-1} (vI_j + \overline{C}_{ij}^l + D_{ij}^l + B_{ij}^l + \overline{H}_{ij}^l)^{-1}. \end{aligned}$$

where $Y_{ij}^l(v) = (vI_j + \overline{C}_{ij}^l + D_{ij}^l + B_{ij}^l + \overline{H}_{ij}^l)^{-1} \triangle_j \overline{C}_{ij}^l$. But

$$vI_j + \overline{C}_{ij}^l + D_{ij}^l + B_{ij}^l + \overline{H}_{ij}^l = diag(v + \mu_2 r + \mu_1 l + f_{ijlr,i-1jl+1r} + f_{ijlr,ij-1lr+1}, 0 \le r \le N_2^j),$$

hence

$$(vI_j + \overline{C}_{ij}^l + D_{ij}^l + B_{ij}^l + \overline{H}_{ij}^l)^{-1} = diag([t + \mu_2 r + \mu_1 l + f_{ijlr,i-1jl+1r} + f_{ijlr,ij-1lr+1}]^{-1}, 0 \le r \le N_2^j).$$

The off-diagonal form of $\triangle_j \overline{C}_{ij}^l$ implies that $[Y_{ij}^l(v)]^r \equiv 0$ for all integers $r > N_2^j$. Hence, using the same technique as above we obtain, for $r \le s$,

$$[I_j - Y_{ij}^l(v)]_{rs}^{-1} = \mu_2^{s-r} \frac{s!}{r!} \prod_{q=r}^{s-1} (v + \mu_2 q + \mu_1 l + f_{ijlq,i-1jl+1q} + f_{ijlq,ij-1lq+1})^{-1},$$

with all other elements being equal to zero. It follows that

$$(A.4) \quad [M_{ij}^{ll}]_{rs} = \begin{cases} \mu_2^{s-r} \frac{s!}{r!} \prod_{k=r}^{s} (v + \mu_2 q + \mu_1 l + f_{ijlq,i-1jl+1q} + f_{ijlq,ij-1lq+1})^{-1} & \text{if } r \le s \\ 0 & \text{otherwise,} \end{cases}$$

hence from $(A.2)$ we deduce that,

$$\begin{aligned} [R_{ij}^{lh}(v)]_{rs} &= \left[\prod_{k=l}^{h-1} M_{ij}^{kk}(v) B_{ij}^{k+1} \right]_{rs} \\ &= \sum_{i_1=0}^{N_1^i} \sum_{i_2=0}^{N_1^i} \cdots \sum_{i_{h-l-1}=0}^{N_1^i} [M_{ij}^{ll}(v) B_{ij}^{l+1}]_{ri_1} [M_{ij}^{l+1l+1}(v) B_{ij}^{l+2}]_{i_1 i_2} \cdots [M_{ij}^{h-1h}(v) B_{ij}^{h-1}]_{i_{h-l-1}s} \\ &= \sum_{i_1=r}^{s} \sum_{i_2=i_1}^{s} \cdots \sum_{i_{h-l-1}=i_{h-l-2}}^{s} [M_{ij}^{ll}(v)]_{ri_1} [B_{ij}^{l+1}]_{i_1 i_1} \cdots [M_{ij}^{h-1h}(v)]_{i_{h-l-1}s} [B_{ij}^{h-1}]_{ss} \\ &= \sum_{i_1=r}^{s} \sum_{i_2=i_1}^{s} \cdots \sum_{i_{h-l-1}=i_{h-l-2}}^{s} \mu_1^{h-l} \mu_2^{s-r} \frac{h!s!}{l!r!} \prod_{k=0}^{h-l} \prod_{p=i_k}^{i_{k+1}} f(v, \mu_1, \mu_2, i, j, l+k, p) \end{aligned}$$

where $i_0 = r$ and $i_{h-l} = s$.

Finally, by substituting (A.4) and the above equation in $(A.3)$, we obtain, if $r \leq s$

$$\begin{aligned}
[F_{ij}^{lh}(v)]_{rs} &= \sum_{k=0}^{s} [R_{ij}^{lh}(v)]_{rk} [M_{ij}^{hh}]_{ks} \\
&= \sum_{k=r}^{s} [R_{ij}^{lh}(v)]_{rk} \mu_2^{s-k} \frac{s!}{k!} \prod_{p=k}^{s} (v + \mu_2 p + \mu_1 h + f_{ijhp,i-1jh+11p} + f_{ijhp,ij-1hp+1})^{-1} \\
&= \sum_{r \leq i_1 \leq i_2 \leq .. \leq i_{h-l} \leq s} \mu_1^{h-l} \mu_2^{s-r} \frac{h!s!}{l!r!} \times \prod_{k=0}^{h-l} \prod_{p=i_k}^{i_{k+1}} f(v, \mu_1, \mu_2, i, j, l+k, p) \qquad \text{(A.5)}
\end{aligned}$$

where $i_0 = r$ and $i_{h-l+1} = s$ □

Proof of (12)

¿From $(A.2)$ and $(A.3)$ we have

$$\begin{aligned}
[F_{ij}^{lh}]_{rs} &= \left[\prod_{k=l}^{h-1} (M_{ij}(v) \bigtriangleup_{ij} B_{ij})^{kk+1} M_{ij}^{hh}(v) \right]_{rs} \\
&= \left[\prod_{k=l}^{h-2} (M_{ij}(v) \bigtriangleup_{ij} B_{ij})^{kk+1} (M_{ij}(v) \bigtriangleup_{ij} B_{ij})^{h-1h} M_{ij}^{hh}(v) \right]_{rs} \\
&= \left[\prod_{k=l}^{h-1} (M_{ij}(v) \bigtriangleup_{ij} B_{ij})^{kk+1} M_{ij}^{h-1h-1}(v) B_{ij}^{hh} M_{ij}^{hh}(v) \right]_{rs} \\
&= \sum_{p=r}^{s} \left[\prod_{k=l}^{h-1} (M_{ij}(v) \bigtriangleup_{ij} B_{ij})^{kk+1} M_{ij}^{h-1h-1}(v) B_{ij}^{hh} \right]_{rp} \left[B_{ij}^{hh} M_{ij}^{hh}(v) \right]_{ps} \\
&= \sum_{p=r}^{s} \left[M_{ij}(v)^{hh} \right]_{ps} \left[B_{ij}^{hh} \right]_{pp} \left[F_{ij}^{lh-1}(v) \right]_{rp}
\end{aligned}$$

$(A.4)$ and (7) complete the proof.

Proof of (13)

For $m, n = 0, ..., (N_1^i + 1)(N_2^j + 1) - 1$, let l, h and r, s be, respectively, the quotient and rest of the Euclidean division of m, n by $N_2^j + 1$. We have

$$\begin{aligned}
[D_{i+1j}(1,0)]_{mn} &= [D_{i+1j}(1,0)]_{l(N_1^i+1)+r, h(N_2^j+j)+s} \\
&= [D_{i+1j}(1,0)]_{rs}^{lh} \\
&= \begin{cases} [D_{i+1j}^{l-1}]_{rs} & \text{if } l = h \text{ and } l \geq 1 \\ [\Theta_{ij}^{10}]_{rs} & \text{if } l = h = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

because

$$D_{i+1j}(1,0) = \begin{pmatrix} \Theta_{ij}^{10} & 0 \\ 0 & D_{i+1j} \end{pmatrix},$$

so

$$[D_{i+1j}(1,0)]_{mn} = \begin{cases} f_{i+1jl-1r,ijlr} & \text{if } l = h \text{ and } r = s \\ 0 & \text{otherwise .} \end{cases} \qquad (*)$$

We also have,

$$[F_{ij}(v)]_{mn} = [F_{ij}(v)]_{l(N_2^j+1)+r,h(N_2^j+1)+s} = \begin{cases} [F_{ij}^{lh}(v)]_{rs} & \text{if } l \leq h \\ 0 & \text{otherwise.} \end{cases} \qquad (**)$$

Using (*) and (**), we obtain

$$\begin{aligned}
[F_{ij}(v)D_{i+1j}(1,0)]_{mn} &= [F_{ij}(v)D_{i+1j}(1,0)]_{l(N_2^j+1)+r,h(N_2^j+1)+s} \\
&= [F_{ij}(v)D_{i+1j}(1,0)]_{rs}^{lh} \\
&= [F_{ij}^{lh}(v)[D_{i+1j}(1,0)]^{hh}]_{rs} \\
&= \begin{cases} [F_{ij}^{lh}(v)D_{i+1j}^{h-1}]_{rs} & \text{if } h \geq 1 \text{ and } l = h \\ [F_{ij}^{l0}(v)\Theta_{ij}^{10}]_{rs} & \text{if } h = l = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} [F_{ij}^{lh}(v)]_{rs}[D_{i+1j}^{h-1}]_{ss} & \text{if } h \geq 1 \text{ and } l = h \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} C_{ij}(t,l,h,r,s)f_{i+1jh-1s,ijls} & \text{if } l \leq h, \text{ and } h \geq 1 \\ & 0 \leq r \leq s \leq N_2^j \\ 0 & \text{sinon.} \end{cases}
\end{aligned}$$

□

Proof of (14)

As before we let l, h and $r, s-1$ be, respectively, the quotients and rests of the Euclidean division of m by $N_2^j + 1$ and n by $N_2^j + 1$,

$$\begin{aligned}
[H_{ij}]_{mn} &= [H_{ij}^{lh}]_{l(N_2^j+1)+r,h(N_2^j+1)+s} \\
&= [H_{ij}^{lh}]_{rs} \\
&= \begin{cases} [H_{ij}^{l}]_{rs} & \text{if } l = h \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} f_{ij+1lr-1,ijlr} & \text{if } m = n = l(N_2^j+1)+r \text{ , } 1 \leq r \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Now, if l, h and $r, s-1$ are, respectively, the quotients and rests of the Euclidean division of m by $N_2^j + 1$ and $n - N_1^i - 1$ by N_2^j, we have

$$
\begin{aligned}
[H_{ij}T_{ij}]_{mn} &= \sum_{k=0}^{(N_1^i+1)(N_2^j+1)-1} [H_{ij}]_{mk}[T_{ij}]_{kn} \\
&= [H_{ij}]_{mm}[T_{ij}]_{mn} \\
&= \begin{cases} f_{ij+1lr-1,ijlr} & \text{if } m = l(N_2^j+1)+r, n = N_1^i + lN_2^j + r \text{ and } 1 \leq r \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
[F_{ij}(v)H_{ij}T_{ij}]_{mn} &= \begin{cases} [F_{ij}(v)]_{l(N_2^j+1)+r,h(N_2^j+1)+s} f_{ij+1hs-1,ijhs} & \text{if } m = l(N_2^j+1)+r, \\ & n = N_1^i + hN_2^j + s \\ & \text{and } s \geq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} [F_{ij}^{lh}(v)]_{rs} f_{ij+1hs-1,ijhs} & \text{if } m = l(N_2^j+1)+r,\ n = N_1^i + hN_2^j + s_- \quad \text{and } s \geq 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} C_{ij}(t,l,h,r,s) f_{ij+1hs-1,ijhs} & \text{if } m = l(N_2^j+1)+r \\ & \text{and } n = N_1^i + hN_2^j + r \\ & \text{with } r \leq s, s \geq 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

□

Proof of (17) and (18)

Let $h > l$ and $s \geq r$. From $(A.2)$ and $(A.3)$, we have

$$
\begin{aligned}
[F_{ij}^{lh}(v)]_{rs} &= \left[\left(\prod_{k=l}^{h-1}(M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right) M_{ij}^{hh}(v)\right]_{rs} \\
&= \sum_{q=r}^{s}\sum_{p=r}^{q}[(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{pq} [M_{ij}^{hh}(v)]_{qs} \\
&= \sum_{q=r}^{s}[(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rr} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{rq} [M_{ij}^{hh}(v)]_{qs} \\
&+ \sum_{q=r+1}^{s}\sum_{p=r+1}^{q}[(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{pq} [M_{ij}^{hh}(v)]_{qs} \\
&= \sum_{q=r}^{s}[(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rr} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1})\right]_{rq} [M_{ij}^{hh}(v)]_{qs} \\
&+ \sum_{p=r+1}^{s}[(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} \sum_{q=p}^{s} \left[\prod_{k=l+1}^{h-l} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{pq} [M_{ij}^{hh}(v)]_{qs}
\end{aligned}
$$

but, if $r \leq s$, we have by using $(A.4)$, that,

$$[(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} = \mu_1(l+1)\mu_2^{s-r}\frac{s!}{r!}\prod_{q=r}^{s}[t+\mu_1 l+\mu_2 q+f_{ijlq,i-1jl+1q}+f_{ijlq,ij-1lq+1}]^{-1}$$

Hence, if $(l,r)=(0,0)$ and $h>0$, we obtain

$$\text{(A.6)}\quad [F_{ij}^{0h}(v)]_{0s}$$
$$= \frac{\mu_1}{t}\left\{[F_{ij}^{1h}(v)]_{0s} + \sum_{p=1}^{s}\mu_2^p p!\prod_{k=1}^{p}(v+\mu_2 k+f_{ijlk,i-1jl+1k}+f_{ijlk,ij-1lk+1})^{-1}[F_{ij}^{1h}]_{ps}\right\}$$
$$= \frac{\mu_1}{t}\left\{\sum_{p=0}^{s}\mu_2^p p!\prod_{k=1}^{p}(v+\mu_2 k+f_{ijlk,i-1jl+1k}+f_{ijlk,ij-1lk+1})^{-1}[F_{ij}^{1h}(v)]_{ps}\right\}.$$

In addition, we see from $(A.5)$, that $lim_{t\to 0}[F_{ij}^{lh}(v)]_{rs}$ exists if $(l,r)\neq(0,0)$. Therefore, using the second and third members in $(A.6)$ it can be shown that $lim_{t\to 0}(tC_{ij}(t,0,h,0,s))$ exists and is equal to (19). Finally (18) is easily obtained by passing to the limit in (8) when $h=r=0$. □

Acknowledgment

H. Elmaroufy is indebted to Professor Lahcen Omari for helpful discussions and useful comments. The support of Burjassot faculty for Mathematical Sciences is Gratefully Acknowledged for its reception and hospitality.

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