THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

On the Pricing of Barrier Options and Related Problems

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Abstract

We treat the problem of computing the theoretical value of barrier options within the framework of Black-Scholes model. First we give a brief introduction to the theory of option pricing. A rigorous mathematical model of a financial market based on Brownian motion and stochastic calculus is described.

Next we consider probabilistic methods to compute certain laws of the first hitting time and the first exit time of a Brownian motion with drift. The results obtained are used to derive price formulas of continuous barrier options with corresponding rebates. The numerical properties of the formulas are examined.

We then study the pricing of discrete barrier options and extend an approximation method by Broadie, Glasserman and Kou (see [BGK]). The method is based on Siegmund's corrected heavy traffic approximation.

We also design a numerical method to estimate the Wiener measure of certain cylinder sets. The method is employed to compute the theoretical value of discrete barrier options.

The method is based on the so called trinomial method (or the explicit finite difference method for the heat equation). Different aspects of the trinomial method are investigated. In particular, we consider the rate of convergence for the trinomial method. We present results which indicate that the convergence rate depends on two factors, namely the smoothness of the initial value (or the payoff function) and the moments for the increments of the trinomial distributed random walk.

Keywords: barrier options, discrete barrier options, rebate options, Brownian motion, heavy traffic approximation, random walk, trinomial method, explicit finite difference method, heat equation.

AMS 2000 Mathematics subject classification. 91B28, 60J65, 60G50, 65M06, 65C50.

Preface

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To my mother

About ECMI

This licentiate thesis was written to conclude a five semester ECMI (European Consortium for Mathematics in Industry) programme in applied mathematics. This programme includes a block of core courses covering several areas of applied mathematics and computing science and a block of specialisation courses within a selected field. The final part of the programme is to work with a mathematical problem that emanates from the industry.

Contents

Chapter 1

Introduction

In the beginning of the seventies Fischer Black and Myron Scholes published their now very famous article "The Pricing of Options and Corporate Liabilities". Based on the principle that, on a rational market, there are no possibilities to make sure profits, Black and Scholes derived a theoretical price for the European call option (see [BS]). Since then the popularity as well as the number of options and other financial derivatives have increased considerably. In particular, path dependent options have received a notable amount of attention in both academic and trade literature. In this report we will examine in detail the pricing of one particular path dependent option, namely the barrier option.

Although the goal is to develop the theory from its foundations, this report is mainly intended for readers who are familiar with the basics of Brownian motion and stochastic calculus. For an introduction to this subject, we recommend [KS]. However, certain relevant concepts and results about Brownian motion are collected in two appendices.

This report is structured as follows. In Chapter 2 we give a brief introduction to the theory of option pricing. We build up a rigorous mathematical model of a financial market based on Brownian motion and stochastic calculus. We also define barrier options and discuss certain economical aspects of barrier options.

Chapter 3 is devoted to derive analytical formulas for the value of continuous barrier options. The formulas are based on certain laws involving the first hitting time and the first exit time of a Brownian motion with drift. The aim with Chapter 3 is to present, in a unified framework, many previously published results.

In Chapter 4 we consider discrete barrier options. In contrast to continuous barrier options the value of a discrete barrier option does not in general possess a closed form price formula. One common approach to price discrete barrier options is to use an approximation method proposed by Broadie, Glasserman and Kou (see [BGK]). In Chapter 4 we discuss an extension of this method.

The purpose of the final chapter, Chapter 5, is to describe a numerical method to estimate the Wiener measure of certain cylinder sets. The method will be useful to calculate the theoretical value of discrete barrier options.

The algorithm is based on the so called trinomial method (or the explicit finite difference method for the heat equation). Consequently, we begin Chapter 5 with an investigation of the trinomial method. We focus especially on the convergence rate for the trinomial method. We believe that this investigation is of independent interest since the trinomial method is so often used in problems related to financial mathematics.

We conclude this thesis with two appendices, which briefly describe the Cameron-Martin's theorem and the strong Markov property.

Finally, we remark that the last three chapters can be read independently of each other.

Chapter 2

A Brief Introduction to the Theory of Option Pricing

2.1 Introduction

In this chapter we will give a brief introduction to the theory of option pricing. We will build up a rigorous mathematical model of a financial market based on Brownian motion and stochastic calculus. For a more comprehensive treatment on option pricing we recommend [HK], [BBC] or [MR]. We will also define barrier options and discuss certain economical aspects of barrier options.

2.2 The Black-Scholes Market and Contingent Claims

Take as given a complete filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ where the sample space Ω is the space of all continuous functions $\omega : [0, \infty) \to \mathbb{R}$. The coordinate mapping process $W_t(\omega) = \omega(t)$, $\omega \in \Omega$, is a Brownian motion with respect to $(P, \{\mathcal{F}_t\}_{t\geq 0})$ and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the *usual conditions*. The usual conditions mean that $\mathcal{F}_0$ contains all null sets of P and that the filtration is right continuous.

Now we can define *the Black-Scholes market*. On this market there are two securities. The price processes for these securities are governed by the following (stochastic) differential equations, viz.

$$\begin{cases} dB_t = rB_t dt, \\ dS_t = \eta S_t\, dt + \sigma S_t\, dW_t\,, \quad 0 \leq t \leq \lambda, \end{cases}$$

where η, r, σ and λ are real constants with $\sigma > 0$ and $\lambda > 0$. Suppose also that $B_0 = 1$ and that S_0 is a constant strictly greater than zero. The stochastic integrals (or differentials) shall be understood in the Itô sense.

We will interpret B_t as the price (in some currency) at time t of a *riskless bond*, with r being the associated riskless interest rate. Moreover, we will interpret S_t as the price (in the same currency as for the bond) at time t of a *risky security* which pays no dividends. The security can for instance be a stock, a commodity or an asset linked to a foreign currency. The constant σ is often referred to as the volatility.

The solutions to the above (stochastic) differential equations are given by

$$B_t = e^{rt} \quad \text{and} \quad S_t = S_0 e^{(\eta - \sigma^2/2)\,t + \sigma W_t},$$

where $t \in [0, \lambda]$. Thus, the price of the risky security follows a so called geometric Brownian motion (with drift).

In the sequel we will always assume that the market is *frictionless*, meaning that the investors are allowed to trade continuously, that there are no transaction costs and that there are no restrictions against selling short. Selling short means selling borrowed assets.

Now suppose that we expand the Black-Scholes market by adding a so called *contingent T-claim*, also known as a *financial derivative* or an *option*. These are assets which are defined in terms of the risky asset and the bond, which in this connection are referred to as the *underlying assets* or the *underlying securities*. We make the following mathematical formalization. (Recall that if T is a stopping time then $\mathcal{F}_T$ denotes the σ-algebra $\{A \in \mathcal{F}\,;\, A \cap \{T \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0\}$).

Definition 2.1. *Let T be a bounded stopping time. A* **contingent T-claim** *is an $\mathcal{F}_T$ measurable and positive random variable X.*

The interpretation of this definition is that the contingent T-claim is a contract which specifies that the stochastic amount X of money is to be paid out to the holder of the contract at time T. In most cases, T is a fixed date, that is, T is a constant.

One of the most important contingent claims is the so called European call option. A European call option with strike price K and time of expiration T, where K and T are constants, is a contract which gives the holder the possibility but not the obligation to buy one share of the risky security at the expiration date T at the prespecified price K. If $S_T \le K$, the contract is worthless at the maturity. If $S_T > K$, the holder can buy one share of the risky security at the price K giving the net profit $S_T - K$. Thus the European call option is equivalent to a contract giving the holder the amount

$$X = \max(S_T - K, 0)$$

at time T.

How much would an investor be willing to pay for a given contingent T-claim X? Remarkably enough, Black and Scholes (see [BS]) asserted that there is a unique rational value for the option, independent of the investor's attitude to risk. In the next three sections we will give an argument leading to this unique rational price.

2.3 Self Financing Portfolios

First some definitions that will be frequently used in the sequel.

Below we let $\mathcal{B}(A)$, where $A \in \mathbb{R}$, denote the smallest σ-algebra containing all open subsets of A.

Definition 2.2. *The stochastic process h is said to be* **progressively measurable** *with respect to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq\lambda}$ if, for each $t \in [0,\lambda]$ and $B \in \mathcal{B}(\mathbb{R})$, the set*

$$\{(s,\omega)\,;\, 0 \leq s \leq t,\, \omega \in \Omega,\, h_s(\omega) \in B\}$$

belongs to $\mathcal{B}(\,[0,t]\,) \otimes \mathcal{F}_t$.

Definition 2.3. *Suppose P and $\tilde{P}$ are equivalent probability measures. Let $\mathcal{L}^p(\,[0,\lambda],\tilde{P}\,)$ denote the set of all $\{\mathcal{F}_t\}_{0\leq t\leq\lambda}$ progressively measurable processes h such that*

$$E^{\tilde{P}}\Big[\int_0^\lambda |h_t|^p dt\Big] < \infty,$$

where $E^{\tilde{P}}$ stands for expectation with respect to $\tilde{P}$.

Let $\mathcal{L}^p_{loc}\,(\,[0,\lambda],\tilde{P}\,)$ denote the set of all $\{\mathcal{F}_t\}_{0\leq t\leq\lambda}$ progressively measurable processes h such that

$$\int_0^\lambda |h_t|^p dt < \infty \quad \tilde{P} - a.s. \tag{2.1}$$

Thus, for all $h \in \mathcal{L}^2_{loc}(\,[0,\lambda],P\,)$ we have that the Itô integral

$$\int_0^\lambda h_s\, dW_s$$

is well defined. Moreover, this integral belongs to $L^2(P)$ if $h \in \mathcal{L}^2(\,[0,\lambda],P\,)$ (see [KS]). Note also that the condition in equation (2.1) is invariant with respect to an equivalent change of (probability) measure.

Referring to the previous section we will next define portfolios.

Definition 2.4. *A* **portfolio** *(or a* **trading strategy***) ϕ is a stochastic process $\phi_t = (\phi^0_t,\phi^1_t)$, $0 \leq t \leq \lambda$, where ϕ^0 and ϕ^1 are progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq\lambda}$. The* **value process** *$V(\phi)$ corresponding to the portfolio ϕ is defined by*

$$V_t(\phi) = \phi^0_t B_t + \phi^1_t S_t,$$

where $t \in [0,\lambda]$.

The random variables ϕ_t^0 and ϕ_t^1 are interpreted as the number of shares of bonds and risky assets, respectively, held in the portfolio at time t.

Recall that if $T : \Omega \mapsto [0, \lambda]$ is a stopping time and $(h_t)_{0 \leq t \leq \lambda}$ is a stochastic process, then

$$h_T(\omega) = h_{T(\omega)}(\omega), \quad \omega \in \Omega.$$

Thus, in particular,

$$V_T(\phi) = \phi_T^0 B_T + \phi_T^1 S_T.$$

Next we will consider portfolios where all the changes in the portfolio values are due to capital gains.

Definition 2.5. *Let $\phi = (\phi^0, \phi^1)$ be a portfolio such that $\phi^0 \in \mathcal{L}^1_{loc}([0, \lambda], P)$ and $\phi^1 S \in \mathcal{L}^2_{loc}([0, \lambda], P)$. The portfolio ϕ is said to be* **self-financing** *if*

$$V_t(\phi) - V_0(\phi) = \int_0^t \phi_s^0 dB_s + \int_0^t \phi_s^1 dS_s \tag{2.2}$$

for all $t \in [0, \lambda]$.

To motivate Definition 2.5, suppose that all trading occur at discrete times $t = t_k,\ k = 0, 1, \dots, n$. Then the gain $G_{t_k} = V_{t_k} - V_{t_0}$ at time t_k is given by the equation

$$G_{t_k} = \sum_{i=0}^{k-1} \phi_{t_i}^0 \left(B_{t_{i+1}} - B_{t_i} \right) + \sum_{i=0}^{k-1} \phi_{t_i}^1 \left(S_{t_{i+1}} - S_{t_i} \right),$$

provided all changes in the portfolio's value are due to capital gains. By letting $\max_k(t_{k+1} - t_k)$ go to zero we are lead to Definition 2.5.

It is often convenient to work with discounted prices, meaning that the prices are expressed in terms of the bond instead of in terms of the monetary unit. For this reason we will introduce a *discounted price process* Y and a *discounted value process* V^Y by setting

$$Y_t = S_t / B_t \quad \text{and} \quad V_t^Y(\phi) = V_t(\phi)/B_t = \phi_t^0 + \phi_t^1 Y_t$$

for each $t \in [0, \lambda]$.

The next theorem gives us an equivalent description of a self-financing trading strategy.

Proposition 2.1. *Let ϕ be a trading strategy such that $\phi^0 \in \mathcal{L}^1_{loc}([0, \lambda], P)$ and $\phi^1 S \in \mathcal{L}^2_{loc}([0, \lambda], P)$. Then ϕ is self-financing if and only if the discounted value process satisfies*

$$V_t^Y(\phi) - V_0^Y(\phi) = \int_0^t \phi_s^1 dY_s,$$

for all $t \in [0, \lambda]$.

Proof. Let for conciseness $V_t = V_t(\phi)$ where ϕ is a portfolio such that $\phi^0 \in \mathcal{L}^1_{loc}([0,\lambda], P)$ and $\phi^1 S \in \mathcal{L}^2_{loc}([0,\lambda], P)$. By applying Itô's rule (see [KS], p.153) we obtain

$$\begin{aligned} dV_t =& d(B_t V_t^Y) \\ =& B_t dV_t^Y + V_t^Y dB_t \\ =& B_t dV_t^Y + \phi_t^0 dB_t + \phi_t^1 Y_t dB_t \end{aligned}$$

for any $t \in [0,\lambda]$. Since $dS_t = d(B_t Y_t) = B_t dY_t + Y_t dB_t$ we get

$$dV_t = B_t\,(dV_t^Y - \phi_t^1 dY_t) + \phi_t^0 dB_t + \phi_t^1 dS_t. \tag{2.3}$$

Proposition 2.1 now follows at once from equation (2.3). □

2.4 Arbitrage

A fundamental concept underlying the Black-Scholes theory is that of *arbitrage.*

Definition 2.6. *An* **arbitrage opportunity** *or an* **arbitrage portfolio** *is a self-financing portfolio ϕ such that the corresponding value process has the following properties,*

$$V_0(\phi) = 0,\ V_\lambda(\phi) \geq 0 \quad and \quad P(\,V_\lambda(\phi) > 0\,) > 0. \tag{2.4}$$

As can be seen from equation (2.4), an arbitrage portfolio is a riskless way to make money, or in the terminology of Björk, "a deterministic money making machine", see [BBC]. Of course, a rational market will be free of arbitrage opportunities.

Actually, on the Black-Scholes market one can construct arbitrage portfolios (see [HP]). To get a reliable model of a security market we must therefore exclude such examples. One way to achieve this is to put constraints on the trading with the risky asset. Before we present such a constraint, we will introduce the so called *martingale measure* in the Black-Scholes market.

Let Q be a measure on $\mathcal{F}$, defined by the Radon-Nikodym derivate $dQ = ZdP$, where

$$Z = e^{-\frac{1}{2}\alpha\lambda^2 + \alpha W_\lambda}$$

with $\alpha = (r-\eta)/\sigma$. Note that $Z > 0$ which implies that Q and P are equivalent. If we now for each $t \leq \lambda$ set

$$W_t^\alpha = W_t - \alpha t,$$

then the Cameron-Martin's theorem, see Appendix A, yields that $\{W_t^\alpha\}_{0\leq t\leq\lambda}$ is a Brownian motion with respect to $(Q, \{\mathcal{F}_t\}_{0\leq t\leq\lambda})$. Moreover, since

$$dY_t = (\eta - r)Y_t dt + \sigma Y_t dW_t = \sigma Y_t dW_t^\alpha, \tag{2.5}$$

we see that the discounted price process Y is a martingale with respect to $(Q, \{\mathcal{F}_t\}_{0\le t\le \lambda})$. For this reason the measure Q is often called the martingale measure for the Black-Scholes market.

We are now in the position to define a class of trading strategies without any arbitrage portfolio.

Definition 2.7. *A portfolio ϕ which satisfies*

$$\phi^1 Y \in \mathcal{L}^2\,(\,[0,\lambda], Q\,)$$

will be called **admissible***.*

Theorem 2.1. *There exists no admissible arbitrage portfolio.*

Proof. Suppose that ϕ is a self-financing and admissible portfolio. By using the self-financing property, Proposition 2.1 and equation (2.5) we get

$$dV_t^Y(\phi) = \phi_t^1 dY_t = \sigma \phi_t^1 Y_t dW_t^\alpha$$

for any $t \in [0, \lambda]$. Since the trading strategy is admissible we get that V^Y is a martingale with respect to $(Q, \{\mathcal{F}_t\}_{0\le t\le \lambda})$. Hence

$$V_0(\phi) = V_0^Y(\phi) = E^Q\big[V_\lambda^Y(\phi)\big].$$

Thus if $V_0(\phi) = 0$ then $E^Q[V_\lambda(\phi)] = 0$. Since P and Q are equivalent we can draw the desired conclusion that ϕ cannot satisfy equation (2.4). □

Once again, consider the contingent T-claim X with $T \le \lambda$. Suppose also that there is an admissible and self-financing portfolio ϕ such that $V_T(\phi) = X$. If the claim is not priced according to the value of the portfolio at any time $t \le T$, then there is a riskless profit on the extended market consisting of the contingent claim, the risky security and the bond. This leads us to the following important definition.

Definition 2.8. *Suppose there is a self-financing and admissible portfolio ϕ such that*

$$V_T(\phi) = X\,.$$

The **theoretical price** *v at time $t = 0$ corresponding to a contingent T-claim X is defined by $v = V_0(\phi)$. The portfolio ϕ is called a* **hedging** *or* **replicating portfolio** *for the claim X.*

It is of course crucial that there are hedging portfolios. We will deal with this question in the next section.

2.5 Hedging Portfolios

This section deals with the problem to hedge a contingent T-claim X, which is square integrable with respect to Q. Our goal is to find an admissible portfolio ϕ such that

$$\begin{cases} dV_t^Y(\phi) = \phi_t^1 dY_t \ , \quad 0 \leq t \leq \lambda = \operatorname{ess\,sup} T, \\ V_T^Y(\phi) = X^Y, \end{cases} \tag{2.6}$$

where $X^Y = X/B_T$ with $X \in L^2(Q)$.

Consider the process M given by

$$M_t = E^Q\big[\, X^Y \,|\, \mathcal{F}_t \,\big], \quad 0 \leq t \leq \lambda.$$

It is evident that M is a square integrable $(Q, \{\mathcal{F}_t\}_{0 \leq t \leq \lambda})$-martingale. The representation theorem for Brownian martingales now gives a process $h \in \mathcal{L}^2(\,[0,\lambda], Q\,)$ such that

$$M_t = M_0 + \int_0^t h_s dW_s^\alpha \ , \quad 0 \leq t \leq \lambda, \quad Q - \text{a.s.} \tag{2.7}$$

Now let

$$\phi_t^1 = h_t/(\sigma Y_t) \quad \text{and} \quad \phi_t^0 = M_t - \phi_t^1 Y_t$$

for each $0 \leq t \leq \lambda$. Since $\phi^0 \in \mathcal{L}^1_{loc}(\,[0,\lambda], P\,)$ and $\phi^1 Y \in \mathcal{L}^2(\,[0,\lambda], Q\,)$ we can conclude that $\phi = (\phi^0, \phi^1)$ is an admissible portfolio. Moreover, the corresponding discounted value process equals

$$V_t^Y(\phi) = \phi_t^0 + \phi_t^1 Y_t = M_t.$$

This implies

$$V_T^Y = X^Y \quad \text{and} \quad dV_t^Y(\phi) = h_t dW_t^\alpha = \phi_t^1 dY_t,$$

where we have in the last equality used equation (2.5). In other words, ϕ is the desired replicating portfolio.

From this result we now get that the theoretical value v of the contract X is given by

$$v = V_0(\phi) = V_0^Y(\phi) = E^Q\big[X^Y\big] = E^Q\big[e^{-rT}X\big].$$

We can summarize this as follows.

Theorem 2.2. *The theoretical price v of a contingent T-claim $X \in L^2(Q)$ at time $t = 0$ is given by*

$$v = E^Q\big[e^{-rT}X\big],$$

where Q is defined by

$$dQ = e^{-\frac{1}{2}\alpha\lambda^2 + \alpha W_\lambda} dP$$

with $\lambda = \operatorname{ess\,sup} T$.

Moreover, the Q-dynamics for the price process S is given by

$$S_t = S_0 \, e^{(r-\sigma^2/2)\,t + \sigma W_t^\alpha}, \quad 0 \le t \le \lambda,$$

where $\{W_t^\alpha\}_{0 \le t \le \lambda}$ is a $(Q, \{\mathcal{F}_t\}_{0 \le t \le \lambda})$-Brownian motion.

2.6 Dividends

So far we have assumed that the the risky security pays no dividends. The same assumption was made in the original paper by Black and Scholes, but it is not difficult to extend the Black-Scholes theory to cover dividend paying securities as well.

There are several different ways to model dividends. In this section we will consider a model proposed by Samuelson in [Sa]. In Samuelson's model the dividends are paid out continuously at a rate which is proportional to the asset price. To be more specific, if we let the random variable D_t denote the total dividend amount paid out during the time interval $[0, t]$, then

$$D_t = \int_0^t q S_s ds, \quad 0 \le t \le \lambda,$$

where q is a constant. The constant q is known as the *dividend rate* or the *dividend yield.*

The model is applicable to options on foreign currencies (see [GK]) and commodities (see [Hull]) but not to stocks. The dividends to a stock are most often paid out at discrete times and consequently, the dividends process $\{D_t\}_{0 \le t \le \lambda}$ corresponding to a stock is not absolutely continuous. For further details about discrete dividends, see [HJ].

In Samuelson's model our previous definition of a self-financing portfolio will no longer be relevant, since we now receive dividends. A more appropriate definition would be to say that a trading strategy ϕ is *self-financing* if the corresponding value process $V(\phi)$, which is defined as before, satisfies

$$V_t(\phi) - V_0(\phi) = \int_0^t \phi_s^0 dB_s + \int_0^t \phi_s^1 dS_s + \int_0^t \phi_s^1 dD_s, \ 0 \le t \le \lambda.$$

Thus, in a self-financing portfolio the only external funds invested in the portfolio come from the dividend payments, which are, on the other hand, used in full.

From this definition one can proceed as in the previous sections. The modified definitions and computations are straightforward, we obtain the following generalization of Theorem 2.2.

Theorem 2.3. *If the underlying asset pays dividends at a constant rate q, then theoretical price v of a contingent T-claim $X \in L^2(Q)$ at time $t = 0$ is given by*

$$v = E^Q\big[e^{-rT}X\big],$$

where Q is defined by

$$dQ = e^{-\frac{1}{2}\alpha_q\lambda^2 + \alpha_q W_\lambda} dP \tag{2.8}$$

with $\lambda = \operatorname{ess\,sup} T$ and $\alpha_q = (r - q - \eta)/\sigma$.

Moreover, the Q-dynamics for the price process S is given by

$$S_t = S_0\, e^{(r-q-\sigma^2/2)\,t + \sigma W_t^{\alpha_q}}, \quad 0 \le t \le \lambda,$$

where $W_t^{\alpha_q}$ is a $(Q, \{\mathcal{F}_t\}_{0\le t\le\lambda})$-Brownian motion.

2.7 The Black-Scholes Formula

In this section we will consider two important examples of contingent claims, namely the European call option and the European put option. The call option is already defined in Section 2.2. A European put option with strike price K and time of maturity T, where K and T are constants, entitles the holder to *sell* one share of the risky security S at the expiration date T at the prespecified price K. Thus, at maturity the put value equals

$$\max(K - S_T, 0).$$

Our next project is to derive the theoretical value of the European call and put option. First however, we will introduce some definitions and prove a lemma that will be frequently used in this work.

Define a process S^μ by setting

$$S_t^\mu = S_0 e^{\mu t + \sigma W_t}, \quad 0 \le t \le T,$$

where $\mu = r - q - \sigma^2/2$.

Lemma 2.1. *Let $\tilde{P}$ be a measure on $(\Omega, \mathcal{F})$ defined by*

$$d\tilde{P} = e^{-\frac{1}{2}\sigma^2 T + \sigma W_T} dP,$$

where T and σ are positive constants. Then, given $K \in \mathbb{R}$ and $A \in \mathcal{F}_T$,

$$e^{-rT}E\big[(S_T^\mu - K)1_A\big] = S_0 e^{-qT}\tilde{P}(A) - Ke^{-rT}P(A).$$

Moreover,

$$\tilde{P}(A) = P(A - \varsigma), \tag{2.9}$$

where $A - \varsigma = \{\omega \in \Omega\,;\, \omega + \varsigma \in A\}$ and the function ς is given by $\varsigma(t) = \sigma t,\ t \ge 0$.

Proof. The first part of the lemma follows at once from the definition of $\tilde{P}$. To prove equation (2.9), let $\tilde{W}_{(\cdot)}(\omega) = \omega(\cdot) - \varsigma(\cdot)$. Note that

$$\tilde{P}(A) = \tilde{P}(\tilde{W} \in A - \varsigma).$$

According to Appendix A $\{\tilde{W}_t\}_{0 \leq t \leq T}$ is a Brownian motion with respect to $(\tilde{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$, which yields

$$\tilde{P}(\tilde{W} \in A - \varsigma) = P(W \in A - \varsigma)$$

since $A - \varsigma \in \mathcal{F}_T$. It is evident that $P(W \in A - \varsigma) = P(A - \varsigma)$ and consequently,

$$\tilde{P}(A) = P(A - \varsigma).$$

□

In what follows we let Φ denote the standard normal distribution function, i.e.

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}}.$$

Now we can establish the theoretical value of a European call and put option.

Theorem 2.4. *The theoretical value v_c of a European call option with strike price K and time to expiration T is given by*

$$v_c = S_0 e^{-qT} \Phi(d_1) - K e^{-rT} \Phi(d_2), \tag{2.10}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The theoretical value v_p of a European put option with strike price K and time to expiration T is given by

$$v_p = K e^{-rT} \Phi(-d_2) - S_0 e^{-qT} \Phi(-d_1). \tag{2.11}$$

In the special case $q = 0$ the price formula in equation (2.10) is called the Black-Scholes formula.

Proof. Let us begin with the call option. From Theorem 2.3 we get that

$$\begin{aligned} v_c &= E^Q\big[e^{-rT}\max(S_T-K,0)\big] \\ &= e^{-rT}E\big[\,(S_T^{\mu}-K)1_A\,\big]\,, \end{aligned}$$

where

$$A=\{\mu T+\sigma W_T\geq k\}$$

with $k=\ln(K/S_0)$. The preceding lemma combined with the scaling property for Brownian motion now yields

$$\begin{aligned} v_c =& S_0e^{-qT}P\big(\,(\mu+\sigma^2)\,T+\sigma W_T\geq k\,\big)-Ke^{-rT}P\big(\,\mu T+\sigma W_T\geq k\,\big) \\ =& S_0e^{-qT}P\big(\,\frac{(\mu+\sigma^2)\sqrt{T}}{\sigma}+W_1\geq\frac{k}{\sigma\sqrt{T}}\,\big) \\ &-Ke^{-rT}P\big(\,\frac{\mu\sqrt{T}}{\sigma}+W_1\geq\frac{k}{\sigma\sqrt{T}}\,\big) \\ =& S_0e^{-qT}P(\,W_1\geq -d_1\,)-Ke^{-rT}P(\,W_1\geq -d_2\,), \end{aligned}$$

where d_1 and d_2 are defined as above. The symmetry of the normal distribution now gives the desired result.

Next consider the put option and note that

$$\begin{aligned} v_p &= e^{-rT}E^Q\big[\max(K-S_T,0)\big] \\ &= -e^{-rT}E\big[\,(S_T^{\mu}-K)1_B\,\big]\,, \end{aligned}$$

where $B=\{\mu T+\sigma W_T\leq k\}$. If we proceed as in the previous case we obtain

$$\begin{aligned} v_p &= Ke^{-rT}P\big(\,\mu T+\sigma W_T\leq k\,\big)-S_0e^{-qT}P\big(\,(\mu+\sigma^2)\,T+\sigma W_T\leq k\,\big) \\ &= Ke^{-rT}P(\,W_1\leq -d_2\,)-S_0e^{-qT}P(\,W_1\leq -d_1\,) \\ &= Ke^{-rT}\Phi(-d_2)-S_0e^{-qT}\Phi(-d_1). \end{aligned}$$

□

2.8 Barrier Options

The term barrier option refers to an option with a payoff depending on whether or not the underlying asset price is above or below a prespecified barrier (or barriers) during the time the option is alive. In the next two subsections we will give a more precise definition of those barrier options which will be treated in this work. Finally, in the third subsection we will briefly discuss certain other barrier options.

2.8.1 Knock-out and Knock-in Call/Put Options

To begin with we will consider *knock-out* options. The special feature of a knock-out option is that it will be extinguished ('knocked-out') if the underlying asset price breaches some barrier (or barriers) prior to the expiration date. The class of all knock-out options can be subdivided into two different subclasses, depending on the number of barriers of the option. In the first mentioned subclass we have mainly four different contracts, namely *up-and-out call*, *up-and-out put*, *down-and-out call* and *down-and-out put option*. The word 'up' refers to the fact that the asset price must travel *up*wards in order to hit the barrier, i.e. the barrier is initially above the asset price, while 'down' means that the barrier can only be hit from above. In the subclass of knock-out options with two barriers, usually referred to as *double-barrier* knock-out options, we find the *double-barrier knock-out call* and the *double-barrier knock-out put*.

The payoff of an up-and-out call option with strike price K, barrier H and expiry date T, where K, T and H are constants, is

$$\max(S_T - K, 0)1_{\{\max_{t\in[0,T]} S_t < H\}}.$$

In other words, at maturity T an up-and-out call option has the same value as a plain European call option provided that the underlying asset price never breaches the barrier during the lifetime of the option. An up-and-out put has the payoff

$$\max(K - S_T, 0)1_{\{\max_{t\in[0,T]} S_t < H\}}$$

at maturity T.

The down-and-out option depends on the minimum of the asset price. Especially we have that the value of a down-and-out call equals

$$\max(S_T - K, 0)1_{\{\min_{t\in[0,T]} S_t > H\}}$$

at maturity T. The definition of a down-and-out put option is straight forward.

The double barrier knock-out option depends on both the minimum and the maximum of the asset price. The payoff at the maturity date of a double barrier knock-out call, with barriers H_l and H_u, equals

$$\max(S_T - K, 0)1_{\{\min_{t\in[0,T]} S_t > H_l, \max_{t\in[0,T]} S_t < H_u\}}.$$

Now the payoff at maturity of a double barrier knock-out put is evident.

Next we turn our attention to so called *knock-in* options. A knock-in option will expire without value if the barrier (barriers) is (are) *never* breached by the underlying asset during the lifetime of the option. For instance the holder of an *up-and-in call* option will receive, at the maturity date T, the amount

$$\max(S_T - K, 0)1_{\{\max_{t\in[0,T]} S_t \geq H\}}.$$

In a similar way one also defines the barrier options *up-and-in put, down-and-in call/put* and *double barrier knock-in call/put options.*

So far we have assumed that the underlying security price is continuously monitored against the barrier. However, interestingly enough, for many traded barrier options the barrier is monitored only at specific dates. These options are usually referred to as *discrete* barrier options. For instance a *discrete* up-and-out call possesses the payoff

$$\max(S_T - K, 0)1_{\{\max_{t\in M} S_t < H\}}$$

at the maturity T, where the set $M = \{t_1, t_2, \dots, t_m\}$ is referred to as the *monitoring dates* or *price-fixing dates.* The price-fixing dates are mostly equidistant in time, that is $M = \{\Delta t, 2\Delta t, \dots, m\Delta t\}$ where $\Delta t = T/m$ for some positive integer m. Discrete barrier options are sometimes also referred to as barrier options with a discrete barrier. In a similar way, barrier options, where the underlying security is continuously monitored against the barrier, are sometimes referred to as a barrier option with a *continuous* barrier or as a *continuous* barrier option.

2.8.2 Rebate or Binary Barrier Options

Rebate options are often combined with knock-out or knock-in call/ put options. The purpose of adding rebate options to knock-out and knock-in options is to compensate for the loss that occurs when the knock-out option is 'knocked out' or the knock-in option is never 'knocked in'.

To begin with we will define the rebate option belonging to a knock-out option. A holder of this contract will receive a prespecified positive amount R provided that the underlying asset price crosses the barrier before or at the maturity date of the knock-out option. The pay out will occur at the same time as the barrier is crossed. If the barrier never is hit the rebate option will expire without value.

Thus the payoff of the rebate option belonging to a continuous up-and-out or down-and-out option with barrier H and time of expiration T can be written as

$$R\,1_{\{\tau(H)\leq T\}},$$

where

$$\tau(H) = \begin{cases} \inf\{t > 0\,;\, S_t > H\} & \text{if } S_0 < H, \\ \inf\{t > 0\,;\, S_t < H\} & \text{if } S_0 > H. \end{cases}$$

That is, if $t = 0$, $\tau(H)$ is the time elapsed until, if ever[1], the asset price hits the barrier H.

For a double barrier knock-out option with barriers H_l and H_u the corresponding rebate option will pay

$$R\,1_{\{\,\tau(H_l)\,\wedge\,\tau(H_u)\,\leq\,T\,\}}$$

at time $\tau(H_l) \wedge \tau(H_u)$, where $(\cdot) \wedge (*) = \min((\cdot), (*))$.

A rebate option must not necessarily be combined with a knock-out option and in this case the rebate option is usually called a *binary* (or *digital*) *knock-in option* or an *American binary* (or *digital*) *option*. Here the term binary or digital is used since the payoff is either a fixed amount or nothing at all.

The rebate option belonging to a knock-in option will instead pay out a prespecified positive amount R only if the underlying does not breach the barrier before the expiry date T of the knock-in option. The payoff at maturity is therefore

$$R\,1_{\{\tau(H)>T\}}$$

in the single barrier case, while if there are two barriers the terminal payoff equals

$$R\,1_{\{\,\tau(H_l)\,\wedge\,\tau(H_u)\,>\,T\,\}}.$$

In both cases the payoff takes place at time T. The options are sometimes also referred to as *binary* (or *digital*) *knock-out options*.

There are discrete versions of the rebate options as well. The definitions of these options are straightforward from the definitions of the continuous counterparts, i.e. the payoffs at maturity are obtained by replacing $\tau(H)$ by $\tau_M(H)$, where $\tau_M(H)$ is defined by

$$\tau_M(H) = \begin{cases} \inf\{t \in M\,;\, S_t > H\} & \text{if } S_0 < H, \\ \inf\{t \in M\,;\, S_t < H\} & \text{if } S_0 > H. \end{cases}$$

2.8.3 Some Other Barrier Options

The barrier options we have described so far are perhaps the most traded barrier options, but they are far from the only ones. For example, *partial barrier options* have lately grown in popularity. For such barrier options the barrier is only activated during some period of the option's. For further discussion about these options we refer to [HK] and [HK2].

A barrier option with a time dependent barrier is called a *moving barrier option*. Pricing formulas of barrier options with an exponential barrier are

[1] We use the convention $\inf \emptyset = \infty$.

discussed in [R] and [KI]. More general techniques to price continuous moving barrier options are treated in [RZ] while discrete moving barrier options can be priced using an algorithm presented in this report (see Chapter 5).

A barrier option depending on two underlying assets is called a *two-asset barrier option*. The barrier crossing event in a two-asset barrier option is often triggered by one of the assets while the final pay out depends on the other asset. Two-asset barrier options are examined in [Ch].

Other interesting barrier options are the so called *Parisian barrier options*. These options are "delayed" barrier options based on the age of excursion of the underlying price process beyond a given barrier. The Parisian up-and-out call for instance, will expire without value if the underlying price process raises above the barrier and stays continuously above the barrier for a time interval longer than a specified delay. Parisian options were introduced in [CJY] and [CCJ].

2.9 Why Barrier Options?

As already mentioned above, barrier options, especially knock-out and knock-in call/put options, have become increasingly popular in the last few years. The reason for this is simply that knock-out and knock-in call/put options are cheaper than the corresponding contracts without any barriers. If an investor finds it unlikely that the underlying asset will fall below a certain price level, it is natural to buy a knock-out option with the barrier at that same level. The difference in price between the knock-out option and the ordinary option can be substantial, especially when the volatility is high. Thus, using barrier options, investors can avoid paying for the scenarios they feel are unlikely.

However, these benefits may imply a risk. Barrier options can be very sensitive to price changes of the underlying asset. For instance, consider an up-and-out call option and suppose that the underlying price is just beneath the barrier and that the option is close to maturity. If a small short term price spike occurs, the option will expire without value. On the other hand, if the asset price remains constant until maturity the option can become very valuable. Thus, investing in barrier options and, especially, knock-out options is sometimes combined with a large risk. For a more comprehensive treatment about this topic we refer to [Li] and the references therein.

Chapter 3

Analytical Formulas for the Value of Continuous Barrier Options

3.1 Introduction

In [M] Merton gives the first known price formula of a barrier option considering a continuous down-and-out call option with zero rebate written on a non-dividend paying security. Subsequently Cox and Rubinstein, [CR], extended the down-and-out call formula to include rebate as well. A complete list for the value of all continuous single barrier options with rebate has recently been presented in [RR] and [R].

Several people have also analysed the continuous double barrier option. Kunitomo and Ikeda, [KI], calculated its value using the Levy formula[1]. Hui, [Hui2], solved the pricing problem using separation of variables, Pelsser, [P], derived the value with the help of contour integration while Geman and Yor in [GY] employed a method based on the Laplace transform. The rebate options corresponding to a double barrier option have been examined in [Hui] and [P].

To the best of our knowledge, with the exception of the formula for the value of the rebate option corresponding to a double barrier knock-out option, all results presented in this chapter have been published earlier. The aim with this chapter is to present, in a unified framework, many previously published price formulas for continuous barrier options. We will also discuss numerical characteristics of the formulas obtained.

The presentation is based on three classical results from probability theory, namely the Cameron-Martin's theorem, the strong Markov property for Brownian motion and the optional sampling theorem. For a reader not fa-

[1] The Levy formula is a formula for the transition law of a Brownian motion with two absorbing boundaries.

miliar with these results, we have two appendices in the end of this work which treat the Cameron-Martin's theorem and the strong Markov property. For a description of the optional sampling theorem we recommend [KS] or [D].

The remainder of this chapter is structured as follows. In the next section we will derive analytical expressions for certain distributions which involve stopping times associated with a Brownian motion. With the aid of these results we will in Section 3 determine the value of all barrier options with zero rebate. Finally, in Section 4, we will derive the value of the rebate options.

3.2 Hitting and Exit Times

In this section we will compute certain laws involving the first hitting time and the first exit time of a Brownian motion with drift. We let the probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be defined as in Section 2.2. If $b \in \mathbb{R}$ the first hitting time of b, hereafter denoted $\lambda(b)$, is defined by

$$\lambda(b) = \inf\{t > 0\,;\, W_t = b\}.$$

It can be proved that $\lambda(b)$ is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ ([KS], p.7). A random time Λ of the form

$$\Lambda = \inf\{t > 0\,;\, W_t \notin (b_1, b_2)\}, \quad b_1 < 0 < b_2,$$

is usually referred to as the first exit time of the open interval (b_1, b_2). Note that Λ can also be written as $\Lambda = \lambda(b_1) \wedge \lambda(b_2)$.

Moreover, we introduce a collection of probability measures $\{P^\theta\,;\, \theta \in \mathbb{R}\}$ given by

$$P^\theta(A) = E\big[e^{-\frac{1}{2}\theta^2 + \theta W_1} 1_A\big],$$

defined for all $A \in \mathcal{F}$. According to Cameron-Martin's theorem we have that the process W^θ, given by

$$W_t^\theta = W_t - \theta t, \quad 0 \leq t \leq 1,$$

is a Brownian motion with respect to $(P^\theta, \{\mathcal{F}_t\}_{0\leq t\leq 1})$. This fact will frequently be used in the sequel.

The subject of the first part of this section is the following distribution,

$$G_+(a, b_1, b_2; \theta) = P^\theta(\, W_1 \leq a,\, \lambda(b_2) < \lambda(b_1),\, \lambda(b_2) \leq 1\,),$$

defined for $b_1 < 0 < b_2$, $a \leq b_2$ and all $\theta \in \mathbb{R}$. The key result in this section is the following lemma, the proof of which is based on an idea described in [An].

Lemma 3.1. *Suppose $b_1 < 0 < b_2$, $a \leq b_2$ and $\theta \in \mathbb{R}$. Set $\lambda^{(0)} = 0$ and $\rho^{(0)} = 0$ and define recursively the stopping times*

$$\lambda^{(n)} = \inf\{t > \rho^{(n-1)}\,;\, W_t = b_2\}$$

and

$$\rho^{(n)} = \inf\{t > \lambda^{(n-1)}\,;\, W_t = b_1\}$$

for $n \geq 1$. Then for any $n \geq 1$

$$\begin{aligned} G_+(a, b_1, b_2; \theta) = &\sum_{i=1}^{n} \big(P^\theta(\, W_1 \leq a,\, \lambda^{(2i-1)} \leq 1\,) - P^\theta(\, W_1 \leq a,\, \lambda^{(2i)} \leq 1\,)\big) \\ &+ P^\theta(\, W_1 \leq a,\, \lambda^{(2n+1)} \leq 1,\, \lambda^{(1)} < \rho^{(1)}\,). \end{aligned} \tag{3.1}$$

Proof. Firstly, let $A = \{\lambda^{(1)} < \rho^{(1)}\}$ and $B_n = \{\lambda^{(n)} \leq 1\}$. Note that for all $\omega \in A^c$ we have $\lambda^{(1)}(\omega) = \lambda^{(2)}(\omega)$, which implies $\rho^{(2)}(\omega) = \rho^{(3)}(\omega)$, which in turn implies $\lambda^{(3)}(\omega) = \lambda^{(4)}(\omega)$ and so forth. Hence, by induction on n it can be shown that for all $\omega \in A^c$ we have $\lambda^{(2n-1)}(\omega) = \lambda^{(2n)}(\omega)$ for any $n \geq 1$, and, accordingly from this

$$1_{B_{2n-1}} 1_{A^c} = 1_{B_{2n}} 1_{A^c} \tag{3.2}$$

for every $n \geq 1$. By a similar argument one can prove that for all $\omega \in A$ and any $n \geq 1$ we have $\lambda^{(2n)} = \lambda^{(2n+1)}$, and, hence,

$$1_{B_{2n}} 1_A = 1_{B_{2n+1}} 1_A \tag{3.3}$$

for every $n \geq 1$. Next observe that for any given sets C_1 and C_2 we have

$$1_{C_1} 1_{C_2} = 1_{C_1} - 1_{C_1} 1_{C_2^c}. \tag{3.4}$$

Successive applications of the equations (3.2), (3.3) and (3.4) yield

$$\begin{aligned} 1_{B_1} 1_A &= 1_{B_1} - 1_{B_1} 1_{A^c} \\ &= 1_{B_1} - 1_{B_2} 1_{A^c} \\ &= 1_{B_1} - 1_{B_2} + 1_{B_2} 1_A \\ &= 1_{B_1} - 1_{B_2} + 1_{B_3} 1_A \\ &\vdots \\ &= 1_{B_1} - 1_{B_2} + 1_{B_3} - \ldots - 1_{B_{2n}} + 1_{B_{2n+1}} 1_A \end{aligned}$$

for any $n \geq 1$. By integrating both sides over $\{W_1 \leq a\}$ with respect to P^θ we obtain equation (3.1). □

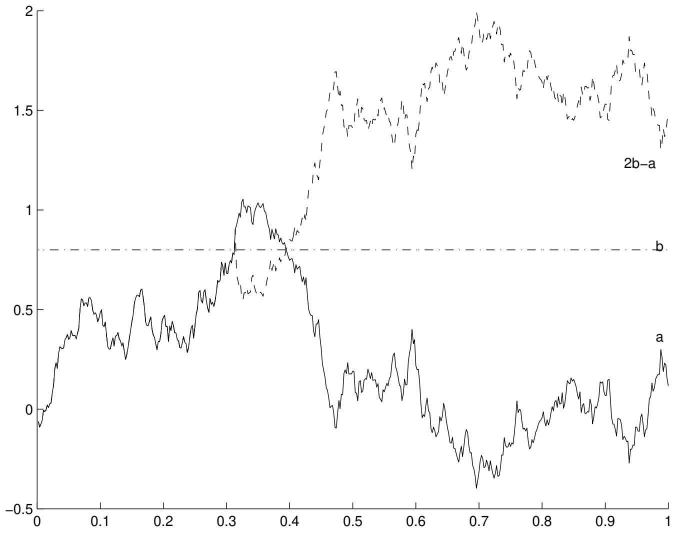

Figure 3.1: Two paths of a Brownian motion.

To evaluate the terms in the sum in equation (3.1) we will use the *reflection principle*. This principle can be described as follows. Consider a Brownian path that crosses the barrier b before time 1 and is below a at time 1, where $a \leq b$ and $b > 0$ (an example of such a path is the solid line in Figure 3.1). Due to the symmetry with respect to b of a Brownian motion starting at b, the probability of this event is the same as the probability of travelling from b to a point above $2b - a$ (the dashed line in Figure 3.1 is the Brownian path reflected in b). The argument above makes the following plausible,

$$P(\,W_1 < a,\, \lambda(b) < 1\,) = P(\,W_1 > 2b - a, \lambda(b) < 1\,),$$

since $\{W_1 > 2b - a\} \subset \{\lambda(b) < 1\}$ we now get

$$P(\,W_1 < a,\, \lambda(b) < 1\,) = P(\,W_1 > 2b - a\,).$$

Hence

$$P(\,W_1 \leq a,\, \lambda(b) \leq 1\,) = \Phi(a - 2b) \tag{3.5}$$

because $P(W_1 = a) = 0$ for any $a \in \mathbb{R}$.

Of course, equation (3.5) still requires a rigorous proof. In the next lemma we will give proof of a generalization of the reflection principle based on the strong Markov property for Brownian motion.

Lemma 3.2. *Let $\lambda^{(i)}$ be defined as in Lemma 3.1 and suppose $a \le b_2$. For any $i \ge 1$ we have*

$$P(\, W_1 \le a,\, \lambda^{(2i-1)} \le 1\,) = \Phi(a - 2\alpha_1^{(i)}) \tag{3.6}$$

and

$$P(\, W_1 \le a,\, \lambda^{(2i)} \le 1\,) = \Phi(a - 2\alpha_2^{(i)}), \tag{3.7}$$

where $\alpha_1^{(i)} = i(b_2 - b_1) + b_1$ and $\alpha_2^{(i)} = i(b_2 - b_1)$.

Proof. We follow [KS], p.95 and p.98. Let the operator U be defined as in Appendix B. Firstly, fix a positive number $t \le 1$ and note that the symmetry of Brownian motion implies

$$\begin{aligned} \big(U_{1-t}1_{(-\infty,a]}\big)(b) &= P(\, b + W_{1-t} \le a\,) \\ &= P(\, b + W_{1-t} \ge 2b - a\,) \\ &= \big(U_{1-t}1_{[2b-a,\infty)}\big)(b) \end{aligned} \tag{3.8}$$

for any real numbers a and b.

The stopping time $\tau = \lambda^{(2i-1)} \wedge 1$ is obviously bounded for any $i \ge 1$. The strong Markov property in combination with equation (3.8) now implies for $\omega \in \{\lambda^{(2i-1)} < 1\}$,

$$\begin{aligned} E\big[1_{\{W_1 \le a\}} \mid \mathcal{F}_\tau\big](\omega) &= \big(U_{1-\tau(\omega)}1_{(-\infty,a]}\big)\big(W_{\tau(\omega)}(\omega)\big) \\ &= \big(U_{1-\tau(\omega)}1_{(-\infty,a]}\big)(b_2) \\ &= \big(U_{1-\tau(\omega)}1_{[2b_2-a,\infty)}\big)(b_2) \\ &= \big(U_{1-\tau(\omega)}1_{[2b_2-a,\infty)}\big)\big(W_{\tau(\omega)}(\omega)\big) \\ &= E\big[1_{\{W_1 \ge 2b_2 - a\}} \mid \mathcal{F}_\tau\big](\omega). \end{aligned}$$

By integrating over $\{\tau < 1\} = \{\lambda^{(2i-1)} < 1\}$ we see that

$$P(\, W_1 \le a,\, \lambda^{(2i-1)} < 1\,) = P(\, W_1 \ge 2b_2 - a,\, \lambda^{(2i-1)} < 1\,). \tag{3.9}$$

Note that $\{\lambda^{(2i-1)} = 1\} \subset \{W_1 = b_2\}$ and thus $P(\,\lambda^{(2i-1)} = 1\,) = 0$. In combination with equation (3.9) this yields

$$P(\, W_1 \le a,\, \lambda^{(2i-1)} \le 1\,) = P(\, W_1 \ge 2b_2 - a,\, \lambda^{(2i-1)} \le 1\,).$$

Since $b_2 \le 2b_2 - a$ we find

$$\{\, W_1 \ge 2b_2 - a,\, \lambda^{(2i-1)} \le 1\,\} = \{\, W_1 \ge 2b_2 - a,\, \rho^{(2i-2)} \le 1\,\}$$

and therefore

$$P(\,W_1 \le a,\, \lambda^{(2i-1)} \le 1\,) = P(\,W_1 \ge 2b_2 - a,\, \rho^{(2i-2)} \le 1\,). \tag{3.10}$$

Using the fact that $2b_2 - a \ge b_1$ and the symmetry of Brownian motion, equation (3.10) gives

$$\begin{aligned} P(\,W_1 \ge 2b_2 - a, \rho^{(2i-2)} \le 1\,) \\ = P(\,W_1 \le 2(b_1 - b_2) + a,\, \lambda^{(2i-3)} \le 1\,) \end{aligned} \tag{3.11}$$

for $i \ge 2$. The equation (3.6) now follows by induction on i.

By replacing $\lambda^{(2i-1)}$ by $\lambda^{(2i)}$ and $\rho^{(2i-2)}$ by $\rho^{(2i-1)}$ in the equations (3.10) and (3.11) we also get (3.7). □

Next we will extend Lemma 3.2 to the case when the drift $\theta \neq 0$.

Lemma 3.3. *Let $\lambda^{(i)}$ be defined as in Lemma 3.1 and suppose $a \le b_2$. For any $i \ge 1$ and any $\theta \in \mathbb{R}$ we have*

$$P^\theta(\,W_1 \le a,\, \lambda^{(2i-1)} \le 1\,) = e^{2\theta\alpha_1^{(i)}}\Phi(a - 2\alpha_1^{(i)} - \theta) \tag{3.12}$$

and

$$P^\theta(\,W_1 \le a,\, \lambda^{(2i)} \le 1\,) = e^{2\theta\alpha_2^{(i)}}\Phi(a - 2\alpha_2^{(i)} - \theta) \tag{3.13}$$

where $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ are defined as in Lemma 3.2.

Proof. Fix an integer $j \ge 1$. Observe that

$$\begin{aligned} P^\theta(\,W_1 \le a,\, \lambda^{(j)} \le 1\,) =& E\big[\exp(-\frac{1}{2}\theta^2 + \theta W_1) 1_{\{W_1 \le a,\, \lambda^{(j)} \le 1\}}\big] \\ =& \int_{-\infty}^{a} \exp(-\frac{1}{2}\theta^2 + \theta x) P(W_1 \in dx,\, \lambda^{(j)} \le 1). \end{aligned}$$

It is now convenient to introduce a constant α, defined by

$$\alpha = \begin{cases} \alpha_1^{(i)},\ i = (j+1)/2, & \text{if } j \text{ is an odd number,} \\ \alpha_2^{(i)},\ i = j/2, & \text{if } j \text{ is an even number,} \end{cases}$$

From Lemma 3.2 we get

$$\frac{d}{dx} P(\,W_1 \le x,\, \lambda^{(j)} \le 1\,) = \varphi(x - 2\alpha),$$

where $\varphi(x) = \frac{d}{dx}\Phi(x)$. Thus

$$
\begin{aligned}
P^\theta(\, W_1 \le a,\, \lambda^{(j)} \le 1\,) =& \int_{-\infty}^{a} \exp(-\frac{1}{2}\,\theta^2 + \theta x)\,\varphi(x-2\alpha)dx \\
=& \int_{-\infty}^{a} \exp\big(-\frac{1}{2}\,\theta^2 + \theta x - \frac{1}{2}(x-2\alpha)^2\big)\frac{dx}{\sqrt{2\pi}} \\
=& \int_{-\infty}^{a} \exp\big(2\theta\alpha - \frac{1}{2}(x-2\alpha-\theta)^2\big)\frac{dx}{\sqrt{2\pi}} \\
=& e^{2\theta\alpha}\int_{-\infty}^{a} \varphi(x-2\alpha-\theta)dx \\
=& e^{2\theta\alpha}\,\Phi(a-2\alpha-\theta),
\end{aligned}
$$

which proves the lemma. □

Next we will focus on the remainder term

$$
P^\theta(\, W_1 \le a,\, \lambda^{(2n+1)} \le 1,\, \lambda^{(1)} < \rho^{(1)}\,)
$$

in the expression of G_+, given in equation (3.1). Observe that

$$
\begin{aligned}
P^\theta(\, W_1 \le a,\, \lambda^{(2n+1)} \le 1,\, \lambda^{(1)} < \rho^{(1)}\,) &\le P^\theta(\, W_1 \le a,\, \lambda^{(2n+1)} \le 1\,) \\
&= e^{2\theta\alpha_1^{(n+1)}}\,\Phi(a-2\alpha_1^{(n+1)}-\theta),
\end{aligned}
$$

according to equation (3.12). Moreover, from [KS], p.112, we get that the following inequality

$$
\Phi(x) \le -\frac{1}{x}e^{-x^2/2}
$$

is valid for any $x < 0$. Thus the remainder term is bounded by

$$
\begin{aligned}
&P^\theta(\, W_1 \le a, \lambda^{(2n+1)} \le 1,\, \lambda^{(1)} < \rho^{(1)}\,) \\
&\qquad \le \frac{-1}{a-2\alpha_1^{(n+1)}-\theta}\,\exp\big(2\theta\alpha_1^{(n+1)} - \frac{(a-2\alpha_1^{(n+1)}-\theta)^2}{2}\big)
\end{aligned}
$$

if $a < 2\alpha_1^{(n+1)} + \theta$, which implies that, given $\delta < 2$,

$$
P^\theta(W_1 \le a,\, \lambda_1^{(2n+1)} \le 1,\, \lambda^{(1)} < \rho^{(1)}) = o(e^{-n^\delta})
$$

as n tends to infinity. This result and Lemmas 3.1 and 3.3 imply the next lemma.

Lemma 3.4. *Suppose $b_1 < 0 < b_2$, $a \le b_2$ and $\delta < 2$. If $\alpha_1^{(i)} = i(b_2 - b_1) + b_1$ and $\alpha_2^{(i)} = i(b_2 - b_1)$ then*

$$G_+(a, b_1, b_2; \theta) = \sum_{i=1}^{n} \left(e^{2\alpha_1^{(i)}\theta} \Phi(a - 2\alpha_1^{(i)} - \theta) - e^{2\alpha_2^{(i)}\theta} \Phi(a - 2\alpha_2^{(i)} - \theta) \right) + R_{n+1}$$

where $R_{n+1} = o(e^{-n^\delta})$, $n \to \infty$, or more precisely

$$|R_{n+1}| \le e^{2\theta\alpha_1^{(n+1)}} \Phi(a - 2\alpha_1^{(n+1)} - \theta).$$

Lemma 3.4 shows how the formula for the distribution G_+ should be implemented in order to be able to control the truncation error. Set

$$p_1^{(i)} = e^{2\alpha_1^{(i)}\theta}\, \Phi(a - 2\alpha_1^{(i)} - \theta) \quad \text{and} \quad p_2^{(i)} = e^{2\alpha_2^{(i)}\theta}\, \Phi(a - 2\alpha_2^{(i)} - \theta).$$

Lemma 3.4 now yields

$$\Big|\, G_+(a, b_1, b_2; \theta) - \sum_{\{i\,;\, p_1^{(i)} \ge \epsilon\}} (p_1^{(i)} - p_2^{(i)}) \,\Big| < \epsilon$$

Thus, if the desired accuracy is set to ϵ, then one has to add the terms $(p_1^{(i)} - p_2^{(i)})$, $i = 1, 2, \ldots$ until $p_1^{(i)} < \epsilon$. The result will then have the desired accuracy.

The remaining part of this section is devoted to introduce and to determine certain transition distributions that will be useful in the sequel. First recall that for any $\theta \in \mathbb{R}$,

$$G_+(a, b_1, b_2; \theta) = P^\theta(\, W_1 \le a,\, \lambda(b_2) < \lambda(b_1), \lambda(b_2) \le 1\,)$$

if $b_1 < 0 < b_2$ and $a \le b_2$. Now let for any $\theta \in \mathbb{R}$,

$$G_-(a, b_1, b_2; \theta) = P^\theta(\, W_1 \ge a,\, \lambda(b_1) < \lambda(b_2),\, \lambda(b_1) \le 1\,)$$

if $b_1 < 0 < b_2$ and $a \ge b_1$. Furthermore, given $b_1 < 0 < b_2$, set

$$G_{1+}(b_1, b_2; \theta) = P^\theta(\, \lambda(b_2) < \lambda(b_1),\, \lambda(b_2) \le 1\,),$$

$$G_{1-}(b_1, b_2; \theta) = P^\theta(\, \lambda(b_1) < \lambda(b_2),\, \lambda(b_1) \le 1\,)$$

and

$$G_2(a_1, a_2, b_1, b_2; \theta) = P^\theta(\, a_1 < W_1 \le a_2,\, \lambda(b_1) \wedge \lambda(b_2) > 1\,)$$

if $b_1 \le a_1 \le a_2 \le b_2$. These distributions can be expressed with the aid of the function G_+, as we will show in the next lemma.

Lemma 3.5. *Let the functions G_-, G_{1+}, G_{1-} and G_2 be defined as above. Then*

$$
\begin{aligned}
G_-(a, b_1, b_2; \theta) &= G_+(-a, -b_1, -b_2; -\theta)\,, \\
G_{1+}(b_1, b_2; \theta) &= \Phi(\theta - b_2) \\
&\quad + G_+(b_2, b_1, b_2; \theta) - G_-(b_2, b_1, b_2; \theta)\,, \\
G_{1-}(b_1, b_2; \theta) &= G_{1+}(-b_1, -b_2; -\theta)
\end{aligned}
$$

and

$$
\begin{aligned}
G_2(a_1, a_2, b_1, b_2; \theta) &= \Phi(a_2 - \theta) - \Phi(a_1 - \theta) \\
&\quad - G_+(a_2, b_1, b_2; \theta) + G_+(a_1, b_1, b_2; \theta) \\
&\quad - G_-(a_2, b_1, b_2; \theta) + G_-(a_1, b_1, b_2; \theta).
\end{aligned} \tag{3.14}
$$

Proof. The expression for G_- follows at once from the symmetry of Brownian motion.

Below, we let $\lambda_1 = \lambda(b_1)$ and $\lambda_2 = \lambda(b_2)$. To prove the second equation, note that

$$
\begin{aligned}
G_{1+}(b_1, b_2; \theta) =& P^\theta(\, W_1 \le b_2,\, \lambda_2 < \lambda_1,\, \lambda_2 \le 1\,) \\
&+ P^\theta(\, W_1 > b_2,\, \lambda_2 < \lambda_1,\, \lambda_2 \le 1\,) \\
=& G_+(b_2, b_1, b_2; \theta) \\
&+ P^\theta(\, W_1 > b_2,\, \lambda_2 < \lambda_1\,)
\end{aligned} \tag{3.15}
$$

since $\{W_2 > b_2\} \subset \{\lambda_2 \le 1\}$. It is obvious that $P^\theta(\, W_1 > b_1,\, \lambda_1 = \lambda_2) = 0$ and, accordingly from this,

$$
\begin{aligned}
P^\theta(\, W_1 > b_2,\, \lambda_2 < \lambda_1\,) =& P^\theta(\, W_1 > b_2\,) \\
&- P^\theta(\, W_1 > b_2,\, \lambda_1 < \lambda_2\,) \\
=& P^\theta(\, W_1 > b_2\,) \\
&- P^\theta(\, W_1 > b_2,\, \lambda_1 < \lambda_2,\, \lambda_1 \le 1\,)
\end{aligned} \tag{3.16}
$$

because $\{\lambda_1 < \lambda_2\} \cap \{W_1 > b_2\} \subset \{\lambda_1 \le 1\}$. The equations (3.15) and (3.16) now yield the expression for G_{1+}. The expression for G_{1-} follows from symmetry.

It remains to determine G_2. Observe that

$$
\begin{aligned}
G_2(a_1, a_2, b_1, b_2; \theta) =& P^\theta(\, a_1 < W_1 \le a_2\,) \\
&- P^\theta(\, a_1 < W_1 \le a_2,\, \lambda_1 \wedge \lambda_2 \le 1\,) \\
=& P^\theta(\, W_1 \le a_2) - P^\theta(W_1 \le a_1\,) \\
&- P^\theta(\, W_1 \le a_2,\, \lambda_1 \wedge \lambda_2 \le 1\,) \\
&+ P^\theta(\, W_1 \le a_1,\, \lambda_1 \wedge \lambda_2 \le 1\,).
\end{aligned}
$$

Equation (3.14) now follows from the fact that

$$\begin{aligned}P^\theta(\, W_1 \le a,\, \lambda_1 \wedge \lambda_2 \le 1\,) =& P^\theta(\, W_1 \le a,\, \lambda_2 < \lambda_1,\, \lambda_2 \le\, 1) \\ &+ P^\theta(\, W_1 \le a,\, \lambda_1 < \lambda_2,\, \lambda_1 \le 1\,)\end{aligned}$$

for any number a. $\square$

Next consider the following distributions,

$$F_+(a, b; \theta) = P^\theta(\, W_1 \le a, \lambda(b) >\, 1),$$

where $a \le b$ and $b > 0$, and

$$F_-(a, b; \theta) = P^\theta(\, W_1 \ge a, \lambda(b) >\, 1),$$

where $a \ge b$ and $b < 0$.

Lemma 3.6. *If F_+ and F_- are defined as above, then*

$$F_+(a, b; \theta) = \Phi(a - \theta) - e^{2b\theta}\Phi(a - 2b - \theta)$$

and

$$F_-(a, b; \theta) = F_+(-a, -b; -\theta).$$

Proof. From equation (3.12) with $i = 1$ we have that

$$P^\theta(\, W_1 \le a,\, \lambda(b) \le 1\,) = e^{2b\theta}\Phi(a - 2b - \theta).$$

Thus

$$\begin{aligned}F_+(a, b; \theta) =& P^\theta(\, W_1 \le a,\, \lambda(b) > 1\,) \\ =& P^\theta(\, W_1 \le a\,) - P^\theta(\, W_1 \le a,\, \lambda(b) \le\, 1) \\ =& \Phi(a - \theta) - e^{2b\theta}\Phi(a - 2b - \theta).\end{aligned}$$

The last part of the lemma follows from symmetry. $\square$

3.3 Barrier Options with Zero Rebate

The purpose of this section is to calculate the theoretical value of continuous barrier options with zero rebate. The rebate options will be treated in the next section.

Because the market is assumed to be free of arbitrage, the following relation must hold

$$v = v_{ki} + v_{ko},$$

where v denotes the theoretical value of a (call/put) option and $[v_{ki}/v_{ko}]$ denotes the value of a [knock-in/knock-out] (call/put) option with zero rebate and with the same option parameters as the (call/put) option. Moreover, the barrier options are presumed to have the barriers at the same level. Since the theoretical values of calls and puts are known (cf Theorem 2.4) it is enough to solely price knock-in *or* knock-out options. We will henceforth focus on knock-out options.

Next we will make some comments about notation. If nothing else is stated we will, throughout this chapter, use the same notation as in Chapter 2. This means that the constants K, R and T denote strike price, rebate and time of expiration, respectively. If the barrier option has only one barrier, it will be denoted by H. Moreover, for a double barrier option, we will denote the lower barrier by H_l and the upper barrier by H_u. Finally, r, q and σ denote the risk free interest rate, dividend yield, and the volatility of the underlying security, respectively.

In what follows it will always be assumed that the knock-out option under consideration has not been knocked-out at time zero. For example, if we consider a down-and-out option, we will implicitly assume that $S_0 > H$ while if we consider an up-and-out option we will take for granted that $S_0 < H$. For double barrier options it will always be assumed that the inequalities $H_l < S_0 < H_u$ are satisfied.

Recall from Chapter 2 that the process S^μ is defined by

$$S_t^\mu = S_0 e^{\mu t + \sigma W_t}, \quad 0 \le t \le T,$$

where $\mu = r - q - \sigma^2/2$. Recall moreover that the measure $\tilde{P}$ is given by

$$\tilde{P}(A) = E\big[e^{-\frac{1}{2}\sigma^2 T + \sigma W_T} 1_A\big], \quad A \in \mathcal{F}.$$

In what follows, let

$$\tau^\mu(H) = \inf\{t > 0\,;\, S_t^\mu = H\}.$$

The next lemma will bridge the gap between Lemma 2.1 and the results obtained in the previous section.

Lemma 3.7. *Let χ be a constant equal to 1 or -1. For any strictly positive numbers T, K, H_l and H_u, which satisfy $H_l < S_0 < H_u$, the following holds*

$$\begin{aligned} &P(\,\chi S_T^\mu \le \chi K, \tau^\mu(H_l) > T, \tau^\mu(H_u) > T\,) \\ &\qquad = P^{\theta_0}(\,\chi W_1 \le \chi c,\, \lambda(d_l) > 1,\, \lambda(d_u) > 1\,), \end{aligned} \tag{3.17}$$

where $\theta_0 = \frac{\mu\sqrt{T}}{\sigma}$, $c = f(K)$, $d_l = f(H_l)$ and $d_u = f(H_u)$ with

$$f(x) = \frac{\ln(x/S_0)}{\sigma\sqrt{T}}\,, \quad x > 0.$$

Furthermore,

$$\tilde{P}(\,\chi S_T^{\mu} \leq \chi K, \tau^{\mu}(H_l) > T, \tau^{\mu}(H_u) > T\,)$$
$$= P^{\theta_1}(\,\chi W_1 \leq \chi c,\ \lambda(d_l) > 1,\ \lambda(d_u) > 1\,), \tag{3.18}$$

where $\theta_1 = \frac{(\mu+\sigma^2)\sqrt{T}}{\sigma}$.

Proof. The scaling property for Brownian motion (see [KS], p.104) implies that the stochastic process

$$\{\sqrt{T}\,W(\frac{t}{T})\}_{t\geq 0}$$

is a Brownian motion under P.

Set $k = \ln(K/S_0)$, $h_l = \ln(H_l/S_0)$, $h_u = \ln(H_u/S_0)$ and let $\eta \in \mathbb{R}$. The scaling property and Cameron-Martin's theorem now yield

$$P(\,\chi(\eta T + \sigma W_T) \leq \chi k,\ \min_{0\leq t\leq T}(\eta t + \sigma W_t) > h_l,\ \max_{0\leq t\leq T}(\eta t + \sigma W_t) < h_u\,)$$
$$= P(\,\chi(\theta + W_1) \leq \chi c,\ \min_{0\leq t\leq 1}(\theta t + W_t) > d_l\ \max_{0\leq t\leq 1}(\theta t + W_t) < d_u\,)$$
$$= P^{\theta}(\,\chi W_1 \leq \chi c,\ \lambda(d_l) > 1,\ \lambda(d_u) > 1\,), \tag{3.19}$$

where c, d_l and d_u are defined as above and where

$$\theta = \frac{\eta\sqrt{T}}{\sigma}.$$

Equation (3.17) now follows by setting $\eta = \mu$ in equation (3.19). From Lemma 2.1 we see that if we replace η by $\mu+\sigma^2$ in equation (3.19) we obtain equation (3.18). $\square$

We are now in the position to establish the theoretical value of all barrier options. We will begin with the single barrier options.

Theorem 3.1. *Let $d = f(H)$, where the function f is defined as in Lemma 3.7, and let c, θ_0 and θ_1 be defined as in Lemma 3.7.*

The theoretical value v_{doc} at time $t = 0$ of a down-and-out call is given by

$$v_{doc} = S_0 e^{-qT} F_-(\max(c,d), d; \theta_1) - K e^{-rT} F_-(\max(c,d), d; \theta_0). \tag{3.20}$$

If $K > H$ then the theoretical value v_{dop} at time $t = 0$ of a down-and-out put can be written as

$$v_{dop} = K e^{-rT}\big(F_-(d,d;\theta_0) - F_-(c,d;\theta_0)\big)$$
$$- S_0 e^{-qT}\big(F_-(d,d;\theta_1) - F_-(c,d;\theta_1)\big)\,. \tag{3.21}$$

If $K \leq H$, $v_{dop} = 0$.

If $K < H$ *then the theoretical value* v_{uoc} *at time* $t = 0$ *of an up-and-out call can be expressed as*

$$\begin{aligned} v_{uoc} = S_0 e^{-qT} \big(F_+(d,d;\theta_1) - F_+(c,d;\theta_1) \big) \\ - K e^{-rT} \big(F_+(d,d;\theta_0) - F_+(c,d;\theta_0) \big) . \end{aligned} \tag{3.22}$$

If $K \geq H$, $v_{uoc} = 0$.

The theoretical value v_{uop} *at time* $t = 0$ *of an up-and-out put option is given by*

$$v_{uop} = K e^{-rT} F_+(\min(c,d), d; \theta_0) - S_0 e^{-qT} F_+(\min(c,d), d; \theta_1). \tag{3.23}$$

Proof. Let us begin with the down-and-out call option. According to Theorem 2.3 and Lemma 2.1 we have

$$\begin{aligned} v_{doc} =& E^Q \big[e^{-rT} \max(S_T - K, 0) 1_{\{ \min_{t\in[0,T]} S_t > H \}} \big] \\ =& e^{-rT} E \big[\max(S_T^\mu - K, 0) 1_{\{ \min_{t\in[0,T]} S_t^\mu > H \}} \big] \\ =& e^{-rT} E \big[(S_T^\mu - K) 1_{\{ S_T^\mu \geq K, \min_{t\in[0,T]} S_t^\mu > H \}} \big] \\ =& S_0 e^{-qT} \tilde{P}(S_T^\mu \geq K,\, \tau^\mu(H) > T) \\ & - K e^{-rT} P(S_T^\mu \geq K,\, \tau^\mu(H) > T). \end{aligned}$$

By letting $H_u \to \infty$ in equation (3.17) we obtain

$$\begin{aligned} P(S_T^\mu \geq K,\, \tau^\mu(H) > T) =& P^{\theta_0}(W_1 \geq c,\, \lambda(d) > 1) \\ =& F_-(\max(c,d), d; \theta_0). \end{aligned}$$

Similarly, by letting $H_u \to \infty$ in equation (3.18) we get

$$\tilde{P}(S_T^\mu \geq K,\, \tau^\mu(H) > T) = F_-(\max(c,d), d; \theta_1),$$

which gives equation (3.20). The theoretical value v_{uop} of an up-and-out put option can be obtained in a similar way. In fact,

$$\begin{aligned} v_{uop} =& E^Q \big[e^{-rT} \max(K - S_T, 0) 1_{\{\max_{t\in[0,T]} S_t < H\}} \big] \\ =& e^{-rT} E \big[(K - S_T^\mu) 1_{\{ S_T^\mu \leq K, \max_{t\in[0,T]} S_t^\mu < H \}} \big] \\ =& K e^{-rT} P(S_T^\mu \leq K,\, \tau^\mu(H) > T) \\ & - S_0 e^{-qT} \tilde{P}(S_T^\mu \leq K,\, \tau^\mu(H) > T) \\ =& K e^{-rT} F_+(\min(c,d), d; \theta_0) - S_0 e^{-qT} F_+(\min(c,d), d; \theta_1). \end{aligned}$$

Next we consider the down-and-out put option. From Theorem 2.3 and Lemma 2.1 we have that the theoretical value for this option equals

$$
\begin{aligned}
v_{dop} =& E^Q\left[e^{-rT}\max(K-S_T,0)1_{\{\min_{t\in[0,T]} S_t > H\}}\right] \\
=& Ke^{-rT}P(\, S_T^\mu \le K,\, \tau^\mu(H) > T\,) - S_0 e^{-qT}\tilde{P}(\, S_T^\mu \le K,\, \tau^\mu(H) > T\,) \\
=& Ke^{-rT}P^{\theta_0}(W_1 \le c,\, \lambda(d) > 1\,) - S_0 e^{-qT} P^{\theta_1}(W_1 \le c,\, \lambda(d) > 1\,).
\end{aligned}
$$

Equation (3.21) now follows since, for every $\theta \in \mathbb{R}$,

$$
\begin{aligned}
P^\theta(W_1 \le c,\, \lambda(d) > 1\,) =& P^\theta(W_1 \ge d,\, \lambda(d) > 1\,) \\
& - P^\theta(W_1 \ge c,\, \lambda(d) > 1\,) \qquad (3.24)\\
=& F_-(d,d;\theta) - F_-(c,d;\theta),
\end{aligned}
$$

provided $c > d$, or equivalently, $K > H$. If $K \le H$ then it follows at once that $v_{dop} = 0$.

Equation (3.22) can be derived in the same fashion as equation (3.21). □

The next theorem gives a closed price formula for the value of a knock-out double barrier option.

Theorem 3.2. *Let c, d_l, d_u, θ_0 and θ_1 be defined as in Lemma 3.7.*

If $K < H_u$ then the theoretical value v_{koc} at time $t = 0$ of a double-barrier knock-out call option is given by

$$
\begin{aligned}
v_{koc} =& S_0 e^{-qT} G_2(\max(c,d_l), d_u, d_l, d_u; \theta_1) \\
& - Ke^{-rT} G_2(\max(c,d_l), d_u, d_l, d_u; \theta_0)\,.
\end{aligned}
$$

If $K \ge H_u$, $v_{koc} = 0$.

If $K > H_l$ the theoretical value v_{kop} of a double barrier knock-out put option at time $t = 0$ is given by

$$
\begin{aligned}
v_{kop} =& Ke^{-rT} G_2(d_l, \min(c,d_u), d_l, d_u; \theta_0) \\
& - S_0 e^{-qT} G_2(d_l, \min(c,d_u), d_l, d_u; \theta_1)\,.
\end{aligned}
$$

If $K \le H_l$, $v_{kop} = 0$.

Proof. The result follows immediately from Lemma 2.1 and Lemma 3.7. □

We will conclude this section with a discussion about different representations for the value of a double barrier option. It is possible to represent the value of a double barrier option in at least two different ways, namely either as Fourier sine series or as series of standard normal distributions, as above. The Fourier solution can be obtained in several different ways. The perhaps most straightforward method would be to solve the *Black-Scholes partial differential equation* (see [MR] p.118), with corresponding boundary condition using separation of variables. This method has been discussed in [Hui2]. Another way to establish this solution is by employing contour integration (see [P]).

The numerical characteristics for the two different solutions have been compared in [HLY]. In that paper it is recommended that in a trading system, the solution involving standard normal distributions should be used, that is, the solution given in Theorem 3.2 where G_2 is computed with the aid of Lemmas 3.4 and 3.5. The argument is that cancellation errors can appear in the Fourier series which may lead to substantial errors in the resulting theoretical value.

3.4 Rebate Options

In this final section of this chapter we will calculate the theoretical value of continuous rebate options. Let us first consider the rebate option belonging to a down-and-in barrier option, that is, a binary knock-out option with $S_0 > H$. The theoretical value v_{bko} of this option is, according to the discussion in Chapter 1, given by

$$\begin{aligned} v_{bko} &= E^Q\left[e^{-rT} R 1_{\{\tau(H)>T\}}\right] \\ &= Re^{-rT} P(\tau^\mu(H) > T), \end{aligned}$$

where τ^μ is defined as above, that is, $\tau^\mu(H) = \inf\{t > 0\,;\, S_t^\mu = H\}$ where $S_t^\mu = S_0 e^{\mu t+\sigma W_t}$ with $\mu = r - q - \sigma^2/2$. Set, as above, $\theta_0 = \mu\sqrt{T}/\sigma$ and $d = f(H)$, where f is defined as in Lemma 3.7. By applying Lemma 3.7 we get

$$\begin{aligned} v_{bko} &= Re^{-rT} P^{\theta_0}(\lambda(d) > 1) \\ &= Re^{-rT}\, F_-(d, d; \theta_0). \end{aligned}$$

We now turn our attention to the rebate option corresponding to a down-and-out option, that is, the binary knock-in option with $S_0 > H$. The payoff at maturity of this contract is, according to the discussion in Chapter 1, given by

$$R 1_{\{\tau(H)\le T\}}.$$

The pay out will occur at the same time the barrier is reached, which is at time $\tau(H)$. Since $\tau(H)$ is not a bounded stopping time we see that this contract does not fit into the theory that was developed in the previous chapter (cf Definition 2.1). One can however easily get around this problem by noting that the claim above is, from a financial point of view, equivalent with a claim X that pays $X = R1_{\{\tau(H)\leq T\}}$ at time Σ, where $\Sigma = \tau(H) \wedge T$. The random variable X is $\mathcal{F}_\Sigma$ measurable and since Σ is bounded we have that X is a contingent Σ-claim. Thus, the theoretical value of a binary knock-in option (where the barrier is below the initial asset price) equals at time $t = 0$

$$\begin{aligned} v_{bki} &= E^Q\big[Re^{-r\Sigma}\,1_{\{\tau(H)\leq T\}}\big] \\ &= E^Q\big[Re^{-r\tau(H)}\,1_{\{\tau(H)\leq T\}}\big] \\ &= RE\big[e^{-r\tau^\mu(H)}\,1_{\{\tau^\mu(H)\leq T\}}\big]. \end{aligned} \tag{3.25}$$

In order to calculate the above expectation, the next lemma will be useful.

Lemma 3.8. *Let $T > 0$, $\eta \in \mathbb{R}$ and let Z be a martingale with respect to $(P, \{\mathcal{F}_t\}_{0\leq t\leq T})$. If Λ is a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\Lambda \leq T$ P-a.s. and X is a bounded $\mathcal{F}_\Lambda$ measurable random variable, then*

$$E\big[XZ_\Lambda\big] = E\big[XZ_T\big].$$

Proof. Since $\Lambda \leq T$ P-a.s., the optional sampling theorem (see [KS], p.19) gives

$$E\big[Z_T \mid \mathcal{F}_\Lambda\big] = Z_\Lambda.$$

Moreover,

$$XE\big[Z_T \mid \mathcal{F}_\Lambda\big] = E\big[XZ_T \mid \mathcal{F}_\Lambda\big]$$

because X is $\mathcal{F}_\Lambda$ measurable and bounded, and therefore

$$E\big[XZ_T \mid \mathcal{F}_\Lambda\big] = XZ_\Lambda.$$

By integration the desired result follows at once. $\square$

As above, let $h = \ln(H/S_0)$. From the scaling property for a Brownian motion we get for any $t \in [0, T]$,

$$\begin{aligned} P(\,\tau^\mu(H) \leq t\,) &= P\big(\min_{0\leq s\leq t}(\mu s + \sigma W_s) \leq h\big) \\ &= P\big(\min_{0\leq s\leq t/T}(\theta_0 s + W_s) \leq d\big) \\ &= P^{\theta_0}(\,\lambda(d) \leq t/T\,), \end{aligned} \tag{3.26}$$

where, as before, $\theta_0 = \mu\sqrt{T}/\sigma$ and $d = f(H)$. The equations (3.25) and (3.26) yield that the value of a binary knock-in option at time $t = 0$ equals

$$v_{bki} = R\,E^{\theta_0}\big[e^{-rT\lambda}\,1_{\{\lambda\leq 1\}}\big],$$

where $\lambda = \lambda(d)$ and E^{θ_0} denotes expectation with respect to P^{θ_0}.

Now we introduce the stopping time $\Lambda = \lambda \wedge 1$. This stopping time is obviously bounded and satisfies

$$e^{-rT\lambda} 1_{\{\lambda \leq 1\}} = e^{-rT\Lambda} 1_{\{\Lambda < 1\}} \quad P\text{-a.s.}$$

and, hence,

$$v_{bki} = R\,E^{\theta_0}\big[e^{-rT\Lambda} 1_{\{\Lambda<1\}}\big].$$

The random variable $e^{-rT\Lambda} 1_{\{\Lambda<1\}}$ is $\mathcal{F}_\Lambda$ measurable and bounded. Lemma 3.8 together with the fact that the process $\{\exp(-\frac{1}{2}\theta_0^2 t + \theta_0 W_t)\}_{t\geq 0}$ is a $(P, \{\mathcal{F}_t\}_{t\geq 0})$-martingale gives

$$\begin{aligned}
v_{bki} &= R\,E\big[\,e^{-\frac{1}{2}\theta_0^2+\theta_0 W_1}\,e^{-rT\Lambda} 1_{\{\Lambda<1\}}\big] \\
&= R\,E\big[\,e^{-\frac{1}{2}\theta_0^2\Lambda+\theta_0 W_\Lambda - rT\Lambda} 1_{\{\Lambda<1\}}\big] \\
&= R\,E\big[\,e^{-\frac{1}{2}\theta_2^2\Lambda+\theta_2 W_\Lambda + (\theta_0-\theta_2)W_\Lambda} 1_{\{\Lambda<1\}}\big],
\end{aligned}$$

where $\theta_2 = \sqrt{\theta_0^2 + 2rT}$. Note that $W_{\Lambda(\omega)}(\omega) = d$ if $\omega \in \{\Lambda < 1\}$. We now apply Lemma 3.8 once more and conclude that

$$\begin{aligned}
v_{bki} &= R\,\exp((\theta_0-\theta_2)\,d)\,E\big[e^{-\frac{1}{2}\theta_2^2\Lambda+\theta_2 W_\Lambda} 1_{\{\Lambda<1\}}\big] \\
&= R\,\exp((\theta_0-\theta_2)\,d)\,P^{\theta_2}(\Lambda < 1).
\end{aligned}$$

Moreover,

$$P^{\theta_2}(\Lambda < 1) = P^{\theta_2}(\lambda \leq 1) = 1 - F_-(d, d, \theta_2)$$

and we finally have that

$$v_{bki} = R\exp((\theta_0-\theta_2)\,d)(1 - F_-(d, d; \theta_2)).$$

The values of all the other rebate options with one barrier are given in the following theorem.

Theorem 3.3. *The theoretical value v_{bko} at time $t = 0$ of a binary knock-out option with one barrier is given by*

$$v_{bko} = \begin{cases} Re^{-rT} F_+(d, d; \theta_0) & \text{if } S_0 < H, \\ Re^{-rT} F_-(d, d; \theta_0) & \text{if } S_0 > H, \end{cases}$$

where d and θ_0 are defined as in Theorem 3.1 and Lemma 3.7, respectively.

The theoretical value v_{bki} at time $t = 0$ of a binary knock-in option with one barrier is given by

$$v_{bki} = \begin{cases} R\exp\big((\theta_0 - \theta_2)\, d\big)\, \big(1 - F_+(d, d, \theta_2)\big) & \text{if } S_0 < H, \\ R\exp\big((\theta_0 - \theta_2)\, d\big)\, \big(1 - F_-(d, d, \theta_2)\big) & \text{if } S_0 > H, \end{cases}$$

where $\theta_2 = \sqrt{\theta_0^2 + 2rT}$.

Proof. The value of the rebate options corresponding to an up-and-in and up-and-out option follows at once from the computations above. □

We will now finish this chapter with a theorem which gives the value of rebate options corresponding to double barrier options.

Theorem 3.4. *The theoretical value v^d_{bko} at time $t = 0$ of a double-barrier binary knock-out option is given by*

$$v^d_{bko} = Re^{-rT}\, G_2(d_l, d_u, d_l, d_u; \theta_0),$$

where d_l, d_u and θ_0 are defined as in Lemma 3.7.

The theoretical value v^d_{bki} at time $t = 0$ of a double-barrier binary knock-in option is given by

$$\begin{aligned} v^d_{bki} &= R\, \exp(\,(\theta_0 - \theta_2)\, d_l)\, G_{1-}(d_l, d_u; \theta_2) \\ &\quad + R\, \exp(\,(\theta_0 - \theta_2)\, d_u)\, G_{1+}(d_l, d_u; \theta_2), \end{aligned}$$

where θ_2 is defined as in Theorem 3.3.

Proof. The first part of the theorem follows at once from Lemma 3.7. To prove the second part, let $\tau_l = \tau(H_l)$ and $\tau_u = \tau(H_u)$. We have

$$\begin{aligned} v^d_{bki} &= E^Q\big[R\exp\big(-r\,(\tau_l \wedge \tau_u)\big)\, 1_{\{\,\tau_l \wedge \tau_u \le T\,\}}\big] \\ &= R\, E^Q\big[\exp(-r\,\tau_l)\, 1_{\{\,\tau_l < \tau_u,\, \tau_l \le T\,\}}\big] \qquad (3.27) \\ &\quad + R\, E^Q\big[\exp(-r\,\tau_u)\, 1_{\{\,\tau_u < \tau_l,\, \tau_u \le T\,\}}\big]\,. \end{aligned}$$

Let for simplicity $\lambda_1 = \lambda(d_l)$ and $\lambda_2 = \lambda(d_u)$. It can easily be shown that (cf equation (3.26))

$$E^Q\big[\exp(-r\,\tau_l)\, 1_{\{\,\tau_l < \tau_u,\, \tau_l \le T\,\}}\big] = E^{\theta_0}\big[\exp(-r\,T\lambda_1)\, 1_{\{\,\lambda_1 < \lambda_2,\, \lambda_1 \le 1\,\}}\big]\,,$$

where E^{θ_0} denotes expectation with respect to P^{θ_0}. Set $\Lambda_1 = \lambda_1 \wedge 1$. It is evident that

$$E^{\theta_0}\big[e^{-rT\lambda_1}\, 1_{\{\lambda_1 \le 1,\, \lambda_1 < \lambda_2\}}\big] = E^{\theta_0}\big[e^{-rT\Lambda_1}\, 1_{\{\Lambda_1 < 1,\, \Lambda_1 < \lambda_2\}}\big].$$

From [KS], p.8, we have that $\{\Lambda_1 < \lambda_2\} \in \mathcal{F}_{\Lambda_1} \cap \mathcal{F}_{\lambda_2}$, which implies $\{\Lambda_1 < 1,\, \Lambda_1 < \lambda_2\} \in \mathcal{F}_{\Lambda_1}$. Thus the random variable $e^{-rT\Lambda_1}\, 1_{\{\Lambda_1<1,\,\Lambda_1<\lambda_2\}}$ is $\mathcal{F}_{\Lambda_1}$ measurable. By applying Lemma 3.8 twice we obtain

$$
\begin{aligned}
E^{\theta_0}\big[e^{-rT\Lambda_1}\, 1_{\{\Lambda_1<1,\,\Lambda_1<\lambda_2\}}\big] &= E\big[e^{-rT\Lambda_1-\frac{1}{2}\theta_0^2\,\Lambda_1+\theta_0 W_{\Lambda_1}}\, 1_{\{\Lambda_1<1\,\Lambda_1<\lambda_2\}}\big] \\
&= E\big[e^{-\frac{1}{2}\theta_2^2\,\Lambda_1+\theta_2 W_{\Lambda_1}+(\theta_0-\theta_2)W_{\Lambda_1}}\, 1_{\{\Lambda_1<1,\,\Lambda_1<\lambda_2\}}\big] \\
&= \exp((\theta_0-\theta_2)\, d_l)\, P^{\theta_2}(\Lambda_1 < 1,\, \Lambda_1 < \lambda_2) \\
&= \exp((\theta_0-\theta_2)\, d_l)\, P^{\theta_2}(\lambda_1 \leq 1,\, \lambda_1 < \lambda_2) \\
&= \exp((\theta_0-\theta_2)\, d_l)\, G_{1-}(d_l, d_u; \theta_2).
\end{aligned}
$$

With a similar method one can also calculate the remaining term in equation (3.27) and from this we get the value of a double-barrier binary knock-in option. □

Chapter 4

Pricing Discrete Barrier Options Using Siegmund's Corrected Heavy Traffic Approximation

4.1 Introduction

In contrast to continuous barrier options the price of a discrete barrier option does not in general possess a closed form price formula. The price can be expressed in terms of the multivariate normal distribution. Here the dimension of the relevant multivariate normal distribution is equal to the number of price fixing dates, which, in most cases, is too large for numerical evaluation.

Therefore, to price discrete barrier options it is natural to employ numerical algorithms or approximation formulas. In this chapter we will focus on approximation formulas, while numerical algorithms will be treated in the next chapter.

Approximation methods have been discussed earlier in the literature. One technique which has given remarkably good results was first proposed by Chuang, [Ch], and, independently, by Broadie, Glasserman and Kou ([BGK] and [BGK2]). The method they developed is based on a result from sequential analysis and queue theory, namely "Siegmund's corrected heavy traffic approximation", which is useful to estimate the joint distribution of a random walk and its maximum.

Broadie et al. derived formulas for some discrete (single) barrier options, but not all. The purpose of this chapter is to continue the work initiated by Broadie et al. and determine approximation formulas for all discrete (single) barrier options.

We will make the same assumptions as in the papers mentioned above by Broadie et al. Thus it will be assumed that the underlying asset can be described as in Section 2.2 and that the price fixing dates are equally spaced in time. We will also use the same notation as in Chapter 2. This means that K, T and H denote strike price, time of expiration and barrier, respectively. The constant m stands for the number of monitoring dates. Finally, we denote r, q and σ the continuously compounded interest rate, continuous dividend yield, and the volatility of the underlying asset, respectively.

The chapter is structured as follows. In Section 4.2 we will present the results of Broadie et al. as well as our main result of this chapter. The latter result will be proved in Sections 4.3 and 4.4, where we will also describe "Siegmund's corrected heavy traffic approximation". In the last section, Section 4.5, we will present some numerical examples.

4.2 Extending the Corrected Barrier Approximation of Broadie, Glasserman and Kou

The most naive approach to approximate the value of a discrete barrier option would be to ignore the fact that the barrier is discrete and price the option as a continuous barrier option with the same barrier. However, numerical examples show that this method can lead to substantial mispricings (see [BGK]) even in the case of daily monitoring. In the paper [BGK] by Broadie et al. it was shown that this simple approximation can be improved just by shifting the barrier. The next theorem is taken from [BGK].

Theorem 4.1. *Let $v^d(H)$ be the price of a discretely monitored knock-in or knock-out down call or up put with barrier H. Let $v(H)$ be the price of the corresponding continuously monitored barrier option. Then, as $m \to \infty$,*

$$v^d(H) = v(He^{\pm\beta\sigma\sqrt{T/m}}) + o(\frac{1}{\sqrt{m}})$$

where $+$ applies if $H > S_0$,$-$ applies if $H < S_0$, and $\beta = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$, with ζ the Riemann zeta function.

Theorem 4.1 can be interpreted as follows. If the price formula of a continuous barrier option is used to value a discrete barrier option, first move the barrier away from the initial asset price by a factor $\exp(\pm\beta\sigma\sqrt{T/m})$, where $+$ applies if $H > S_0$ and $-$ applies if $H < S_0$. Numerical results presented in the same paper indicate that the shift of the barrier gives surprisingly good approximations for moderate to large values of m if the initial asset price is not too close to the barrier. In addition, the suggested approximation has the great benefits of being easy to implement and fast to evaluate[1].

[1] more than 10.000 valuations per second on an Intel Pentium 133 MHz processor according to [BGK].

The proof of Theorem 4.1 in [BGK] was based on certain results developed in an article written by Sigmund and Yuh, see [SY]. In this chapter we will use the same results to determine approximation formulas for discrete down-and-out/in put and up-and-out/in call options. Note that none of these options are included in Theorem 4.1.

Before we present our main result of this chapter, we recall that F_+ denotes the following distribution function

$$F_+(a,b;\theta) = P(\,\theta + W_1 \leq a,\ \max_{0\leq t\leq 1}(\theta t + W_t) \leq b\,),$$

where $a \leq b$, $b > 0$ and $\theta \in \mathbb{R}$. Moreover, if $a \geq b$, $b < 0$ and $\theta \in \mathbb{R}$ then we let

$$F_-(a,b;\theta) = P(\,\theta + W_1 \geq a,\ \min_{0\leq t\leq 1}(\theta t + W_t) \geq b\,).$$

According to the previous chapter we have that, given $a \leq b$ and $b > 0$,

$$F_+(a,b;\theta) = \Phi(a-\theta) - e^{2b\theta}\Phi(a-2b-\theta)$$

and, given $a \geq b$ and $b < 0$,

$$F_-(a,b;\theta) = F_+(-a,-b;-\theta).$$

Theorem 4.2. *Let v^d_{uoc} be the theoretical value at time $t = 0$ of a discretely monitored up-and-out call option. Then, as $m \to \infty$,*

$$\begin{aligned} v^d_{uoc} =& S_0 e^{-qT}\big(F_+(d, d+\beta/\sqrt{m};\theta_1) - F_+(c, d+\beta/\sqrt{m};\theta_1)\big) \\ &- Ke^{-rT}\big(F_+(d, d+\beta/\sqrt{m};\theta_0) - F_+(c, d+\beta/\sqrt{m};\theta_0)\big) \\ &+ o(\frac{1}{\sqrt{m}}), \end{aligned}$$

provided $K < H$, where

$$c = \frac{ln(K/S_0)}{\sigma\sqrt{T}},\ d = \frac{ln(H/S_0)}{\sigma\sqrt{T}},\ \theta_0 = \frac{(r-q-\sigma^2/2)\sqrt{T}}{\sigma}$$

and $\theta_1 = \theta_0 + \sigma\sqrt{T}$. If $K \geq H$, $v^d_{uoc} = 0$.

For a corresponding down-and-out put option the theoretical value v^d_{dop} at time $t = 0$ is given by

$$\begin{aligned} v^d_{dop} =& Ke^{-rT}\big(F_-(d, d-\beta/\sqrt{m};\theta_0) - F_-(c, d-\beta/\sqrt{m};\theta_0)\big) \\ &- S_0 e^{-qT}\big(F_-(d, d-\beta/\sqrt{m};\theta_1) - F_-(c, d-\beta/\sqrt{m};\theta_1)\big) \\ &+ o(\frac{1}{\sqrt{m}}), \quad m \to \infty, \end{aligned}$$

provided $K > H$. If $K \leq H$, $v^d_{dop} = 0$.

Remarkably enough, one will not get the above approximations by simply shifting the barrier as in Theorem 4.1 (cf equations (3.21) and (3.22)).

The value of the corresponding knock-in options can now be approximated by using the relation

$$v_{ki}^d = v - v_{ko}^d,$$

where v denotes the theoretical value of a (call/put) option and $[v_{ki}^d/v_{ko}^d]$ denotes the value of a discrete [knock-in/knock-out] (call/put) option with zero rebate and with the same option parameters as the (call/put) option. Moreover, the barrier options are presumed to have the barrier at the same level.

We will prove Theorem 4.2 in the next two sections.

4.3 The Value of a Discrete Barrier Option

To begin with, consider a discrete up-and-out call option where the price fixing dates are given by the set

$$M = \{\Delta t,\, 2\Delta t, \dots, m\Delta t\}\ , \quad \Delta t = T/m.$$

The value v_{uoc}^d at time $t = 0$ for this option is, according to Theorem 2.3, given by

$$\begin{aligned} v_{uoc}^d =& E^Q\big[e^{-rT}\max(S_T - K, 0)\, 1_{\{\, S_t < H,\ t\in M\,\}}\big] \\ =& e^{-rT} E\big[\,(S_T^\mu - K)\, 1_{\{\, S_T^\mu > K,\, S_t^\mu < H,\ t\in M\,\}}\big], \end{aligned}$$

where, for each $t \in [0, T]$, $S_t^\mu = S_0 \exp(\mu t + \sigma W_t)$ and $\mu = r - q - \sigma^2/2$. Set $k = \ln(K/S_0)$ and $h = \ln(H/S_0)$. By applying Lemma 2.1 we obtain

$$\begin{aligned} v_{uoc}^d =& S_0 e^{-qT} \tilde{P}(\, S_T^\mu > K,\, \max_{t\in M} S_t^\mu < H\,) \\ & - K e^{-rT} P(\, S_T^\mu > K,\, \max_{t\in M} S_t^\mu < H\,) \\ =& S_0 e^{-qT} P\big(\,(\mu + \sigma^2)T + \sigma W_T > k,\, \max_{t\in M}\, \big((\mu+\sigma^2)\,t + \sigma W_t\big) < h\,\big) \\ & - K e^{-rT} P\big(\,\mu T + \sigma W_T > k,\, \max_{t\in M}\,(\mu t + \sigma W_t) < h\,\big). \end{aligned}$$

The scaling property for Brownian motion now yields for any $\eta \in \mathbb{R}$,

$$\begin{aligned} & P\big(\,\eta T + \sigma W_T > k,\, \max_{t\in M}\,(\eta t + \sigma W_t) < h\,\big) \\ & = P\Big(\,\frac{\eta\sqrt{T}}{\sigma} + W_1 > \frac{k}{\sigma\sqrt{T}},\ \max_{t\in M}\,\big(\,\frac{\eta t}{\sigma\sqrt{T}} + W(\frac{t}{T})\big) < \frac{h}{\sigma\sqrt{T}}\,\Big) \qquad (4.1) \\ & = P\big(\,\theta + W_1 > c,\, \max_{t\in A}\,(\theta t + W_t) < d\,\big), \end{aligned}$$

where

$$\theta = \frac{\eta\sqrt{T}}{\sigma}, \; c = \frac{k}{\sigma\sqrt{T}}, \; d = \frac{h}{\sigma\sqrt{T}}$$

and $A = \{1/m, 2/m, \dots, 1\}$. So if we replace η in equation (4.1) by $\mu + \sigma^2$ and μ, respectively, we see that the value of a discrete up-and-out call can be expressed as

$$\begin{aligned} v^d_{uoc} =& S_0 e^{-qT} P\big(\theta_1 + W_1 > c, \max_{t\in A}(\theta_1 t + W_t) < d\big) \\ & - K e^{-rT} P\big(\theta_0 + W_1 > c, \max_{t\in A}(\theta_0 t + W_t) < d\big), \end{aligned} \tag{4.2}$$

where

$$\theta_0 = \frac{\mu\sqrt{T}}{\sigma} \quad \text{and} \quad \theta_1 = \frac{(\mu+\sigma^2)\sqrt{T}}{\sigma} = \theta_0 + \sigma\sqrt{T}.$$

We will now introduce the discrete counterpart to F_+ (cf Section 4.2). Set for $b > 0$, $a < b$ and $\theta \in \mathbb{R}$,

$$F^{(m)}_+(a,b;\theta) = P\big(\theta + W_1 \le a, \max_{t\in A}(\theta t + W_t) \le b\big).$$

Note that, given $b > 0$, $a < b$,

$$P\big(\theta + W_1 > a, \max_{t\in A}(\theta t + W_t) < b\big) = F^{(m)}_+(b,b;\theta) - F^{(m)}_+(a,b;\theta). \tag{4.3}$$

In combination with equation (4.2) this gives the following theorem.

Theorem 4.3. *Let v^d_{uoc} be the theoretical value at time $t = 0$ of a discretely monitored up-and-out call option. Then*

$$\begin{aligned} v^d_{uoc} =& S_0 e^{-qT}\left(F^{(m)}_+(d,d;\theta_1) - F^{(m)}_+(c,d;\theta_1)\right) \\ & - K e^{-rT}\left(F^{(m)}_+(d,d;\theta_0) - F^{(m)}_+(c,d;\theta_0)\right), \end{aligned}$$

provided $K < H$, where c, d, θ_0 and θ_1 are defined as in Theorem 4.2. If $K \ge H$, $v^d_{uoc} = 0$.

We now turn our attention to discrete *down-and-out puts*. The theoretical value v^d_{dop} of this option is given by

$$v^d_{dop} = E^Q\big[e^{-rT}\max(K - S_T, 0)\,1_{\{S_t > H,\, t\in M\}}\big].$$

If we proceed as above, using Lemma 2.1 and the scaling property, we get

$$
\begin{aligned}
v^d_{dop} =& e^{-rT} E\big[\, (K - S^{\mu}_T)\, 1_{\{\, S^{\mu}_T < K,\, S^{\mu}_t > H,\ t\in M\,\}}\big] \\
=& K e^{-rT} P\big(\, \mu T + \sigma W_T < k,\ \min_{t\in M} (\mu t + \sigma W_t) > h\,\big) \\
&- S_0 e^{-qT} P\Big(\, (\mu+\sigma^2)T + \sigma W_T < k,\ \min_{t\in M} \big(\, (\mu+\sigma^2)t + \sigma W_t\big) > h\Big) \\
=& K e^{-rT} P\big(\theta_0 + W_1 < c,\ \min_{t\in A} (\theta_0 t + W_t) > d\big) \\
&- S_0 e^{-qT} P\big(\theta_1 + W_1 < c,\ \min_{t\in A} (\theta_1 t + W_t) > d\big).
\end{aligned}
$$

The symmetry of Brownian motion gives

$$
\begin{aligned}
&P(\,\theta + W_1 < a,\ \min_{t\in A}(\theta t + W_t) > b\,) \\
&\qquad\qquad = P(\,-\theta + W_1 > -a,\ \max_{t\in A}(-\theta t + W_t) < -b\,).
\end{aligned}
$$

for any numbers a and b. The results above in conjunction with equation (4.3) yield the following theorem.

Theorem 4.4. *Let v^d_{dop} be the theoretical value at time $t = 0$ of a discretely monitored down-and-out put option. Then*

$$
\begin{aligned}
v^d_{dop} =& K e^{-rT} \big(F^{(m)}_+(-d,-d;-\theta_0) - F^{(m)}_+(-c,-d;-\theta_0)\big) \\
&- S_0 e^{-qT} \big(F^{(m)}_+(-d,-d;-\theta_1) - F^{(m)}_+(-c,-d;-\theta_1)\big),
\end{aligned}
$$

provided $K > H$, where c, d, θ_0 and θ_1 are defined as in Theorem 4.2. If $K \le H$, $v^d_{dop} = 0$.

In the next section we will discuss an approximation of the function $F^{(m)}_+$.

4.4 Siegmund's Corrected Heavy Traffic Approximation

The following result is often referred to as *Siegmund's corrected heavy traffic approximation.*

Theorem 4.5. *If $b > 0$, $a \le b$ and $\theta \in \mathbb{R}$, then, as $m \to \infty$,*

$$
\begin{aligned}
&P(\,\theta\sqrt{m} + W_m < a\sqrt{m},\ \lambda^{\theta}_m < m\,) \\
&\qquad\qquad = e^{2\theta(b+\beta/\sqrt{m})}\Phi(a - 2(b+\beta/\sqrt{m}) - \theta) + o(\frac{1}{\sqrt{m}})
\end{aligned}
$$

where

$$\lambda_m^\theta = \inf\{n \in \mathbb{N};\ \frac{\theta n}{\sqrt{m}} + W_n > b\sqrt{m}\}$$

and where β *is defined as in Theorem 4.1.*

The generic term 'heavy traffic approximation' is taken from queue theory. Roughly speaking, the term refers to approximations of distributions which involves functionals of a random walk. Theorem 4.5 is such an approximation, originally proved in a more general setting but the formulation given here is enough for our purposes. For more details about this we recommend the reader to [SY] or [As]. For a proof of Theorem 4.5, see [Si2] p.220-224.

It would be of great interest to know more about the error term $o(1/\sqrt{m})$ in Siegmund's approximation. In several examples, the approximation error is small (see [Si] and [BGK]). We can also quote Asmussen "Numerical studies indicate that the ... approximation is superior to all other known" (see [As], p.276).

The value of the constant β originally comes from the following expression

$$\beta = \frac{E[\,W_{\lambda_+}^2\,]}{2E[W_{\lambda_+}]},$$

where $\lambda_+ = \inf\{n \in \mathbb{N};\ W_n > 0\}$. The result $\beta = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$ can be found in a paper by Chernoff (see [Che]). For additional information, see [Si2] or [Lo]. It can be shown that

$$\frac{E[\,W_{\lambda_+}^2\,]}{2E[W_{\lambda_+}]} = \lim_{m\to\infty} E[W_{\lambda_m^\theta} - b\sqrt{m}\,;\ \lambda_m^\theta < \infty]$$

for any $\theta \in \mathbb{R}$ (see [Si2], p.215). Thus, the constant β may be viewed as an approximation to the average of the amount by which the random walk $\{\theta n + W_n\}_{n\in\mathbb{N}}$ exceeds the boundary $b\sqrt{m}$ the first time the random walk is above the boundary.

Our aim is now to find an approximation of $F_+^{(m)}$ based on Theorem 4.5. Note that since $a \le b$ we have

$$P(\,\theta\sqrt{m} + W_m < a\sqrt{m},\ \lambda_m^\theta < m\,) = P(\,\theta\sqrt{m} + W_m < a\sqrt{m},\ \lambda_m^\theta \le m\,)\,.$$

Moreover, the scaling property yields

$$\begin{aligned} P(\,\theta\sqrt{m}+W_m &< a\sqrt{m},\ \lambda_m^\theta \le m\,) \\ &= P\Big(\,\theta\sqrt{m} + W_m < a\sqrt{m},\ \max_{n=1,\dots,m}\big(\frac{\theta n}{\sqrt{m}} + W_n\big) > b\sqrt{m}\Big) \\ &= P\big(\,\theta + W_1 < a,\ \max_{t\in A}(\theta t + W_t) > b\,\big). \end{aligned}$$

where, as before, $A = \{1/m, 2/m, \dots, 1\}$. Hence

$$\begin{aligned} P\big(\,\theta + W_1 < a,\, \max_{t\in A}(\theta t + W_t) > b\,\big) \\ = e^{2\theta(b+\beta/\sqrt{m})}\Phi(a - 2(b+\beta/\sqrt{m}) - \theta) + o(\frac{1}{\sqrt{m}}) \end{aligned}$$

as $m \to \infty$. From this we obtain an approximation of $F_+^{(m)}$, namely

$$\begin{aligned} F_+^{(m)}(a, b; \theta) =& P\big(\,\theta + W_1 < a\,\big) \\ & - P\big(\,\theta + W_1 < a,\, \max_{t\in A}(\theta t + W_t) > b\,\big) \\ =& \Phi(a - \theta) \\ & - e^{2\theta(b+\beta/\sqrt{m})}\Phi(a - 2(b+\beta/\sqrt{m}) - \theta) + o(\frac{1}{\sqrt{m}}) \end{aligned}$$

as $m \to \infty$. If we compare this expression of $F_+^{(m)}$ with F_+ in Section 4.2, then we see that

$$F_+^{(m)}(a, b; \theta) = F_+(a, b + \beta/\sqrt{m}; \theta) + o(\frac{1}{\sqrt{m}}), \quad \text{as } m \to \infty. \tag{4.4}$$

Thus to calculate the probability of the event $[\,\theta + W_1 \leq a;\, \max_{t\in A}(\theta t + W_t) \leq b\,]$ using the formula for a continuous barrier, one should first lift the barrier $\beta/\sqrt{m}$ units upwards. This compensates for the fact that when the random walk $\{\theta t + W_t,\, t = 1/m, 2/m, \dots, 1\}$ breaches the barrier, it exceeds it on average with $\beta/\sqrt{m}$ units.

Equation (4.4) in conjunction with the Theorems 4.3 and 4.4 completes the proof of Theorem 4.2. To see how the approximation performs, we will now present some numerical examples and compare them with other methods.

4.5 Numerical Examples

Let us first consider the value of discrete up-and-out calls with different barrier levels but with the other parameters fixed. Table 4.1 shows the value obtained by using different methods. In the first column in the table we have the level of the barrier. The values of the other option parameters are in the caption. The second column contains the value of the corresponding continuous barrier option. In the third column we have used the formula in Theorem 4.2, with $o(1/\sqrt{m})$ set to zero.

The values in the fourth column are obtained by using a method proposed in [BGK]. Even if up-and-out call options were not included in Theorem 4.1, Broadie et al. suggested that one can approximate the value of an up-and-out call option by a similar shift, that is, lifting the barrier upwards by a

Barrier	Continuous Barrier (1)	H (2)	BGK (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
155	12.775	12.891	12.905	12.894	0.9	0.0	0.1
150	12.240	12.426	12.448	12.431	1.5	0.0	0.1
145	11.395	11.676	11.707	11.684	2.5	0.1	0.2
140	10.144	10.541	10.581	10.551	3.9	0.1	0.3
135	8.433	8.947	8.994	8.959	5.9	0.1	0.4
130	6.314	6.909	6.959	6.922	8.8	0.2	0.5
125	4.012	4.605	4.649	4.616	13.0	0.2	0.7
120	1.938	2.410	2.442	2.418	19.8	0.3	1.0
115	0.545	0.803	0.819	0.807	32.5	0.5	1.5
112	0.127	0.257	0.264	0.260	51.1	1.2	1.6

Table 4.1: Up-and-out call options price results, varying H. The option parameters are $S_0 = 110$, $K = 100$, $\sigma = 0.3$, $r = 0.10$, $q = 0.0$, $T = 0.2$ and $m = 50$. If one assumes that there are 250 trading days per year, then $m = 50$ corresponds to daily monitoring.

factor $\exp(\beta\sigma\sqrt{T/m})$ and then use the formula for the value of a continuous up-and-out call.

In the fifth column we have collected prices obtained by a so called trinomial method presented in [BGK2] (the errors of the trinomial prices are according to the same article approximately ± 0.001). Finally, in the last three columns we have the relative error measured in percentage for the different approximations.

Note the surprisingly great differences in price between the discrete and the corresponding continuous barrier option. So it is worth to emphasize that one should not neglect the fact that some barrier options are discretely and not continuously monitored. We also see that the approximation derived in this chapter yields good results, and that the accuracy of the result is dependent of how close the barrier is to the initial price. This method also performs better than the approximation proposed in [BGK].

In Table 4.2 we have varied the number of price fixing dates as well. In this example, however, we use a different trinomial method than in the previous case. The method is fully described in the next chapter. The error seems to be ± 0.001 here as well.

As is to be expected, the approximation developed in this paper degrade as the number of monitoring times decreases. In the extreme case with the barrier very close to the initial asset price, the method even performs worse than the approximation proposed by [BGK]. One could remark though, that in the extreme case none of the methods work especially well.

In the final example, presented in Table 4.3, we have examined how the

m	Barrier	Continuous Barrier (1)	H (2)	BGK (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
25	130	6.314	7.124	7.221	7.148	11.7	0.3	1.0
	125	4.012	4.829	4.918	4.851	17.3	0.5	1.4
	120	1.938	2.600	2.669	2.616	25.9	0.6	1.9
	115	0.545	0.916	0.950	0.925	41.1	0.9	2.8
	112	0.127	0.320	0.336	0.329	61.4	3.0	2.0
5	130	6.314	7.837	8.286	7.934	20.4	1.2	4.4
	125	4.012	5.622	6.062	5.721	29.9	1.7	5.9
	120	1.938	3.326	3.683	3.409	43.1	2.5	8.0
	115	0.545	1.404	1.624	1.481	63.2	5.2	9.6
	112	0.127	0.622	0.751	0.708	82.1	12.3	6.0

Table 4.2: Up-and-out call options price results, varying H and m. The option parameters are $S_0 = 110,\ K = 100,\ \sigma = 0.3,\ r = 0.10,\ q = 0.0$ and $T = 0.2$.

Panel	Barrier	Continuous Barrier (1)	H (2)	BGK (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
A	155	6.798	7.270	7.290	7.274	6.6	0.1	0.2
	140	2.916	3.251	3.265	3.254	10.4	0.1	0.3
	125	0.566	0.693	0.699	0.695	18.6	0.2	0.6
B	140	3.766	4.516	4.578	4.531	16.9	0.3	1.0
	130	1.576	2.086	2.130	2.097	24.9	0.5	1.6
	120	0.331	0.541	0.561	0.546	39.4	0.9	2.8
C	140	7.171	8.277	8.354	8.296	13.6	0.2	0.7
	130	3.653	4.550	4.608	4.565	20.0	0.3	0.9
	120	1.110	1.629	1.659	1.637	32.2	0.5	1.4

Table 4.3: Up-and-out call options price results, varying K, σ and T. The option parameters are $S_0 = 110$, $r = 0.1$ and $q = 0.0$ for all panels. Panel A has $K = 100$, $\sigma = 0.3$, $T = 1$ and $m = 250$ (daily monitoring). Panel B has $K = 100$, $\sigma = 0.6$, $T = 0.2$ and $m = 50$ (daily monitoring). Panel C has $K = 90$, $\sigma = 0.6$, $T = 0.2$ and $m = 50$ (daily monitoring)

other parameters influence the accuracy of the approximation.

It is of course not possible to draw any certain conclusions from just numerical examples. But the results presented here indicate that the approximation gives good results for large values of m and if the barrier is not too close to the initial asset price.

Chapter 5

Estimating the Wiener Measure of Cylinder Sets Using the Trinomial Method

5.1 Introduction

The aim with this chapter is to design a numerical method to determine the Wiener measure of a certain cylinder set. To be more precise, if we let the probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be defined as in Chapter 2, then our aim is to find a good estimate of the probability

$$P(A_c) = P(W_{T_1} \in I_1, W_{T_2} \in I_2, \dots, W_{T_m} \in I_m), \tag{5.1}$$

where, for each $i = 1, 2, \dots, m$, $T_i = iT$, $T > 0$, and where I_i is the open interval (a_i, b_i), $-\infty \leq a_i < b_i \leq \infty$. In other words, our aim is to find an approximation of the probability that a Brownian motion at time T_i is greater than a_i and smaller than b_i for each $i = 1, 2, \dots, m$.

It is also possible to interpret the probability above in terms of a Gaussian random walk. If $\{S_n\}_{n=1}^{\infty}$ denotes a Gaussian random walk, say

$$S_n = \sum_{i=1}^{n} \zeta_i,$$

where $\{\zeta_i\}_{i=1}^{\infty}$ is a sequence of independent and normal distributed random variables with mean 0 and variance σ^2, then the probability of the event $[S_i \in I_i,\ i = 1, 2, \dots, m]$ is equal to $P(A_c)$, with $T = \sigma^2$.

The problem of computing the probability in equation (5.1) arises in different fields of mathematics, for instance sequential analysis, insurance risk theory, queue theory and financial mathematics (see [Si2], [As] and [BGK2]). In this chapter we will show a financial application of the numerical method

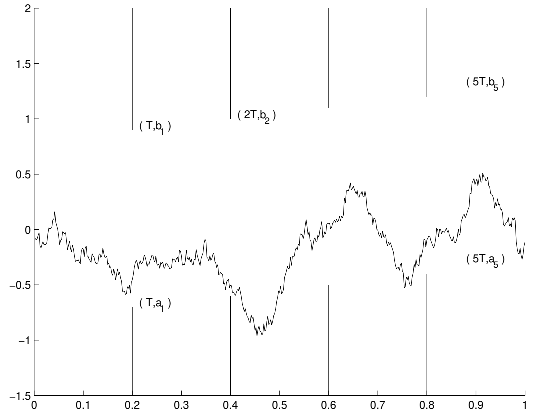

Figure 5.1: A Brownian path that goes through five windows.

mentioned above. More precise, we will give an application to the pricing of discrete barrier options.

Methods to evaluate the probability $P(A_c)$ and related problems have been discussed in the literature before. Siegmund ([Si2], p.49) considers numerical algorithms to compute the probability that a Gaussian random walk hits a barrier before a certain time. Broadie et al. [BGK2] develop a method to estimate the price of discrete barrier options. This method will be discussed in greater details in Section 5.6. Several people have also analysed the more general problem of computing the Wiener integral

$$\int_{\Omega} f(\omega) dP(\omega),$$

where P is the Wiener measure. The numerical evaluation of the above integral was probably first studied by Cameron [C] in 1951. More recently in 1999 Steinbauer [St] has developed a quadrature formula to estimate the Wiener integral. In [St] it is also given a detailed discussion about numerical methods to compute Wiener integrals. For additional information about Monte Carlo methods, see [BBG].

In this thesis we will develop a method based on the so called trinomial method (or the explicit finite difference method for the heat equation). Our approach to this problem can be described as follows. Let for $i = 1, 2 \ldots, m$ the function $\chi_i : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the interval I_i, that is

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \in I_i, \\ 0 & \text{otherwise.} \end{cases}$$

Set also $v_m = \chi_m$ and define recursively

$$v_{i-1}(x) = \big(U_T(v_i \, \chi_i) \big)(x) \ , \quad x \in \mathbb{R}, \ 1 \le i \le m,$$

where the operators U_t, $t \geq 0$, are defined by

$$(U_t f)(x) = E\big[f(x + W_t)\big], \quad x \in \mathbb{R},$$

for every bounded Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ (cf Appendix B). By applying Theorem 1 in Appendix B we see that

$$v_{i-1}(W_{T_{i-1}}) = E\big[v_i(W_{T_i})\,\chi_i(W_{T_i}) \,|\, \mathcal{F}_{T_{i-1}}\big]$$

for any $i \leq i \leq m$ and hence

$$\begin{aligned}
P(A_c) =& E\Big[\prod_{i=0}^{m} \chi_i\left(W_{T_i}\right)\Big] \\
=& E\Big[\prod_{i=0}^{m-1} \chi_i\left(W_{T_i}\right) E[v_m\left(W_{T_m}\right)\chi_m(W_{T_m}) \,|\, \mathcal{F}_{T_{m-1}}]\Big] \\
=& E\Big[\prod_{i=0}^{m-1} \chi_i\left(W_{T_i}\right) v_{m-1}\big(W_{T_{m-1}}\big)\Big] \\
=& E\Big[\prod_{i=0}^{m-2} \chi_i\left(W_{T_i}\right) E[v_{m-1}\left(W_{T_{m-1}}\right)\chi_{m-1}(W_{T_{m-1}}) \,|\, \mathcal{F}_{T_{m-2}}]\Big] \\
=& E\Big[\prod_{i=0}^{m-2} \chi_i\left(W_{T_i}\right) v_{m-2}\big(W_{T_{m-2}}\big)\Big] \\
&\vdots \\
=& v_0(0).
\end{aligned}$$

Thus, one way to compute the probability $P(A_c) = v_0(0)$ is to determine the functions v_i for $i = 0, 1, \ldots, m-1$. We are thereby led to the problem to compute the function $x \mapsto (U_T f)(x)$ for a given function f and fixed T. One well-known approach to solve this last mentioned problem is given by the trinomial method.

The trinomial method is in some sense based on the central limit theorem. The idea is to replace the Brownian motion with a random walk, where the terms in the random walk are lattice random variables with (at the most) three possible outcomes. To be more precise, let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed (abbr. i.i.d.) random variables with mean 0 and variance Δt, where $\Delta t = T/n$ for some positive integer n. Moreover, assume that

$$P(X_1 \in \{-\Delta x, 0, \Delta x\}) = 1,$$

where $\Delta x > 0$. The constants Δt and Δx are dependent of each other (this dependence will be discussed in greater details in the next section). Let $[a]$

denote the greatest integer $\leq a$ and set[1]

$$Y_t^{(n,T)} = \sum_{i=1}^{[nt/T]} X_i, \quad t \geq 0,\ n \in \mathbb{N}_+.$$

The central limit theorem then tells us that the sequence of trinomial distributed random variables $\{Y_T^{(n,T)}\}_{n=1}^{\infty}$ will converge in distribution to a normal distributed random variable with mean 0 and variance T, that is, the sequence will converge in distribution to W_T. This gives us reason to believe that $U_T^{(n,T)} f$, where

$$(U_t^{(n,T)} f)(x) = E\big[f(x + Y_t^{(n,T)})\big], \quad t \geq 0,\ n \in \mathbb{N}_+,\ x \in \mathbb{R},$$

might be a good approximation of $U_T f$, and that is precisely what we are going to investigate in the Sections 5.2-5.4.

The discussion above gives that a first ansatz to a numerical method to compute the probability $P(A_c)$ is given by

$$\begin{cases} \tilde{v}_m = \chi_m, \\ \tilde{v}_{i-1} = U_T^{(n,T)}\,(\,\tilde{v}_i\,\chi_i), \quad \text{for } 1 \leq i \leq m. \end{cases} \tag{5.2}$$

The quantity $\tilde{v}_0(0)$ will then be an estimation of $P(A_c)$. With a minor change we can improve the method above considerably, as will be seen in Sections 5.5 and 5.6.

The chapter is organized as follows. In Section 5.2 we will develop the relations between the heat equation and the operators U and $U^{(n,T)}$. Furthermore, in Section 5.3 we will discuss stability and also prove a convergence result for the sequence $\{U_T^{(n,T)} f\}_{n=1}^{\infty}$. Based on some results from the theory of Besov spaces and the rate of convergence in the central limit theorem, we will in Section 5.4 give some sharp estimates of the convergence rate for the error which appears when the function $U_T f$ is approximated by $U_T^{(n,T)} f$. In Section 5.5 we will return to our original problem of estimating the Wiener measure for cylinder sets. With the aid of the observations done in Sections 5.2-5.4 we will improve the algorithm presented in equation (5.2). In Section 5.6 we will show how the algorithm can be used to price discrete barrier options and present some numerical examples and comparisons. Finally, in Section 5.7 we will make some suggestions for future research.

5.2 The Heat Equation and the Explicit Finite Difference Method

There is a rich interplay between the heat equation and Brownian motion, the study of which goes back already to Bachelier in 1900, [Ba]. In particular,

[1] We use the convention $\sum_{i=k}^{k-1} x_i = 0$ for $k \in \mathbb{Z}$.

the function p, defined by the relation

$$p(t,x) = \frac{d}{dx} P(W_t \le x), \quad t > 0,\ x \in \mathbb{R},$$

is a solution to the heat equation, i.e. p satisfies the following partial differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2} \tag{5.3}$$

for $t > 0$ and all $x \in \mathbb{R}$.

With the aid of this observation we will now construct other solutions to the heat equation. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel measurable function (abbr. $f \in L^\infty$) and define for each x and $t \ge 0$

$$u(t,x) = (U_t f)(x). \tag{5.4}$$

where the operators $\{U_t, t \ge 0\}$ are defined as in the introduction, i.e.

$$(U_t f)(x) = E\big[f(x + W_t)\big].$$

By using the symmetry of Brownian motion and changing variables we have for all $t > 0$,

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} f(x+\xi) P(W_t \in d\xi) \\ &= \int_{\mathbb{R}} f(x-\xi) p(t,\xi) d\xi \\ &= \int_{\mathbb{R}} f(\xi) p(t, x-\xi) d\xi. \end{aligned}$$

The function $(t,x) \mapsto p(t, x-\xi)$ solves the heat equation for each fixed ξ. By differentiating under the integral sign we now get that u is a solution of the heat equation, too. If, in addition, f is continuous then the bounded convergence theorem yields

$$\begin{aligned} \lim_{\substack{t \searrow 0 \\ y \to x}} u(t,y) &= \lim_{\substack{t \searrow 0 \\ y \to x}} E\big[\, f(y + W_t)\big] \\ &= f(x), \end{aligned}$$

since $W_t \to 0$ P-a.s. as t tends to zero.

To sum up, u is continuous in $[0,T] \times \mathbb{R}$ and satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} & \text{in } (0,T] \times \mathbb{R}, \\ u|_{t=0} = f & \text{on } \mathbb{R}, \end{cases}$$

for any continuous and bounded function f and any given $T > 0$. A function which satisfies these conditions is said to be a solution to the *Cauchy* (or *initial value*) *problem for the heat equation*. The function f is often referred to as *the initial value*.

There are infinitely many solutions to the Cauchy problem for the heat equation (see [J], p. 171-173), but if we restrict ourselves to merely consider uniformly bounded solutions then there only exists one solution (see [KS], p. 255). This solution must be given by equation (5.4), since the function $(t, x) \mapsto (U_t f)(x)$ is obviously uniformly bounded by $\sup_{x \in \mathbb{R}} |f(x)|$.

There are many numerical methods to solve the initial value problem for the heat equation. One method which is known as *the explicit finite difference method* has great similarities with the trinomial method. This relationship is the topic of the remaining part of this section.

The idea behind the explicit finite difference method is to replace the derivatives in the heat equation with the following finite difference approximations,

$$\frac{\partial u}{\partial t}(t, x) \approx \frac{u(t + \Delta t, x) - u(t, x)}{\Delta t}$$

and

$$\frac{\partial^2 u}{\partial x^2}(t, x) \approx \frac{u(t, x + \Delta x) - 2u(t, x) + u(t, x - \Delta x)}{(\Delta x)^2},$$

where Δx is some positive number and $\Delta t = T/n$ for some strictly positive integer n. The constants Δx and Δt can be thought of as being the same numbers as in the introduction. One then looks for a function $\tilde{u}$ satisfying

$$\frac{\tilde{u}(t + \Delta t, x) - \tilde{u}(t, x)}{\Delta t} = \frac{1}{2} \frac{\tilde{u}(t, x + \Delta x) - 2\tilde{u}(t, x) + \tilde{u}(t, x - \Delta x)}{(\Delta x)^2} \tag{5.5}$$

for all $(t, x) \in \Lambda$, where

$$\Lambda = \{(t, x)\,;\, (t, x) = (i\Delta t, j\Delta x),\ i = 0, 1, \dots, n,\ j \in \mathbb{Z}\}.$$

The set Λ will henceforth be referred to as the mesh. Moreover, we let $\tilde{u}(0, x) = f(x)$ for all x satisfying $(0, x) \in \Lambda$. On the mesh we will for simplicity write $\tilde{u}_{i,j} = \tilde{u}(i\Delta t, j\Delta x)$.

By introducing the parameter

$$\lambda = \frac{\Delta t}{2(\Delta x)^2}$$

the equation (5.5) gives

$$\tilde{u}_{i+1,j} - \tilde{u}_{i,j} = \lambda\, \tilde{u}_{i,j+1} - 2\lambda\, \tilde{u}_{i,j} + \lambda\, \tilde{u}_{i,j-1}$$

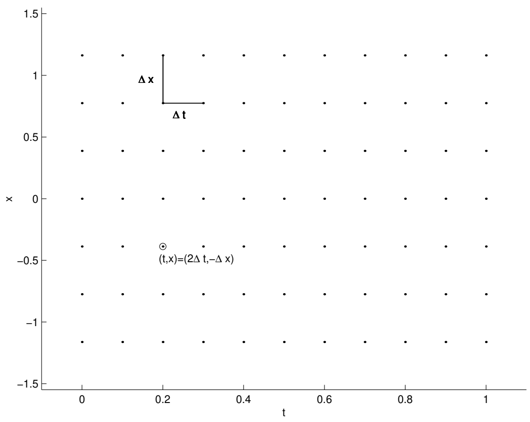

Figure 5.2: The mesh, $T = 1$, $n = 10$ and $\lambda = 1/3$.

or, stated equivalently,

$$\tilde{u}_{i+1,j} = \lambda\, \tilde{u}_{i,j+1} + (1 - 2\lambda)\, \tilde{u}_{i,j} + \lambda\, \tilde{u}_{i,j-1}. \tag{5.6}$$

The above equation has for certain values of λ a probabilistic interpretation. To see this, let the random variables $\{X_i\}_{i=1}^{\infty}$ be defined as in the Section 5.1, which means that $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with mean 0, variance Δt and with the property $P(X_1 \in \{-\Delta x, 0, \Delta x\}) = 1$. If we let

$$p_0 = P(X_1 = 0) \quad \text{and} \quad p_1 = P(X_1 = \Delta x) = P(X_1 = -\Delta x),$$

then $E[X_1] = 0$ and, moreover,

$$2p_1(\Delta x)^2 = \Delta t$$

since the variance is equal to Δt. Thus, provided $\lambda \in (0, 1/2]$ it is possible to define

$$p_1 = \lambda \quad \text{and} \quad p_0 = 1 - 2\lambda.$$

Consider once again the recurrence equation (5.6). From the discussion above we see that, if $\lambda \in (0, 1/2]$, then this equation may be written as

$$\tilde{u}_{i+1,j} = E\big[\tilde{u}(i\Delta t, j\Delta x + X_1)\big]\,.$$

Using the relation (5.6) once more, we have

$$\tilde{u}_{i+2,j} = E\big[\tilde{u}(i\Delta t, j\Delta x + X_1 + X_2)\big]$$

and hence by repeated application of equation (5.6) we get

$$\tilde{u}_{i+l,j} = E\big[\tilde{u}(i\Delta t, j\Delta x + \sum_{k=1}^{l} X_k)\big]$$

for any integer l such that $0 \le l \le n - i$. We recall that $\tilde{u}(0,x) = f(x)$. By setting $i = 0$ we therefore obtain

$$\tilde{u}_{l,j} = E\big[f(j\Delta x + \sum_{k=1}^{l} X_k)\big].$$

The expression above for $\tilde{u}$ may be identified as

$$\tilde{u}_{l,j} = (U^{(n,T)}_{l\Delta t} f)(j\Delta x),$$

where the operators $\{U^{(n,T)}_t, t \ge 0,\ n \in \mathbb{N}_+\}$ are defined as in the introduction, that is

$$(U^{(n,T)}_t f)(x) = E\big[f(x + \sum_{k=1}^{[nt/T]} X_k)\big].$$

Up to now we have only considered the case $\lambda \in (0, 1/2]$. If $\lambda > 1/2$ it will no longer be possible to give a probabilistic interpretation of the explicit finite difference method. However, for these values of λ the method will no longer be neither *stable* nor *convergent*. These concepts, stability and convergence, are the subject of the next section.

5.3 Stability and Convergence

In this section we will study two different aspects of the collection $\{U^{(n,T)}_t,\ t \ge 0,\ n \in \mathbb{N}_+\}$. For conciseness, throughout this section we will write $U^{(n)}$ instead of $U^{(n,T)}$. We will establish a convergence result for the sequence $\{U^{(n)}_T f\}_{n=1}^{\infty}$ for a large class of functions f, but to begin with, we will present an example which illustrates the importance of the fact that the operators $U^{(n)}_t$, $t \ge 0$, $n \in \mathbb{N}_+$, are uniformly bounded in L^∞, that is, they are bounded with respect to the norm $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. The example we are going to show is taken from [T].

Consider the explicit finite difference solution $\tilde{u}$, defined on the mesh $\{(t,x)\,;\,(t,x) = (i\Delta t, j\Delta x),\ i = 0,1,\dots,n,\ j \in \mathbb{Z}\}$ with $n\Delta t = T$, given by the following Cauchy problem,

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} & \text{in } (0,T] \times \mathbb{R}, \\ u|_{t=0} = f & \text{on } \mathbb{R}. \end{cases}$$

We let $f(x) = \epsilon \cos(x\pi/\Delta x)$ where ϵ is a small positive number. We can think of f as a function representing a small round off error. From equation (5.6) we know that

$$\begin{aligned}\tilde{u}_{1,j} &= \lambda f(\,(j+1)\Delta x\,) + (1-2\lambda)\, f(\,j\Delta x\,) + \lambda f(\,(j-1)\Delta x\,)\\ &= \big(\lambda(-1)^{j+1} + (1-2\lambda)(-1)^j + \lambda(-1)^{j-1}\big)\epsilon\\ &= (1-4\lambda)(-1)^j \epsilon\end{aligned}$$

or, more generally

$$\tilde{u}_{i,j} = (1-4\lambda)^i(-1)^j\epsilon.$$

If $\lambda > 1/2$ then the discrete supremum norm of the approximative solution, i.e.

$$\sup_{0\le i\le n, j\in\mathbb{Z}} |\tilde{u}_{i,j}| = (4\lambda-1)^n\epsilon,$$

will tend to infinity as $n \to \infty$ even if $T = n\Delta t$ is bounded. Thus very small perturbations of the initial data, for instance caused by round off errors, may lead to such big changes in the discrete solution that it becomes useless.

On the other hand, if $\lambda \in (0, 1/2]$ then the solution $\tilde{u}$ above is bounded by ϵ. In fact, provided $\lambda \in (0, 1/2]$ then for any bounded initial value the corresponding explicit finite difference solution is bounded. This will be evident if we recall that the explicit finite difference solution $\tilde{u}$ can be written as

$$\tilde{u}_{i,j} = (U^{(n)}_{i\Delta t}\, g)(j\Delta x),$$

where g is the initial value. Clearly,

$$\|U^{(n)}_t g\|_\infty \le \|g\|_\infty \tag{5.7}$$

for all $t \ge 0$ and $n \in \mathbb{N}_+$. Thus, in contrast to the case when $\lambda > 1/2$, we see that the solution $\tilde{u}$ is uniformly bounded by $\|g\|_\infty$. The fact that the operators $U^{(n)}_t$ satisfy the inequality in equation (5.7) is sometimes referred to as *stability* in L^∞.

We will next turn our attention to the limit of the sequence $\{U^{(n)}_T f\}_{n=1}^\infty$. Introduce an i.i.d. sequence of random variables $\{Z_i\}_{i=1}^\infty$ such that

$$P(Z_1 = 0) = 1 - 2\lambda \quad \text{and} \quad P(Z_1 = 1) = P(Z_1 = -1) = \lambda.$$

It is evident that Z_1 has variance 2λ and that $X_1 =_d \Delta x Z_1$, where $\zeta_1 =_d \zeta_2$ means that the random variables ζ_1 and ζ_2 have the same probability laws. Next suppose that f is a bounded and continuous function and recall that

$\lambda = \Delta t/(2(\Delta x)^2)$, where $\Delta t = T/n$. Thus, as a consequence of the central limit theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} (U_T^{(n)} f)(x) &= \lim_{n\to\infty} E\big[\, f\big(x + \Delta x \sum_{i=1}^{n} Z_i\,\big)\,\big] \\
&= \lim_{n\to\infty} E\big[\, f\big(x + \sqrt{\frac{T}{2n\lambda}} \sum_{i=1}^{n} Z_i\,\big)\,\big] \qquad (5.8)\\
&= (U_T f)(x)
\end{aligned}$$

for every $x \in \mathbb{R}$.

In many applications it is essential that we can be assured that the sequence $\{U_T^{(n)} f\}_{n=1}^{\infty}$ will converge pointwise to $U_T f$ even for *discontinuous* initial values f. In order to generalize equation (5.8) to a larger set of functions than solely continuous functions the next theorem, known as *Skorohod's representation theorem*, will be useful.

Theorem 5.1. *Let $\{\zeta_i\}_{i\geq 1}$ be a sequence of independent and identical distributed random variables with mean 0 and variance 1, and let $S_n = \sum_{i=1}^{n} \zeta_i$. There is a sequence of $\mathcal{F}_t$ stopping times $\{\tau_i\}_{i\geq 0}$ such that $S_n =_d W(\tau_n)$ and $\tau_i - \tau_{i-1}, i = 1, 2, \ldots$ are independent and identically distributed with mean 1.*

For a proof of this theorem, see [D], p.404.

The next theorem is partly based on a proof given in [D], p.405.

Theorem 5.2. *Suppose $f \in L^{\infty}$. Let ν denote the Lebesgue measure on $\mathbb{R}$ and let D_f denote the set*

$$D_f = \{x \in \mathbb{R};\ f \textit{ is discontinuous at } x\}.$$

If $\nu(D_f) = 0$ then

$$\lim_{n\to\infty} (U_T^{(n)} f)(x) = (U_T f)(x) \qquad (5.9)$$

for any $x \in \mathbb{R}$.

Proof. Let $Y_n = \sum_{i=1}^{n} X_i$. The definition of λ gives that $\sqrt{\Delta t} = \Delta x \sqrt{2\lambda}$ and thus

$$\frac{1}{\sqrt{\Delta t}} Y_n =_d \frac{1}{\sqrt{2\lambda}} Z_1 + \ldots + \frac{1}{\sqrt{2\lambda}} Z_n,$$

where Z_i, $i = 1, \ldots, n$ are defined as above. The terms in the random walk

$$\{\frac{1}{\sqrt{\Delta t}}\, Y_n\}_{n=1}^{\infty}$$

have therefore mean 0 and variance 1, so according to Theorem 5.1 there exist stopping times $\{\tau_i\}_{i=0}^{\infty}$ such that

$$\frac{1}{\sqrt{\Delta t}} Y_n =_d W(\tau_n)$$

for every $n \geq 0$ and such that $(\tau_i - \tau_{i-1})$, $i = 1, 2, \ldots$ is an i.i.d. with mean 1. The scaling property for Brownian motion implies

$$Y_n =_d W(T \frac{\tau_n}{n})$$

since $\Delta t = T/n$. Consequently

$$\begin{aligned}(U_T^{(n)} f)(x) &= E\big[f(x + Y_n)\big] \\ &= E\big[f\big(x + W(T \frac{\tau_n}{n})\big)\big] .\end{aligned} \tag{5.10}$$

According to the strong law of large numbers we have

$$\begin{aligned}\frac{\tau_n}{n} &= \frac{1}{n} \sum_{i=1}^{n} (\tau_i - \tau_{i-1}) \\ &\to 1\end{aligned}$$

P-a.s. as n tends to infinity, and, as a consequence of this,

$$W(T \frac{\tau_n}{n}) \to W(T) \qquad P-\text{a.s. as } n \to \infty.$$

Because $\nu(D_f) = 0$, $P(x + W_T \in D_f) = 0$ for each fixed x and therefore

$$f\big(x + W(T \frac{\tau_n}{n})\big) \to f\big(x + W(T)\big) \tag{5.11}$$

P-a.s. for each fixed x as $n \to \infty$.

The equations (5.10) and (5.11) in combination with the bounded convergence theorem imply

$$\lim_{n \to \infty} (U_T^{(n)} f)(x) = E\big[f(x + W_T)\big] = (U_T f)(x)$$

for all x, which is the desired conclusion. □

One can note that the above result is the best possible in the sense that one can construct a function f such that $\nu(D_f) = \epsilon$, for an arbitrary number $\epsilon > 0$, and $(U_T^{(n)} f)(x) \to a \neq (U_T f)(x)$, for some $x \in \mathbb{R}$. Take for instance the function $f = \chi_A$, where χ_A is the characteristic function of the interval

$$A = \big\{x\,;\,(0, x) \in \bigcup_{n=1}^{\infty} \Lambda_n\big\} \cap \big(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\big).$$

The set Λ_n above denotes the mesh for a fixed n, i.e.

$$\Lambda_n = \big\{ (t,x)\,;\,(t,x) = \big(i\frac{T}{n}, j\sqrt{\frac{T}{2n\lambda}}\,\big),\ i = 0,1,2,\dots,n,\ j \in \mathbb{Z} \big\}$$

(cf Section 5.2). It is evident that f is discontinuous at every point in the interval $(-\epsilon/2, \epsilon/2)$ and it follows that $\nu(D_f) = \epsilon$. From the definition of Λ_n we have for every integer $n \geq 1$

$$(U_T^{(n)} f)(0) = P(\,|\sum_{i=1}^{n} X_i| < \epsilon/2\,).$$

and hence, according to the central limit theorem,

$$\lim_{n\to\infty} (U_T^{(n)} f)(0) = P(|W_T| < \epsilon/2) > 0.$$

However, $(U_T f)(0) = 0$ as $\nu(A) = 0$, which completes our example.

5.4 Rate of Convergence

This section deals with questions concerning the rate of convergence of the sequence $U_T^{(n,T)} f$, $n = 1,2\dots$. The question we will try to answer is as follows. What conditions on f and X_1 imply that the difference

$$(U_T^{(n,T)} f)(x) - (U_T f)(x) \tag{5.12}$$

for fixed $x \in \mathbb{R}$ equals $o(n^{-\alpha})$ or $O(n^{-\alpha})$ for some $\alpha > 0$ as n tends to infinity. This question has been treated in the literature several times. For instance Berry and Esseen (see [Be] and [Es]) consider the special case when f is piecewise constant and von Bahr [Bahr] when f is a polynomial. Leimar and Reisen [LR] analyse the case when $f(x) = \max(e^x - K, 0)$, with $K > 0$, and $\lambda = 1/2$. Kreiss et al. [KTW] and Heston et al. [HZ] examine the dependence between the smoothness of f and the convergence rate of the difference in equation (5.12). Butzer et al. [BHW] investigate the convergence rate when the initial values f are differentiable functions. Finally, Löfström [Lö] presents sharp estimates on the difference in equation (5.12) uniformly for all $x \in \mathbb{R}$ if f belongs to a so called Besov space.

Before we comment these papers any further we will introduce some definitions. The next definition has its origin in the theory about finite difference methods (see [T], p.43).

Definition 5.1. *Let ζ be any random variable such that $E\big[\zeta^2\big] < \infty$ and let $t = Var(\zeta)$. We will say that ζ is* **consistent** *of order μ, where μ is an integer, if*

$$E\big[\,e^{i\xi\zeta}\,\big] = E\big[\,e^{i\xi W_t}\,\big] + O(\xi^{\mu+2})\,, \quad \xi \to 0,$$

where i is the imaginary unit and $\xi \in \mathbb{R}$.

It can be proved that if ζ has an absolute moment of order $\mu+1$, i.e. $E[\,|\zeta|^{\mu+1}\,]<\infty$, and ζ is consistent of order μ then $E[\,\zeta^k\,]=E[\,W_t^k\,]$, where $t=Var(\zeta)$, for all positive integers $k\le\mu+1$ (see [D], p.101).

If ζ is consistent of order μ then it is also consistent of any order less than μ. For this reason one sometimes say that ζ is *exactly* consistent of order μ if ζ is consistent of order μ but not consistent of order $\mu+1$.

Consider next the random variable X, where $X=X_1$ with X_1 defined as before. Recall from Section 5.2 that

$$P(X=0)=1-2\lambda \quad\text{and}\quad P(X=\Delta x)=P(X=-\Delta x)=\lambda$$

where $0<\lambda=\Delta t/(2\Delta x^2)\le 1/2$. The random variable X can be exactly consistent of order 2 or 4 depending on the value of λ. To see this, note that from Taylor's formula we have as $\xi\to 0$,

$$\begin{aligned}
E\left[\,e^{i\xi X}\,\right] &= 1-2\lambda+2\lambda\,\cos(\xi\Delta x)\\
&= 1-2\lambda\,\frac{(\xi\Delta x)^2}{2}+2\lambda\,\frac{(\xi\Delta x)^4}{24}+O(\xi^6)\\
&= 1-\frac{\xi^2\Delta t}{2}+\frac{\xi^4\,(\Delta t)^2}{48\,\lambda}+O(\xi^6)\\
&= \exp(-\Delta t\,\xi^2/2)-\frac{1}{8}\xi^4\,(\Delta t)^2\,(1-\frac{1}{6\lambda})+O(\xi^6).
\end{aligned}$$

Since $E[\,\exp(i\,\xi\,W(\Delta t))\,]=\exp(-\Delta t\,\xi^2/2)$ we see that X is exactly consistent of order 2 if $\lambda\in(0,1/6)\cup(1/6,1/2]$ but if $\lambda=1/6$ then X is exactly consistent of order 4.

A consequence of the discussion above is that the trinomial method computes exactly when the initial value f is a polynomial of degree $\mu+1$, where μ is the consistency number for X.

Next we will introduce certain Banach spaces known as *Besov spaces* and below denoted by B^s_∞, $s>0$. The Besov spaces are subspaces of the Banach space C_0, where C_0 denotes the class of all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{x\to\pm\infty} f(x)=0$$

equipped with the norm $\|f\|_{C_0}=\max_{x\in\mathbb{R}}|f(x)|$.

The norm in the Besov space B^s_∞, henceforth denoted $\|\cdot\|_{B^s_\infty}$, is given as follows. Set $s=k+\gamma$, where k is a nonnegative integer and $0<\gamma\le 1$. If $0<\gamma<1$ then

$$\|f\|_{B^{k+\gamma}_\infty}=\|f\|_{C_0}+\sup_{h>0}\frac{1}{h^\gamma}\,\|D^kf(\cdot+h)-D^kf(\cdot)\|_{C_0}\,,$$

where $D^k f$ denotes the k:th derivative of f. If $\gamma = 1$ we set

$$\|f\|_{B^{k+1}_\infty} = \|f\|_{C_0} + \sup_{h>0} \frac{1}{h} \|D^k f(\cdot + h) - 2D^k f(\cdot) + D^k f(\cdot - h)\|_{C_0}.$$

In the literature there exist many other equivalent definitions of the norm in the Besov space B^s_∞. The definition here is taken from [BTW].

Thus, if $f \in B^s_\infty$ for some *integer* $s > 0$, which implies $\gamma = 1$ and $k = s-1$, then f must have a k:th derivative where $D^k f$ satisfies a *Zygmund condition*, i.e. there is a constant C such that for each $x, h \in \mathbb{R}$ we have

$$|D^k f(x+h) - 2D^k f(x) + D^k f(x-h)| \leq C|h|.$$

If $0 < \gamma < 1$ and $f \in B^{k+\gamma}_\infty$ then $D^k f$ exists and satisfies a *Hölder condition* with exponent γ, i.e. there is a constant C such that for every $x, h \in \mathbb{R}$ we have

$$|\, D^k f(x+h) - D^k f(x)\,| \leq C|h|^\gamma.$$

One property for the Besov spaces states that if $s_1 < s_2$ then $B^{s_1}_\infty \supset B^{s_2}_\infty$ (see [BL], p.142). Thus the functions in $B^{s_2}_\infty$ are generally smoother than the functions in $B^{s_1}_\infty$. Much more can of course be said about Besov spaces, but we refer the interested reader to [BL], Chapters 6 and 7, and the references therein.

We shall introduce yet another Banach space, below denoted A^s_∞, $s > 0$. The space A^s_∞ is a subspace to C_0. In order to define the norm in this space, let the operators $V^{(\Delta x)}_t$, where $t \geq 0$, be given by

$$(V^{(\Delta x)}_t f)(x) = E\big[f(x + \sum_{i=1}^n X_i)\big], \quad \text{where } n = \Big[\frac{t}{2(\Delta x)^2 \lambda}\Big],$$

for all $x \in \mathbb{R}$ and $f \in L^\infty$. It is clear that $(V^{(\Delta x)}_t f)(x) = (U^{(n,T)}_t f)(x)$, provided $2\lambda(\Delta x)^2 = T/n$. The norm in A^s_∞ is defined by

$$\|f\|_{A^s_\infty} = \|f\|_{C_0} + \sup_{0<\Delta x<1} \sup_{t \in R_{\Delta x}} (\Delta x)^{-s} \|V^{(\Delta x)}_t f - U_t f\|_{C_0},$$

where $R_{\Delta x} = \{\Delta t, 2\Delta t, \dots\}$ with $\Delta t = 2\lambda(\Delta x)^2$.

The following striking result is due to Löfström (see [Lö], p.408).

Theorem 5.3. *Suppose that X_1 is consistent of order μ. Then*

$$A^\varsigma_\infty = B^\varsigma_\infty\,, \quad 0 < \varsigma \leq \mu,$$

with equivalent norms. Moreover, if $f \in C_0$ and

$$\sup_{t \in R_{\Delta x}} \|\, V^{(\Delta x)}_t f - U_t f\,\|_{C_0} = o\big((\Delta x)^\mu\big), \quad \text{as } \Delta x \to 0,$$

then $f \equiv 0$.

Thus, the convergence rate is closely related to the smoothness of the initial value f and to the moments of X_1. In particular, if $f \in B^s_\infty$ and if X_1 is consistent of order μ, Theorem 5.3 yields that there is for each $\varsigma \leq \min(\mu, s)$ a constant C, independent of f and n, such that

$$\| U_T^{(n,T)} f - U_T f \|_{C_0} \leq \frac{C}{n^{\varsigma/2}} \|f\|_{B^\varsigma_\infty}, \quad n \geq 2\lambda T.$$

Consequently, if $f \in B^s_\infty$ and if X_1 is consistent of order μ then

$$|\, (U_T^{(n,T)} f)(x) - (U_T f)(x) \,| = O(n^{-\alpha}), \quad \text{as } n \to \infty, \tag{5.13}$$

where

$$\alpha = \frac{1}{2} \min(\mu, s).$$

In particular, if Df belongs to C_0 and satisfies a Zygmund condition, then

$$|\, (U_T^{(n,T)} f)(x) - (U_T f)(x) \,| = O(1/n), \quad \text{as } n \to \infty,$$

for any $x \in \mathbb{R}$ and any $\lambda \in (0, 1/2]$. If, in addition, $D^3 f \in C_0$ and $D^3 f$ satisfies a Zygmund condition and, furthermore, $\lambda = 1/6$, then the convergence rate is quadratic, i.e.

$$|\, (U_T^{(n,T)} f)(x) - (U_T f)(x) \,| = O(1/n^2), \quad \text{as } n \to \infty.$$

In the literature there are results similar to the one in the equation (5.13) derived from Taylor expansions of the initial value f or $U_t f$. See for instance the work by Butzer et al. [BHW] or Heston and Zhou [HZ]. However, the results by Butzer et al. and Heston et al. require more local regularity of the initial value than in Theorem 5.3.

Much more can be said about the relation between the smoothness of the initial value and the rate of convergence for the trinomial method. We recommend [KTW] for a further discussion.

Next we will focus on the convergence rate when the initial value f is discontinuous. Since such functions are not included in the Besov space B^s_∞ for any s we have no use of Theorem 5.3. For this problem the works of Berry and Esseen in the fourties are of great value (see [Be] or [Es]). By using methods from Fourier analysis they were able to show the following famous theorem.

Theorem 5.4. *Let $\{\zeta_i, i = 1, 2, \dots\}$ be a sequence of i.i.d. random variables with mean 0, variance 1 and finite absolute third moment. There is a constant C only depending on the third absolute moment such that the distribution function $F_n(x) = P(\frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_i \leq x)$ satisfies*

$$|F_n(x) - \Phi(x)| \leq C/\sqrt{n} \tag{5.14}$$

for any x, where Φ is the standard normal distribution function.

Let χ_A denote the characteristic function for the set $A \subseteq \mathbb{R}$ and set $f_a = \chi_{(-\infty,a]}$. Moreover, let $U_T^{(n)} = U_T^{(n,T)}$. As a consequence of equation (5.14) we get

$$|\,(U_T^{(n)} f_a)(x) - (U_T f_a)(x)\,| = O(1/\sqrt{n}), \quad \text{as } n \to \infty, \tag{5.15}$$

for any x. It is possible to show that the convergence rate in (5.15) cannot be better than $n^{-1/2}$ for the special case $\lambda = 1/2$. More precise, if $\lambda = 1/2$ then

$$|\,(U_T^{(2n)} f_a)(a) - (U_T f_a)(a)\,| \sim \frac{1}{2\sqrt{\pi n}} \tag{5.16}$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$ as n tends to infinity. We will now finish this section with a proof of equation (5.16). The proof is taken from [Es].

It is evident that

$$(U_T^{(2n)} f_a)(a) = P(Y_{2n} \leq 0) = \frac{1}{2}\big(1 + P(Y_{2n} = 0)\big).$$

where $Y_n = \sum_{i=1}^n X_i$. If the random walk Y ends up at 0 after $2n$ steps it has jumped upwards n times and downwards the same number of times. Since there are $\binom{2n}{n}$ numbers of such outcomes, each having probability 2^{-2n}, we have

$$P(Y_{2n} = 0) = \binom{2n}{n} 2^{-2n}.$$

Stirling's Formula tells us

$$n! \sim n^n e^{-n} \sqrt{2\pi n}.$$

Thus

$$P(Y_{2n} = 0) = \frac{(2n)!}{(n!)^2}\, 2^{-2n} \sim \frac{(2n)^{2n}\,\sqrt{4\pi n}}{n^{2n}\,2\pi n}\, 2^{-2n} = \frac{1}{\sqrt{\pi n}}$$

and therefore

$$|\,(U_T^{(2n)} f_a)(a) - (U_T f_a)(a)\,| \sim \frac{1}{2\sqrt{\pi n}}.$$

We are now ready to deal with the main problem of this chapter.

5.5 Estimating the Wiener Measure of Cylinder Sets Using the Trinomial Method

In this section we will return to the problem of estimating the Wiener measure of the cylinder set A_c, where $A_c = \{\omega \in \Omega\,;\, \omega(iT) \in I_i,\, i = 1, 2, \dots, m\}$

with $I_i = (a_i, b_i)$ and $-\infty \leq a_i < b_i \leq \infty$ for $i = 1, 2, \ldots, m$. Recall from Section 5.1 that $P(A_c) = v_0(0)$, where v_0 is given by the following recursion scheme

$$\begin{cases} v_m = \chi_m, \\ v_{i-1} = U_T\,(v_i\,\chi_i), \quad \text{for } 1 \leq i \leq m. \end{cases}$$

Above χ_i denotes the characteristic function of the interval I_i.

First, since

$$\begin{aligned} v_{m-1}(x) &= (U_T \chi_m)(x) \\ &= \Phi\Big(\frac{x - b_m}{\sqrt{T}}\Big) - \Phi\Big(\frac{x - a_m}{\sqrt{T}}\Big), \end{aligned} \tag{5.17}$$

the function $v_{m-1}(x)$ can easily be evaluated at arbitrary points x by using some approximation of the normal distribution Φ (see for instance [Hull], p.242). So in the following we will assume that we start the recursion above from the function v_{m-1}.

As described already in Section 5.1, for each $i = 1, 2, \ldots, m-1$ it is natural to estimate the function v_{i-1} by $U_T^{(n,T)}\,(v_i\,\chi_i)$. There is, however, one disadvantage with this approach. For any $i = 1, \ldots, m-1$ the function $v_i\,\chi_i$ is discontinuous at the boundary points of the interval I_i and, according to the discussion in the preceding section, a discontinuous initial value f may cause a slow convergence of the sequence $\{U_T^{(n,T)} f\}_{n=1}^{\infty}$ as n tends to infinity. But suppose for a moment that f can be written as

$$f = \phi - g, \tag{5.18}$$

where g is a function such that $U_T\,g$ can easily be evaluated analytically and ϕ is in some sense a smooth function. Then the discussion in Section 5.4 gives us strong reasons to believe that one will obtain a better estimate of $U_T f$ by using

$$U_T^{(n,T)}\phi - U_T g$$

instead of $U_T^{(n,T)} f$. Our next aim is to show how the functions $v_i\chi_i$, $i = 1, \ldots, m-1$ can be decomposed as in equation (5.18).

For the sake of simplicity, assume that I_i, $i = 1, 2, \ldots, m-1$, are bounded intervals, that is $a_i > -\infty$ and $b_i < \infty$. We will return to the special case when some of the intervals may be unbounded later on in this section.

Fix i such that $1 \leq i \leq m-1$. Note that the function

$$(t, x) \mapsto (U_t\,(v_{i+1}\,\chi_{i+1})\,)(x)\,, \quad t > 0,\ x \in \mathbb{R}$$

is a solution to heat equation and therefore must be infinitely differentiable with respect to both t as well as x (see [KS], p.254). In particular, we have

$v_i \in C^\infty(\mathbb{R})$, where $C^\infty(A)$ denotes the class of all infinitely differentiable functions on the open set $A \subseteq \mathbb{R}$.

Next, consider the functions

$$\psi_a(x) = e^{\gamma_a\,(x-a_i)} \sum_{k=0}^{d} \frac{\alpha_k}{k!}\,(x-a_i)^k$$

and

$$\psi_b(x) = e^{\gamma_b\,(x-b_i)} \sum_{k=0}^{d} \frac{\beta_k}{k!}\,(x-b_i)^k\,.$$

Here we assume that d is a positive integer. The constants γ_a and γ_b can be thought of as a positive and a negative number, respectively. However, we will for the moment put no restrictions on γ_a or γ_b. The coeffecients α_k and β_k above are chosen such that ψ_a and ψ_b equal the function v_i and its first d derivatives at the points a_i and b_i, respectively. Thus $v_i^{(k)}(a_i) = \psi_a^{(k)}(a_i)$ and $v_i^{(k)}(b_i) = \psi_b^{(k)}(b_i)$ for each $k = 0, 1, \ldots d$. The Leibnitz rule gives

$$\frac{d^k\psi_a}{dx^k}(a_i) = \sum_{j=0}^{k} \binom{k}{j} \gamma_a^{k-j}\alpha_j.$$

Thus we can define α_k recursively by

$$\alpha_k = v_i^{(k)}(a_i) - \sum_{j=0}^{k-1} \binom{k}{j} \gamma_a^{k-j}\alpha_j\,, \quad k = 0, 1, \ldots, d.$$

Similarly, β_k is given by

$$\beta_k = v_i^{(k)}(b_i) - \sum_{j=0}^{k-1} \binom{k}{j} \gamma_b^{k-j}\beta_j\,, \quad k = 0, 1, \ldots, d.$$

In practice, however, we will not (or rather cannot) differentiate the function v_i in order to estimate the coefficients α_k or β_k. Instead we will use numerical differentiation. This step will be described in greater details later on in this section.

Now set

$$g_i = \psi_a\,\chi_{(-\infty,a_i]} + \psi_b\,\chi_{[b_i,\infty)},$$

where χ_A denotes the characteristic function of the interval A, and

$$\phi_i = v_i\chi_i + g_i.$$

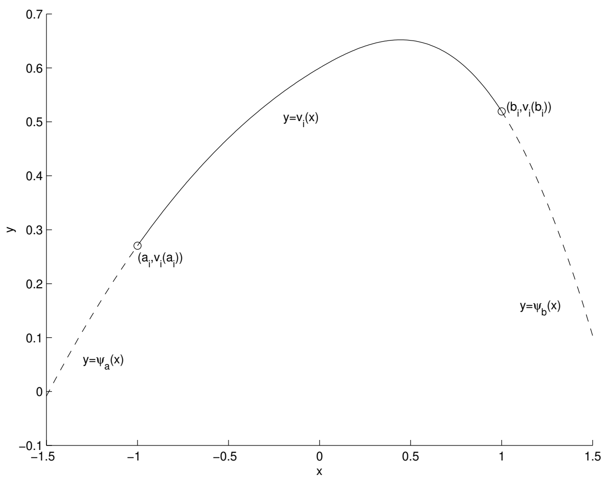

Figure 5.3: The functions v_i, ψ_a and ψ_b. The solid line is the graph of $x \mapsto v_i(x)$ for $x \in I_i$. The dashed lines are the graphs of $x \mapsto \psi_a(x)$ and $x \mapsto \psi_b(x)$ for $x \leq a_i$ and $x \geq b_i$, respectively.

Hence

$$\phi_i(x) = \begin{cases} \psi_a(x) & \text{if } x \leq a_i, \\ v_i(x) & \text{if } a_i < x < b_i, \\ \psi_b(x) & \text{if } x \geq b_i. \end{cases}$$

The function ϕ_i is obviously d times differentiable and, furthermore, since the d:th derivative $\phi_i^{(d)}$ is continuous and belongs to $C^\infty(\mathbb{R} \setminus \{a_i, b_i\})$, the function $\phi_i^{(d)}$ satisfies a local Zygmund condition, i.e. for each fixed $M > 0$ there exists a constant C such that for each $x, x+h \in [-M, M]$ we have

$$|\phi_i^{(d)}(x+h) - 2\phi_i^{(d)}(x) + \phi_i^{(d)}(x-h)| \leq C|h|.$$

If, in addition, we assume that $\gamma_a > 0$ and $\gamma_b < 0$ then $\phi_i^{(d)} \in C_0$ and thus $\phi_i \in B_\infty^{d+1}$. Since $U_T g_i$ can be evaluated using the normal distribution and elementary functions (see Lemma 5.1 below), we have obtained the desired decomposition

$$v_i \chi_i = \phi_i - g_i$$

as in equation (5.18). Moreover, note that if $d \geq 3$ then $\phi_i \in B_\infty^4$. Thus, if $d = 3$ and $\lambda = 1/6$ then we get $|(U_T^{(n,T)}\phi_i)(x) - (U_T\phi_i)(x)| = O(1/n^2)$ as $n \to \infty$.

To sum up, the method we propose can be described as follows. The Wiener probability $P(A_c) \approx \tilde{v}_0(0)$ where $\tilde{v}_0(0)$ is determined by the following recursion scheme (where, of course, we only calculate the functions $\tilde{v}_i$, $i = 1, 2, \dots, m-1$ at the mesh points).

$$\begin{cases} \tilde{v}_{m-1}(\cdot) = \Phi\Big(\frac{(\cdot)-b_m}{\sqrt{T}}\Big) - \Phi\Big(\frac{(\cdot)-a_m}{\sqrt{T}}\Big), \\ \tilde{v}_{i-1} = U_T^{(n,T)}\left(\tilde{v}_i \chi_i + g_i\right) - U_T g_i, \qquad \text{for } 1 \le i \le m-1. \end{cases} \tag{5.19}$$

Here for each $i = 1, 2, \dots, m-1$ the function g_i is defined in analogy to the above, the coefficients $\{\alpha_k\}_{k=0}^d$ and $\{\beta_k\}_{k=0}^d$ are chosen such that $\psi_a^{(k)}(a_i) = \tilde{v}_i^{(k)}(a_i)$ and $\psi_b^{(k)}(b_i) = \tilde{v}_i^{(k)}(b_i)$ for $k = 0, 1, \dots d$. As mentioned above, the derivatives of $\tilde{v}_i$ will be determined by numerical differentiation (see below).

Some remarks are in order before we derive an analytical expression of $U_T g_i$. It remains to determine the parameters λ, d, γ_a and γ_b. We will deal with this problem in the next section.

As we have already mentioned, since the function $\tilde{v}_i$ is merely calculated at discrete points we will need to find a method to estimate the derivatives of $\tilde{v}_i$ at the boundary points of I_i. Since we will solely use the algorithm in equation (5.19) for $d \le 3$ it suffices to estimate the first three derivatives. A natural approach to this problem is to differentiate an appropriate interpolations polynomial. If we let j_a be the smallest integer such that $j_a \Delta x > a_i$ and let q_a be the (interpolation) polynomial of degree 3 which satisfies

$$q_a\big((j_a + l)\Delta x\big) = \tilde{v}_i\big((j_a + l)\Delta x\big)\,, \quad l = -2, -1, 0, 1,$$

then the first three derivatives at the boundary point a_i can be estimated by

$$\frac{d^k \tilde{v}_i}{dx^k}(a_i) \approx \frac{d^k q_a}{dx^k}(a_i)\,, \quad k = 0, 1, 2, 3.$$

In a similar way we obtain approximations for the derivatives at the boundary point b_i from the equations

$$\frac{d^k \tilde{v}_i}{dx^k}(b_i) \approx \frac{d^k q_b}{dx^k}(b_i)\,, \quad k = 0, 1, 2, 3,$$

where q_b denotes the polynomial of degree 3 which satisfies

$$q_b\big((j_b + l)\Delta x\big) = \tilde{v}_i\big((j_b + l)\Delta x\big)\,, \quad l = -1, 0, 1, 2,$$

and where j_b is the greatest integer such that $j_b \Delta x < b_i$.

So far we have assumed that $a_i > -\infty$ and $b_i < \infty$. If $a_i = -\infty$ or $b_i = \infty$ for some i, then we simply let $g_i(x) = \psi_b(x)\chi_{[b_i,\infty)}$ or $g_i(x) = \psi_a(x)\chi_{(-\infty,a_i]}$, respectively, in the equation (5.19).

Using the following lemma, the functions $U_T g_i$, $i = 1, 2, \dots, m-1$ can be evaluated in an efficient way.

Lemma 5.1. *Let* $\varphi(x) = \frac{d}{dx}\Phi(x)$. *If*

$$g = \psi_a\,\chi_{(-\infty,a\,]} + \psi_b\,\chi_{[\,b,\infty)}$$

where ψ_a *and* ψ_b *are defined by*

$$\psi_a(x) = e^{\gamma_a\,(x-a)} \sum_{k=0}^{d} \frac{\alpha_k}{k!}\,(x-a)^k$$

and

$$\psi_b(x) = e^{\gamma_b\,(x-b)} \sum_{k=0}^{d} \frac{\beta_k}{k!}\,(x-b)^k$$

then

$$\begin{aligned}(U_T\,g)(x) =& e^{\gamma_a(x-a)+\gamma_a^2\,T/2} \sum_{k=0}^{d} \Big(\,\frac{\hat\alpha_k}{k!}\,M_k\big(\frac{a-x}{\sqrt{T}} - \gamma_a\sqrt{T}\,\big)\Big)\\ &+ e^{\gamma_b(x-b)+\gamma_b^2\,T/2} \sum_{k=0}^{d} \Big(\,\frac{\hat\beta_k}{k!}\,(-1)^k M_k\big(\frac{x-b}{\sqrt{T}} + \gamma_b\sqrt{T}\,\big)\Big)\end{aligned}$$

where

$$\hat\alpha_k = T^{k/2} \sum_{i=0}^{d-k} \frac{\alpha_{i+k}}{i!}(x-a+\gamma_a T)^i$$

and

$$\hat\beta_k = T^{k/2} \sum_{i=0}^{d-k} \frac{\beta_{i+k}}{i!}(x-b+\gamma_b T)^i$$

and where the functions M_k *are defined recursively by*

$$M_k(y) = \begin{cases} \Phi(y) & \text{if } k=0,\\ -\varphi(y) & \text{if } k=1,\\ y^{k-1}\,M_1(y) + (k-1)\,M_{k-2}(y) & \text{if } k=2,3,\dots,d. \end{cases} \tag{5.20}$$

Proof. Let $\hat\alpha_k$ be defined as above and let $\hat\gamma_a = \gamma_a\sqrt{T}$. If we set $\hat\psi_a(\xi) = \psi_a(x+\sqrt{T}\xi)$ then

$$\hat\psi_a(\xi) = e^{\gamma_a(x-a)+\hat\gamma_a\xi} \sum_{k=0}^{d} \frac{\hat\alpha_k}{k!}(\xi-\hat\gamma_a)^k,$$

since for any $k = 0, 1, \dots, d$,

$$\Big(\frac{d^k}{d\xi^k} \sum_{i=0}^{d} \frac{\alpha_i}{i!} (x + \sqrt{T}\xi - a)^i \Big)\Big|_{\xi=\hat{\gamma}_a} = T^{k/2} \sum_{i=0}^{d-k} \frac{\alpha_{i+k}}{i!} (x + \sqrt{T}\hat{\gamma}_a - a)^i$$
$$= \hat{\alpha}_k.$$

Let $h_a = (a - x)/\sqrt{T}$. The scaling property for Brownian motion and the definition of $\hat{\psi}_a$ give

$$\begin{aligned} \big(U_T\,(\psi_a\,\chi_{(-\infty,a]}\,)\big)(x) =& E\big[\psi_a(x+W_T)\,\chi_{(-\infty,a]}(x+W_T)\,\big] \\ =& E\big[\hat{\psi}_a(W_1)\,\chi_{(-\infty,h_a]}(W_1)\,\big] \\ =& e^{\gamma_a(x-a)} \sum_{k=0}^{d} \frac{\hat{\alpha}_k}{k!} \int_{-\infty}^{h_a} e^{\hat{\gamma}_a\xi}\,(\xi - \hat{\gamma}_a)^k \varphi(\xi)d\xi. \end{aligned}$$

Note moreover that

$$\begin{aligned} \int_{-\infty}^{h_a} e^{\hat{\gamma}_a\xi}\,(\xi - \hat{\gamma}_a)^k \varphi(\xi)d\xi &= \int_{-\infty}^{h_a} (\xi - \hat{\gamma}_a)^k\, e^{\hat{\gamma}_a^2/2}\, \varphi(\xi - \hat{\gamma}_a)d\xi \\ &= e^{\hat{\gamma}_a^2/2} \int_{-\infty}^{h_a-\hat{\gamma}_a} \xi^k\, \varphi(\xi)\, d\xi. \end{aligned}$$

Thus, if we set

$$M_k(y) = \int_{-\infty}^{y} \xi^k \varphi(\xi)d\xi \tag{5.21}$$

for each integer $k \geq 0$, then

$$\big(U_T\,(\psi_a\,\chi_{(-\infty,a]}\,)\,\big)(x) = e^{\gamma_a(x-a)+\hat{\gamma}_a^2/2} \sum_{k=0}^{d} \frac{\hat{\alpha}_k}{k!}\, M_k(h_a - \hat{\gamma}_a).$$

By using a similar argument as above we get

$$\big(U_T\,(\psi_b\,\chi_{[b,\infty)}\,)\,\big)(x) = e^{\gamma_b(x-b)+\hat{\gamma}_b^2/2} \sum_{k=0}^{d} \frac{\hat{\beta}_k}{k!} \int_{h_b-\hat{\gamma}_b}^{\infty} \xi^k \varphi(\xi)d\xi,$$

where $h_b = (b - x)/\sqrt{T}$ and $\hat{\gamma}_b = \gamma_b\sqrt{T}$ and where the $\hat{\beta}_k$:s are defined as in the proposition. From the symmetry of the normal density we now conclude

$$\big(U_T\,(\psi_b\,\chi_{[b,\infty)}\,)\,\big)(x) = e^{\gamma_b(x-b)+\hat{\gamma}_b^2/2} \sum_{k=0}^{d} \frac{\hat{\beta}_k}{k!} (-1)^k\, M_k(\hat{\gamma}_b - h_b).$$

It remains to show that the functions M_k, $k = 0, 1, \ldots, d$, satisfy equation (5.20). It is evident that $M_0(y) = \Phi(y)$. Since $\frac{d}{d\xi}\varphi(\xi) = -\xi\varphi(\xi)$ we also have $M_1(y) = -\varphi(y)$. Partial integration now yields for $k \geq 2$

$$\begin{aligned} M_k(y) &= -\xi^{k-1}\,\varphi(\xi)\Big|_{\xi=y} + (k-1)\int_{-\infty}^{y} \xi^{k-2}\,\varphi(\xi)d\xi \\ &= y^{k-1}\,M_1(y) + (k-1)\,M_{k-2}(y)\,. \end{aligned}$$

□

Let us make some comments about the computional complexity before we finish this section. Note that for each fixed $i = 1, 2, \ldots, m-1$ $O(n)$ computations are required to calculate the function $U_T g_i$ in equation (5.19). On the other hand, for each $i = 1, 2 \ldots, m-1$, the number of computations to evaluate the functions $U_T^{(n,T)}(\tilde{v}_i\,\chi_i + g_i)$ or $U_T^{(n,T)}(\tilde{v}_i\,\chi_i)$ is of order $O(n^2)$. Thus, the correction term $U_T g_i$ added to the trinomial method will not to any appreciable extent extend the computional time. However, since the algorithm in equation (5.19) requires several evaluations of polynomials it is possible to improve the performance of the algorithm by using Horner's Scheme (see [RW], p.372), which is an efficient way to evaluate a polynomial.

5.6 Numerical Examples and Conclusions

In this section we will compute the value of a discrete double barrier option using the method described in the previous section. The theoretical price v at time $t = 0$ of a discrete double barrier knock-out call with expiry date T_m is given by

$$v = e^{-rT_m} E^Q\big[\max(\,S_{T_m} - K,\,0)\,1_{\{\,H_l < S_t < H_u\,,\,t \in M\,\}}\big].$$

Assume for the sake of simplicity, that the barriers H_l and H_u and strike price K satisfy $H_l \leq K \leq H_u$. Moreover, assume that the monitoring dates $M = \{T_1, T_2, \ldots, T_m\}$ are equally spaced in time with $T = T_i - T_{i-1}$ and set for each $i = 1, 2, \ldots, m$,

$$a_i = \frac{\ln(H_l/S_0) - \mu T_i}{\sigma} \quad \text{and} \quad b_i = \frac{\ln(H_u/S_0) - \mu T_i}{\sigma}$$

where $\mu = r - q - \sigma^2/2$. Introduce the cylinder set

$$A_c = \{\,a_i < W_{T_i} < b_i,\, i = 1, 2, \ldots, m-1\}.$$

The Markov property now yields

$$\begin{aligned} v =& e^{-rT_m} E^Q\big[\max(\,S_{T_m} - K,\,0)\,1_{\{\,H_l < S_t < H_u\,,\,t \in M\,\}}\big] \\ =& e^{-rT_{m-1}} E\big[\,E[\,e^{-rT}\max(\,S_0 e^{\mu T_m + \sigma W_{T_m}} - K, 0)\,1_{\{a_m < W_{T_m} < b_m\}}|\mathcal{F}_{T_{m-1}}]\,1_{A_c}\,\big] \\ =& e^{-rT_{m-1}} E\big[\,(U_T f)(W_{T_{m-1}})\,1_{A_c}\,\big]\,, \end{aligned}$$

where $f(x) = e^{-rT}\max(S_0 e^{\mu T_m+\sigma x} - K, 0)\,\chi_{(\,a_m, b_m)}(x)$. So if we modify the recursion scheme in equation (5.19) by letting

$$\tilde{v}_{m-1}(x) = (U_T f)(x),$$

then $e^{-rT_{m-1}}\tilde{v}_0(0)$ will be our approximation to the theoretical price v.

First of all we need to derive an explicit expression for $\tilde{v}_{m-1}$. To this end, let

$$s(x) = S_0 e^{\mu T_{m-1}+\sigma x}.$$

Moreover, introduce the sets $B_x = \{\,a_m - x < W_T < b_m - x\}$ and $C_x = \{W_T \geq c - x\}$ where $c = (\ln(K/S_0) - \mu T_m)/\sigma$. Using Lemma 2.1 a last time in this paper yields

$$\begin{aligned}\tilde{v}_{m-1}(x) =& e^{-rT} E\big[\max(s(x)\,e^{\mu T+\sigma W_T} - K, 0)\,1_{B_x}\big]\\ =& e^{-rT} E\big[(s(x)\,e^{\mu T+\sigma W_T} - K)\,1_{B_x}\,1_{C_x}\big]\\ =& s(x)e^{-qT}P(B_x \cap C_x - \varsigma) - Ke^{-rT}P(B_x \cap C_x),\end{aligned}$$

where $\varsigma(t) = \sigma t$. Hence

$$\begin{aligned}\tilde{v}_{m-1}(x) =& s(x)e^{-qT}\Big(\Phi\big(\frac{x+\sigma T - b_m}{\sqrt{T}}\big) - \Phi\big(\frac{x+\sigma T - c}{\sqrt{T}}\big)\Big)\\ &- Ke^{-rT}\Big(\Phi\big(\frac{x - b_m}{\sqrt{T}}\big) - \Phi\big(\frac{x-c}{\sqrt{T}}\big)\Big)\end{aligned}$$

since $a_m \leq c \leq b_m$.

We are now in the position to present some examples. In Figure 5.4 we present the difference between $\tilde{v}_0(0)$ and v as a function of n for different values on d. In this first example we have chosen λ to be $1/3$ and $\gamma_a = \gamma_b = 0$. The option price is approximately 1.2624 (cf the straight line in Figure 5.4). Consider first the case when we use just the basic trinomial method and do not add (or withdraw) any polynomial. The corresponding price is denoted $d = -1$ in Figure 5.4. We see that the convergence is slow and oscillating. If we add a polynomial of degree $d = 0$ the convergence is more regular but the rate of convergence seems to be more or less the same. In contrast to these examples, when d is equal to 1, which corresponds to differentiable initial values, we get a faster and smoother convergence. When $d = 2$ or 3 the convergence rate do not increase. In fact, it becomes slower.

Figure 5.4 reflects very well the convergence behaviour for the proposed method for all choices of $\lambda \in (0, 1/2]$ *except* $\lambda = 1/6$, that is, when X_1 is consistent of order 4.

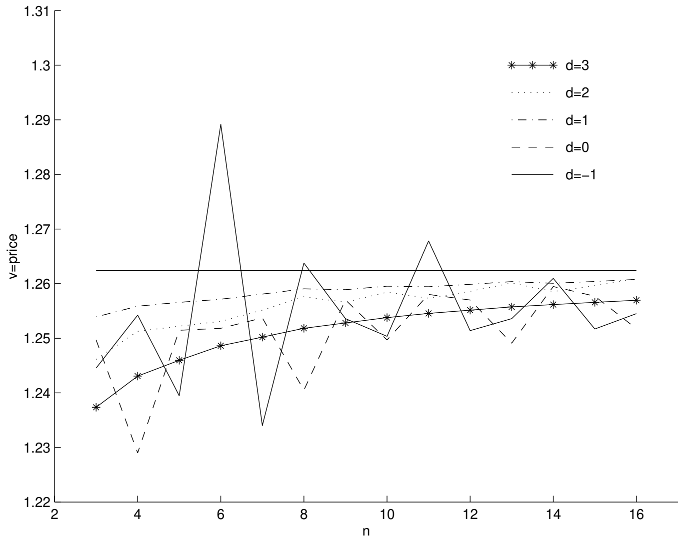

Figure 5.4: Convergence rate for the proposed method when $\lambda = 1/3$ and $\gamma_a = \gamma_b = 0$. The option parameters are $S_0 = 100$, $K = 90$, $H_l = 80$, $H_u = 120$, $\sigma = 0.3$, $r = 0.1$, $q = 0.0$, $T_m = 1$ year and $m = 50$ (number of monitoring times, corresponds to weekly monitoring).

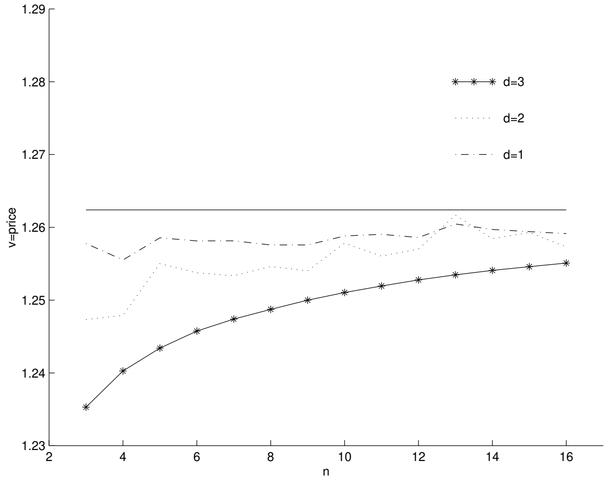

Figure 5.5: Convergence rate for the proposed method when $\lambda = 1/2$. The other parameters are as in Figure 5.4.

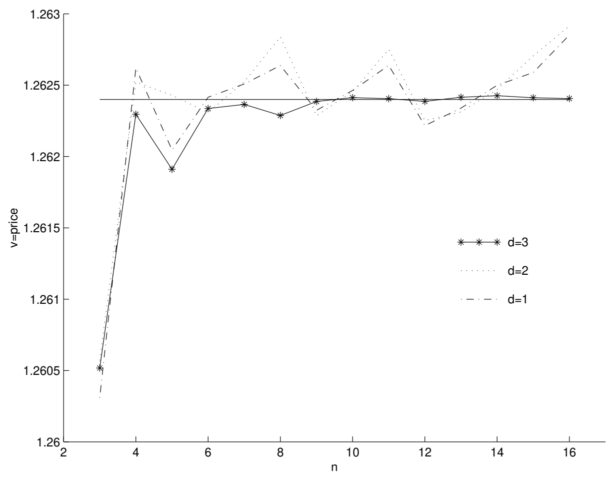

Figure 5.6: Convergence rate for the proposed method when $\lambda = 1/6$. The other parameters are as in Figure 5.4.

In the next two figures we present the convergence rate for $d = 1, 2, 3$ and $\lambda = 1/2$ (the binomial method) and $\lambda = 1/6$, respectively. The option parameters are the same as in the previous example.

In the special case $\lambda = 1/2$ we see in Figure 5.5 that the convergence pattern is roughly the same as in the case $\lambda = 1/3$. On the other hand, if $\lambda = 1/6$ we see in Figure 5.6 that the method obtain the best convergence rate when $d = 3$. In Table 5.1 we have collected the prices obtained for different values of λ and d. The table clearly illustrates that the fastest convergence occurs when $d = 3$ and $\lambda = 1/6$. Finally, Figure 5.7 shows how the smoothing of the initial value improves the convergence rate.

In the next example we shall investigate how the values of γ_a and γ_b influence the error, or rather, if there is a difference in the convergence rate in the two cases $\gamma_a = \gamma_b = 0$ and $\gamma_a > 0$, $\gamma_b < 0$. From a theoretical point there is distinct difference between these cases. If $\gamma_a > 0$ and $\gamma_b < 0$, then the function g (cf Section 5.5) is bounded, whereas if $\gamma_a = \gamma_b = 0$ then $g(x) = O(x^d)$ as x tends to infinity.

Recall that the density function for the standard normal distribution decreases as $O(e^{-x^2/2})$ as x tends to infinity. Thus we believe that the growth in g has hardly any greater impact on the rate of convergence, as the example in Table 5.2 indicates. Unfortunately, we have not been able to prove this. Hopefully future research will bring an answer to this problem. But still, we suggest that the algorithm in equation (5.19) should be used with the parameter values $d = 3$, $\lambda = 1/6$ and $\gamma_a = \gamma_b = 0$. Setting $\gamma_a = \gamma_b = 0$ has one practical advantage, the algorithm is easier to implement.

In the final example we have compared our method with an algorithm

n	$d=1,\ \lambda=1/3$	$d=1,\ \lambda=1/2$	$d=3,\ \lambda=1/6$
3	1.2539	1.2578	1.2605
4	1.2559	1.2555	1.2623
5	1.2566	1.2586	1.2619
6	1.2572	1.2581	1.2623
7	1.2581	1.2581	1.2624
8	1.2591	1.2576	1.2623
9	1.2589	1.2576	1.2624
10	1.2595	1.2588	1.2624
11	1.2594	1.2590	1.2624
12	1.2599	1.2586	1.2624
13	1.2604	1.2605	1.2624
14	1.2601	1.2597	1.2624
15	1.2604	1.2594	1.2624
16	1.2608	1.2591	1.2624

Table 5.1: Convergence rate for the proposed method for different values on λ and d. The option parameters are as in Figure 5.4.

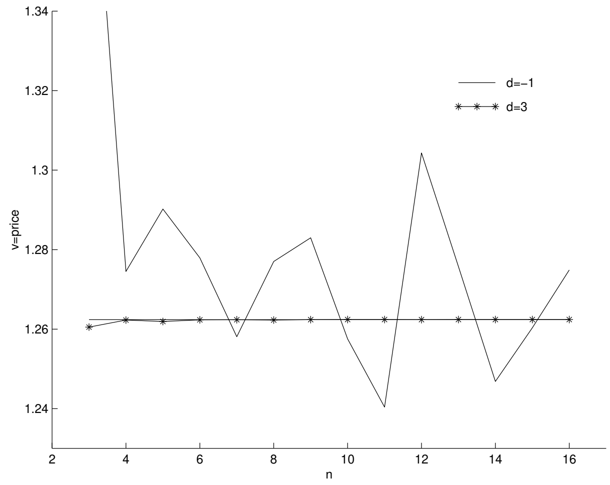

Figure 5.7: Convergence rate for the proposed method when $\lambda = 1/6$. The other parameters are as in Figure 5.4.

n	$\gamma_a = 0.0$	$\gamma_a = 0.1$	$\gamma_a = 1$	$\gamma_a = 10$
4	1.2605	1.2605	1.2606	1.2608
5	1.2623	1.2623	1.2623	1.2625
6	1.2619	1.2619	1.2619	1.2620
7	1.2623	1.2623	1.2624	1.2624
8	1.2624	1.2624	1.2624	1.2624
9	1.2623	1.2623	1.2623	1.2623
10	1.2624	1.2624	1.2624	1.2624
11	1.2624	1.2624	1.2624	1.2624
12	1.2624	1.2624	1.2624	1.2624

Table 5.2: Convergence rate for the proposed method for different values on γ_a and γ_b. The value on d, λ and γ_b are 3, 1/6 and $-\gamma_a$, respectively. The option parameters are as in Figure 5.4.

N	BGK	BGK (2-pt Extrapol.)	N	H
256	9.4969		40	9.4895
504	9.4935	9.4899	60	9.4907
1240	9.4919	9.4907	80	9.4907
2308	9.4912	9.4905	100	9.4906
4524	9.4909	9.4905	120	9.4905
8632	9.4907	9.4905	140	9.4905

Table 5.3: The value of a discrete down-and-out call, the option parameters are $S_0 = K = 100$, $H = 95$, $T_4 = 0.2$ year (time of maturity), $\sigma = 0.6$, $r = 0.1$, and $q = 0.0$. There are 4 monitoring dates which are equally spaced in time, i.e. the monitoring dates are given by $\{\Delta t, 2\Delta t, 3\Delta t, 4\Delta t\}$ where $\Delta t = T_4/4$. N denotes the total number of iterations, i.e. $N = 4n$. The trinomial parameters in H are $\gamma_a = \gamma_b = 0$, $d = 3$ and $\lambda = 1/6$.

developed by Broadie et al. (see [BGK2]) which is designed to estimate the value of discrete (single) barrier options. For simplicity we henceforth call this method the BGK method. The BGK method has similarities with the overshoot method presented in Chapter 4. A discrete barrier at place H is replaced by a discrete barrier at place $H\exp(\pm 0.5\sigma\Delta x)$, with $+$ for an upper barrier and $-$ for a lower barrier. The factor 0.5 is the trinomial analog of the factor β described in Chapter 4. Subsequently the theoretical value is computed using the trinomial method on a mesh with the property that certain nodes on the mesh coincide with the new barrier. Numerical experiments in that paper indicate that the convergence rate for the method is $O(1/N)$. In order to increase the convergence rate a Richardson interpolation is used. For further details, see [BGK2].

Table 5.3 shows results from the different methods. The values in the second and third column are taken from a numerical example in [BGK2]. The BGK method has been used with as well as without Richardson extrapolation. In the final column we have the values from the method presented in this work. As we can see, in this example the method presented in this paper outperforms the BGK method.

5.7 Suggestions for Future Research

We shall conclude this chapter with some suggestions for future research and potential improvements of the algorithm. The research presented in this chapter leaves certain questions unanswered. It would be of great interest to know how the estimations of the derivatives $\tilde{v}_i^{(k)}$, $i = 1, 2, \ldots, m-1$, influence the total error, i.e. the difference between $\tilde{v}_0(0)$ and $P(A_c)$. A better insight in this problem may lead to better methods to estimate the derivatives.

It would also be of great value to prove certain modifications of Theorem 5.3. In our application we are perhaps more interested in pointwise estimates of the error rather than estimates in the supremum norm. It seems plausible that the convergence rate for

$$(U_T^{(n,T)} f)(x) - (U_T f)(x) \tag{5.22}$$

for some fixed x mainly depends on f around some neigbourhood of x. Therefore it may be possible to derive sharp point-wise bounds for the difference in equation (5.22) without having to assume that the initial value f is in C_0 (for instance).

Appendix A

The purpose of this appendix is to state the Cameron-Martin theorem, which is frequently used in this work, and to this end we need some definitions. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be our probability space, defined as in Section 2.2. In addition, assume $\eta : [0,T] \to \mathbb{R}$ is a deterministic function in $L^2[0.T]$, i.e $\int_{[0,T]} \eta^2(t)dt < \infty$ where T is a fixed positive number. Set

$$Z_t = \exp(-\frac{1}{2}\int_0^t \eta^2(s)ds + \int_0^t \eta(s)dW_s)$$

for $t \in [0,T]$ and introduce a new measure, denoted P^η, defined by

$$P^\eta(A) = E^P[1_A Z_T] \ , \quad A \in \mathcal{F}.$$

Because $Z_T > 0$ the measure P^η must be equivalent to P. Furthermore, Z is a martingale with respect to $(P, \mathcal{F}_t; 0 \leq t \leq T)$ (see [KS], p. 191 and p. 199). As a consequence of the martingale property we get

$$\begin{aligned} P^\eta(\Omega) &= E^P[Z_T] \\ &= E^P[Z_0] \\ &= 1 \end{aligned}$$

and it follows that P^η is a probability measure. Cameron-Martin's theorem now tells us, how one can construct a P^η-Brownian motion with the aid of a P-Brownian motion W.

Theorem 1. *Let*

$$W_t^\eta = W_t - \int_0^t \eta(s)ds$$

for $t \in [0,T]$ where η is defined as above, then W^η is a Brownian motion with respect to $(P^\eta, \mathcal{F}_t; 0 \leq t \leq T)$.

For a proof of this theorem, see [KS], p. 190.

Appendix B

The topic of this appendix is the strong Markov property for Brownian motion. As the name suggests, the strong Markov property is a generalization of the Markov property. Intuitively the Markov property says "if $s > 0$ then $W_{t+s} - W_s$, $t > 0$ is a Brownian motion, independent of what happened before time s". The strong Markov property says that this also holds for bounded stopping times, that is, "if Λ is a bounded stopping time then $W_{t+\Lambda} - W_\Lambda$, $t > 0$ is a Brownian motion, independent of what happened before time Λ".

To be able to give a more rigorous description of the strong Markov property, we will need some definitions. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be our probability space, defined as in Chapter 1. Introduce also a collection of operators $\{U_t\}_{t\geq 0}$ defined on the set of all bounded Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ in the following way

$$(U_t f)(x) = E\big[f(x + W_t)\big].$$

The strong Markov property for Brownian motion can now be described as follows,

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Borel measurable function. If Λ is a stopping time with respect to $\{\mathcal{F}_t\}$ such that $\Lambda \leq T$, where T is a fixed positive number, then*

$$E[f(W_T) \mid \mathcal{F}_\Lambda](\omega) = (U_{T-\Lambda(\omega)} f)(W_{\Lambda(\omega)}(\omega)) \quad P\text{-}a.s.$$

For a proof of this theorem, see [KS], Section 2.6.

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