Explicit isoperimetric constants, phase transitions in the random-cluster and Potts models, and Bernoullicity

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Abstract

The random-cluster model is a dependent percolation model that has applications in the study of Ising and Potts models. In this paper, several new results for the random-cluster model with cluster parameter $q \geq 1$ are obtained. These include an explicit pointwise dynamical construction of random-cluster measures for arbitrary graphs, and for unimodular transitive graphs, lack of percolation for the free random-cluster measure at the lower critical value on nonamenable graphs, and a number of inequalities for the critical values. Some of these inequalities lead to considerations of isoperimetric constants in certain hyperbolic graphs, and the first nontrivial explicit calculations of such constants are obtained. Applications to the Potts model include Bernoullicity in the $\mathbb{Z}^d$ case at all temperatures, and non-robust phase transition in the case of nonamenable regular graphs.

1 Introduction

One of the most important and much-studied dependent percolation models today is the random-cluster model. It was introduced in 1972 by Fortuin and Kasteleyn [24], and after a decade and a half of relative silence, the model was revived in the late 1980s with the influential papers by Swendsen and Wang [57], Edwards and Sokal [19], and Aizenman, Chayes, Chayes, and Newman [2]. Since then, the random-cluster model has served as a major tool in studying Ising and Potts models, and has also been studied in its own right by several authors. This paper is an investigation of a number of aspects of the random-cluster model, and we shall obtain results that are intrinsic to the model itself as well as applications to the Potts model. We shall work mainly in settings with more general graph structures (Cayley graphs, transitive graphs, etc.) than the usual $\mathbb{Z}^d$ setting, but some of our results are new and interesting also in the $\mathbb{Z}^d$ case. Our main results are the following:

- An explicit pointwise **dynamical construction** of random-cluster measures is obtained (Section 3).

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- The Potts model on $\mathbb{Z}^d$ (and more generally on amenable Cayley graphs) is shown to satisfy a mixing condition known as **Bernoullicity** (Section 4) at all temperatures. Our proof is the simplest to date even for those cases where the result was known previously.

- For unimodular transitive nonamenable graphs, we extend the result of Benjamini, Lyons, Peres and Schramm [5, 6] for i.i.d. percolation to show **lack of percolation** for the free random-cluster measure at the lower critical value on nonamenable graphs (Theorem 5.2).

- We consider **four** (possibly different) **critical values** for the random-cluster model on a given Cayley graph, and we sort out how these can relate to each other (Sections 5 and 6). For this purpose, we give the first explicit nontrivial calculations of positive **isoperimetric constants**.

- The random-cluster model is exploited to show that the Potts model on nonamenable regular graphs for entire intervals of temperatures exhibits phase transition but not so-called **robust phase transition** (Section 7).

We shall begin by recalling, in Section 2, some basics concerning random-cluster and Potts models. After that, we come in Section 3 to the dynamical construction of random-cluster measures. It is a kind of "backwards coupling" construction, inspired by the Propp-Wilson algorithm [51]. The construction is then applied to obtain the Bernoullicity result in Section 4. Another point of the dynamical construction is to obtain natural couplings between random-cluster measures in various situations. Such a coupling is used to prove the lack of percolation result in Section 5. Most of our results on the relations between the four critical values p_c^{wired}, p_c^{free}, p_u^{wired}, and p_u^{free} are fairly immediate. An exception is the result that $p_u^{\text{wired}}(q) < p_c^{\text{free}}(q)$ can occur for some graphs and some q, which requires careful analysis (carried out in Section 6) of the random-cluster model on certain tilings of the hyperbolic plane. Parts of this analysis are based on the Peierls-type comparison methods of Jonasson and Steif [41] and Jonasson [40], and these methods are also exploited in Section 7 to obtain our result on non-robust phase transition for the Potts model.

Since it is of independent interest, we give here our result on isoperimetric constants. A regular euclidean polygon of $d^\dagger$ sides has interior angles $\pi(1-2/d^\dagger)$. In order for such polygons to form a tessellation of the plane with d polygons meeting at each vertex, we must have $\pi(1-2/d^\dagger) = 2\pi/d$, i.e., $1/d+1/d^\dagger = 1/2$, or, equivalently, $(d-2)(d^\dagger-2) = 4$. In all such cases, tessellations have been well known since antiquity. In the hyperbolic plane, the interior angles can take any value in $\big(0, \pi(1-2/d)\big)$, whence a tessellation exists only if $1/d+1/d^\dagger < 1/2$, or, equivalently, $(d-2)(d^\dagger-2) < 4$; again, this condition is sufficient as well, as has been known since the 19th century.

Let $G = (V, E)$ be the planar graph formed by the edges and vertices of one of these regular tessellations by polygons with $d^\dagger$ sides and with degree d at each vertex. Given a finite set $K \subset V$, write $\partial_E K$ for the set of edges with exactly one endpoint in K. We prove in Theorem 6.1 that

$$\inf\left\{\frac{|\partial_E K|}{|K|}\,;\ K \subset V \text{ finite and nonempty}\right\} = (d-2)\sqrt{1-\frac{4}{(d-2)(d^\dagger-2)}}\,.$$

This should be compared to the regular tree of degree d, where the left-hand side is equal to $d-2$.

2 Background

In the following subsections we recall some preliminaries on random-cluster and Potts models, and a little bit about stochastic domination and various classes of infinite graph structures. General references for Sections 2.1–2.4 are Häggström [34] and Georgii, Häggström and Maes [27], whereas for Section 2.5 we refer to Benjamini et al. [5].

2.1 Random-cluster and Potts models on finite graphs

Let $G = (V, E) = \big((V(G), E(G)\big)$ be a finite graph. An edge $e \in E$ connecting two vertices $x, y \in V$ is also denoted $[x, y]$. An element ξ of $\{0, 1\}^E$ will be identified with the subgraph of G that has vertex set V and edge set $\{e \in E\,;\ \xi(e) = 1\}$. An edge e with $\xi(e) = 1$ ($\xi(e) = 0$) is said to be open (closed). Of central importance to us will be the number of connected components of ξ, which will be denoted $\|\xi\|$. We emphasize that in the definition of $\|\xi\|$, isolated vertices in ξ also count as connected components.

The **random-cluster measure** $\mathsf{RC} := \mathsf{RC}^G_{p,q}$ (sub- and superscripts will be dropped whenever possible) with parameters $p \in [0, 1]$ and $q > 0$, is the probability measure on $\{0, 1\}^E$ that to each $\xi \in \{0, 1\}^E$ assigns probability

$$\mathsf{RC}(\xi) := \frac{q^{\|\xi\|}}{Z} \prod_{e\in E} p^{\xi(e)}(1-p)^{1-\xi(e)}\,, \tag{1}$$

where $Z := Z^G_{p,q} := \sum_{\xi\in\{0,1\}^E} q^{\|\xi\|} \prod_{e\in E} p^{\xi(e)}(1-p)^{1-\xi(e)}$ is a normalizing constant. It is easy to see that if X is a $\{0, 1\}^E$-valued random object with distribution RC, then we have, for each $e = [x, y] \in E$ and each $\xi \in \{0, 1\}^{E\setminus\{e\}}$, that

$$\mathsf{RC}\big(X(e) = 1 \mid X(E \setminus \{e\}) = \xi\big) = \begin{cases} p & \text{if } x \leftrightarrow y, \\ \frac{p}{p+(1-p)q} & \text{otherwise,} \end{cases} \tag{2}$$

where $x \leftrightarrow y$ is the event that there is an open path (i.e., a path of open edges) from x to y in $X(E \setminus \{e\})$. Here, $X(E')$ denotes the restriction of X to E' for $E' \subseteq E$.

When $q = 1$, we see that all edges are independently open and closed with respective probabilities p and $1 - p$, so that we get the usual i.i.d. bond percolation model on G, which we refer to as **Bernoulli**(p) percolation. All other choices of q yield dependence between the edges. Throughout the paper, **we shall assume that** $q \geq 1$. This conforms with most other studies of the random-cluster model, and there are two reasons for doing this. First, when $q \geq 1$, the conditional probability in (2) becomes increasing not only in p but also in ξ, and this allows a set of very powerful stochastic domination arguments that are not available for $q < 1$. Second, it is only random-cluster measures with $q \in \{2, 3, \dots\}$ that have proved to be useful the analysis of Potts models, which we now go on to describe.

Given the finite graph G and an integer $q \geq 2$, the q-state Potts model provides a model for picking an element $\omega \in \{1, \dots, q\}^V$ in a random but correlated way. The values $1, \dots, q$ attainable at each vertex $x \in V$ are called **spins**. Fix the so-called **inverse temperature** parameter $\beta \geq 0$, and define the **Gibbs measure for the q-state Potts model on G at inverse temperature** β, denoted $\mathsf{Pt} := \mathsf{Pt}^G_{q,\beta}$, as the probability measure that to each $\omega \in \{1, \dots, q\}^V$ assigns probability

$$\mathsf{Pt}(\omega) := \frac{1}{Z} \exp\left(-2\beta \sum_{[x,y]\in E} \mathbf{1}_{\{\omega(x)\neq\omega(y)\}}\right),$$

where Z is another normalizing constant (different from the one in (1)). The main link between random-cluster and Potts models is the following well-known result.

Proposition 2.1. *Fix the finite graph G, an integer $q \geq 2$, and $p \in [0,1]$. Pick a random edge configuration $X \in \{0,1\}^E$ according to the random-cluster measure $\mathsf{RC}^G_{p,q}$. Then, for each connected component $\mathcal{C}$ of X, pick a spin uniformly from $\{1,\ldots,q\}$, and assign this spin to all vertices of $\mathcal{C}$. Do this independently for different connected components. The $\{1,\ldots,q\}^V$-valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\mathsf{Pt}^G_{q,\beta}$ for the q-state Potts model on G at inverse temperature $\beta := -\frac{1}{2}\log(1-p)$.*

This provides a way of reformulating problems about pairwise dependencies in the Potts model into problems about connectivity probabilities in the random-cluster model. Aizenman et al. [2] exploited such ideas to obtain results about the phase transition behavior of the Potts model. Since then the random-cluster model has been an important tool for studying the Potts model; see, e.g., [34] for a list of references.

2.2 Stochastic domination and weak convergence

Let E be any finite or countably infinite set. (The reason for denoting it by E is that in our applications, it will be an edge set.) For two configurations $\xi, \xi' \in \{0,1\}^E$, we write $\xi \preccurlyeq \xi'$ if $\xi(e) \leq \xi'(e)$ for all $e \in E$. A function $f : \{0,1\}^E \to \mathbf{R}$ is said to be increasing if $f(\xi) \leq f(\eta)$ whenever $\xi \preccurlyeq \eta$. For two probability measures μ and μ' on $\{0,1\}$, we say that μ is **stochastically dominated** by μ', writing $\mu \overset{\mathcal{D}}{\preccurlyeq} \mu'$, if

$$\int_{\{0,1\}^E} f\,d\mu \leq \int_{\{0,1\}^E} f\,d\mu' \tag{3}$$

for all bounded increasing f. By Strassen's Theorem, this is equivalent to the existence of a coupling $\mathbf{P}$ of two $\{0,1\}^E$-valued random objects X and X', with respective distributions μ and μ', such that $\mathbf{P}(X \preccurlyeq X') = 1$. We call such a coupling a **witness** of the stochastic domination (3).

A useful tool for establishing stochastic domination is Holley's inequality (see [34] or [27]). Since the conditional distribution in (2) is increasing both in ξ and in p (recall that we only consider random-cluster measures with $q \geq 1$), Holley's inequality applies to show that, for any finite graph $G = (V,E)$,

$$\mathsf{RC}^G_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{RC}^G_{p_2,q}$$

whenever $p_1 \leq p_2$. Similarly we get, for conditional distributions, that

$$\mathsf{RC}^G_{p,q}\big(X \in \cdot \mid X(E') = \xi\big) \overset{\mathcal{D}}{\preccurlyeq} \mathsf{RC}^G_{p,q}\big(X \in \cdot \mid X(E') = \xi'\big) \tag{4}$$

whenever $E' \subseteq E$ and $\xi \preccurlyeq \xi'$.

We shall also be considering weak convergence of probability measures on $\{0,1\}^E$. For such probability measures $\mu_1, \mu_2, \ldots$ and μ, we say that μ is the (weak) limit of μ_i as $i \to \infty$ if $\lim_{i\to\infty} \mu_i(A) = \mu(A)$ for all cylinder events A.

2.3 The random-cluster model on infinite graphs

Let $G = (V, E)$ be an infinite, locally finite graph. The definition (1) of random-cluster measures does not work in this case, because there are uncountably many different configurations $\xi \in \{0,1\}^E$, and each one ought to have probability 0. Instead, there are two other approaches to defining random-cluster measures on infinite graphs: one via limiting procedures, and one via local specifications (Dobrushin-Lanford-Ruelle, or DLR, equations). We shall sketch the first approach, and then explain how it relates to the second.

Let $V_1, V_2, \ldots$ be a sequence of finite vertex sets increasing to V in the sense that $V_1 \subset V_2 \subset \ldots$ and $\bigcup_{i=1}^{\infty} V_i = V$. For any finite $K \subseteq V$, define

$$E(K) := \{[x,y] \in E\,;\ x, y \in K\},$$

set $E_i := E(V_i)$ and note that $E_1, E_2, \ldots$ increases to E in the same sense that $V_1, V_2, \ldots$ increases to V. Set ∂V_i to be the (inner) boundary of V_i, i.e.,

$$\partial V_i := \{v \in V_i\,;\ \exists\,[x,y] \in E \setminus E_i \text{ with } x = v\}\,.$$

Also set $G_i := (V_i, E_i)$, and let $\mathsf{FRC}_{p,q}^{G,i}$ be the probability measure on $\{0,1\}^E$ corresponding to picking $X \in \{0,1\}^E$ by letting $X(E_i)$ have distribution $\mathsf{RC}_{p,q}^{G_i}$ and setting $X(e) := 0$ for all $e \in E \setminus E_i$. Since the projection of $\mathsf{FRC}_{p,q}^{G,i}$ on $\{0,1\}^{E \setminus E_i}$ is deterministic, we can also view $\mathsf{FRC}_{p,q}^{G,i}$ as a measure on $\{0,1\}^{E_i}$, in which case it coincides with $\mathsf{RC}_{p,q}^{G_i}$. By applying (4) to the graph G_i with $E' := E_i \setminus E_{i-1}$ and $\xi \equiv 0$, we get that

$$\mathsf{FRC}_{p,q}^{G,i-1} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}_{p,q}^{G,i}\,,$$

so that

$$\mathsf{FRC}_{p,q}^{G,1} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}_{p,q}^{G,2} \overset{\mathcal{D}}{\preccurlyeq} \cdots\,. \tag{5}$$

This implies the existence of a limiting probability measure $\mathsf{FRC}_{p,q}^{G}$ on $\{0,1\}^E$. This limit is independent of the choice of $\{V_i\}_{i=1}^{\infty}$, and we call it the random-cluster measure on G with **free boundary condition** (hence the F in FRC) and parameters p and q.

Next, define $\mathsf{WRC}_{p,q}^{G,i}$ as the probability measure on $\{0,1\}^E$ corresponding to first setting $X(E \setminus E_i) \equiv 1$, and then picking $X(E)$ in such a way that

$$\mathsf{WRC}_{p,q}^{G,i}\big(X(E_i) = \xi\big) = \frac{q^{\|\xi\|^*}}{Z} \prod_{e \in E_i} p^{\xi(e)}(1-p)^{1-\xi(e)}$$

where $\|\xi\|^*$ is the number of connected components of ξ **that do not intersect** ∂V_i. Similarly as in (5), we get

$$\mathsf{WRC}_{p,q}^{G,1} \overset{\mathcal{D}}{\succcurlyeq} \mathsf{WRC}_{p,q}^{G,2} \overset{\mathcal{D}}{\succcurlyeq} \cdots$$

(note the reverse inequalities), and thus also a limiting measure $\mathsf{WRC}_{p,q}^{G}$ which we call the random-cluster measure on G with **wired boundary condition** and parameters p and q.

We now briefly discuss how the above relates to the DLR approach to the random-cluster model on infinite graphs. It is natural to expect that the limiting measures $\mathsf{FRC}^G_{p,q}$ and $\mathsf{WRC}^G_{p,q}$ should satisfy some analogue of (2). Indeed, $\mathsf{FRC}^G_{p,q}$ admits conditional probabilities satisfying

$$\mathsf{FRC}^G_{p,q}\big(X(e) = 1 \mid X(E \setminus \{e\}) = \xi\big) = \begin{cases} p & \text{if } x \leftrightarrow y, \\ \frac{p}{p+(1-p)q} & \text{otherwise} \end{cases} \tag{6}$$

for any $e \in E$ and any $\xi \in \{0,1\}^{E\setminus\{e\}}$, where the event $x \leftrightarrow y$ is defined as in (2). Although $\mathsf{WRC}^G_{p,q}$ does *not*, in general, satisfy the same local specification, it satisfies

$$\mathsf{WRC}^G_{p,q}\big(X(e) = 1 \mid X(E \setminus \{e\}) = \xi\big) = \begin{cases} p & \text{if } x \overset{\infty}{\longleftrightarrow} y, \\ \frac{p}{p+(1-p)q} & \text{otherwise,} \end{cases} \tag{7}$$

where $x \overset{\infty}{\longleftrightarrow} y$ denotes the event that either ξ contains a path from x to y, or it contains an infinite self-avoiding path starting at x and an infinite self-avoiding path starting at y. In other words, $x \overset{\infty}{\longleftrightarrow} y$ is the same event as $x \leftrightarrow y$, except that in $x \overset{\infty}{\longleftrightarrow} y$ the path from x to y is allowed to go "via infinity". We think of this as a kind of compactification of the graph. These facts are stated in [27, Theorem 6.17]. (That $\mathsf{FRC}^G_{p,q}$ satisfies (6) is due to [13]. The fact that $\mathsf{WRC}^G_{p,q}$ satisfies (7) can be proved analogously. Other proofs of (7) can be found in [40] and in [27].) We call a probability measure on $\{0,1\}^E$ a **DLR random-cluster measure** (resp., a **DLR wired-random-cluster measure**) with the given parameters p and q if it satisfies the local specifications in (6) (resp., in (7)). (These local specifications are usually given on any finite edge-set, rather than on a single edge. However, single-edge specifications are enough; see, e.g., [27, Theorem 6.18].) It turns out that $\mathsf{FRC}^G_{p,q}$ and $\mathsf{WRC}^G_{p,q}$ play the following special role in the class of DLR random-cluster and wired-random-cluster measures: If μ is any DLR random-cluster measure or DLR wired-random-cluster measure for G with parameters p and q, then

$$\mathsf{FRC}^G_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \mu \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^G_{p,q}\,.$$

We finally mention that (provided G is connected) the specifications (6) and (7) differ with positive probability if and only if the event of having more than one infinite connected component has positive probability. By an application of the uniqueness theorem of Burton and Keane [14], we get in the case where G is the usual $\mathbb{Z}^d$ lattice (and more generally when G is a transitive amenable graph — see Section 2.5 for a definition) that the number of infinite clusters is at most one, FRC-a.s. as well as WRC-a.s. It follows that in this case, both FRC and WRC are simultaneously DLR random-cluster measures and DLR wired-random-cluster measures.

2.4 The Potts model on infinite graphs

Let $G = (V, E)$ be infinite and locally finite, and let $\{G_i := (V_i, E_i)\}_{i=1}^{\infty}$ be as in the previous subsection. For $q \in \{2, 3, \dots\}$ and $\beta \geq 0$, define probability measures $\left\{\mathsf{FPt}^{G,i}_{q,\beta}\right\}_{i=1}^{\infty}$ on $\{1, \dots, q\}^V$ in such a way that the projection of $\mathsf{FPt}^{G,i}_{q,\beta}$ on $\{1, \dots, q\}^{V_i}$ equals $\mathsf{Pt}^{G_i}_{q,\beta}$, and the spins on $V \setminus V_i$ are i.i.d. uniformly distributed on $\{1, \dots, q\}$ and independent of the spins on V_i. It turns out that $\mathsf{FPt}^{G,i}_{q,\beta}$ has a limiting distribution $\mathsf{FPt}^{G}_{q,\beta}$ as $i \to \infty$.

Also, for a fixed spin $r \in \{1, \dots, q\}$, define $\mathsf{WPt}^{G,i}_{q,\beta,r}$ to be the distribution corresponding to picking $X \in \{1, \dots, q\}^V$ by letting $X(V \setminus V_i) \equiv r$, and letting $X(V_i)$ be distributed according to $\mathsf{Pt}^{G_i}_{q,\beta}$ *conditioned on the event that* $X(\partial V_i) \equiv r$. Again, $\mathsf{WPt}^{G,i}_{q,\beta,r}$ has a limiting distribution as $i \to \infty$, which we denote $\mathsf{WPt}^{G}_{q,\beta,r}$.

The existence of the limiting distributions $\mathsf{FPt}^{G}_{q,\beta}$ and $\mathsf{WPt}^{G}_{q,\beta,r}$ are nontrivial results, and in fact the shortest route to proving them goes via the stochastic monotonicity arguments for the random-cluster model outlined in Section 2.3, and proving Propositions 2.2 and 2.3 below.

A probability measure μ on $\{1, \dots, q\}^V$ is said to be a Gibbs measure (in the DLR sense) for the q-state Potts model on G at inverse temperature β, if it admits conditional distributions such that for all $v \in V$, all $r \in \{1, \dots, q\}$, and all $\omega \in \{1, \dots, q\}^{V \setminus \{v\}}$, we have

$$\mu\big(X(v) = r \mid X(V \setminus \{v\}) = \omega\big) = \frac{1}{Z} \exp\Big(-2\beta \sum_{[v,y] \in E} \mathbf{1}_{\{\omega(v) \neq \omega(y)\}}\Big), \tag{8}$$

where the normalizing constant Z may depend on v and ω but not on r. It turns out that $\mathsf{FPt}^{G}_{q,\beta}$ and $\mathsf{WPt}^{G}_{q,\beta,r}$ are both Gibbs measures in this sense.

The following extensions of Proposition 2.1 provide the relations between FRC and WRC on one hand, and FPt and WPt on the other.

Proposition 2.2. *Fix the infinite locally finite graph G, an integer $q \geq 2$, and $p \in [0, 1]$. Pick a random edge configuration $X \in \{0, 1\}^E$ according to $\mathsf{FRC}^{G}_{p,q}$. Then, for each connected component $\mathcal{C}$ of X, pick a spin uniformly from $\{1, \dots, q\}$, and assign this spin to all vertices of $\mathcal{C}$. Do this independently for different connected components. The $\{1, \dots, q\}^V$-valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\mathsf{FPt}^{G}_{q,\beta}$ for the q-state Potts model on G at inverse temperature $\beta := -\frac{1}{2} \log(1 - p)$.*

Proposition 2.3. *Let G, p and q be as in Proposition 2.2. Pick a random edge configuration $X \in \{0, 1\}^E$ according to the random-cluster measure $\mathsf{WRC}^{G}_{p,q}$. Then, for each* **finite** *connected component $\mathcal{C}$ of X, pick a spin uniformly from $\{1, \dots, q\}$, and assign this spin to all vertices of $\mathcal{C}$. Do this independently for different connected components. Finally assign value r to all vertices of infinite connected components. The $\{1, \dots, q\}^V$-valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\mathsf{WPt}^{G}_{q,\beta,r}$ for the q-state Potts model on G at inverse temperature $\beta := -\frac{1}{2} \log(1 - p)$.*

We make one final remark about the construction of FPt and WPt. Clearly, the prescribed behavior of $\mathsf{FPt}^{G,i}_{q,\beta}$ and $\mathsf{WPt}^{G,i}_{q,\beta,r}$ on $V \setminus V_i$ is of no importance for the limiting measures $\mathsf{FPt}^{G}_{q,\beta}$ and $\mathsf{WPt}^{G}_{q,\beta,r}$. The point of the precise way in which we defined this behavior is to admit further variants of Propositions 2.2 and 2.3. The statement in Proposition 2.2 remains true if we replace $\mathsf{FRC}^{G}_{p,q}$ and $\mathsf{FPt}^{G}_{q,\beta}$ by $\mathsf{FRC}^{G,i}_{p,q}$ and $\mathsf{FPt}^{G,i}_{q,\beta}$. In Proposition 2.3, we may replace $\mathsf{WRC}^{G}_{p,q}$ and $\mathsf{WPt}^{G}_{q,\beta,r}$ by $\mathsf{WRC}^{G,i}_{p,q}$ and $\mathsf{WPt}^{G,i}_{q,\beta,r}$, provided that the spin assignment procedure is modified in such a way that spin r is assigned to all vertices of all connected components that intersect $(V \setminus V_i) \cup \partial V_i$ (rather than just of all infinite connected components).

2.5 Some classes of infinite graphs

The class of all infinite locally finite graphs is often too large to obtain the most interesting results for the random-cluster model (and other stochastic models on graphs; see, e.g., [44] for a survey), and indeed most of our results will concern more restrictive classes of graphs. Here we recall some such classes.

Let, as usual, $G = (V, E)$ be an infinite locally finite graph. The number of edges incident to a vertex x is called the **degree** of x. The graph G is said to be **regular** if every vertex has the same degree.

A **graph automorphism** of G is a bijective mapping $\gamma : V \to V$ with the property that for all $x, y \in V$, we have $[\gamma x, \gamma y] \in E$ if and only if $[x, y] \in E$. Write $\mathrm{Aut}(G)$ for the set of all graph automorphisms of G. To each $\gamma \in \mathrm{Aut}(G)$, there is a corresponding mapping $\tilde{\gamma} : E \to E$ defined by $\tilde{\gamma}[x, y] := [\gamma x, \gamma y]$. The graph G is said to be **transitive** if and only if for any $x, y \in V$ there exists $\gamma \in \mathrm{Aut}(G)$ such that $\gamma x = y$. More generally, G is said to be **quasi-transitive** if V can be partitioned into finitely many sets $V_1, \ldots, V_k$ such that for any $i \in \{1, \ldots, k\}$ and any $x, y \in V_i$, there exists a $\gamma \in \mathrm{Aut}(G)$ such that $\gamma x = y$.

A probability measure μ on $\{0, 1\}^E$ is said to be **automorphism invariant** if for any n, any $e_1, \ldots, e_n \in E$, any $i_1, \ldots, i_n \in \{0, 1\}$, and any graph automorphism γ we have

$$\mu\big(X(e_1) = i_1, \ldots, X(e_n) = i_n\big) = \mu\big(X(\tilde{\gamma}(e_1)) = i_1, \ldots, X(\tilde{\gamma}(e_n)) = i_n\big) .$$

It follows from the construction of the free and wired random-cluster measures FRC and WRC (in particular from the independence of the choice of $\{G_i = (V_i, E_i)\}_{i=1}^{\infty}$) that both measures are automorphism invariant. It turns out that automorphism invariance has far-reaching consequences for percolation processes on various classes of transitive graphs; see, e.g., [14], [33], [5], and [45].

Two important properties, that may or may not hold for a given quasi-transitive graph, are amenability and unimodularity, which we review next. We say that a graph G is **amenable** if

$$\inf \frac{|\partial W|}{|W|} = 0 ,$$

where the infimum ranges over all finite $W \subset V$, and $|\cdot|$ denotes cardinality. There are various alternative definitions of amenability of a graph, which coincide for transitive graphs (and more generally for graphs of bounded degree), but not in general.

For any graph G and $x \in V$, define the **stabilizer** $S(x)$ as the set of graph automorphisms that fix x, i.e.,

$$S(x) := \{\gamma \in \mathrm{Aut}(G)\, ;\ \gamma x = x\} .$$

For $x, y \in V$, define

$$S(x)y := \{z \in V\, ;\ \exists \gamma \in S(x) \text{ such that } \gamma y = z\} .$$

When $\mathrm{Aut}(G)$ is given the weak topology generated by its action on V, all stabilizers are compact because G is locally finite and connected. We say that G is **unimodular** if for all x, y in the same orbit of $\mathrm{Aut}(G)$ (in the transitive case, this just means for all $x, y \in V$), we have the symmetry

$$|S(x)y| = |S(y)x| .$$

An important class of transitive graphs is the class of Cayley graphs. If Γ is a finitely generated group with symmetric generating set $\{g_1, \dots, g_n\}$, then the **Cayley graph** associated with Γ and that particular set of generators is the (unoriented) graph $G = (V, E)$ with vertex set $V := \Gamma$, and edge set

$$E := \{[x, y];\ x, y \in \Gamma, \exists i \in \{1, \dots, n\} \text{ such that } xg_i = y\} .$$

(The word "symmetric" here means that the inverse of each element of the set is also an element.) Most examples of graphs that have been studied in percolation theory are Cayley graphs. These include $\mathbb{Z}^d$ (which, with a slight abuse of notation, is short for the graph with vertex set $\mathbb{Z}^d$ and edges connecting pairs of vertices at Euclidean distance 1 from each other), and the regular tree $\mathbf{T}_n$ in which every vertex has exactly $n+1$ neighbours. The graph $\mathbb{Z}^d$ is amenable, while $\mathbf{T}_n$ is nonamenable for $n \geq 2$. Also studied are certain tilings of the hyperbolic plane (see Section 6), and further examples can be obtained, e.g., by taking Cartesian products of other Cayley graphs (such as $\mathbf{T}_n \times \mathbb{Z}$, the much-studied example of Grimmett and Newman [31]).

All Cayley graphs are unimodular. An example, due to Trofimov [59], of a transitive graph that is nonunimodular (and hence not a Cayley graph) may be obtained by taking the binary tree $\mathbf{T}_2$, fixing a so-called topological end ζ (loosely speaking, a direction to infinity in the tree), and adding an edge between each vertex and its ζ-grandparent.

We shall have occasion to use the following (Unimodular) Mass-Transport Principle:

Theorem 2.4. *Let G be a transitive unimodular graph. Let $o \in V$ be an arbitrary base point. Suppose that $\phi : V \times V \to [0, \infty]$ is invariant under the diagonal action of* $\mathrm{Aut}(G)$. *Then*

$$\sum_{x \in V} \phi(o, x) = \sum_{x \in V} \phi(x, o) \,..mtp$$

See [5] for a discussion of this principle and for a proof.

3 A dynamical construction

Let $G = (V, E)$ be infinite and locally finite, and let $\{G_i := (V_i, E_i)\}_{i=1}^{\infty}$ be as in Section 2. We know from Section 2.3 that

$$\mathsf{FRC}_{p,q}^{G,1} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}_{p,q}^{G,2} \overset{\mathcal{D}}{\preccurlyeq} \cdots \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}_{p,q}^{G} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}_{p,q}^{G} \overset{\mathcal{D}}{\preccurlyeq} \cdots \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}_{p,q}^{G,2} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}_{p,q}^{G,1} . \tag{9}$$

Other well-known stochastic inequalities are that for $p_1 \leq p_2$ and $i \in \{1, 2, \dots\}$, we have

$$\mathsf{FRC}_{p_1,q}^{G,i} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}_{p_2,q}^{G,i} , \tag{10}$$

$$\mathsf{FRC}_{p_1,q}^{G} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}_{p_2,q}^{G} , \tag{11}$$

$$\mathsf{WRC}^{G,i}_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^{G,i}_{p_2,q}\,, \tag{12}$$

and

$$\mathsf{WRC}^{G}_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^{G}_{p_2,q}\,. \tag{13}$$

For all of the above stochastic inequalities, it is desirable to find some natural construction of couplings that witness them. What we shall construct in this section is a coupling of all of the above probability measures (for all $p \in [0,1]$, $q \geq 1$ and $i \in \{1,2,\dots\}$) *simultaneously* that provides witnesses to the stochastic inequalities (9)–(13) above. Some additional pleasing aspects of the construction are the following.

(A1) Not only are FRC and WRC automorphism invariant separately, but also their joint behaviour in our coupling is automorphism invariant. This remains true also if we consider these measures simultaneously for different parameter values.

(A2) If G is obtained as an automorphism-invariant percolation process on another graph H, then the construction is easily set up in such a way that the joint distribution of G and the random-cluster measures on G becomes an automorphism-invariant process on H. (See [37] for an example where an analogous property turns out to be important in the context of Ising models with external field on percolation clusters.)

Nevertheless, there are still some desirable aspects of couplings of random-cluster processes for which we do not know whether or not they hold for our construction; see Question 3.1 at the end of this section.

The construction is based on time dynamics for the random-cluster model. Such time dynamics have previously been considered, e.g., by Bezuidenhout, Grimmett and Kesten [10] and by Grimmett [29], for the random-cluster model on $\mathbb{Z}^d$. To some extent our construction will resemble Grimmett's analysis. However, one feature of our construction that differs from Grimmett's is that the dynamics are run "from the past" rather than "into the future", along the lines of the very popular CFTP (coupling from the past) algorithm of Propp and Wilson [51]; see also [58] for an early treatment of dynamics from the past, and [18] for a survey putting the ideas in a more general mathematical context. For the case of finite graphs, CFTP was applied to simulate the random-cluster model in [51]. Simulation on infinite graphs would require additional arguments, but our purpose is not simulation; rather, it is to gain some theoretical information. For models other than the random-cluster model, CFTP ideas have been extended to the setting of infinite graphs in [9], [38] and [37], but in all those cases the interaction of the dynamics had a strictly local character, which is not the case in our context. Another feature of our construction is the simultaneity in the parameter space. Such simultaneity, which is related to the level-set representations of Higuchi [39], appears in both [29] and [51]; Propp and Wilson use the term "omnithermal" to denote this particular feature of the construction.

Let us start with a simple finite case: how do we construct a $\{0,1\}^{E_i}$-valued random element with distribution $\mathsf{RC}^{G_i}_{p,q}$ (equivalently, with distribution $\mathsf{FRC}^{G,i}_{p,q}$)? If we are content with getting something that has only approximately the right distribution, then the following dynamical approach works fine: Define some ergodic Markov chain whose unique equilibrium distribution is $\mathsf{RC}^{G_i}_{p,q}$, and run it for time T starting from an arbitrary initial state ξ. If T is large enough, then the distribution of the final state is close to $\mathsf{RC}_{p,q}^{G_i}$, regardless of the choice of ξ.

In particular, we may proceed as follows. To each edge $e \in E_i$, we independently assign an i.i.d. sequence $(\phi_1^e, \phi_2^e, \dots)$ of exponential random variables with mean 1, and an independent i.i.d. sequence $(U_1^e, U_2^e, \dots)$ of uniform $[0,1]$ random variables. For $e \in E_i$ and $k = 1, 2, \dots$, let $\tau_k^e := \phi_1^e + \dots + \phi_k^e$, so that $(\tau_1^e, \tau_2^e, \dots)$ are the jump times of a unit rate Poisson process. Now define a $\{0,1\}^{E_i}$-valued continuous-time Markov chain $\{{}^{\xi}\bar{X}_{p,q}^{G_i}(t)\}_{t\geq 0}$ with starting state ${}^{\xi}\bar{X}_{p,q}^{G_i}(0) := \xi$ and evolution as follows. For $e := [x,y] \in E_i$, the value of ${}^{\xi}\bar{X}_{p,q}^{G_i}(t)(e)$ does not change other than (possibly) at the times $\tau_1^e, \tau_2^e, \dots$, at which times it takes the value

$$
{}^{\xi}\bar{X}_{p,q}^{G_i}(\tau_k^e)(e) := \begin{cases} 1 & \text{if } U_k^e < p \text{ and } x \leftrightarrow y \text{ in } {}^{\xi}\bar{X}_{p,q}^{G_i}(\tau_k^e)(E_i \setminus \{e\}) \\ 1 & \text{if } U_k^e < \frac{p}{p+(1-p)q} \text{ and } \neg \left(x \leftrightarrow y \text{ in } {}^{\xi}\bar{X}_{p,q}^{G_i}(\tau_k^e)(E_i \setminus \{e\})\right) \\ 0 & \text{otherwise.} \end{cases} \tag{14}
$$

(Note that a.s., $\tau_k^e \neq \tau_j^{e'}$ for all j,k when $e \neq e'$.) It is easy to see that this Markov chain is irreducible and reversible with $\mathsf{RC}_{p,q}^{G_i}$ as stationary distribution, so that indeed ${}^{\xi}\bar{X}_{p,q}^{G_i}(t)$ converges in distribution to $\mathsf{RC}_{p,q}^{G_i}$ as $t \to \infty$. Note also that since $p \geq \frac{p}{p+(1-p)q}$, the chain preserves the partial order $\preccurlyeq$ on $\{0,1\}^{E_i}$; in other words, for all $t \geq 0$ we have

$$
{}^{\xi}\bar{X}_{p,q}^{G_i}(t) \preccurlyeq {}^{\eta}\bar{X}_{p,q}^{G_i}(t) \quad \text{whenever} \quad \xi \preccurlyeq \eta\,. \tag{15}
$$

To get a $\{0,1\}^{E_i}$-valued random object whose distribution is precisely $\mathsf{RC}_{p,q}^{G_i}$, we need to consider some limit as $t \to \infty$. On the other hand, ${}^{\xi}\bar{X}_{p,q}^{G_i}(t)$ does not converge in any a.s. sense, so this may appear not to be feasible.

The solution, which turns the convergence in distribution into a.s. convergence, is to run the dynamics from the past up to time 0, rather than from time 0 into the future. For $T \geq 0$, define the $\{0,1\}^{E_i}$-valued continuous-time Markov chain

$$
\left\{ {}^{\text{free}}_{-T}X_{p,q}^{G_i}(t) \right\}_{t\in[-T,0]}
$$

with starting state ${}^{\text{free}}_{-T}X_{p,q}^{G_i}(-T) \equiv 0$ and the following evolution, similar to the one of ${}^{\xi}\bar{X}_{p,q}^{G_i}$. The value at an edge $e := [x,y] \in E_i$ changes only at times $(\dots, -\tau_2^e, -\tau_1^e)$, when it takes the value

$$
{}^{\text{free}}_{-T}X_{p,q}^{G_i}(-\tau_k^e)(e) := \begin{cases} 1 & \text{if } U_k^e < p \text{ and } x \leftrightarrow y \text{ in } {}^{\text{free}}_{-T}X_{p,q}^{G_i}(-\tau_k^e)(E_i \setminus \{e\}) \\ 1 & \text{if } U_k^e < \frac{p}{p+(1-p)q} \text{ and } \neg \left(x \leftrightarrow y \text{ in } {}^{\text{free}}_{-T}X_{p,q}^{G_i}(-\tau_k^e)(E_i \setminus \{e\})\right) \\ 0 & \text{otherwise,} \end{cases} \tag{16}
$$

like in (14). We have, for $0 \leq T_1 \leq T_2$, that

$$
{}^{\text{free}}_{-T_1}X_{p,q}^{G_i}(0) \preccurlyeq {}^{\text{free}}_{-T_2}X_{p,q}^{G_i}(0)
$$

(essentially because of (15)), so by monotonicity ${}^{\text{free}}_{-T}X_{p,q}^{G_i}(0)$ has an a.s. limit ${}^{\text{free}}X_{p,q}^{G_i} \in \{0,1\}^{E_i}$, defined by setting ${}^{\text{free}}X_{p,q}^{G_i}(e) := \lim_{T\to\infty} {}^{\text{free}}_{-T}X_{p,q}^{G_i}(0)(e)$ for each $e \in E_i$.

Clearly, ${}^{\text{free}}_{-T}X^{G_i}_{p,q}(0)$ has the same distribution as ${}^{\xi}\bar{X}^{G_i}_{p,q}(T)$ with $\xi \equiv 0$, so ${}^{\text{free}}_{-T}X^{G_i}_{p,q}(0)$ converges in distribution to $\mathsf{RC}^{G_i}_{p,q}$ as $T \to \infty$. Hence ${}^{\text{free}}X^{G_i}_{p,q}$ has distribution $\mathsf{RC}^{G_i}_{p,q}$, and if we furthermore define ${}^{\text{free}}X^{G,i}_{p,q} \in \{0,1\}^E$ by setting

$$ {}^{\text{free}}X^{G,i}_{p,q}(e) := \begin{cases} {}^{\text{free}}X^{G_i}_{p,q}(e) & \text{for } e \in E_i \\ 0 & \text{otherwise} \end{cases} $$

for each $e \in E$, then ${}^{\text{free}}X^{G,i}_{p,q}$ has distribution $\mathsf{FRC}^{G,i}_{p,q}$.

Now suppose that we have defined the random variables $(\phi^e_1, \phi^e_2, \dots)$ and $(U^e_1, U^e_2, \dots)$ for all $e \in E$ (and not just all $e \in E_i$) in the obvious way. By another application of the order-preserving property (15), we get that

$$ {}^{\text{free}}X^{G,1}_{p,q} \preccurlyeq {}^{\text{free}}X^{G,2}_{p,q} \preccurlyeq \dots $$

so that the limiting object ${}^{\text{free}}X^{G}_{p,q}$, defined by taking ${}^{\text{free}}X^{G}_{p,q}(e) := \lim_{i\to\infty} {}^{\text{free}}X^{G,i}_{p,q}(e)$, exists. For any cylinder set $A \in \{0,1\}^E$, we have

$$ \mathbf{P}\left({}^{\text{free}}X^{G}_{p,q} \in A\right) = \lim_{i\to\infty} \mathbf{P}\left({}^{\text{free}}X^{G,i}_{p,q} \in A\right) = \lim_{i\to\infty} \mathsf{FRC}^{G,i}_{p,q}(A) = \mathsf{FRC}^{G}_{p,q}(A) \qquad (17) $$

so that ${}^{\text{free}}X^{G}_{p,q}$ has distribution $\mathsf{FRC}^{G}_{p,q}$. Thus, to summarize the construction so far, what we have is a coupling of $\{0,1\}^E$-valued random objects ${}^{\text{free}}X^{G,1}_{p,q}, {}^{\text{free}}X^{G,2}_{p,q}, \dots$ and ${}^{\text{free}}X^{G}_{p,q}$ that witnesses the stochastic inequalities in the first half of (9).

Next, we go on to construct, in analogous fashion, the corresponding objects for wired random-cluster measures. For $T \geq 0$, define the $\{0,1\}^E$-valued continuous-time Markov chain

$$ \left\{ {}^{\text{wired}}_{-T}X^{G,i}_{p,q}(t) \right\}_{t\in[-T,0]} $$

with starting configuration ${}^{\text{wired}}_{-T}X^{G,i}_{p,q}(-T) \equiv 1$. Edges $e \in E \setminus E_i$ remain in state 1 forever, while the value of an edge $e := [x,y] \in E_i$ is updated at times $(\dots, -\tau^e_2, -\tau^e_1)$, when it takes the value

$$ {}^{\text{wired}}_{-T}X^{G,i}_{p,q}(-\tau^e_k)(e) := \begin{cases} 1 & \text{if } U^e_k < p \text{ and } A(x,y,i,p,q,e,k) \\ 1 & \text{if } U^e_k < \frac{p}{p+(1-p)q} \text{ and } \neg A(x,y,i,p,q,e,k) \\ 0 & \text{otherwise;} \end{cases} \qquad (18) $$

here, $A(x,y,i,p,q,e,k)$ is the event $\left\{ x \overset{\partial V_i}{\longleftrightarrow} y \text{ in } {}^{\text{wired}}_{-T}X^{G,i}_{p,q}(-\tau^e_k)(E_i \setminus \{e\}) \right\}$, where $x \overset{\partial V_i}{\longleftrightarrow} y$ is the event that either

(a) there is an open path from x to y (not using e), or

(b) both x and y have open paths (not using e) to ∂V_i.

It is immediate from the definition of $\mathsf{WRC}^{G,i}_{p,q}$ that the conditional $\mathsf{WRC}^{G,i}_{p,q}$-probability that an edge $e := [x,y] \in E_i$ is open, given the status of all other edges, is p or $p/[p + (1-p)q]$, depending on whether or not the event $x \overset{\partial V_i}{\longleftrightarrow} y$ happens. It follows that the distribution of ${}^{\text{wired}}_{-T}X^{G,i}_{p,q}(0)$ tends to $\mathsf{WRC}^{G,i}_{p,q}$ as $T \to \infty$. Moreover, the dynamics in (18) preserves $\preccurlyeq$ similarly as in (15), implying that

$$ {}^{\text{wired}}_{-T_1}X^{G,i}_{p,q} \succcurlyeq {}^{\text{wired}}_{-T_2}X^{G,i}_{p,q} $$

whenever $0 \le T_1 \le T_2$. This establishes the existence of a limiting $\{0,1\}^E$-valued random object ${}^{\text{wired}}X_{p,q}^{G,i}$ defined by ${}^{\text{wired}}X_{p,q}^{G,i}(e) := \lim_{T\to\infty} {}^{\text{wired}}X_{p,q}^{G,i}(0)(e)$ for each $e \in E$. Clearly, ${}^{\text{wired}}X_{p,q}^{G,i}$ has distribution $\mathsf{WRC}_{p,q}^{G,i}$. Another use of the $\preccurlyeq$-preserving property of the dynamics (18) shows that

$$ {}^{\text{wired}}X_{p,q}^{G,1} \succcurlyeq {}^{\text{wired}}X_{p,q}^{G,2} \succcurlyeq \cdots , $$

so that we have a limiting object ${}^{\text{wired}}X_{p,q}^{G} \in \{0,1\}^E$ defined by setting ${}^{\text{wired}}X_{p,q}^{G}(e) := \lim_{i\to\infty} {}^{\text{wired}}X_{p,q}^{G,i}(e)$ for each $e \in E$. By arguing as in (17), we get that ${}^{\text{wired}}X_{p,q}^{G}$ has distribution $\mathsf{WRC}_{p,q}^{G}$. The random objects ${}^{\text{wired}}X_{p,q}^{G,1}, {}^{\text{wired}}X_{p,q}^{G,2}, \ldots$ and ${}^{\text{wired}}X_{p,q}^{G}$ witness the stochastic inequalities in the second half of (9).

In order to fully establish that we have a witness to (9), it remains to show that ${}^{\text{free}}X_{p,q}^{G}$ and ${}^{\text{wired}}X_{p,q}^{G}$ witness the middle inequality in (9), i.e., we need to show that ${}^{\text{free}}X_{p,q}^{G} \preccurlyeq {}^{\text{wired}}X_{p,q}^{G}$. From the observations that the right-hand sides of (16) and (18) are increasing in the configurations on $E_i \setminus \{e\}$, and that for each such configuration the right-hand side of (18) is greater than that of (16), we get that

$$ {}^{\text{free}}_{-T}X_{p,q}^{G,i}(t) \preccurlyeq {}^{\text{wired}}_{-T}X_{p,q}^{G,i}(t) $$

for any $i \in \{1,2,\ldots\}$, $T \ge 0$ and $t \in [-T,0]$. By taking $t := 0$, letting $T \to \infty$ and then $i \to \infty$, we get

$$ {}^{\text{free}}X_{p,q}^{G} \preccurlyeq {}^{\text{wired}}X_{p,q}^{G} \tag{19} $$

as desired. Hence our coupling is a witness to all the inequalities in (9).

It remains to be demonstrated that the coupling is also a witness to the inequalities (10)–(13). Note first that the right-hand sides of (16) and (18) are increasing not only in the configurations on $E_i \setminus \{e\}$, but also in p. It follows that for $p_1 \le p_2$ we have

$$ {}^{\text{free}}_{-T}X_{p_1,q}^{G,i}(t) \preccurlyeq {}^{\text{free}}_{-T}X_{p_2,q}^{G,i}(t) $$

and

$$ {}^{\text{wired}}_{-T}X_{p_1,q}^{G,i}(t) \preccurlyeq {}^{\text{wired}}_{-T}X_{p_2,q}^{G,i}(t) $$

for all $i \in \{1,2,\ldots\}$, $T \ge 0$ and $t \in [-T,0]$. Taking $t := 0$ and letting $T \to \infty$ yields

$$ {}^{\text{free}}X_{p_1,q}^{G,i} \preccurlyeq {}^{\text{free}}X_{p_2,q}^{G,i} $$

and

$$ {}^{\text{wired}}X_{p_1,q}^{G,i} \preccurlyeq {}^{\text{wired}}X_{p_2,q}^{G,i} , $$

witnessing (10) and (12). Letting $i \to \infty$, we get

$$ {}^{\text{free}}X_{p_1,q}^{G} \preccurlyeq {}^{\text{free}}X_{p_2,q}^{G} \tag{20} $$

and

$$ {}^{\text{wired}}X_{p_1,q}^{G} \preccurlyeq {}^{\text{wired}}X_{p_2,q}^{G} , \tag{21} $$

finally witnessing (11) and (13). In fact, examination also shows that as long as $p_1 \le p_2$ and $p_1/[(1-p_1)q_1] \le p_2/[(1-p_2)q_2]$, we have

$$ {}^{\text{free}}X^G_{p_1,q_1} \preccurlyeq {}^{\text{free}}X^G_{p_2,q_2}\,, \tag{22} $$

$$ {}^{\text{wired}}X^G_{p_1,q_1} \preccurlyeq {}^{\text{wired}}X^G_{p_2,q_2}\,, \tag{23} $$

and

$$ {}^{\text{free}}X^G_{p_1,q_1} \preccurlyeq {}^{\text{wired}}X^G_{p_2,q_2}\,, \tag{24} $$

witnessing more general well-known stochastic inequalities [23].

Property (A1) of the coupling is obvious from the construction. In order for (A2) to be true, we need only define random variables $\{\phi^e_k, U^e_k\}_{e\in E(H),i=1,2,\ldots}$ for all edges in H and to take them to be independent of the percolation process that yields G from H.

We end this section with a discussion of an open problem concerning our coupling. For $p_1 < p_2$, define

$$ \Delta_q(p_1,p_2) := \min\left\{p_2 - p_1,\ \frac{p_2}{p_2+(1-p_2)q} - \frac{p_1}{p_1+(1-p_1)q}\right\} $$

and note that $\Delta_q(p_1,p_2) > 0$. For $e \in E$ and $\xi \in \{0,1\}^{E\setminus\{e\}}$, write $A(\xi,e,p,q)$ for the event that ${}^{\text{free}}X^G_{p,q}(E\setminus\{e\}) = \xi$. From the fact that $\mathsf{FRC}^G_{p,q}$ is a DLR random-cluster measure, it follows that for any $e \in E$ and (almost) any $\xi,\eta \in \{0,1\}^{E\setminus\{e\}}$ such that $\xi \preccurlyeq \eta$, we have

$$ \mathbf{P}\left({}^{\text{free}}X^G_{p_2,q}(e) = 1 \mid A(\eta,e,p_2,q)\right) - \mathbf{P}\left({}^{\text{free}}X^G_{p_1,q}(e) = 1 \mid A(\xi,e,p_1,q)\right) \ge \Delta_q(p_1,p_2) \tag{25} $$

(and similarly for wired random-cluster measures; everything we say in relation to Question 3.1 applies as well to the wired case as to the free). From this, one is easily seduced into thinking that

$$ \mathbf{P}\left({}^{\text{free}}X^G_{p_2,q}(e) = 1\,,\ {}^{\text{free}}X^G_{p_1,q}(e) = 0 \mid A(\eta,e,p_2,q)\cap A(\xi,e,p_1,q)\right) \ge \Delta_q(p_1,p_2) \tag{26} $$

but to conclude this directly from (25) is unwarranted, because conditioning on ξ and η jointly is not the same as conditioning on them separately. It is nevertheless natural to ask whether something like (26) is true. In particular, the following question asks for a weaker property.

Question 3.1. *For $p_1 < p_2$ and $q \ge 1$, does there exist an $\varepsilon > 0$ (depending on p_1, p_2 and q) such that for any $e \in E$ and (almost) any $(\xi,\eta) \in (\{0,1\}^{E\setminus\{e\}})^2$, we have*

$$ \mathbf{P}\left({}^{\text{free}}X^G_{p_2,q}(e) = 1\,,\ {}^{\text{free}}X^G_{p_1,q}(e) = 0 \mid A(\eta,e,p_2,q)\cap A(\xi,e,p_1,q)\right) \ge \varepsilon\,? $$

A positive answer to this question (for our coupling or for some other automorphism-invariant witness to the stochastic inequality $\mathsf{FRC}^G_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^G_{p_2,q}$) is precisely the missing ingredient that prevented the authors of [35] from extending their uniqueness monotonicity result for i.i.d. percolation ($q = 1$) for unimodular transitive graphs to the more general case $q \ge 1$ (i.e., from proving (31) and (32)). Such a positive answer might perhaps also be an ingredient in applying the reasoning of Schonmann [53] in order to remove the unimodularity assumption in these results. The following weaker property would also suffice in the transitive unimodular setting:

Question 3.2. *For $p_1 < p_2$ and $q \geq 1$, does there exist an $\varepsilon > 0$ (depending on p_1, p_2 and q) such that for any $e \in E$ and (almost) any $(\xi, \eta) \in (\{0,1\}^{E \setminus \{e\}})^2$, we have*

$$\mathbf{P}\left({}^{\text{free}}X^G_{p_1,q}(e) = 1 \mid A(\eta, e, p_2, q) \cap A(\xi, e, p_1, q)\right) \geq \varepsilon\,?$$

4 Bernoullicity

Let Γ be a closed subgroup of $\mathrm{Aut}(G)$ with $G = (V, E)$ being any connected graph. We shall be most interested in the case that G is the Cayley graph of Γ and in the case that $\Gamma = \mathrm{Aut}(G)$ and G is quasi-transitive. Let S and T be arbitrary state spaces. For $\gamma \in \Gamma$, define the map $\theta_\gamma : S^V \to S^V$ (or $\theta_\gamma : T^V \to T^V$) by setting $\theta_\gamma \omega(x) := \omega(\gamma^{-1}x)$ for each $x \in V$. A measurable mapping $f : (S^V, \mu) \to (T^V, \nu)$ is said to be **Γ-equivariant** if it commutes with these actions of Γ, i.e., if $f(\theta_\gamma \omega) = \theta_\gamma\big(f(\omega)\big)$ for all $\gamma \in \Gamma$ and μ-a.e. $\omega \in S^V$; it is called **measure-preserving** if $\nu = \mu \circ f^{-1}$. The action of Γ on (S^V, μ) is called **free** if for μ-a.e. $x \in S^V$, the only element in Γ that leaves x fixed is the identity.

We say that a probability measure ν on T^V is a **Γ-factor of an i.i.d. process** if there exists a T^V-valued random element X with distribution ν, a state space S, an S^V-valued random element Y with distribution μ, and a Γ-equivariant measure-preserving mapping $f : (S^V, \mu) \to (T^V, \nu)$ such that

(i) Y is an i.i.d. process, and

(ii) $X = f(Y)$.

In case G is the Cayley graph of Γ, if S can be taken to be finite and f can be taken to be an invertible mapping, then (Γ, ν) is said to be **Bernoulli**, a mixing property of fundamental importance in ergodic theory. We refer to [49, p. 127] for the definition of "Bernoulli" for more general Γ. We shall prove, using the dynamical construction in Section 3, that Bernoullicity holds for the wired Potts model on $\mathbb{Z}^d$, and more generally on amenable quasi-transitive graphs:

Theorem 4.1. *For any Cayley graph G of an amenable group Γ or for any quasi-transitive amenable graph G with automorphism group Γ, any $q \in \{2, 3, \dots\}$ and $r \in \{1, \dots, q\}$, and any $\beta \geq 0$, the Gibbs measure $\mathsf{WPt}^G_{q,\beta,r}$ is Bernoulli with respect to the action of Γ.*

For the $\mathbb{Z}^d$ case, this was previously known only for the cases where either $q = 2$ (Ising model) or β is sufficiently small; see, e.g., [48], [43] and [56]. For the Ising model result on amenable graphs, see [1], while for a proof of a stronger property than Bernoullicity in the case of β small, using CFTP ideas, see [38]. The paper [37] uses ideas similar to ours to prove that the Ising model is Bernoulli.

We call an i.i.d. process (S^V, μ) **standard** if S is a standard Borel space and the marginal of μ on S is Borel. Ornstein and Weiss [49] show that when Γ is amenable and discrete, then (Γ, ν) is Bernoulli iff it is a free Γ-factor of a standard i.i.d. process. More generally, we have the following result:

Lemma 4.2. *Let V be a countable set and Γ be a closed subgroup of the symmetric group on V. Suppose that all orbits of the Γ-action on V are infinite and that Γ is amenable, unimodular, and not the union of an increasing sequence of compact proper subgroups of Γ. Further, suppose that for each $x \in V$, the Γ-stabilizer of x is compact. Then every free Γ-factor of a standard i.i.d. process (S^V, μ) is Bernoulli.*

Proof. Let Z_n be i.i.d. Poisson processes on Γ. Let W be a selection of one point from each orbit of the action of Γ on V. Given $v \in V$, let $X_n(v)$ be the number of points in Z_n that take o to v for $v \in V$, where $\{o\} = W \cap \Gamma v$. Since Γ is a countable union of translates of stabilizers, each stabilizer has positive Haar measure, so that $X_n(v)$ is a nontrivial Poisson random variable. Also, the random variables $\langle X_n(v); n \geq 1, v \in V\rangle$ are mutually independent. It is not hard to see that $\langle X_n\rangle$ is a free Γ-factor of $\langle Z_n\rangle$. By [49], X_n is Bernoulli for each n. Therefore $\langle X_n;\ n \geq 1\rangle$ is Bernoulli by [49, Theorem III.6.5]. Since every standard i.i.d. process (S^V, μ) is a Γ-factor of $\langle X_n;\ n \geq 1\rangle$, it follows that every free Γ-factor of (S^V, μ) is also Bernoulli. □

We also need the following fact:

Lemma 4.3. *If G is a quasi-transitive amenable graph, then* $\mathrm{Aut}(G)$ *is amenable, unimodular, and not the union of an increasing sequence of compact proper subgroups.*

Proof. $\mathrm{Aut}(G)$ is amenable and unimodular by results of Soardi and Woess [55] and Salvatori [52]; see also [5] for another proof. Furthermore, in this case $\mathrm{Aut}(G)$ is compactly generated by, say, the set $\Delta := \{\gamma \in \mathrm{Aut}(G);\ d(o, \gamma o) \leq 2r + 1\}$, where r is such that every vertex of G is within distance r of some vertex in $\mathrm{Aut}(G)o$ and $d(\cdot, \cdot)$ denotes distance in G. Thus, if Γ_n are compact increasing subgroups of $\mathrm{Aut}(G)$ whose union is $\mathrm{Aut}(G)$, we have $\bigcap(\Delta \setminus \Gamma_n) = \emptyset$, whence for some n, we have $\Delta \subseteq \Gamma_n$. Since Δ generates $\mathrm{Aut}(G)$, it follows that $\Gamma_n = \mathrm{Aut}(G)$. □

Because of the above, Theorem 4.1 is established once the following lemma is proved:

Lemma 4.4. *For any graph G, any subgroup Γ of* $\mathrm{Aut}(G)$*, any $q \in \{2, 3, \dots\}$ and $r \in \{1, \dots, q\}$, and any $\beta \geq 0$, the Gibbs measure $\mathsf{WPt}^G_{q,\beta,r}$ is a Γ-factor of an i.i.d. process. If Γ is countable and every element of Γ other than the identity moves an infinite number of vertices, then the action of Γ on $\mathsf{WPt}^G_{q,\beta,r}$ is free.*

Proof. Let the degree of G be d. For each $x \in V$, let $N_x = \{Z^x_1, \dots, Z^x_d\}$ be the set of neighbors of x in any fixed order.

Take

$$S := \big\{[0, \infty) \times [0, 1]\big\}^{\{1,2,\dots\}\times\{1,\dots,d\}} \times [0, 1]^d \times [0, 1] \times \{1, \dots, q\}\,.$$

Let

$$\Big\{\phi^j_k(x), U^j_k(x), U^j_*(x), U^*(x), \sigma(x)\,;\ k = 1, 2, \dots\,,\ j = 1, \dots, d,\ x \in V\Big\}$$

be independent random variables with $\phi^j_k(x)$ exponential of mean 1, $U^j_k(x)$, $U^j_*(x)$, and $U^*(x)$ uniform $[0, 1]$, and $\sigma(x)$ uniform on $\{1, \dots, q\}$. For each $x \in V$, put

$$Y(x) := \Big(\big(\phi^j_k(x), U^j_k(x)\big)_{k=1,2,\dots,j=1,\dots,d}, \big(U^j_*(x)\big)_{j=1,\dots,d}, U^*(x), \sigma(x)\Big)\,.$$

Set $p := 1 - e^{-2\beta}$, and construct a $\{0, 1\}^E$-valued edge configuration $X^G_{p,q}$ with distribution $\mathsf{WRC}^G_{p,q}$ by the dynamical construction in Section 3, where for each $e \in E$ we take

$$(\phi^e_k, U^e_k)_{k=1,2,\dots} := \big(\phi^j_k(x), U^j_k(x)\big)_{k=1,2,\dots}\,, \tag{27}$$

where $x \in V$ and $j \in \{1, \ldots, d\}$ are chosen in such a way that $e = [x, Z_j^x]$, and, if we denote $y := Z_j^x$ and j' is such that $x = Z_{j'}^y$, then $U_*^j(x) < U_*^{j'}(y)$. This choice of x and j is a.s. unique.

From $X_{p,q}^G$, we obtain the desired spin configuration $X \in \{1, \ldots, q\}^V$ with distribution $\mathsf{WPt}_{q,\beta,r}^G$ by assigning spins to the connected components of $X_{p,q}^G$ as in Proposition 2.3: All vertices in infinite connected components in $X_{p,q}^G$ are assigned value r, whereas the vertices of each finite connected component $\mathcal{C}$ are assigned value $\sigma(x)$, where x is the vertex in $\mathcal{C}$ that minimizes $U^*(x)$. It is obvious that this mapping $Y \mapsto X$ from S^V to $\{1, \ldots, q\}^V$ is $\mathrm{Aut}(G)$-equivariant, and that the resulting spin configuration has distribution $\mathsf{WPt}_{q,\beta,r}^G$. Hence $\mathsf{WPt}_{q,\beta,r}^G$ is a factor of an i.i.d. process.

To see that the action of Γ is free under the additional hypotheses stated in the lemma, it suffices to show that for any $\gamma \in \Gamma$ other than the identity, $\mathbf{P}[\theta_\gamma X = X] = 0$. From the hypotheses, we may find an infinite set W of vertices such that $\gamma x \notin W$ for all $x \in W$ and $\gamma x \neq \gamma y$ for distinct $x, y \in W$. Because of (8), there is some $c < 1$ such that for any $x_1, \ldots, x_n \in W$, we have $\mathbf{P}[X(x_i) = X(\gamma^{-1} x_i) \text{ for all } i = 1, \ldots, n] \leq c^n$. Therefore $\mathbf{P}[\theta_\gamma X = X] = 0$. □

5 The four critical values

Let, as usual, $G = (V, E)$ be infinite and locally finite. A probability measure μ on $\{0,1\}^E$ is said to be **insertion tolerant** if for any $e \in E$ and almost every $\xi \in \{0,1\}^{E \setminus \{e\}}$, the conditional μ-probability that e is open given the configuration ξ on $\{0,1\}^{E \setminus \{e\}}$, is strictly positive. Newman and Schulman [47] showed that for any automorphism-invariant insertion-tolerant percolation process on $\mathbb{Z}^d$, the number of infinite clusters is a.s. either 0, 1 or ∞. It has been observed by several authors (see, e.g., [7]) that this result (as well as its proof in [47]) extends to the class of transitive connected graphs.

Suppose that G is transitive and connected. The Newman-Schulman result then applies to $\mathsf{FRC}_{p,q}^G$ and $\mathsf{WRC}_{p,q}^G$ because (6) and (7) imply that the two measures are insertion tolerant whenever $p \in (0,1)$. Furthermore, $\mathsf{FRC}_{p,q}^G$ and $\mathsf{WRC}_{p,q}^G$ are ergodic; this was proved in [13] for FRC and in [12] for both measures. A simple proof of the stronger property that the tail σ-field is trivial appears in [44]. Ergodicity also follows easily from Lemma 4.4. In any case, ergodicity shows that for each fixed p and q, the number of infinite clusters is an a.s. constant (which, however, need not be the same for $\mathsf{FRC}_{p,q}^G$ and for $\mathsf{WRC}_{p,q}^G$). Hence, given G and q, the set $[0,1]$ of possible values for p can be partitioned into three sets, according to whether the $\mathsf{FRC}_{p,q}^G$-a.s. number of infinite clusters is 0, 1 or ∞, and similarly for $\mathsf{WRC}_{p,q}^G$. From (11) and (13) we can immediately deduce that the set of p for which the number of infinite clusters is 0 is an interval containing 0. In other words, there exist critical values $p_c^{\text{free}} := p_c^{\text{free}}(G,q)$ and $p_c^{\text{wired}} := p_c^{\text{wired}}(G,q)$ such that

$$\mathsf{FRC}_{p,q}^G(\exists \text{ at least one infinite cluster}) = \begin{cases} 0 & \text{for } p < p_c^{\text{free}}, \\ 1 & \text{for } p > p_c^{\text{free}} \end{cases} \tag{28}$$

and

$$\mathsf{WRC}_{p,q}^G(\exists \text{ at least one infinite cluster}) = \begin{cases} 0 & \text{for } p < p_c^{\text{wired}}, \\ 1 & \text{for } p > p_c^{\text{wired}}. \end{cases} \tag{29}$$

The question of how the interval above $p_c^{\rm free}$ ($p_c^{\rm wired}$) is split up according to whether the number of infinite clusters is 1 or ∞ is more intricate. Does it split nicely into two intervals, or are the sets more complicated? A proof is given in [44, Proposition 5.2] that in the transitive unimodular case, the uniqueness sets are simply intervals. Presumably, this holds whenever G is transitive. A similar proof shows that

$$\mathsf{FRC}^G_{p,q}(\exists \text{ a unique infinite cluster}) \le \mathsf{WRC}^G_{p,q}(\exists \text{ a unique infinite cluster}). \qquad (30)$$

Thus, if G is transitive and unimodular, there exist critical values $p_u^{\rm free} := p_u^{\rm free}(G,q)$ and $p_u^{\rm wired} := p_u^{\rm wired}(G,q)$ such that

$$\mathsf{FRC}^G_{p,q}(\exists \text{ a unique infinite cluster}) = \begin{cases} 0 & \text{for } p < p_u^{\rm free}, \\ 1 & \text{for } p > p_u^{\rm free} \end{cases} \qquad (31)$$

and

$$\mathsf{WRC}^G_{p,q}(\exists \text{ a unique infinite cluster}) = \begin{cases} 0 & \text{for } p < p_u^{\rm wired}, \\ 1 & \text{for } p > p_u^{\rm wired}. \end{cases} \qquad (32)$$

We conjecture the following strengthening, analogous to the simultaneous uniqueness results of [3, 35, 36]:

Conjecture 5.1. *Let $G = (V,E)$ be connected, transitive and unimodular. For a configuration $\xi \in \{0,1\}^E$, write $N(\xi)$ for the number of infinite clusters in ξ. Let $\mathcal{D}$ be the set of quadruples (p_1, p_2, q_1, q_2) such that*

$$p_1 \le p_2 \quad \text{and} \quad \frac{p_1}{(1-p_1)q_1} \le \frac{p_2}{(1-p_2)q_2},$$

with at least one of these inequalities being strict. In the notation of Section 3, we have a.s. for all quadruples $(p_1,p_2,q_1,q_2) \in \mathcal{D}$ simultaneously, each infinite cluster of Y contains $N(X)$ infinite clusters of X, where X and Y may be any of the following three random variables:

(i) $X = {}^{\rm free}X^G_{p_1,q_1}$ *and* $Y = {}^{\rm free}X^G_{p_2,q_2}$,

(ii) $X = {}^{\rm wired}X^G_{p_1,q_1}$ *and* $Y = {}^{\rm wired}X^G_{p_2,q_2}$,

(iii) $X = {}^{\rm free}X^G_{p_1,q_1}$ *and* $Y = {}^{\rm wired}X^G_{p_2,q_2}$.

Summarizing (28), (29), (31), and (32), we have (for G connected, transitive and unimodular) four critical values $p_c^{\rm wired}, p_c^{\rm free}, p_u^{\rm wired}, p_u^{\rm free} \in [0,1]$ such that the $\mathsf{FRC}^G_{p,q}$-a.s. number of infinite clusters equals

$$\begin{cases} 0 & \text{for } p < p_c^{\rm free}, \\ \infty & \text{for } p \in (p_c^{\rm free}, p_u^{\rm free}), \\ 1 & \text{for } p > p_u^{\rm free} \end{cases}$$

and the $\mathsf{WRC}^G_{p,q}$-a.s. number of infinite clusters equals

$$\begin{cases} 0 & \text{for } p < p_c^{\rm wired}, \\ \infty & \text{for } p \in (p_c^{\rm wired}, p_u^{\rm wired}), \\ 1 & \text{for } p > p_u^{\rm wired}. \end{cases}$$

We now consider whether there is an infinite cluster at p_c. It is known that there is none for $q = 1$ on nonamenable transitive unimodular graphs [5, 6]. On the other hand, it is known that there can be infinite clusters for $q > 2$ on the Cayley graphs $\mathbf{T}_n$ for $n \geq 2$ with respect to the wired random cluster measure; see [16, 32]. While we do not have a criterion that settles the question completely, the invariance of our coupling allows us to prove the following:

Theorem 5.2. *Let G be a transitive nonamenable unimodular graph and $q \geq 1$. Then there is no infinite cluster $\mathsf{FRC}^G_{p_c^{\mathrm{free}}(q),q}$-a.s. Also, the following are equivalent:*

(i) There is no infinite cluster $\mathsf{WRC}^G_{p_c^{\mathrm{wired}}(q),q}$-a.s.

(ii) For every edge $e \in G$, the function $p \mapsto \mathsf{WRC}^G_{p,q}\big(X(e) = 1\big)$ is continuous from the left at $p_c^{\mathrm{wired}}(q)$.

(iii) $\mathsf{FRC}^G_{p_c^{\mathrm{wired}}(q),q} = \mathsf{WRC}^G_{p_c^{\mathrm{wired}}(q),q}$.

Proof. Fix an edge $e = [x, y]$. Let $Q^{\mathrm{free}}(p) := \mathsf{FRC}^G_{p,q}\big(X(e) = 1\big)$ and $Q^{\mathrm{wired}}(p) := \mathsf{WRC}^G_{p,q}\big(X(e) = 1\big)$. Now $Q^{\mathrm{free}}(p) = \lim_{i \to \infty} \mathsf{FRC}^{G,i}_{p,q}\big(X(e) = 1\big)$ by definition. Since the latter probabilities are rational functions in p and q, they are continuous at all p. Since they are increasing in p and increase to their limit, we obtain left-continuity of $Q^{\mathrm{free}}(p)$ at all p.

For $p < p_c^{\mathrm{free}}(q)$, the probability that

$$^{\mathrm{free}}X^G_{p_c^{\mathrm{free}}(q),q}(e) = 1 \text{ and } {}^{\mathrm{free}}X^G_{p,q}(e) = 0$$

in our dynamical coupling equals $Q^{\mathrm{free}}\big(p_c^{\mathrm{free}}(q)\big) - Q^{\mathrm{free}}(p)$. Thus, left-continuity of $Q^{\mathrm{free}}(p)$ at $p = p_c^{\mathrm{free}}(q)$ implies that this probability tends to 0 as $p \uparrow p_c^{\mathrm{free}}(q)$. Therefore, minor modifications of the proofs in [6] show that there is no infinite cluster $\mathsf{FRC}^G_{p_c^{\mathrm{free}}(q),q}$-a.s. (It is in these proofs, which we do not repeat, that the invariance of our coupling is used.)

Similarly, (ii) implies (i). That (i) implies (iii) is due to [2]; the reasoning is sketched in the proof of Proposition 6.11 below. Finally, if (iii) holds, then $\mathsf{FRC}_{p,q} = \mathsf{WRC}_{p,q}$ for all $p \leq p_c^{\mathrm{wired}}$ by [2] again. Therefore $Q^{\mathrm{wired}}(p) = Q^{\mathrm{free}}(p)$ for $p \leq p_c^{\mathrm{wired}}$ and the continuity of $Q^{\mathrm{free}}(p)$ implies (ii) as above. ◻

Given a spin configuration $\omega \in \{1, \ldots, q\}^V$ and an edge configuration $\xi \in \{0, 1\}^E$, we may partition V into **connected single-spin components**, meaning that x and y are in the same connected single-spin components if and only if there is a path from x to y in which all vertices have the same spin and all edges are open. The following facts relate the four critical values to corresponding phenomena for the Potts model.

Proposition 5.3. *Let G be any graph, $\beta > 0$, $q \geq 1$, and $p := 1 - e^{-2\beta}$.*

(i) There is no infinite cluster $\mathsf{WRC}_{p,q}$-a.s. iff there is a unique Gibbs measure for the Potts model with the corresponding parameters (q, β).

(ii) Let $\omega \in \{1, \ldots, q\}^V$ be chosen according to $\mathsf{FPt}_{q,\beta}$ and independently $\xi \in \{0, 1\}^E$ be chosen according to Bernoulli(p) percolation. There is a unique infinite cluster $\mathsf{FRC}_{p,q}$-a.s. iff (ω, ξ) a.s. produces a unique infinite connected single-spin component.

(iii) *Let $\omega \in \{1, \dots, q\}^V$ be chosen according to $\mathsf{WPt}_{q,\beta}$ and independently $\xi \in \{0,1\}^E$ be chosen according to Bernoulli(p) percolation. There is a unique infinite cluster $\mathsf{WRC}_{p,q}$-a.s. iff (ω, ξ) a.s. produces a unique infinite connected single-spin component.*

(iv) *If there is a unique infinite cluster $\mathsf{FRC}_{p,q}$-a.s., then $\mathsf{FPt}_{q,\beta}$ is not extremal among all Gibbs measures.*

(v) *Suppose that G is transitive. If there is a unique infinite cluster $\mathsf{FRC}_{p,q}$-a.s., then $\mathsf{FPt}_{q,\beta}$ is not extremal among invariant Gibbs measures. If G is also unimodular, then the converse holds; in fact, if there is not a unique infinite cluster $\mathsf{FRC}_{p,q}$-a.s., then $\mathsf{FPt}_{q,\beta}$ is ergodic, i.e., extremal among all invariant probability measures.*

Proof. Part (i) is essentially due to Aizenman et al. [2] (or see [27]). Parts (ii) and (iii) follow immediately from the coupling of random-cluster and Potts models underlying Propositions 2.2 and 2.3.

Part (iv) follows from (ii): If there is a unique infinite component $\mathsf{FRC}_{p,q}$-a.s., then let (ω, ξ) have the distribution $\mathsf{FPt}_{q,\beta} \times \mathsf{RC}_{p,1}$, as in (ii). By (ii), we may define $r(\omega, \xi)$ to be the spin of the unique infinite single-spin component determined by (ω, ξ). Let $\mathcal{C}(\omega)$ be the collection of maximal connected subgraphs of G whose vertices have a common spin; this does not depend on ξ. Then $r(\omega, \xi)$ is the spin of the unique graph K in $\mathcal{C}(\omega)$ such that $K \cap \xi$ contains an infinite component with positive probability (in ξ). In particular, $r(\omega, \xi)$ depends only on ω a.s.; and it is a tail random variable that is not trivial. Thus, we have shown that the tail σ-field of $\mathsf{FPt}_{q,\beta}$ is not trivial. This is equivalent to nonextremality among all Gibbs measures by [26, Theorem 7.7].

The first part of (v) is due to [54, Theorem 4.2]. The converse of (iv) is not true in general, as is well-known on trees (see, e.g., [11] or [21]). However, if G is transitive and unimodular and if there is not a unique infinite cluster $\mathsf{FRC}_{p,q}$-a.s., then by [45, Theorem 4.1 and Lemma 6.4], $\mathsf{FPt}_{q,\beta}$ is ergodic, which is the same as extremal among all invariant measures. □

How do the four critical values relate to each other? From the definitions it is immediate that

$$p_c^{\text{free}} \le p_u^{\text{free}} \tag{33}$$

and

$$p_c^{\text{wired}} \le p_u^{\text{wired}} . \tag{34}$$

By the middle inequality in (9) and (30), we also have

$$p_c^{\text{wired}} \le p_c^{\text{free}} \tag{35}$$

and

$$p_u^{\text{wired}} \le p_u^{\text{free}} . \tag{36}$$

All of (33)–(36) can in fact be equalities; this happens, e.g., whenever G is amenable. To see this for (33) and (34), just note the well-known fact that the Burton-Keane [14, 25] encounter-point argument (for showing uniqueness of the infinite cluster under the insertion tolerance condition) goes through in the amenable setting. For (35) and (36), see Grimmett [29] and Jonasson [40], where it is shown that for all $q \geq 1$, there are at most countably many p such that $\mathsf{FRC}^G_{p,q} \neq \mathsf{WRC}^G_{p,q}$.

The inequalities can also be strict. To get examples with strict inequalities in (33) and (34), one can take $q := 1$ and G to be any of the nonamenable transitive unimodular graphs that are known to have a "middle phase" for i.i.d. percolation (i.e., a set of values of p that give rise to infinitely many infinite clusters); see, e.g., [44]. Using the ideas in the proof of [44, Proposition 5.2] and the inequalities (22) and (23), it is not hard to show that one can take q to be slightly larger than 1 in all such examples. For an example where the inequality in (35) is strict, we can simply take G to be the regular tree $\mathbf{T}_n$ with $n \geq 2$ and $q > 2$ (see, e.g., Häggström [32]), or take any nonamenable example with q sufficiently large (see the proof of Theorem 1.2(a) in [40]). Finally, for an example where (36) is strict, we refer to Section 6.

The inequalities (33)–(36) say nothing about the relation between p_u^{wired} and p_c^{free}. Here it is possible to get a strict inequality in either direction. Examples with $p_c^{\mathrm{free}} < p_u^{\mathrm{wired}}$ can be obtained in the same way as for (33) and (34); see Proposition 6.20 below for explicit bounds on certain graphs. To get an example with the reverse inequality $p_u^{\mathrm{wired}} < p_c^{\mathrm{free}}$ is more intricate and is the topic of the next section. Note that any such example also gives strict inequality in (35) and in (36).

Question 5.4. *We say that G has* **one end** *if the complement of every finite subset has exactly one infinite component. If G is any nonamenable quasi-transitive graph with one end and $q \geq 1$, are all the inequalities (33)–(36) necessarily strict?*

Of course, when $q = 1$, a famous conjecture of [7] asserts a positive answer.

6 Isoperimetric constants and the critical values

In order to show that there is a Cayley graph G with $p_u^{\mathrm{wired}}(q) < p_c^{\mathrm{free}}(q)$ for some $q > 1$, we shall need an estimate of an isoperimetric constant. In fact, we are able to calculate precisely the necessary isoperimetric constants for planar regular graphs whose dual is also regular (in this case, either the graph or its dual is a Cayley graph; see [17]). Planar duality and Euler's formula will be essential for this. If G is a graph drawn in the plane in such a way that edges do not cross and such that each bounded set in the plane contains only finitely many vertices of G, then G is said to be **properly embedded**. We shall always assume without mention that planar graphs are properly embedded. (The graphs we shall consider can be embedded in the hyperbolic plane more geometrically than in the euclidean plane, but topologically and combinatorially, this is not different from euclidean embddings.) If G is a planar (multi)graph, then the **planar dual** $G^\dagger$ of G (really, of this particular embedding of G) is the (multi)graph formed as follows: The vertices of $G^\dagger$ are the faces formed by G. Two faces of $G^\dagger$ are joined by an edge precisely when they share an edge in G. Thus, $E(G)$ and $E(G^\dagger)$ are in a natural one-to-one correspondence. Furthermore, if one draws each vertex of $G^\dagger$ in the interior of the corresponding face of G and each edge of $G^\dagger$ so that it crosses the corresponding edge of G, then the dual of $G^\dagger$ is G. We shall always assume that G and its planar dual $G^\dagger$ are locally finite.

We shall make use of the following isoperimetric constants. For $K \subseteq V$, recall that $E(K) := \{[x,y] \in E\,;\ x,y \in K\}$ and set $E^*(K) := \{[x,y] \in E\,;\ x \in K \text{ or } y \in K\}$.

Define $\partial_E K := E^*(K) \setminus E(K)$ and $G(K) := \big(K, E(K)\big)$. Write

$$\iota'_E(G) := \lim_{N\to\infty} \inf\left\{\frac{|\partial_E K|}{|K|}\,;\ K \subset V,\ G(K) \text{ connected},\ N \le |K| < \infty\right\},$$

$$\beta(G) := \lim_{N\to\infty} \inf\left\{\frac{|K|}{|E(K)|}\,;\ K \subset V,\ G(K) \text{ connected},\ N \le |K| < \infty\right\},$$

$$\delta(G) := \lim_{N\to\infty} \sup\left\{\frac{|K|}{|E^*(K)|}\,;\ K \subset V,\ G(K) \text{ connected},\ N \le |K| < \infty\right\}.$$

We write d_G for the degree of vertices in G when G is regular. We remark that $\beta(G) = 2/\alpha(G)$, with $\alpha(G)$ defined as in [5], except that α was defined with an infimum, rather than a liminf. In any case, when G is regular,

$$\beta(G) = \frac{2}{d_G - \iota'_E(G)} \tag{37}$$

and

$$\delta(G) = \frac{2}{d_G + \iota'_E(G)}\,. \tag{38}$$

It is shown in [5] that when G is transitive,

$$\iota'_E(G) = \inf\left\{\frac{|\partial_E K|}{|K|}\,;\ K \subset V \text{ finite and nonempty}\right\}.$$

(The right-hand side is denoted $\iota_E(G)$ there.) Thus, when G is transitive, we have that

$$\delta(G) = \sup\left\{\frac{|K|}{|E^*(K)|}\,;\ K \subset V \text{ finite and nonempty}\right\}. \tag{39}$$

Recall from Section 2.5 that G is called quasi-transitive if the vertex set of G decomposes into a finite number of orbits under the action of $\mathrm{Aut}(G)$. Note that G is quasi-transitive iff $G^\dagger$ is quasi-transitive.

The estimate that we shall need is embodied in Corollary 6.5, but the precise combinatorial calculation is the following.

Theorem 6.1. *If G is a planar regular graph with regular dual $G^\dagger$, then*

$$\iota'_E(G) = (d_G - 2)\sqrt{1 - \frac{4}{(d_G - 2)(d_{G^\dagger} - 2)}}\,.$$

Remark 6.2. In this case, G and $G^\dagger$ are transitive. This is folklore. Since we have been unable to find a suitable reference, we include a proof here. First, recall the existence of tessellations by congruent polygons. It is easy to see that the edge graphs of any two such tessellations of the same type are isomorphic, by going out ring by ring around a starting polygon. Now we assert that any (proper) tessellation of a plane with degree d and codegree $d^\dagger$ has an edge graph that is isomorphic to the edge graph of the corresponding tessellation above. In case $(d-2)(d^\dagger - 2) = 4$, we replace each face by a congruent copy of a flat polygon; in the other case, replace it by a congruent copy of a regular hyperbolic polygon of $d^\dagger$ sides and interior angles $2\pi/d$. Glue these together along the edges. We get a Riemannian surface of curvature 0 or -1, correspondingly, that is simply connected. Riemann's theorem says that the surface is isometric to either the euclidean plane or the hyperbolic plane. That is, we now have a tessellation by congruent polygons. (One could also prove the existence statement in a similar manner.)

Remark 6.3. Contrary to what one might first expect, we believe that combinatorial balls never give the best isoperimetric constants when G is a nonamenable planar Cayley graph, e.g., $\liminf_n |\partial_E B_n|/|B_n| > \iota'_E(G)$, where B_n is the ball of radius n about o. In many cases, this follows from the formulas of [22]. For example, suppose that

$$\theta := \lim_{n\to\infty} \frac{|B_{n+1}|}{|B_n|}$$

exists. (This is not the case for all Cayley graphs; see [28].) Then

$$\theta - 1 \le \liminf_{n\to\infty} |\partial_E B_n|/|B_n| \,.$$

Thus, if $\theta - 1 > \iota'_E(G)$, then we may conclude that $\liminf_{n\to\infty} |\partial_E B_n|/|B_n| > \iota'_E(G)$. For example, if $d_{G^\dagger} = 6$, in which case G is always a Cayley graph (see [17]), then Theorem 6.1 shows that $\iota'_E(G) = \sqrt{(d_G-2)(d_G-3)}$. On the other hand, [22] shows that

$$\sum_{n\ge 0} |B_n \setminus B_{n-1}| z^n = \frac{z^3+2z^2+2z+1}{z^3+(2-d_G)(z^2+z)+1} = \frac{z^2+z+1}{z^2+(1-d_G)z+1}$$
$$= \frac{z^2+z+1}{(1-\gamma z)(1-\gamma^{-1}z)} = (z^2+z+1)\sum_{n\ge 0}(\gamma z)^n \sum_{m\ge 0}(\gamma^{-1}z)^m \,,$$

where γ is the smallest positive root of $z^2+(1-d_G)z+1$. Therefore,

$$|B_n \setminus B_{n-1}| = \gamma^n(3-\gamma^{-2n-2}-\gamma^{-2n}-\gamma^{-2n+2})/(1-\gamma^{-2})$$

for $n \ge 1$. Thus, $\theta = \gamma$ exists and $\theta - 1 = \sqrt{d_G-3}\big(\sqrt{d_G-3}+\sqrt{d_G+1}\big)/2$, which is easily verified to be larger than $\iota'_E(G)$.

Theorem 6.1 follows from applying the following identity to G and $G^\dagger$, then solving the resulting two equations using (37) and (38):

Theorem 6.4. *For any planar regular graph G with regular dual, we have*

$$\beta(G) + \delta(G^\dagger) = 1 \,.$$

Proof. Note first that the constant $\iota'_E(G)$ is unchanged if, in its definition, we require K to be connected and simply connected when K is regarded as a union of closed faces of $G^\dagger$ in the plane. This is because filling in holes increases $|K|$ and decreases $|\partial_E K|$. Since G is regular, the same holds for $\beta(G)$ by (37). Likewise, the assumed regularity of $G^\dagger$ and (38) imply a comparable statement for $\delta(G^\dagger)$. In fact, we shall need a refinement of this idea for $\delta(G^\dagger)$. Namely, given a finite connected set K in $V(G^\dagger)$, regard each element of K as a face of G and let $K' \subset V$ be the set of vertices bounding these faces. Let $\widehat{K}$ be the set of all faces in $G^\dagger$ that lie in the interior of the outermost cycle formed by $E(K')$. Then again $|\widehat{K}| \ge |K|$ and $|\partial_E \widehat{K}| \le |\partial_E K|$, so that $\delta(G^\dagger)$ can be approached arbitrarily closely by such sets $\widehat{K}$. Note that $|E^*(\widehat{K})| = |E((\widehat{K})')|$.

Now let $\epsilon > 0$ and let K be a finite connected set in $V(G^\dagger)$ such that $|E^*(K)| > 1/\epsilon$, $|K|/|E^*(K)| \ge \delta(G^\dagger) - \epsilon$, and $|E(K')| = |E^*(K)|$, where K' is defined as above. Since the number of faces of the graph $G(K')$ is at least $|K| + 1$, Euler's formula applied to the graph $G(K')$ gives

$$|K'|/|E(K')| + |K|/|E^*(K)| \le 1 + 1/|E^*(K)| < 1 + \epsilon\,. \tag{40}$$

Our choice of K then implies that

$$|K'|/|E(K')| + \delta(G^\dagger) \le 1 + 2\epsilon\,.$$

Since $G(K')$ is connected and $|K'| \to \infty$ when $\epsilon \to 0$, it follows that $\beta(G) + \delta(G^\dagger) \le 1$.

To prove that $\beta(G) + \delta(G^\dagger) \ge 1$, let $\epsilon > 0$. Let $K \subset V$ be connected and simply connected (when regarded as a union of closed faces of $G^\dagger$ in the plane) such that $|K|/|E(K)| \le \beta(G) + \epsilon$. Let K^f be the set of vertices in $G^\dagger$ corresponding to the faces of $G(K)$. Since $|E^*(K^f)| \le |E(K)|$ and the number of faces of the graph $G(K)$ is precisely $|K^f| + 1$, we have

$$|K|/|E(K)| + |K^f|/|E^*(K^f)| \ge |K|/|E(K)| + |K^f|/|E(K)| = 1 + 1/|E(K)| \ge 1$$

by Euler's formula applied to the graph $G(K)$. (In case K^f is empty, a comparable calculation shows that $|K|/|E(K)| \ge 1$.) In light of (39), it follows that

$$\beta(G) + \delta(G^\dagger) + \epsilon \ge \beta(G) + \epsilon + |K^f|/|E^*(K^f)| \ge |K|/|E(K)| + |K^f|/|E^*(K^f)| \ge 1\,.$$

Since ϵ is arbitrary, the desired inequality follows. □

From (37) and (38), we see that $\beta(G) > \delta(G)$ when G is regular and $\iota'_E(G) > 0$. Thus, we obtain the following inequality.

Corollary 6.5. *If G is a planar regular graph with nonamenable regular dual, then*

$$\beta(G) + \beta(G^\dagger) > 1\,.$$

The proof of Theorem 6.4 appears not to give any idea of which finite sets $K \subset V$ yield quotients $|\partial_E K|/|K|$ close to $\iota'_E(G)$. However, Y. Peres has deduced the following from a closer examination of the proof. As in the proof of Theorem 6.4, we shall write K' for the set of vertices incident to the faces corresponding to K, for both $K \subset V$ and for $K \subset V^\dagger$. Likewise, $\widehat{K}$ denotes the faces inside the outermost cycle of $E(K')$. According to the reasoning of the first paragraph of the proof of Theorem 6.4 and (40), we have

$$|(\widehat{K})'|/|E((\widehat{K})')| + |K|/|E^*(K)| \le |(\widehat{K})'|/|E((\widehat{K})')| + |\widehat{K}|/|E^*(\widehat{K})| \le 1 + 1/|E((\widehat{K})')|\,. \tag{41}$$

Proposition 6.6. *Let G be a planar regular graph with regular dual $G^\dagger$. Let $K_0 \subset V$ be an arbitrary finite connected set and recursively define $L_n := (\widehat{K}_n)' \subset V^\dagger$ and $K_{n+1} := (\widehat{L}_n)' \subset V$. Then $|\partial_E K_n|/|K_n| \to \iota'_E(G)$ and $|\partial_E L_n|/|L_n| \to \iota'_E(G^\dagger)$.*

Proof of Proposition 6.6. The amenable case is trivial, so assume that G is nonamenable. Write

$$\kappa_n := |\partial_E K_n|/|K_n| - \iota'_E(G)$$

and

$$\lambda_n := |\partial_E L_n|/|L_n| - \iota'_E(G^\dagger)\,.$$

Also write $d := d_G$, $d^\dagger := d_{G^\dagger}$, $\iota := \iota'_E(G)$, and $\iota^\dagger := \iota'_E(G^\dagger)$. We may rewrite (41) as

$$\frac{2}{d^\dagger - |\partial_E L_n|/|L_n|} + \frac{2}{d + |\partial_E K_n|/|K_n|} \le 1 + \frac{1}{|E(L_n)|}\,,$$

or, again, as

$$\frac{2}{d^\dagger - \iota^\dagger - \lambda_n} + \frac{2}{d + \iota + \kappa_n} \le 1 + \frac{1}{|E(L_n)|} = \frac{2}{d^\dagger - \iota^\dagger} + \frac{2}{d + \iota} + \frac{1}{|E(L_n)|}\,,$$

whence

$$\frac{2\lambda_n}{(d^\dagger - \iota^\dagger)(d^\dagger - \iota^\dagger - \lambda_n)} + \frac{2\kappa_n}{(d + \iota)(d + \iota + \kappa_n)} \le \frac{1}{|E(L_n)|}\,.$$

Therefore

$$\begin{aligned} 2\lambda_n &\le \frac{(d^\dagger - \iota^\dagger)(d^\dagger - \iota^\dagger - \lambda_n)}{(d + \iota)(d + \iota + \kappa_n)}(2\kappa_n) + \frac{(d^\dagger - \iota^\dagger)(d^\dagger - \iota^\dagger - \lambda_n)}{|E(L_n)|} \\ &\le \left(\frac{d^\dagger - \iota^\dagger}{d + \iota}\right)^2 2\kappa_n + \frac{(d^\dagger - \iota^\dagger)^2}{|E(L_n)|}\,. \end{aligned}$$

Similarly, we have

$$2\kappa_{n+1} \le \left(\frac{d - \iota}{d^\dagger + \iota^\dagger}\right)^2 2\lambda_n + \frac{(d - \iota)^2}{|E(K_{n+1})|}\,.$$

Putting these together, we obtain

$$2\kappa_{n+1} \le a(2\kappa_n) + b_n\,,$$

where

$$a := \left(\frac{(d - \iota)(d^\dagger - \iota^\dagger)}{(d + \iota)(d^\dagger + \iota^\dagger)}\right)^2$$

and

$$b_n := \left(\frac{(d - \iota)(d^\dagger - \iota^\dagger)}{d^\dagger + \iota^\dagger}\right)^2 \frac{1}{|E(L_n)|} + \frac{(d - \iota)^2}{|E(K_{n+1})|}\,.$$

Therefore

$$2\kappa_n \le 2\kappa_0 a^{n-1} + \sum_{j=0}^{n-2} a^j b_{n-j}\,.$$

Since $a < 1$ and $b_n \to 0$, we obtain $\kappa_n \to 0$. Hence $\lambda_n \to 0$ too. ◻

If μ is a probability measure on $\{0,1\}^E$, we associate a **dual** measure $\mu^\dagger$ on $\{0,1\}^{E^\dagger}$ as follows. Given $e \in E$, let $e^\dagger$ be the edge in $E^\dagger$ that crosses e. Given $\xi \in \{0,1\}^E$, let $\tilde{\xi} \in \{0,1\}^{E^\dagger}$ be the function $e^\dagger \mapsto 1 - \xi(e)$. For a Borel set $A \subset \{0,1\}^E$, write $\tilde{A} := \{\tilde{\xi}\,;\ \xi \in A\}$. Then $\mu^\dagger$ is defined by $\mu(A) = \mu^\dagger(\tilde{A})$. Our next proposition is more or less well-known (see, e.g., [15, 60]), but perhaps has not been stated in this particular form before. For completeness, we provide the simple proof here.

Proposition 6.7. *For any planar graph G, $\mathsf{FRC}^G_{p,q}$ is dual to $\mathsf{WRC}^{G^\dagger}_{p',q}$ if*

$$p' = \frac{(1-p)q}{p+(1-p)q}\,. \tag{42}$$

Proof. Suppose first that G is a finite graph. For any $\xi \in \{0,1\}^E$, write $\tilde\xi := E^\dagger \setminus \xi^\dagger$ as above. Euler's formula applied to the graph $\tilde G := (V^\dagger, \tilde\xi)$ says that

$$|V^\dagger| - |\tilde\xi| + \|\xi\| = 1 + \|\tilde\xi\|$$

since the number of faces of $\tilde G$ is equal to $\|\xi\|$. Thus,

$$\begin{aligned}\mathsf{RC}^G_{p,q}(\xi) &= Z^{-1}p^{|\xi|}(1-p)^{|E\setminus\xi|}q^{\|\xi\|} = Z^{-1}p^{|E^\dagger\setminus\tilde\xi|}(1-p)^{|\tilde\xi|}q^{1+\|\tilde\xi\|-|V^\dagger|+|\tilde\xi|} \\ &= Z^{-1}q^{1-|V^\dagger|}p^{|E^\dagger\setminus\tilde\xi|}[(1-p)q]^{|\tilde\xi|}q^{\|\tilde\xi\|} = \tilde Z^{-1}(1-p')^{|E^\dagger\setminus\tilde\xi|}p'^{|\tilde\xi|}q^{\|\tilde\xi\|} = \mathsf{RC}^{G^\dagger}_{p',q}(\tilde\xi)\,,\end{aligned}$$

where Z and $\tilde Z$ are normalizing constants that do not depend on ξ. Thus, $\mathsf{RC}^G_{p,q}$ is dual to $\mathsf{RC}^{G^\dagger}_{p',q}$.

Now let V_i be a sequence of increasing finite subsets of V such that the faces of $G(V_i)$ are faces of G, except for the outer face, of course. Then we have seen that $\mathsf{RC}^{G_i}_{p,q}$ is dual to $\mathsf{RC}^{G_i^\dagger}_{p',q}$. Thus, the general result follows by taking weak limits. □

In consequence, the methods of Benjamini and Schramm [8] show the following.

Proposition 6.8. *Let G be a planar nonamenable quasi-transitive graph and p' be as in* (42). *In the natural coupling of $\mathsf{FRC}^G_{p,q}$ and $\mathsf{WRC}^{G^\dagger}_{p',q}$, the number of infinite clusters with respect to each is a.s. one of the following:* $(0,1)$, $(1,0)$, *or* (∞,∞).

Write

$$h(x) := x/(1-x)\,. \tag{43}$$

Corollary 6.9. *For any planar nonamenable quasi-transitive graph G,*

$$h\big(p_c^{\mathrm{wired}}(G,q)\big)h\big(p_u^{\mathrm{free}}(G^\dagger,q)\big) = h\big(p_c^{\mathrm{free}}(G,q)\big)h\big(p_u^{\mathrm{wired}}(G^\dagger,q)\big) = q\,,$$

$0 < p_c^{\mathrm{wired}}(G,q) \le p_c^{\mathrm{free}}(G,q) < 1$, *and* $0 < p_u^{\mathrm{wired}}(G,q) \le p_u^{\mathrm{free}}(G,q) < 1$.

Proof. Proposition 6.8 shows that there is no infinite cluster a.s. with respect to the free measure iff there is a unique infinite cluster a.s. with respect to the dual wired measure. Therefore, $p_u^{\mathrm{wired}}(G^\dagger,q) = \big(p_c^{\mathrm{free}}(G,q)\big)'$ in the notation of (42). Some algebra shows that this is the same as $h\big(p_c^{\mathrm{free}}(G,q)\big)h\big(p_u^{\mathrm{wired}}(G^\dagger,q)\big) = q$. A similar proof shows the other equation.

It is well known that $0 < p_c(G) := p_c(G,1) < 1$ under the present assumptions [44]. From (22) and (23), it follows that the same holds for $p_c^{\mathrm{wired}}(G,q)$ and $p_c^{\mathrm{free}}(G,q)$ when $q > 1$. The same now follows for the uniqueness critical points by the equations just established. □

Corollary 6.10. *Let G be a planar nonamenable quasi-transitive graph. Then there is $\mathsf{WRC}^G_{p_u^{\mathrm{wired}},q}$-a.s. a unique infinite cluster.*

Proof. In light of Theorem 5.2, there is no infinite cluster $\mathsf{FRC}^{G^\dagger}_{p_c^{\mathrm{free}}(G^\dagger),q}$-a.s. Hence by Proposition 6.8 and Corollary 6.9, there is a unique infinite cluster $\mathsf{WRC}^G_{p_u^{\mathrm{wired}}(G),q}$-a.s. □

The methods of Grimmett [30] show that for any quasi-transitive G, the critical points $p_c^{\mathrm{wired}}(q)$ and $p_c^{\mathrm{free}}(q)$ are continuous and strictly increasing functions of q as long as $p_c(1) < 1$, and similarly $p_u^{\mathrm{wired}}(q)$ and $p_u^{\mathrm{free}}(q)$ are continuous and strictly increasing functions of q as long as $p_u(1) < 1$. When G is a planar regular graph with regular dual, [8] shows that $p_c(G) < p_u(G) := p_u(G, 1)$, while Theorem 6.18 below shows that $p_u^{\mathrm{wired}}(G, q) < p_c^{\mathrm{free}}(G, q)$ for large q. Thus, there is at least one q for which $p_c^{\mathrm{free}}(G, q) = p_u^{\mathrm{wired}}(G, q)$. If G is isomorphic to its dual, then these critical values are equal to $\sqrt{q}/(\sqrt{q}+1)$ because of Corollary 6.9. We do not know whether this holds for exactly one q; see Question 6.22.

Another natural set of p to examine is the set where $\mathsf{FRC}_p = \mathsf{WRC}_p$ (which is where the corresponding free and wired Potts measures are equal). In general, this is not simply an interval, but let us define

$$p_{\mathrm{F=W}}(G) := \sup\{p < 1\,;\ \mathsf{FRC}_p^G \neq \mathsf{WRC}_p^G\}\,.$$

We say that a graph has **bounded fundamental cycle length** if there is a set of (oriented simple) cycles of the graph with bounded length that generates all cycles by addition and subtraction (in the sense of homology). For example, this is the case for Cayley graphs of finite presentations of groups. Recall that G has one end if the complement of every finite subset has exactly one infinite component.

Proposition 6.11. *Let G be any graph.*

(i) If $p < p_c^{\mathrm{wired}}(G)$, then $\mathsf{FRC}_p^G = \mathsf{WRC}_p^G$.

(ii) If $p_c^{\mathrm{wired}}(G) < p < p_u^{\mathrm{free}}(G)$, then $\mathsf{FRC}_p^G \neq \mathsf{WRC}_p^G$.

(iii) If G is a graph with one end and bounded fundamental cycle length, then $p_{\mathrm{F=W}}(G) < 1$.

(iv) If G is a planar nonamenable quasi-transitive graph with one end, then $p_{\mathrm{F=W}}(G) = p_u^{\mathrm{free}}(G)$.

Proof. Part (i) is due to [2], but we recap the short proof here. It suffices to show that $\mathsf{WRC}_p^G \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}_p^G$ if $p < p_c^{\mathrm{wired}}(G)$, or, more generally, if there is no infinite cluster WRC_p^G-a.s. Since there is no infinite cluster, given any ball B about o and any $\epsilon > 0$, there is a ball B' so that with probability at least $1-\epsilon$, there is a unique maximal set $K \subset B'$ such that all of $\partial_E K$ is closed and $B \subset K$. Given that K is such a set, the configuration restricted to $G(K)$ has the distribution $\mathsf{RC}_p^{G(K)}$, which is dominated by the restriction of FRC_p^G to $G(K)$. In particular, this holds for the restriction of the configuration to B. Since ϵ and B were arbitrary, the result follows.

Part (ii) is due to [40]. Again, the proof is short: If $\mathsf{FRC}_p^G = \mathsf{WRC}_p^G$, then the two measures give the same number of infinite clusters a.s. Furthermore, since the measures are both DLR random-cluster measures and DLR wired random-cluster measures, there is at most one infinite cluster a.s. Hence $p \notin \big(p_c^{\mathrm{wired}}(G), p_u^{\mathrm{free}}(G)\big)$.

Now let G be any graph with one end and bounded fundamental cycle length. Let t be an upper bound for the lengths of a set of fundamental cycles. The fundamental theorem of [4] implies that the t-neighborhood of any minimal cutset is connected. Note that by (22), FRC_p^G dominates Bernoulli(p^*) bond percolation $\mathbf{P}_{p^*}$ with p^* close to 1 when p is close to 1. Choose a coupling (ξ, ω) such that ξ has distribution FRC_p^G, ω has distribution $\mathbf{P}_{p^*}$, and $\xi \supseteq \omega$. Let ω' consist of those edges e such that all bonds within distance t of e are open in ω; define ξ' similarly with respect to ξ. The percolation ω' dominates a Bernoulli percolation with survival parameter close to 1 by [46] or [45, Remark 6.2]. Thus, a.s. the *closed* bonds of ω' do not form any infinite cluster if p^* is sufficiently close to 1. Therefore, when p is close to 1, the same holds for ξ'. Choose p so close to 1 that ξ' has no infinite closed clusters. Let B_r be the ball of radius r about o and $\epsilon > 0$. For any set of sites S, let $K(S)$ be the set of vertices that can be reached from some vertex of S without using an open bond from ξ'. There is a sphere S about o so that the probability of the event $E := \{K(S) \cap B_{r+t} = \emptyset\}$ is at least $1 - \epsilon$. Since S separates B_r from ∞ as a set of vertices, so does $\partial_E S$ as a set of edges. Hence the same holds for the larger set $K(S)$ on the event E. Let L be the t-neighborhood of $\partial_E K(S)$. Then on the event E, L consists of open edges and separates B_r from ∞. Furthermore, some minimal cutset of $\partial_E K(S)$ separates B_r from ∞ and has a connected t-neighborhood, whence L contains a connected open cutset that separates B_r from ∞. Therefore, given E, the configuration ξ restricted to $G(B_r)$ has the distribution WRC_p^{G,B_r} [which means $\mathsf{WRC}_p^{G,i}$ if $G_i = G(B_r)$], which dominates the restriction of WRC_p^G to $G(B_r)$. Since ϵ and r were arbitrary, part (iii) follows.

Finally, if G is planar, nonamenable, and quasi-transitive, consider $p > p_u^{\text{free}}(G)$. Let p' be as in (42). We have $p' < p_c^{\text{wired}}(G^\dagger)$ by Corollary 6.9, whence $\mathsf{FRC}_{p'}^{G^\dagger} = \mathsf{WRC}_{p'}^{G^\dagger}$ by part (i). Because of Proposition 6.7, it follows that $\mathsf{WRC}_p^G = \mathsf{FRC}_p^G$. Hence $p_{\mathrm{F=W}}(G) = p_u^{\text{free}}(G)$ by part (ii). □

Remark 6.12. The argument of the last paragraph shows that as long as there is a unique infinite cluster $\mathsf{FRC}_{p_u^{\text{free}}}$-a.s., then $\mathsf{WRC}_p^G = \mathsf{FRC}_p^G$, which establishes a conjecture of [54, Conjecture 4.2] in the planar nonamenable case.

Question 6.13. *If G is a quasi-transitive graph and $q > 0$, is the set of $p \in [0,1]$ where $\mathsf{FRC}_p^G \neq \mathsf{WRC}_p^G$ an interval?*

The following proposition relating the geometry of the graph to the behavior of the free random-cluster measure uses the ideas of [40, 41].

Proposition 6.14. *Let G be a graph with degrees bounded by d. Write $b := \log\big(p/(1-p)\big)/\log q$ and $b^+ := \max\{b, 0\}$. If $b < \beta(G)$ and*

$$\log q > \frac{1 + \log(d-1)}{\beta(G) - b^+},$$

then $\mathsf{FRC}_{p,q}^G$-a.s., there is no infinite cluster.

Remark 6.15. In this proposition, a better result is obtained by replacing $\beta(G)$ by the corresponding quantity that results when K is required to contain a fixed point o in the definition of $\beta(G)$. The same proof applies.

To prove Proposition 6.14, we shall use the following bound analogous to the well-known bound of Kesten [42] on site-connected clusters.

Lemma 6.16. *Let G be any graph all of whose vertices have degrees at most d. For any fixed $o \in V(G)$, let b_n be the number of connected subgraphs of G that contain o and have exactly n edges. Then $\limsup_{n\to\infty} b_n^{1/n} < e(d-1)$.*

Proof. Let $b_{n,\ell}$ denote the number of connected subgraphs (V', E') of G such that $o \in V'$, $|E'| = n$, and $|E^*(V') \setminus E'| = \ell$. Note that for such a subgraph,

$$\ell \le d|V'| - 2n \le d(n+1) - 2n = (d-2)n + d\,. \tag{44}$$

Let $p := 1/(d-1)$ and consider Bernoulli(p) bond percolation on G. Writing the fact that 1 is at least the probability that the cluster of o is finite and using (44), we obtain

$$1 \ge \sum_{n,\ell} b_{n,\ell} p^n (1-p)^\ell \ge \sum_n b_n p^n (1-p)^{(d-2)n+d}\,.$$

Therefore $\limsup_{n\to\infty} b_n^{1/n} \le 1/[p(1-p)^{d-2}]$. Putting in the chosen value of p gives the result. ◻

Proof of Proposition 6.14. Because of (22), it suffices to prove the claim when $b \ge 0$. So assume that $b \ge 0$.

Let $o \in V(G_i)$ for all i. If ξ is a configuration, write $\xi(o)$ for the component of o determined by ξ. Suppose that G' is a finite connected subgraph of G containing o. If ξ is a configuration in G_i such that $\xi(o) = G'$, then let ξ' be obtained from ξ by closing all edges in $\xi \cap E(G')$. Then $\|\xi'\| = \|\xi\| + |V(G')| - 1$, whence

$$\mathsf{RC}_{p,q}^{G,i}[\xi] = \left(\frac{p}{1-p}\right)^{|\xi\cap E(G')|} q^{-|V(G')|+1}\mathsf{RC}_{p,q}^{G,i}[\xi'] \le \left(\frac{p}{1-p}\right)^{|E(G')|} q^{-|V(G')|+1}\mathsf{RC}_{p,q}^{G,i}[\xi']$$

since $p/(1-p) \ge 1$ in light of the assumption that $0 \le b$. Also, if $\xi_1 \ne \xi_2$ are such that $\xi_1(o) = \xi_2(o) = G'$, then $\xi_1' \ne \xi_2'$. Summing over all ξ such that $\xi(o) = G'$ yields

$$\mathsf{RC}_{p,q}^{G,i}[\xi(o) = G'] \le \left(\frac{p}{1-p}\right)^{|E(G')|} q^{-|V(G')|+1}\,.$$

Now the right-hand side equals $q^{b|E(G')|-|V(G')|+1}$ by the definition of b. Choose $b' \in \big(b, \beta(G)\big)$. Then provided that $|V(G')|$ is sufficiently large, we have that this last is less than $q^{(b-b')|E(G')|+1}$. Therefore, for all large N, we have

$$\mathsf{RC}_{p,q}^{G,i}\big[|E\big(\xi(o)\big)| \ge N\big] \le \sum_{n\ge N} \big(e(d-1)\big)^n q^{(b-b')n+1} < \infty$$

if $\log q > \big(1+\log(d-1)\big)/(b'-b)$, by Lemma 6.16. Therefore, for such q this sum tends to 0 as $N \to \infty$, so that given any $\epsilon > 0$, there is some N_0 such that for all $N \ge N_0$ and for all i, we have $\mathsf{RC}_{p,q}^{G,i}\big[|E\big(\xi(o)\big)| \ge N\big] < \epsilon$. Since the event $\{|E\big(\xi(o)\big)| \ge N\}$ depends only on the edges within distance N of o, it follows that $\mathsf{FRC}_{p,q}^{G}\big[|E\big(\xi(o)\big)| \ge N\big] < \epsilon$. Since $\epsilon > 0$ is arbitrary, we find that $\mathsf{FRC}_{p,q}^{G}\big[|E\big(\xi(o)\big)| = \infty\big] = 0$. ◻

Corollary 6.17. *Let G be a planar quasi-transitive graph whose dual has degrees bounded by $d^\dagger$. Write $b := \log\big(p/(1-p)\big)/\log q$. If $b > 1 - \beta(G^\dagger)$ and*

$$\log q > \frac{1 + \log(d^\dagger - 1)}{b \wedge 1 - 1 + \beta(G^\dagger)}\,,$$

then $\mathsf{WRC}_{p,q}^{G}$-a.s., there is a unique infinite cluster.

Proof. Let p' be as in (42). Then $p'/(1-p') = q(1-p)/p$. Let $b' := \log\big(p'/(1-p')\big)/\log q = 1-b$. By our hypothesis, $b' < \beta(G^\dagger)$ and $\log q > \big(1+\log(d^\dagger-1)\big)/(\beta(G^\dagger)-b^+)$, whence Proposition 6.14 applied to $G^\dagger$ shows that there is no infinite cluster $\mathsf{FRC}^{G^\dagger}_{p',q}$-a.s. Hence the conclusion follows from Proposition 6.8. □

Putting all this together, we arrive at our main result in this section:

Theorem 6.18. *If G is a planar regular nonamenable graph of degree d with regular dual of degree $d^\dagger$, then $p_u^{\text{wired}}(q) < p_c^{\text{free}}(q)$ for*

$$q > \frac{\big(2+\log\big((d-1)(d^\dagger-1)\big)\big)(dd^\dagger-d-d^\dagger)}{\sqrt{(d-2)(d^\dagger-2)(dd^\dagger-2d-2d^\dagger)}}\,.$$

Proof. Let

$$b_0 := \frac{\big(1+\log(d-1)\big)\big(1-\beta(G^\dagger)\big)+\big(1+\log(d^\dagger-1)\big)\beta(G)}{\big(1+\log(d-1)\big)+\big(1+\log(d^\dagger-1)\big)}\,.$$

Then $0 < 1-\beta(G^\dagger) < b_0 < \beta(G) < 1$ because of Corollary 6.5. Furthermore, we have

$$\frac{1+\log(d-1)}{\beta(G)-b_0^+} = \frac{1+\log(d^\dagger-1)}{b_0\wedge 1 - 1+\beta(G^\dagger)} = \frac{\big(2+\log\big((d-1)(d^\dagger-1)\big)\big)(dd^\dagger-d-d^\dagger)}{\sqrt{(d-2)(d^\dagger-2)(dd^\dagger-2d-2d^\dagger)}}$$

since

$$\beta(G) = \frac{d(d^\dagger-2)+\sqrt{(d-2)(d^\dagger-2)(dd^\dagger-2d-2d^\dagger)}}{2(dd^\dagger-d-d^\dagger)}\,,$$

as calculation shows from Theorem 6.1. Thus, there is an interval of p for which b in Proposition 6.14 is close enough to b_0 that the hypotheses of both Proposition 6.14 and Corollary 6.17 are satisfied. This gives the result. □

Remark 6.19. Even when $p_c^{\text{free}} = p_u^{\text{wired}}$, there is no infinite cluster $\mathsf{FRC}_{p,q}$-a.s. and a unique infinite cluster $\mathsf{WRC}_{p,q}$-a.s. for $p = p_c^{\text{free}} = p_u^{\text{wired}}$, in view of Theorem 5.2 and Corollary 6.10.

For comparison, we present the following bounds on when the opposite inequality $p_c^{\text{free}}(q) < p_u^{\text{wired}}(q)$ holds.

Proposition 6.20. *If G is a planar regular nonamenable graph of degree d with regular dual of degree $d^\dagger$, then $p_c^{\text{free}}(q) < p_u^{\text{wired}}(q)$ for $q < dd^\dagger-2d-2d^\dagger$.*

For example, if $d = d^\dagger = 5$, then $p_c^{\text{free}} < p_u^{\text{wired}}$ if $q < 5$, while $p_u^{\text{wired}} < p_c^{\text{free}}$ if $q > (2+4\log 2)\sqrt{5} = 10.67^+$.

Proof. We claim that $p_c^{\text{free}}(G,q) < p_u^{\text{wired}}(G,q)$ if

$$h\big(p_c(G)\big)h\big(p_c(G^\dagger)\big) < 1/q\,, \tag{45}$$

where h is defined as in (43). Indeed,

$$h\big(p_c^{\text{free}}(G,q)\big) \le qh\big(p_c(G)\big) \tag{46}$$

by (22) (applied with $p_1 = p$, $q_1 = 1$, $q_2 = q$, and p_2 chosen so that $h(p) = h(p_2)/q$) and

$$h\big(p_u^{\text{wired}}(G,q)\big) = q/h\big(p_c^{\text{free}}(G^\dagger,q)\big) \geq 1/h\big(p_c(G^\dagger)\big) \tag{47}$$

by Corollary 6.9 and (46) (applied to $G^\dagger$). Therefore, if (45) holds, then (46) and (47) give

$$h\big(p_c^{\text{free}}(G,q)\big) \leq qh\big(p_c(G)\big) < 1/h\big(p_c(G^\dagger)\big) \leq h\big(p_u^{\text{wired}}(G,q)\big)\,,$$

which implies that $p_c^{\text{free}}(G,q) < p_u^{\text{wired}}(G,q)$.

Now $p_c(G) \leq 1/(1+\iota_E(G))$ by [7]. Therefore (45) is implied by $\iota_E(G)\iota_E(G^\dagger) > q$. This is the same as the claimed range of q, as calculation shows from Theorem 6.1. □

Remark 6.21. Note that when $p_c^{\text{free}}(G) < p_u^{\text{wired}}(G)$, we also have $p_c^{\text{wired}}(G) < p_u^{\text{free}}(G)$, and thus we have an interval of p for which $\mathsf{FRC}_p \neq \mathsf{WRC}_p$ (see Proposition 6.11 above). For $q = 2$, the condition in Proposition 6.20 shows that this holds for $d^\dagger = 3$ when $d > 8$, for $d^\dagger = 4$ when $d > 5$, and for all $d^\dagger \geq 5$ when $d \geq 5$. This generalizes the result in [61].

We end the section with some open questions.

Question 6.22. *Let G be a nonamenable quasi-transitive graph. Is the set of q for which $p_u^{\text{wired}}(G,q) < p_c^{\text{free}}(G,q)$ an interval? If G has only one end, is the set of such q nonempty?*

Question 6.23. *If G is nonamenable and quasi-transitive, can there be there any $q > 1$ such that $p_c^{\text{wired}}(G,q) = p_u^{\text{free}}(G,q)$ (so that all four critical values coincide)?*

7 Robust phase transition

A classical question about the Potts model is when it exhibits a phase transition for given q, β and G. We say that a **phase transition** occurs when there is more than one Gibbs measure for the q-state Potts model on G with inverse temperature β. As noted above in Section 5, this happens iff $\mathsf{WPt}^G_{q,\beta,r_1} \neq \mathsf{WPt}^G_{q,\beta,r_2}$ for $r_1 \neq r_2$. This is equivalent to the statement $\mathsf{WPt}^G_{q,\beta,r}(\omega(o) = r) > 1/r$ for any fixed vertex $o \in V$ and also to $\mathsf{WRC}^G_{p,q}(o \leftrightarrow \infty) > 0$, where $p := 1 - e^{-2\beta}$ and $\{o \leftrightarrow \infty\}$ is the event that o is contained in an infinite open cluster. Letting $\{o \leftrightarrow \partial V_i\}$ be the event that o is connected to ∂V_i by a path of open edges in G_i (where G_i, V_i and E_i are as in Section 2), it follows that phase transition in the q-state Potts model with the given parameters is equivalent to $\inf_i \mathsf{WRC}^{G,i}_{p,q}(o \leftrightarrow \partial V_i) > 0$. Hence there exists a critical value β_c (given by $\beta_c = -\frac{1}{2}\log(1 - p_c^{\text{wired}})$) such that we have phase transition for $\beta > \beta_c$, but not for $\beta < \beta_c$.

Pemantle and Steif [50] introduced the stronger concept of robust phase transition. In order to define this, we need to generalize the Potts model slightly: When defining $\mathsf{Pt}^{G_i}_{q,\beta}$, let us allow different interaction along different edges, i.e., replace β with $\mathbf{B} := \{\beta_e\}_{e\in E_i}$. It is then straightforward to modify the measures $\mathsf{Pt}^{G_i}_{q,\mathbf{B}}$ to measures $\mathsf{FPt}^{G,i}_{q,\mathbf{B}}$ and $\mathsf{WPt}^{G,i}_{q,\mathbf{B}}$ in the same way as we did in Section 2 for the case $\mathbf{B} \equiv \beta$. Now for $\epsilon > 0$, define $\mathbf{B}^i_\epsilon$ so that $B^i_\epsilon(e)$ equals ϵ for all edges with one endpoint in ∂V_i and equals β for all other edges. We say that the q-state Potts model with inverse temperature β exhibits a **robust phase transition** if $\inf_i \mathsf{WPt}^{G,i}_{q,\mathbf{B}^i_\epsilon,r}\big(\omega(o) = r\big) > 1/r$ for *all* $\epsilon > 0$.

By making a corresponding extension of the random-cluster model, i.e., by replacing p with $\mathbf{p} := \{p_e\}_{e\in E_i}$, we see that this is equivalent to

$$\inf_i \mathsf{WRC}^{G,i}_{\mathbf{p}^i_s,q}(o \leftrightarrow \partial V_i) > 0$$

for all $s > 0$, where $\mathbf{p}^i_s$ equals $p := 1 - e^{-2\beta}$ for edges that do not have an endpoint in ∂V_i and equals s for those that do. Pemantle and Steif show that when G is a tree and $q \geq 3$, then there is sometimes a phase transition but not a robust phase transition. We extend their result to the following:

Theorem 7.1. *Let G be an infinite regular nonamenable graph with degree d. Then there exists $q_0 < \infty$ such that for $q \geq q_0$ and $e^{2\beta} - 1 \in [q^{2/(d+\iota'_E(G)/2)}, q^{2/(d-\iota'_E(G)/2)}]$, the q-state Potts model on G with inverse temperature β exhibits a phase transition but not a robust phase transition.*

Proof. Fix i. From the definition of $\mathsf{WRC}^{G,i}_{\mathbf{p}^i_s,q}$ it is clear that we may, instead of regarding the vertices outside V_i as connected, regard them as contracted to a single vertex v_0. Now define a new graph H_i by adding an edge between v_0 and all vertices of V_i, i.e., let $V(H_i) := V_i \cup \{v_0\}$ and $E(H_i) := E(V_i) \cup E_0$ where $E_0 := \{[v_0, v]\,;\ v \in V_i\}$. Let $\tilde{\mathbf{p}}$ be s for all edges of E_0 and p for all other edges. Then by conditioning on $\mathsf{RC}^{H_i}_{\tilde{\mathbf{p}},q}(E_0)$ and using Holley's inequality, we see that

$$\mathsf{RC}^{H_i}_{\tilde{\mathbf{p}},q}\big(E^*(V_i)\big) \overset{\mathcal{D}}{\succcurlyeq} \mathsf{WRC}^{G,i}_{\mathbf{p}^i_s,q}. \tag{48}$$

Now the proof of [40], Theorem 1.2(a), shows precisely that for q large enough and s small enough, $\inf_i \mathsf{RC}^{H_i}_{\tilde{\mathbf{p}},q}(A_i) = 0$, where A_i is the event that there is a path of open edges connecting o to ∂V_i without passing through v_0. [The proof is essentially the same as that of Proposition 6.14, but here one has to take the edges of E_0 into account. This is done by a simple application of Markov's inequality.] By (48), it follows that there is no robust phase transition in the corresponding Potts model. On the other hand, [40], Theorem 4.4(a), says that for q large enough, a phase transition occurs. ◻

Let us now briefly consider the case when G is instead an amenable quasi-transitive graph. Then, as noted above, $\mathsf{FRC}_{p,q} = \mathsf{WRC}_{p,q}$ for all but at most countably many values of p. Therefore, if $\mathsf{WRC}_{p,q}(o \leftrightarrow \infty) > 0$, then $\mathsf{FRC}_{p',q}(o \leftrightarrow \infty) > 0$ for all $p' > p$. This strongly suggests that $\inf_i \mathsf{FRC}^{G,i}_{p',q}(o \leftrightarrow \partial V_i) > 0$, and since $\mathsf{FRC}^{G,i}_{p,q}$ is $\mathsf{WRC}^{G,i}_{\mathbf{p}^i_s,q}$ with $s = 0$, this would imply that the critical temperatures for phase transition and robust phase transition coincide. However, as $\mathsf{FRC}^{G,i}_{p,q}$ is increasing in i, it is possible to imagine a scenario where o is not connected to ∂V_i for any i but still connected to infinity in the limit. This does not happen for the WRC measures on any graph ([2], proof of Theorem 2.3(c)), but is unknown for FRC.

Question 7.2. *If G is amenable and quasi-transitive, then is the critical inverse temperature for robust phase transition β_c?*

This question does not address $\beta = \beta_c$. For $G := \mathbf{Z}^d$, $d \geq 2$ and q large, it is well-known that the Potts model at criticality ($\beta = \beta_c$) exhibits phase transition, whereas it was recently shown by van Enter [20] that (still at criticality) there is no robust phase transition.

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