Chains of Frobenius subalgebras of $so(M)$ and the corresponding twists

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Abstract

Chains of extended jordanian twists are studied for the universal enveloping algebras $U(so(M))$. The carrier subalgebra of a canonical chain $\mathcal{F}_{\mathcal{B}_{0\prec p\max}}$ cannot cover the maximal nilpotent subalgebra $N^{+}(so(M))$. We demonstrate that there exist other types of Frobenius subalgebras in $so(M)$ that can be large enough to include $N^{+}(so(M))$. The problem is that the canonical chains $\mathcal{F}_{\mathcal{B}_{0\prec p}}$ do not preserve the primitivity on these new carrier spaces. We show that this difficulty can be overcome and the primitivity can be restored if one changes the basis and passes to the deformed carrier spaces. Finally the twisting elements for the new Frobenius subalgebras are explicitly constructed. This gives rise to a new family of universal R-matrices for orthogonal algebras. For a special case of $g = so(5)$ and its defining representation we present the corresponding matrix solution of the Yang-Baxter equation.

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1 Introduction

Quantizations of triangular Lie bialgebras $\mathbf{L}$ with antisymmetric classical r-matrices $r = -r_{21}$ satisfying the classical Yang-Baxter equation (CYBE) form an important class of triangular Hopf algebras $\mathcal{A}(m, \Delta, S, \eta, \epsilon; \mathcal{R})$, with R-matrix satisfying the unitarity condition $\mathcal{R}_{21}\mathcal{R} = 1$. These quantizations are defined by the twisting elements $\mathcal{F} = \sum f_{(1)} \otimes f_{(2)} \in \mathcal{A} \otimes \mathcal{A}$ that satisfy the twist equations [1]:

$$(\mathcal{F})_{12} (\Delta \otimes \mathrm{id}) \mathcal{F} = (\mathcal{F})_{23} (\mathrm{id} \otimes \Delta) \mathcal{F}, \qquad (\epsilon \otimes \mathrm{id}) \mathcal{F} = (\mathrm{id} \otimes \epsilon) \mathcal{F} = 1. \quad (1)$$

The knowledge of the twisting element is quite important in applications giving (twisted) R-matrix $\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$ and twisted coproduct $\Delta_{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1}$.

The explicit expressions of the twisting elements $\mathcal{F}$ were found in [2], for the carrier algebras $\mathbf{L}$ with special properties of their triangular decompositions. Such carrier subalgebras are the multidimensional analogs of the enlarged Heisenberg algebra and can be found in any simple Lie algebra g of rank greater than 1. In the root system $\Lambda(g)$ one can choose the initial root λ_0 and consider the set π of its constituent roots

$$\begin{array}{l} \pi = \{\lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_0; \quad \lambda' + \lambda_0, \lambda'' + \lambda_0 \notin \Lambda(g)\} \\ \pi = \pi' \cup \pi''; \qquad \pi' = \{\lambda'\}, \pi'' = \{\lambda''\}. \end{array}$$

The subalgebra $\mathbf{L}$ is generated by the elements $\{E_{\lambda_0}, E_\lambda \mid \lambda \in \pi\}$ and the Cartan generator H_{λ_0} dual to λ_0. The solution $\mathcal{F}_{\mathcal{E}\mathcal{J}}$ of the twist equations corresponding to the carrier subalgebra $\mathbf{L}$ is called the extended jordanian twist and can be decomposed into the product of two factors, the jordanian twist $\Phi_{\mathcal{J}}$ and the extension $\Phi_{\mathcal{E}}$:

$$\mathcal{F}_{\mathcal{E}\mathcal{J}} = \Phi_{\mathcal{E}} \cdot \Phi_{\mathcal{J}} = \prod_{\lambda' \in \pi'} \exp\left\{E_{\lambda'} \otimes E_{\lambda_0 - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0}}\right\} \cdot \exp\{H_{\lambda_0} \otimes \sigma_{\lambda_0}\}. \quad (2)$$

Here $\sigma_{\lambda_0} = \ln(1 + E_{\lambda_0})$.

To construct the twists for the higher dimensional carrier subalgebras $\mathbf{L}$ of g we have to consider the subset $\Lambda_{\lambda_0}^{\perp}$ of roots orthogonal to λ_0 and the corresponding subalgebra $g_{\lambda_0}^{\perp} \subset g$. It was shown in [3] that for the classical Lie algebras g one can always find a subalgebra g_1 in $g_{\lambda_0}^{\perp}$ whose generators become primitive after the twist $\mathcal{F}_{\mathcal{E}\mathcal{J}}$. Such primitivization of $g_k \subset g_{k-1}$

(called the matreshka effect) makes it possible to compose chains of extended twists of the type (2) corresponding to the injections $g_p \subset \ldots \subset g_1 \subset g_0 = g$,

$$\begin{array}{l} \mathcal{F}_{\mathcal{B}_{0\prec p}} = \prod_{\lambda' \in \pi'_p} \exp\left\{ E_{\lambda'} \otimes E_{\lambda_0^p - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^p}} \right\} \cdot \exp\{ H_{\lambda_0^p} \otimes \sigma_{\lambda_0^p} \} \cdot \\ \prod_{\lambda' \in \pi'_{p-1}} \exp\left\{ E_{\lambda'} \otimes E_{\lambda_0^{p-1} - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^{p-1}}} \right\} \cdot \exp\{ H_{\lambda_0^{p-1}} \otimes \sigma_{\lambda_0^{p-1}} \} \cdot \\ \ldots \\ \prod_{\lambda' \in \pi'_0} \exp\left\{ E_{\lambda'} \otimes E_{\lambda_0^0 - \lambda'} e^{-\frac{1}{2}\sigma_{\lambda_0^0}} \right\} \cdot \exp\{ H_{\lambda_0^0} \otimes \sigma_{\lambda_0^0} \} . \end{array} \quad (3)$$

In the case $g = sl(n)$ the subalgebras g_k (they remain primitive after the twisting by $\mathcal{F}_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}$) coincide with $g^{\perp}_{\lambda_{k-1}}$. The result is that the maximal canonical chain $\mathcal{F}_{\mathcal{B}_{0\prec p\max}}$ for $g = sl(n)$ is full, its carrier subalgebra contains all the generators of the nilpotent subalgebra $N^+(g)$.

For the orthogonal simple algebras the situation is different. In this case $g^{\perp}_{\lambda_{k-1}} = g_k \oplus sl^{(k)}(2)$ and the coproducts of generators in the space $g^{\perp}_{\lambda_{k-1}} \backslash g_k$ are nontrivially deformed by the twist $\mathcal{F}_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}$. The next extended twist $\mathcal{F}_{\mathcal{E}_k\mathcal{J}_k}$ does not contain these generators in its carrier space. Such chain cannot be full .

The canonical twists (3) correspond to Frobenius subalgebras in g described by the coboundary bilinear forms $\omega^+_p = \sum_{k=0}^{p} E^*_{\lambda_0^k}([\,,\,])$ [4]. In this paper we show that for the orthogonal algebras these forms can be modified. The Frobenius subalgebras can be enlarged in order to include all nonzero root generators from $g^{\perp}_{\lambda_{k-1}} \backslash g_k$.

The problem is how to find the corresponding twists, i.e. to solve the equations (1) for the subalgebra $g^{\perp}_{\lambda_{k-1}}$ that contains the generators with deformed coproducts $\Delta_{\mathcal{F}_{\mathcal{E}_k\mathcal{J}_k}}$.

In [5] it was demonstrated that under certain conditions (while the coproducts in g are nontrivially twisted by $\mathcal{F}$) one can find in $U_{\mathcal{F}}(g)$ the deformed carrier subspace that is primitive and generates a subalgebra of g. Below we show that this effect is in some sense universal. The corresponding deformed spaces for orthogonal algebras can be found for any extended twist $\mathcal{F}_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}$.As a result the canonical chain of twists $\mathcal{F}_{\mathcal{B}_{0\prec p}}$ can be extended using some additional factors (the deformed jordanian twists). For the maximal value of p the corresponding chain $\mathcal{F}_{\mathcal{G}_{0\prec p}}$ becomes full and the corresponding carrier space contains all the generators of $N^+(g)$.

2 Frobenius subalgebras

We consider the orthogonal algebras $g = so\,(M)$ of the series B_N (for $M = 2N+1$) and D_N (for $M = 2N$). The root system $\Lambda(g)$ will be fixed as follows

$$\Lambda(g) = \left\{ \begin{array}{lll} \pm e_i, \ \pm e_j \pm e_k & \text{for} & g = so(2N+1), \\ \pm e_i \pm e_j & \text{for} & g = so(2N) \end{array} \right\} \qquad (4)$$
$$i, j, k = 1, \ldots, N.$$

Let $\mathcal{E}_{i,j}$ be the ordinary matrix units,

$$[\mathcal{E}_{i,j}, \mathcal{E}_{k,l}] = \delta_{jk}\mathcal{E}_{i,l} - \delta_{il}\mathcal{E}_{k,j},$$

and $M_{i,j}$ – the antisymmetric ones,

$$[M_{a,b}, M_{c,d}] = \delta_{bc}M_{a,d} + \delta_{ad}M_{b,c} - \delta_{ac}M_{b,d} - \delta_{bd}M_{a,c}.$$

The generators of $g = so(M)$ can be realized as follows. The Cartan subalgebra $\mathcal{H}(g)$ is generated by

$$H_j = \left(-\frac{i}{2}\right) M_{2j-1,2j}, \qquad j = 1, ..., N. \qquad (5)$$

For Cartan generators we shall also use the notation:

$$H_{j\pm(j+1)} = \left(-\frac{i}{2}\right) \left(M_{2j-1,2j} \pm M_{2j+1,2j+2}\right). \qquad (6)$$

To the roots of $\Lambda(so(M))$ we attribute the generators

$$\left. \begin{array}{l} E_{i+j} = \frac{1}{2}\left(-M_{2i,2j} + iM_{2i,2j-1} + iM_{2i-1,2j} + M_{2i-1,2j-1}\right); \\ E_{i-j} = \frac{1}{2}\left(-M_{2i,2j} - iM_{2i,2j-1} + iM_{2i-1,2j} - M_{2i-1,2j-1}\right); \\ E_{-i+j} = \frac{1}{2}\left(+M_{2i,2j} - iM_{2i,2j-1} + iM_{2i-1,2j} + M_{2i-1,2j-1}\right); \\ E_{-i-j} = \frac{1}{2}\left(+M_{2i,2j} + iM_{2i,2j-1} + iM_{2i-1,2j} - M_{2i-1,2j-1}\right); \end{array} \right\} \quad i < j; \qquad (7)$$

and

$$E_{\pm k} = \frac{1}{\sqrt{2}}\left(\pm M_{2k,2N+1} - iM_{2k-1,2N+1}\right), \qquad k \leq N, \quad \text{for } so\,(2N+1). \quad (8)$$

The Borel subalgebras $B(g)$ are generated by the sets $\{H_i, E_i, E_{i\pm j}\}$ for $g = so(2N+1)$ and $\{H_i, E_{i\pm j}\}$ for $g = so(2N)$. To describe the chains of

Frobenius subalgebras we shall also need the alternative realization of these generators through the ordinary matrix units. To get it let us renumerate the generators:

$$\left.\begin{array}{lll} A_{i,j} & \equiv & -E_{i-j}, \\ A_{i,2N+2-j} & \equiv & -E_{i+j}, \\ A_{i,N+1} & \equiv & -E_{i}, \end{array}\right\} \qquad \text{for} \quad so(2N+1) \tag{9}$$

and

$$\left.\begin{array}{lll} A_{i,j} & \equiv & -E_{i-j}, \\ A_{i,2N+1-j} & \equiv & -E_{i+j}, \end{array}\right\} \qquad \text{for} \quad so(2N). \tag{10}$$

In these terms the Borel subalgebra $B(so(M))$ is spanned by the set $\{A_{i,j}|i \leq j\}$ and we can also use the following matrix realization:

$$\begin{array}{l} H_i = \frac{1}{2}\left(\mathcal{E}_{i,i} - \mathcal{E}_{M+1-i,M+1-i}\right), \\ A_{i,j} = \mathcal{E}_{i,j} - \mathcal{E}_{M+1-j,M+1-i}. \end{array} \tag{11}$$

The canonical chains of twists (3) for orthogonal simple Lie algebras are based on the sequence of injections

$$U(so(M)) \supset U(so(M-4)) \supset \ldots \supset U(so(M-4k)) \supset \ldots \tag{12}$$

Each link of such chains (see (3)) contains the jordanian twist $\Phi_{\mathcal{J}_k}$ =exp$\{H_{\lambda_0^k} \otimes \sigma_{\lambda_0^k}\}$ based on one of the long roots:

$$\lambda_0^k = e_1^k + e_2^k \quad (\text{for both } D_N \text{ and } B_N)). \tag{13}$$

The hyperplane $V_{\lambda_0^k}^{\perp}$ orthogonal to λ_0^k coincides with the root space of the subalgebra $g_{\lambda_0^k}^{\perp} = so\left(M-4\left(k+1\right)\right) \oplus so^{(k+1)}(3)$. For each subalgebra $g_{k+1} = so\left(M-4\left(k+1\right)\right)$ we can again consider independently its root system Λ^{k+1} and choose the next initial root to be

$$\lambda_0^{k+1} = e_1^{k+1} + e_2^{k+1}.$$

On each step we can fix two vector subrepresentations $d^{v(a)}$ in the restriction (to g_{k+1}) of the adjoint representation ad_{g_k}:

$$d_{g_{k+1}}^{v(a)} \subset \mathrm{ad}_{g_k \downarrow g_{k+1}}, \qquad a = 1, 2;$$

It is easy to check that the constituent roots form the weight diagrams for these representations. The representation space for $d^{v(a)}_{g_{k+1}}$ is spanned by the generators

$$\{E_a, E_{a\pm l}\} \text{ with the roots } \left\{e^k_a, e^k_a \pm e^k_l \in \pi_k\right\}$$

for $M-4k=2N+1$ and

$$\{E_{a\pm k}\} \text{ with the roots } \left\{e^k_a \pm e^k_l \in \pi_k\right\}$$

for $M-4k=2N$. In both cases $l=3,\ldots,N$.

For the representations $d^{v(a)}_{g_{k+1}}$ and $d^{v(a)}_{g_{k+1}} \otimes d^{v(b)}_{g_{k+1}}$ the following scalar and tensor invariants, I^a_{M-4k} and $I^{a\otimes b}_{M-4k}$ (with $a,b=1,2$), will be used in the construction of twists and twisted coproducts:

$$\begin{array}{l} I^a_{2N+1} = \frac{1}{2}E^2_a + \sum_{l=3}^N \left(E_{a+l}E_{a-l}\right), \\ I^{a\otimes b}_{2N+1} = E_a \otimes E_b + \sum_{l=3}^N \left(E_{a+l}\otimes E_{b-l} + E_{a-l}\otimes E_{b+l}\right), \\ I^{a\wedge b}_{2N+1} = E_a \wedge E_b + \sum_{l=3}^N \left(E_{a+l}\wedge E_{b-l} + E_{a-l}\wedge E_{b+l}\right), \end{array} \tag{14}$$

$$\begin{array}{l} I^a_{2N} = \sum_{l=3}^N \left(E_{a+l}E_{a-l}\right), \\ I^{a\otimes b}_{2N} = \sum_{l=3}^N \left(E_{a+l}\otimes E_{b-l} + E_{a-l}\otimes E_{b+l}\right), \\ I^{a\wedge b}_{2N} = \sum_{l=3}^N \left(E_{a+l}\wedge E_{b-l} + E_{a-l}\wedge E_{b+l}\right). \end{array} \tag{15}$$

The set of initial roots defines a natural gradation in the root space of the subalgebra $N^+(so(M)) \subset so(M)$:

$$\Lambda\left(N^+(so(M))\right) = \bigcup_{k=0}^{p_{\max}} \left(\lambda^k_0 \cup \pi_k\right), \tag{16}$$

where $p_{\max} = [M/4] + [(M+1)/4]$.

The inverse of the map defined by the classical r-matrix is the Frobenius bilinear form. Let us study the Frobenius subalgebras in $B(so(M))$.

Proposition 1 *Let* $\mathbf{L}$ *be a semidirect sum of a subalgebra* $\mathbf{S}$ *and a commutative ideal* $\mathbf{N}$. *Then* $\mathbf{L}$ *is Frobenius if and only if the following conditions hold:*

i) $\mathbf{L}$ *acts transitively on the space* $\mathbf{N}^*$ *with the generic point* A^**;*

ii) *the stationary subalgebra* $\mathbf{S}_{A^*} = \{s \in \mathbf{S} : A^*([s,x]) = 0, \text{for any } x \in \mathbf{N}\}$ *is Frobenius with a Frobenius homomorphism* $f_0 : \mathbf{S}_{A^*} \longrightarrow \mathbf{C}$.

Moreover, in this case $f = f_0 \oplus A^*$ is a Frobenius homomorphism for $\mathbf{L}$.

This statement can be obtained as a consequence of the Proposition 1 and the Remark after it in [6]. Here is how it can be used in the case of orthogonal simple algebras.

Lemma 1 *Let $\mathbf{L}_1 \subset B(so(M))$ be a subalgebra generated by the set $\{H_1, H_2, A_{i,j}, i = 1, 2\}$. Then $\mathbf{L}_1$ is Frobenius.*

Proof. In $\mathbf{L}_1$ the following subalgebras can be fixed $\mathbf{N}_1 = \{A_{1,j}\}, \mathbf{S}_1 = \{H_1, H_2, A_{2,j}\}$. The generators of the dual space $\mathbf{N}_1^*$ can be identified with the matrices $\{A_{i,1}\}$ defined according to the rule (11) and connected with $A_{1,j}$ through the bilinear form $< A, B >= \mathrm{tr}(A, B)$ or by a Killing form in the general setting. Since $\dim \mathbf{N}_1 = \dim \mathbf{S}_1$ it suffices to find a point $A^* \in \mathbf{N}_1^*$ such that $\mathbf{S}_{A^*} = 0$. One can check directly that $A_0^* = \sum_{i=2}^{n-1} A_{i1}$ satisfies this condition. ♠

This point A_0^* is not unique. If $G(\mathbf{S}_1)$ is the subgroup of $SO(M)$ corresponding to the algebra $\mathbf{S}_1$ then for any $g \in G(\mathbf{S}_1)$ the point $\mathrm{Ad}^*(g)(A_0^*) = (A_0^*)^g$ satisfies the condition $\mathbf{S}_{(A_0^*)^g} = 0$ since $\mathbf{S}_{(A_0^*)^g} = g^{-1}\mathbf{S}_{A_0^*} g = 0$. For our purposes it is convenient to choose $A_0^* = A_{2,1} + A_{M-1,1}$ (One can check that this point satisfies the conditions of the Proposition 1.)

Lemma 2 *Let $\mathbf{L}_{K,M}$ be a subalgebra of $B(so(M))$ generated by the set*

$$\{H_i, A_{i,j} | i = 1, \dots, 2K; j = 1, \dots M; i < j; 2K \leq [M/2]\}.$$

Then $\mathbf{L}_{K,M}$ is Frobenius.

Proof. The algebra $\mathbf{L}_{K,M}$ has the structure of a semidirect sum:

$$\mathbf{L}_{K,M} = \mathbf{S}_{K,M} \dot{\oplus} \mathbf{N}_{K,M},$$

where

$$\mathbf{N}_{K,M} = \{\{H_i \ | i = 1, \dots, 2K;\} \{A_{i,j} | i = 2, \dots, 2K; j = 1, \dots M; i < j\}\}.$$

Evidently $\mathbf{L}_{K,M}$ acts transitively on $\mathbf{N}_{K,M}^*$ with the generic point $A_1^* = A_{2,1} + A_{M-1,1}$. One can easily check that

$$(\mathbf{S}_{K,M})_{(A_0^*)} = \{H_i, A_{i,j} | i = 3, \dots, 2K; i < j\}.$$

Thus

$$(\mathbf{S}_{K,M})_{(A_0^*)} \cong \mathbf{L}_{K-1,M-4} \subset B(so(M-4)).$$

Obvious induction shows that $\mathbf{L}_{K,M}$ is Frobenius due to the Proposition 1. ♠

The algebra $\mathbf{L}_{K,M}$ has the nontrivial second cohomology group $H^2(\mathbf{L}_{K,M})$. The latter contains $\Lambda^2\left(H^*_{K,M}\right)$ where $H^*_{K,M}$ is the space dual to the Cartan subalgebra in $\mathbf{L}_{K,M}$, $H^*_{K,M} = \mathbf{L}_{K,M} \bigcap \mathcal{H}(so(M)) \subset B(so(M))$. It is easy to see that all bilinear forms $H_i^* \bigwedge H_j^*$ are 2-cocycles and not coboundaries. Here $H_i^* \in H^*_{K,M}$. Consequently $A^*_{K,M} + \zeta_{ij} H_i^* \bigwedge H_j^*$ are the nondegenerate 2-cocycles on $\mathbf{L}_{K,M}$. The map $A^*_{K,M}$ is a Frobenius homomorphism, $A^*_{K,M} : \mathbf{L}_{K,M} \longrightarrow \mathbf{C}$, because $A^*_{K,M}([x,y])$ is a nondegenerate bilinear form on $\mathbf{L}_{K,M}$. The induction procedure shows that $A^*_{K,M}$ can be chosen in the following form:

$$A^*_{K,M} = (A_{21} + A_{43} + A_{65} \ldots) + (A_{M-1,1} + A_{M-3,3} + A_{M-5,5} \ldots) . \qquad (17)$$

In the case of the orthogonal simple Lie algebras chains of twists (3) introduced in [3] refer to the Frobenius subalgebras (contained in the corresponding $B(so\,(M))$) with the coboundary nondegenerate bilinear forms [4],

$$\omega_p^+ = \sum_{k=0}^{p} \gamma_k \left(E_{1+2}^{(k)}\right)^* ([\,,\,]) , \qquad (18)$$

where p is the number of links in the chain $\mathcal{F}_{\mathcal{B}_{0\prec p}}$ and the parameters $\gamma_k = 0,1$ indicate that we describe here the set of forms. It is obvious that the forms (18) ignore the subspaces of $so^{(k)}(3)$ subalgebras that appear on each step of the sequence of injections in (16). According to the formula (17) to describe the maximal Frobenius subalgebras in $B(so(M))$ the following form must be considered:

$$\omega_p^{\pm} = \sum_{k=0}^{p} \left(\gamma_k \left(E_{1+2}^k\right)^* - \delta_k \left(E_{1-2}^k\right)^*\right) ([\,,\,]) . \qquad (19)$$

Here both parameters are discrete: $\gamma_k, \delta_k = 0,1$. (Notice that this does not lead to the undesirable terms in the corresponding carrier space because in the Borel subalgebra $B(so\,(M))$ (fixed by the choice of $\left\{\lambda_0^k\right\}$) there are no constituent roots for $e_1^k - e_2^k$. The form $\omega_p^{\pm}$ is considered on $B(so\,(M))$. In terms of the integral root system $\Lambda(so(M))$ (not split in the subsystems $\Lambda^{(k)}$) the generators $E_{1+2}^{(k)}$ and $E_{1-2}^{(k)}$ form the sequence :

$$\left\{E_{1+2}, E_{1-2}, E_{3+4}, E_{3-4}, \ldots, E_{(2p+1)+(2p+2)}, E_{(2p+1),-(2p+2)}\right\} \approx$$
$$\approx \left\{A_{1,M-1}, A_{1,2}, A_{3,M-3}, A_{3,4}, \ldots A_{2p+1,M-(2p+1)}, A_{2p+1,2p+2}, \right\}.$$

Thus we come to the conclusion that there must be two sets of chains of twists for the orthogonal algebras corresponding to the two sets of the coboundary forms (18) and (19). The first set is the canonical chain of twists (3), whose twisting element can be rewritten in terms of invariants (14),(15):

$$\begin{array}{l}
\mathcal{F}_{\mathcal{B}_{0\prec p}} = \exp\left\{I_{M-4p}^{1\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{p}}\right)\right\}\cdot\exp\{H_{1+2}^{(p)}\otimes\sigma_{1+2}^{p}\}\cdot \\
\exp\left\{I_{M-4(p-1)}^{1\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{p-1}}\right)\right\}\cdot\exp\{H_{1+2}^{(p-1)}\otimes\sigma_{1+2}^{p-1}\}\cdot \\
\ldots \\
\exp\left\{I_{M}^{1\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{0}}\right)\right\}\cdot\exp\{H_{1+2}^{(0)}\otimes\sigma_{1+2}^{0}\} \\
= \prod_{k=p}^{0}\exp\left\{I_{M-4k}^{1\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{k}}\right)\right\}\cdot\exp\{H_{1+2}^{(k)}\otimes\sigma_{1+2}^{k}\}
\end{array} \tag{20}$$

(here $\sigma_{\lambda_0^k}$ are denoted by $\sigma_{1+2}^{k} = \ln\left(1+E_{1+2}^{(k)}\right)$ according to (13) and for simplicity we put $\gamma_l = 1$).

When dealing with the forms $\omega_p^{\pm}$ the problem is that in the process of twisting by a chain (20) the costructure of the subalgebras $so^{(k)}(3)$ is considerably changed and the twist equations (1) become extremely difficult to solve.

3 Construction of the full chains of twists

According to the general structure of a chain of twists [3] we can study its links separately. Let us assume that we have constructed the $k-1$ links of a chain and found the matreshka effect. This means that after the chain twisting with $k-1$ links we get the subalgebra $g^{(k)} = so\,(M-4k)$ with primitive generators. We shall show that it is possible to construct the next link of the chain so that the twist will correspond to the enlarged form $\omega_p^{\pm}$ (see (19)). To start the construction of the k-th link we have to choose the initial root λ_0^k (as in (13)) and the subalgebra $\mathbf{L}_{K,M-4k}$ described in Lemma

2 (with $2K = [M/2]$). First we apply the following jordanian twist to the subalgebra $\mathbf{L}_{K,M-4k}$:

$$\Phi_{\mathcal{J}_k} = \exp\left(H^k_{1+2} \otimes \sigma^k_{1+2}\right). \tag{21}$$

This results in the following deformed coproducts:

$$\begin{array}{l}
\Delta_{\mathcal{J}_k}\left(H^k_{1+2}\right) = H^k_{1+2} \otimes e^{-\sigma^k_{1+2}} + 1 \otimes H^k_{1+2}, \\
\Delta_{\mathcal{J}_k}\left(E^k_{1+2}\right) = E^k_{1+2} \otimes e^{\sigma^k_{1+2}} + 1 \otimes E^k_{1+2}, \\
\Delta_{\mathcal{J}_k}\left(E^k_{a\pm l}\right) = E^k_{a\pm l} \otimes e^{\frac{1}{2}\sigma^k_{1+2}} + 1 \otimes E^k_{a\pm l}, \\
\quad l = 3, \ldots, N; \quad a = 1, 2;
\end{array}$$

For $M = 2N + 1$ we also get

$$\Delta_{\mathcal{J}_k}\left(E^k_a\right) = E^k_a \otimes e^{\frac{1}{2}\sigma^k_{1+2}} + 1 \otimes E^k_a,$$

Notice that the generators $\left\{H^k_{1-2}, E^k_{1-2}, E^k_{2-1}\right\}$ remain primitive.

The second twisting factor must be the full canonical extension [2] for the jordanian twist $\Phi_{\mathcal{J}_k}$ (21):

$$\Phi_{\mathcal{E}_k} = \exp\left(I^{1\otimes 2}_{M-4k}\left(1 \otimes e^{-\frac{1}{2}\sigma^k_{1+2}}\right)\right), \tag{22}$$

The successive application of these two factors performs the extended jordanian twisting by the element $\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}$ that leads to the following costructure in $\mathbf{L}_{K,M-4k}$:

$$\begin{array}{l}
\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(H^k_{1+2}\right) = H^k_{1+2} \otimes e^{-\sigma^k_{1+2}} + 1 \otimes H^k_{1+2} - \left(1 \otimes e^{-\frac{3}{2}\sigma^k_{1+2}}\right) I^{1\otimes 2}_{M-4k}, \\
\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(H^k_{1-2}\right) = H^k_{1-2} \otimes 1 + 1 \otimes H^k_{1-2}, \\
\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E^k_{1+2}\right) = E^k_{1+2} \otimes e^{\sigma^k_{1+2}} + 1 \otimes E^k_{1+2}, \\
\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E^k_{1\pm k}\right) = E^k_{1+k} \otimes e^{-\frac{1}{2}\sigma^k_{1+2}} + 1 \otimes E^k_{1+k}, \\
\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E^k_{2\pm k}\right) = E^k_{2+k} \otimes e^{\frac{1}{2}\sigma^k_{1+2}} + e^{\sigma^k_{1+2}} \otimes E^k_{2+k},
\end{array}$$

$$\begin{array}{l}
\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E^k_{1-2}\right) = \\
E^k_{1-2} \otimes 1 + 1 \otimes E^k_{1-2} + \left(1 \otimes e^{-\frac{1}{2}\sigma^k_{1+2}}\right) I^{1\otimes 1}_{M-4k} + I^1_{M-4k} \otimes \left(e^{-\sigma^k_{1+2}} - 1\right), \\
\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E^k_{2-1}\right) = \\
E^k_{2-1} \otimes 1 + 1 \otimes E^k_{2-1} + \left(e^{\sigma^k_{1+2}} - 1\right) \otimes I^2_{M-4k} e^{-\sigma^k_{1+2}} + \left(1 \otimes e^{-\frac{1}{2}\sigma^k_{1+2}}\right) I^{2\otimes 2}_{M-4k}.
\end{array}$$

And in the case of $M=2N+1$ for the short root generators we get:

$$\begin{array}{l}\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E_1^k\right)=E_1^k\otimes e^{-\frac{1}{2}\sigma_{1+2}^k}+1\otimes E_1^k,\\ \Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E_2^k\right)=E_2^k\otimes e^{\frac{1}{2}\sigma_{1+2}^k}+e^{\sigma_{1+2}^k}\otimes E_2^k.\end{array}$$

It would be necessary to have the coproducts for some of the invariants (see (14) and (15))

$$\begin{array}{l}\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(I_{M-4k}^1\right)=\\ I_{M-4k}^1\otimes e^{-\sigma_{1+2}^k}+1\otimes I_{M-4k}^1+I_{M-4k}^{1\otimes 1}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^k}\right),\end{array}\tag{23}$$

$$\begin{array}{l}\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(I_{M-4k}^2e^{-\sigma_{1+2}^k}\right)=\\ I_{M-4k}^2e^{-\sigma_{1+2}^k}\otimes 1+e^{\sigma_{1+2}^k}\otimes I_{M-4k}^2e^{-\sigma_{1+2}^k}+I_{M-4k}^{2\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^k}\right).\end{array}\tag{24}$$

We have two generators of $\mathbf{L}_{K,M-4k}$ that are not yet incorporated in the carrier subalgebra of the twist: H_{1-2}^k and E_{1-2}^k. The coproduct of the latter is deformed. So the canonical jordanian factor cannot be used here. In [5] it was indicated that the reason of the nonprimitivity of the coproduct $\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E_{1-2}^k\right)$ is that the generator E_{1-2}^k belongs to the long series of the initial root $\lambda_0^k=e_1^k+e_2^k$. It was shown there that in such a case the deformed carrier subspace must exist with primitive basic elements. In our situation such "deformed" generators must have the form

$$\begin{array}{l}G_{1-2}^k=E_{1-2}^k-I_{M-4k}^1,\\ G_{2-1}^k=E_{2-1}^k-I_{M-4k}^2e^{-\sigma_{1+2}}.\end{array}\tag{25}$$

Using the coproducts (23, 24) it is easy to check that both G_{1-2}^k and G_{2-1}^k are primitive,

$$\begin{array}{l}\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(G_{1-2}^k\right)=\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E_{1-2}^k\right)-\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(I_{M-4k}^1\right)\\ =G_{1-2}^k\otimes 1+1\otimes G_{1-2}^k,\\ \Delta_{\mathcal{E}_k\mathcal{J}_k}\left(G_{2-1}^k\right)=\Delta_{\mathcal{E}_k\mathcal{J}_k}\left(E_{2-1}^k\right)-\Delta_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}\left(I_{M-4k}^2\right)\left(e^{-\sigma_{1+2}^k}\otimes e^{-\sigma_{1+2}^k}\right)\\ =G_{2-1}^k\otimes 1+1\otimes G_{2-1}^k.\end{array}$$

Together with H_{1-2}^k the elements (25) generate a 3-dimensional space V_G^k of primitive elements in the algebra $U_{\mathcal{E}_k\mathcal{J}_k}\left(so(M-4k)\right)$. Both G_{1-2}^k and G_{2-1}^k commute with $U\left(so\left(M-4(k+1)\right)\right)$ as well as H_{1-2}^k whose dual vector is orthogonal to the roots of $so\left(M-4(k+1)\right)$.

The subspace V_G^k spanned by $\left\{H_{1-2}^k, G_{1-2}^k, G_{2-1}^k\right\}$ is algebraically closed:

$$\begin{aligned}\left[H_{1-2}^k, G_{1-2}^k\right] &= G_{1-2}^k, \\ \left[H_{1-2}^k, G_{2-1}^k\right] &= -G_{2-1}^k, \\ \left[G_{1-2}^k, G_{2-1}^k\right] &= 2H_{1-2}^k.\end{aligned}$$

Let us denote this algebra by $so_G^{(k)}(3)$. Clearly it is primitive, commutes with $U\left(so\left(M-4\left(k+1\right)\right)\right)$ and is realized on a deformed subspace. (This space is not orthogonal to H_{1+2}^k. Moreover, G_{2-1}^k, G_{1-2}^k are not any longer eigenvectors of $\mathrm{ad}_{H_{1+2}^k}$.)

Another subalgebra which remains primitive after the composition $\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}$ of twists (21) and (22) is $so\left(M-4\left(k+1\right)\right)$ (due to the matreshka effect). We come to conclusion that the twisted $U_{\mathcal{E}_k\mathcal{J}_k}\left(so\left(M-4k\right)\right)$ contains the primitive subalgebra $g_{\lambda_0^k}^{\perp}$

$$U_{\mathcal{E}_k\mathcal{J}_k}\left(so\left(M-4k\right)\right) \supset g_{\lambda_0^k}^{\perp} = so\left(M-4\left(k+1\right)\right) \oplus so_G^{(k)}(3).$$

Its Borel subalgebra is $\mathbf{L}_{K,M-4(k+1)} \oplus B(so_G^{(k)}(3))$ and it is Frobenius (see Section 2).

Remember that the subalgebra $g_{\lambda_0^k}^{\perp}$ has a structure of direct sum. Further, twisting by the next factors (such as $\Phi_{\mathcal{E}_{k+s}}\Phi_{\mathcal{J}_{k+s}}$) can not affect the primitive subalgebra $so_G^{(k)}(3)$. Each step produces (in the corresponding $g_{\lambda_0^k}^{\perp}$) the additional subalgebra $so_G^{(k)}(3), \quad k=1,\ldots,p$. The primitive subalgebras that can be found in an orthogonal algebra after the chain twisting (20) with p links contain not only $so\left(M-4\left(p+1\right)\right)$ but also a direct sum of p copies of $so_G(3)$:

$$U_{\mathcal{B}_{0\prec p}}\left(so\left(M\right)\right) \supset \mathcal{D} = \oplus_{k=1}^p\, so_G^{(k)}(3).$$

The main consequence is that in the twisted $U_{\mathcal{B}_{0\prec p}}\left(so\left(M\right)\right)$ one can perform further twist deformations with the carrier subalgebra in $\mathcal{D}$. The most interesting among them are the jordanian twists defined by

$$\Phi_{\mathcal{J}_k}^G = \exp\left(H_{1-2}^k \otimes \sigma_{_G}^k\right) \qquad (26)$$

that can be attributed to any number of copies $so_G(3)$. Here $\sigma_{_G}^k = \ln\left(1+G_{1-2}^k\right)$. Thus in the general expression for the twisting element (20) one can insert in the appropriate k places the additional factors which are the jordanian twisting elements on the deformed carrier spaces. This means that we can perform a substitution

$$\begin{array}{l}\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}\Rightarrow\Phi^G_{\mathcal{J}_k}\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}=\Phi_{\mathcal{G}_k}\\ \exp\left\{I^{1\otimes 2}_{M-4k}\left(1\otimes e^{-\frac{1}{2}\sigma^k_{1+2}}\right)\right\}\cdot\exp\{H^k_{1+2}\otimes\sigma^k_{1+2}\}\Rightarrow\\ \exp\left(H^k_{1-2}\otimes\sigma^k_G\right)\cdot\exp\left\{I^{1\otimes 2}_{M-4k}\left(1\otimes e^{-\frac{1}{2}\sigma^k_{1+2}}\right)\right\}\cdot\exp\left(H^k_{1+2}\otimes\sigma^k_{1+2}\right)\end{array}$$

Thus *the full chain* has the following form

$$\begin{array}{l}\mathcal{F}_{\mathcal{G}_{0\prec p}}=\prod^0_{k=p}\Phi_{\mathcal{G}_k}=\\ \prod^0_{k=p}\exp\left(H^k_{1-2}\otimes\sigma^k_G\right)\cdot\exp\left\{I^{1\otimes 2}_{M-4k}\left(1\otimes e^{-\frac{1}{2}\sigma^k_{1+2}}\right)\right\}\cdot\exp\{H^k_{1+2}\otimes\sigma^k_{1+2}\}.\end{array}\tag{27}$$

This result means that we have constructed the explicit quantizations with a triangular R-matrix

$$\mathcal{R}_{\mathcal{G}_{0\prec p}}=\left(\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)_{21}\left(\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)^{-1}$$

for the following set of classical r-matrices:

$$r_{\mathcal{G}_{0\prec p}}=\sum_{k=0}^{p}\eta_k\left(H^k_{1+2}\wedge E^k_{1+2}+\xi_k H^k_{1-2}\wedge E^k_{1-2}+I^{1\wedge 2}_{M-4k}\right)$$

Here all the parameters are independent and continuous. Elementary computations show that these full chains (27) correspond to the coboundary forms (19). To illustrate these quantizations we present in Appendix the matrix $\mathcal{R}_{\mathcal{G}_{0\prec p}}$ for the algebra $so(5)$ in the defining representation.

In Section 2 we proved that adding $\zeta_{ij}H^*_i\bigwedge H^*_j$ to the forms of type (19) we obtain new non-degenerate 2-cocycles, which are not coboundaries. We can also construct the corresponding twists for these modified cocycles:

$$\omega^{\pm}_p=\sum_{k=0}^{p}\left(\gamma_k\left(E^k_{1+2}\right)^*-\delta_k\left(E^k_{1-2}\right)^*\right)([\,,\,])+\sum_{i,j=0;\,i\neq j}^{p}\zeta_{ij}H^*_i\bigwedge H^*_j.\tag{28}$$

Notice that the subalgebras $so^{(k)}_G(3)$ commute not only with $so\left(M-4(k+1)\right)$ but also with any $\left\{E^{(s)}_{1+2}|s\leq k\right\}$. This means that after having twisted $U\left(so\left(M\right)\right)$ by the chain (27) we obtain $p+1$ pairs of commuting primitive elements $\left\{\sigma_{1+2}^k, \sigma_G^k | k=0,\ldots,p\right\}$. Therefore we can apply the Reshetikhin twist

$$\Phi_{\mathcal{R}} = \exp\left(\zeta_{ij}\sigma_i \otimes \sigma_j\right), \quad \sigma_i \in \left\{\sigma_{1+2}^k, \sigma_G^k | k=0,\ldots,p\right\}. \tag{29}$$

to the algebra $U_{\mathcal{G}_{0\prec p}}(so(M))$.

Thus the element

$$\Phi_{\mathcal{R}}\mathcal{F}_{\mathcal{G}_{0\prec p}}$$

defines also a twist for $U(so(M))$. It leads to the deformed Hopf algebra $U_{\mathcal{R}\mathcal{G}_{0\prec p}}(so(M))$ with the universal element

$$\mathcal{R}_{\mathcal{R}\mathcal{G}_{0\prec p}} = \left(\Phi_{\mathcal{R}}\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)_{21}\left(\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)^{-1}\left(\Phi_{\mathcal{R}}\right)^{-1}$$

and the classical r-matrix

$$\begin{array}{rcl} r_{\mathcal{R}\mathcal{G}_{0\prec p}} & = & \sum_{k=0}^{p}\eta_k\left(H_{1+2}^k \wedge E_{1+2}^k + \xi_k H_{1-2}^k \wedge E_{1-2}^k + I_{M-4k}^{1\wedge 2}\right) \\ & & + \sum_{i,j=0;\, i\neq j}^{p}\zeta_{ij}E_s^i \bigwedge E_t^j; \end{array}$$

$$E_s^k, E_t^k \in \left\{E_{1+2}^k, E_{1-2}^k | k=0,\ldots,p\right\}.$$

The dimensions of the nilpotent subalgebras $N^+(so(M))$ in the sequence $g_{\lambda_0^p}^{\perp} \subset g_{\lambda_0^{p-1}}^{\perp} \subset \ldots \subset g_{\lambda_0^0}^{\perp} \subset g$ are subject to the following simple relation:

$$\dim\left(N^+(so(M))\right) - \dim\left(N^+(so(M-4))\right) = 2\left(\dim d_{so(M-4)}^v + 1\right). \tag{30}$$

Taking this into account we conclude (from the formula (16)) that the chains (27) are full. Furthermore, this means that for $p = p_{\max} = [M/4] + [(M+1)/4]$ the corresponding carrier spaces contain all the generators of the nilpotent subalgebra $N^+(so(M))$. When M is even-odd one can always find in $so(M)$ one independent Cartan generator which cannot be included in the carrier subalgebra of a chain. When M is even-even or odd the total number of jordanian twists in a maximal full chain $\mathcal{F}_{\mathcal{G}_{0\prec p\max}}$ is equal to the rank of $so(M)$. Thus in the latter case the carrier subalgebra is equal to the Borel.

4 Conclusions

The family of explicit twisting elements was constructed for the universal enveloping algebras $\mathcal{A} = U(so(M))$ (series B, and D) with full nilpotent subalgebras $N^+(so(M))$ included in the corresponding carrier spaces.

There is a variety of applications for explicitly known twisting elements $\mathcal{F}$. Using a particular (e.g. fundamental) representation for one of the factors of $\mathcal{A} \otimes \mathcal{A}$ we get from the universal R-matrix the L-operator of the FRT-formalism and this results in explicit relations among the generators of the original universal enveloping algebra and the FRT-generators of the twisted one.

Twisting of the coalgebra in $\mathcal{A}$ induces changes in Clebsch-Gordan coefficients of bases in the tensor products of irreducible representations $c_V \otimes d_W$. The evaluation of these coefficients is given by the direct action of the matrix F (F is the value of the twisting element in the corresponding representation: $F = c_V \otimes d_W(\mathcal{F})$) on the original CG coefficients [7].

Due to the embedding of the simple Lie algebras g into the corresponding Yangians (as Hopf subalgebras) $U(g) \subset \mathcal{Y}(g)$ [8] the Yangian R-matrix $R_{\mathcal{Y}}$ can be twisted by the same $\mathcal{F}$ defined for g [9, 10]. As a result for the case of orthogonal algebra $g = so(M)$ the R-matrix (in the defining representation $d \subset \mathrm{Mat}(M, \mathbf{C}) \otimes \mathrm{Mat}(M, \mathbf{C})$) will be changed:

$$\begin{array}{c} ud\,(1 \otimes 1) + \mathcal{P} - \frac{u}{u-1+M/2}\mathcal{K} \quad \longrightarrow \\ ud\left(\mathcal{F}_{21}\mathcal{F}^{-1}\right) + \mathcal{P} - \frac{u}{u-1+M/2} d\left(\mathcal{F}_{21}\right)\mathcal{K} d\left(\mathcal{F}^{-1}\right) \end{array}$$

(Here u is a spectral parameter and the operator $\mathcal{K}$ is obtained from the permutation $\mathcal{P}$ by transposing its first tensor factor.) For the canonical chains $\mathcal{F} = \mathcal{F}_{\mathcal{B}_{0\prec p}}$ the deformed solutions of YBE were given in the explicit form in [11]. Similarly to the case of canonical chains the twists $\mathcal{F}_{\mathcal{G}_{0\prec p}}$ produce the sets of deformed Yangians and the new integrable hamiltonians (cf. the $sl(2)$ -case [7]).

5 Acknowledgements

This work was supported in part by the Russian Foundation for Basic Research under the grants N 99-01-00101 (P.P.K.) and N 00-01-00500 (V.D.L.), the Royal Swedish Academy of Sciences under the program "Cooperation between Sweden and former USSR", the INTAS grant N 99-01459 (P.P.K.).

6 Appendix

Here we take $g = so(5)$. This is the simplest case where the full chain differs nontrivially from the canonical one [3] and the deformed carrier space is used to construct the twist:

$$\begin{array}{l} \mathcal{F}_{\mathcal{G}_{0\prec p}} = \\ \exp\left(H^0_{1-2} \otimes \sigma^0_{_G}\right) \cdot \exp\left\{I_5^{1\otimes 2}\left(1 \otimes e^{-\frac{1}{2}\sigma^0_{1+2}}\right)\right\} \cdot \exp\{H^0_{1+2} \otimes \sigma^0_{1+2}\}. \end{array} \tag{31}$$

Consider now the corresponding R-matrix $\mathcal{R}_{\mathcal{G}_{0\prec p}} = \left(\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)_{21}\left(\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)^{-1}$ in the defining 5-dimensional representation d of $so(5)$. This means that we use the following matrix realization for the generators of $B(so(5))$:

$$\begin{array}{l} H_i = \frac{1}{2}\left(\mathcal{E}_{i,i} - \mathcal{E}_{M+1-i,M+1-i}\right), \\ A_{i,j} = \mathcal{E}_{i,j} - \mathcal{E}_{M+1-j,M+1-i}; \\ \\ i,j = 1,\ldots,5. \end{array} \tag{32}$$

As a result we get the solution of the matrix Yang-Baxter equation that can be written in terms of tensor products of 5×5 matrix units $\mathcal{E}_{i,j}$:

$$\begin{array}{l} d\left(\mathcal{R}_{\mathcal{G}_{0\prec p}}\right) = R_{\mathcal{G}_0} = \\ (\mathcal{E}_{4,5} - \mathcal{E}_{1,2}) \otimes \left(\frac{1}{2}(\mathcal{E}_{1,1} - \mathcal{E}_{2,2} + \mathcal{E}_{4,4} - \mathcal{E}_{5,5}) + \frac{1}{4}(\mathcal{E}_{4,5} - \mathcal{E}_{1,2}) - \frac{1}{8}\mathcal{E}_{1,5}\right) + \\ (\mathcal{E}_{4,5} + \mathcal{E}_{1,2}) \otimes \left(-\frac{1}{4}(\mathcal{E}_{1,4} + \mathcal{E}_{2,5}) - \frac{1}{8}\mathcal{E}_{1,5}\right) + \\ (\mathcal{E}_{3,5} - \mathcal{E}_{1,3}) \otimes ((\mathcal{E}_{2,3} - \mathcal{E}_{3,4})) + \\ (\mathcal{E}_{3,5} + \mathcal{E}_{1,3}) \otimes \left(-\frac{1}{2}(\mathcal{E}_{3,5} + \mathcal{E}_{1,3})\right) + \\ (\mathcal{E}_{3,4} - \mathcal{E}_{2,3}) \otimes ((\mathcal{E}_{3,5} - \mathcal{E}_{1,3})) + \\ (\mathcal{E}_{2,5} - \mathcal{E}_{1,4}) \otimes \left(\frac{1}{4}(\mathcal{E}_{2,5} - \mathcal{E}_{1,4}) + \frac{1}{2}(\mathcal{E}_{1,1} + \mathcal{E}_{2,2} - \mathcal{E}_{4,4} - \mathcal{E}_{5,5})\right) + \\ (\mathcal{E}_{2,5} + \mathcal{E}_{1,4}) \otimes \left(-\frac{1}{4}(\mathcal{E}_{4,5} + \mathcal{E}_{1,2}) - \frac{1}{4}\mathcal{E}_{1,5}\right) + \\ (\mathcal{E}_{1,1} + \mathcal{E}_{2,2} - \mathcal{E}_{4,4} - \mathcal{E}_{5,5}) \otimes \left(\frac{1}{2}(\mathcal{E}_{1,4} - \mathcal{E}_{2,5})\right) + \\ (\mathcal{E}_{1,1} - \mathcal{E}_{2,2} + \mathcal{E}_{4,4} - \mathcal{E}_{5,5}) \otimes \left(\frac{1}{2}(\mathcal{E}_{1,2} - \mathcal{E}_{4,5})\right) + \\ (\mathcal{E}_{4,4} - \mathcal{E}_{5,5} + \mathcal{E}_{2,4}) \otimes \left(\frac{1}{2}\mathcal{E}_{1,5}\right) + \\ (\mathcal{E}_{1,4}) \otimes \left(-\frac{1}{4}\mathcal{E}_{1,5} - \mathcal{E}_{2,5}\right) + \\ (\mathcal{E}_{1,5}) \otimes \left(-\frac{1}{2}(\mathcal{E}_{1,1} - \mathcal{E}_{2,2}) + \frac{1}{2}\mathcal{E}_{2,4} + \frac{1}{2}\mathcal{E}_{2,5} + \frac{1}{4}\mathcal{E}_{1,4} + \frac{1}{4}\mathcal{E}_{1,5} - \frac{3}{8}\mathcal{E}_{4,5} + \frac{5}{8}\mathcal{E}_{1,2}\right). \end{array}$$

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