# SPECTRAL PROPERTIES IN THE LOW-ENERGY LIMIT OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS $H = -d^2/dx^2 + V$. THE CASE $\langle 1, V1\rangle \neq 0$

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Abstract. In this paper we consider the Schrödinger operator $H = -d^2/dx^2 + V$ in $L^2(\mathbf{R})$, where $V$ satisfies an abstract short-range condition and the (solvability) condition $\langle 1, V1\rangle \neq 0$. Spectral properties of $H$ in the low-energy limit are analyzed. Asymptotic expansions for $R(\zeta) = (H-\zeta)^{-1}$ and the $S$-matrix $S(\lambda)$ are deduced for $\zeta \to 0$ and $\lambda \downarrow 0$, respectively. Depending on the zero-energy properties of $H$, the expansions of $R(\zeta)$ take different forms. Generically, the expansions of $R(\zeta)$ do not contain negative powers; the appearance of negative powers in $\zeta^{1/2}$ is due to the possible presence of zero-energy resonances (half-bound states) or the eigenvalue zero of $H$ (bound state), or both. It is found that there are at most two zero resonances modulo $L^2$-functions.

## 1. Introduction

In this paper we study spectral properties and scattering theory in the low-energy limit of one-dimensional Schrödinger operators $H = H_0 + V$ in $L^2(\mathbf{R})$, where $H_0 = -d^2/dx^2$ and the symmetric operator $V$ belongs to an abstract class of short-range potentials. In a framework of weighted Sobolev spaces $H^{m,s}(\mathbf{R})$ with regularity $m$ and decay $s$, we assume that the potential $V$ is a compact operator from $H^{1,0}$ to $H^{0,\beta}$ for some $\beta > 1$ and that it extends to a compact operator from $H^{1,-s}$ to $H^{-1,\beta-s}$.

The results on the spectral properties of $H$ are expressed in terms of asymptotic expansions for the resolvent $R(\zeta) = (H-\zeta)^{-1}$ and the $S$-matrix $S(\lambda)$, $\lambda = \operatorname{Re}\zeta$, as $\zeta \to 0$; this is the *low-energy limit.* Generally, the expansions take the form

$$R(\zeta) = -\zeta^{-1}B_{-2}^{(\alpha)} - i\zeta^{-1/2}B_{-1}^{(\alpha)} + B_0^{(\alpha)} + i\zeta^{1/2}B_1^{(\alpha)} + \cdots, \tag{1.1}$$

$$S(\lambda) = S_0^{(\alpha)} + i\lambda^{1/2}S_1^{(\alpha)} - \lambda S_2^{(\alpha)} + \cdots, \tag{1.2}$$

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where $\operatorname{Im}\zeta \geq 0$, $\operatorname{Im}\zeta^{1/2} \geq 0$, $\lambda = \operatorname{Re}\zeta > 0$ and $|\zeta| \to 0$. Depending on the zero-energy behaviour of $H$ the expansions take different forms. In (1.1) and (1.2) this is indicated by the upper index $\alpha$. The coefficients in (1.1) and (1.2) can be found explicitly. Expressions for the leading coefficients, at least, are given. The expansion (1.1) is valid in the operator norm in $B(-1, s; 1, -s') = B(H^{-1,s}, H^{1,-s'})$. There is a complicated relation between the order $l$ of the expansion and the required $\beta$ and $s, s'$. Generally, expansions to a high order $l$ reguire large $\beta$ and $s, s'$. The expansion (1.2) is valid in the operator norm in $B(\mathbf{C}^2)$.

The main two steps towards these results are: 1. A complete analysis of the zero-energy properties of $H$. 2. Derivation of the expansions; we demonstrate that the technique of Jensen and Kato [4] developed for the three-dimensional Schrödinger operator can be made to work, despite the additional complications in dimension one (see below).

In comparison with the three-dimensional situation treated in [4], the study of $H$ is more intricated because the kernel of the free resolvent has a square-root singularity in the limit as the energy tends to zero. Due to this property we are forced to make the assumption $\langle 1, V1\rangle \neq 0$, where $\langle\cdot,\cdot\rangle$ denotes the natural duality between $H^{m,s}$ and $H^{-m,-s}$. This assumption is a natural solvability condition but has no physical explanation. Under this assumption we classify the zero-energy properties of $H$. A priori zero can be an eigenvalue of $H$ or a zero resonance for $H$, or both. We have a zero resonance (or half-bound state) if $H\psi = 0$ has a solution $\psi$ in a space slightly larger than $L^2$. In dimension one it is natural to assume that a zero resonance belongs to $L^\infty$. It turns out that essentially three cases may occur: $H$ has no eigenvalue zero and no zero resonance (*zero is a regular point*), $H$ has no eigenvalue zero but has a zero resonance (*zero is an exceptional point of 1st kind*), $H$ has eigenvalue zero but has no zero resonance (*zero is an exceptional point of 2nd kind*), or $H$ has both eigenvalue zero and a zero resonance (*zero is an exceptional point of the 3rd kind*). In all cases we derive expansions of $(1 + R_0(\zeta)V)^{-1}$ and, via the resolvent equation, $R(\zeta)$. As an application, asymptotic expansions of the scattering matrix are obtained as the energy tends to zero. The singularities (negative powers in $\zeta^{1/2}$) in (1.1) occur in the exceptional cases. Generically, there are no negative powers, which explains the terminology of the exceptional cases.

This work complements the works of Murata [14] and Bollé *et al* [1]. Murata considers general elliptic differential operators of Schrödinger type, allowing non-selfadjoint, non-local potentials. Inspired by Vaĭnberg [18], he develops a method, different from the technique of Jensen-Kato, to obtain asymptotic expansions for the resolvent. However, the analysis of the zero-energy properties are not explicit nor complete. For

Schrödinger operators with multiplicative (local) potentials having exponential decay at infinity, Bollé *et al* deduce norm-convergent Taylor (Laurent, respectively) series of the transition operator $(1+uR_0(\zeta)v)^{-1}$ as the spectral parameter $\zeta$ tends to zero, where $R_0(\zeta)=(H_0-\zeta)^{-1}$ is the free resolvent and the (symmetric) weights $u$ and $v$ are defined as $u(x)=|V(x)|^{1/2}$ and $v(x)=u(x)\mathrm{Sign}\,V(x)$. For potentials decaying like $\mathrm{O}(|x|^{-\beta})$ as $|x|\to\infty$ for some $\beta>2$, their zero-energy analysis is complete. In particular, they show that zero *cannot* be an eigenvalue of $H$. We extend this analysis to the abstract class of potentials, relying on the mapping properties only. The classification of the point zero in the spectrum of $H$ resembles the one in dimension three [4] with one exception. In dimension three there is at most *one* zero resonance function modulo $L^2$-functions, whereas in dimension one there are at most *two*. Due to this circumstance, additional subcases arise in the exceptional cases of 1st (three subcases) and 3rd kind (three subcases).

Related results obtained by entirely different methods are found in [2, 3, 8, 15, 19], wherein the leading coefficients in the expansions of the scattering matrix are derived. However, these methods do not allow one to compute higher order terms.

This work is motivated by a number of problems related to the study of the threshold behaviour of resolvents of $2\times 2$ operator-valued matrix Hamiltonians of the form

$$\mathbf{H}(g)=\begin{pmatrix} H_a & 0 \\ 0 & H_b \end{pmatrix}+g\begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix} \tag{1.3}$$

in $\mathcal{H}=\mathcal{H}_a\oplus\mathcal{H}_b$, where $\mathcal{H}_a$ and $\mathcal{H}_b$ are Hilbert spaces. Here $H_a$ and $H_b$ are self-adjoint operators, $V_{ab}\in B(\mathcal{H}_b,\mathcal{H}_a)$, $V_{ba}=V_{ab}^*$ and $g$ is a coupling constant. It is assumed that the threshold $\lambda_b$ of the absolutely continuous spectrum of the component Hamiltonian $H_b$ is situated strictly above such a threshold $\lambda_a$ for $H_a$. There are several possible situations, e.g. the one where $H_b$ has an isolated eigenvalue coinciding with the threshold $\lambda_a$. In an abstract framework in the paper [12], and in mostly fairly singular situations, asymptotic expansions of the resolvent $\mathbf{R}(1;\zeta)=(\mathbf{H}(1)-\zeta)^{-1}$ are derived as $\zeta\to\lambda_a$. Applications of these abstract results to scattering theory of concrete pairs of two-channel Hamiltonians are given in the paper [13]. In particular, Hamiltonians with one-dimensional Schrödinger operators as component Hamiltonians are studied in details and low-energy expansions of the scattering matrix are obtained. These results rely explicitly on asymptotic expansions of the resolvent of the component one-dimensional Schrödinger operators, i.e. the main theorems in the present paper. Moreover, by modifying slightly the proofs in [12] one can study perturbation of eigenvalues and half-bound states of $\mathbf{H}(0)$ embedded at the threshold $\lambda_a$. Detailed results are found in the paper [7], where it is shown that for instance embedded eigenvalues of $\mathbf{H}(0)$ leave the continuous spectrum of $\mathbf{H}(g)$ for $0<|g|<\eta_0$ for some positive $\eta_0$.

These eigenvalues may show up as resonances or discrete eigenvalues. Applications are given to the Friedrichs' model and Schrödinger-type Hamiltonians of the form in (1.3).

The abstract theory in [12] can be applied also to the three-dimensional Schrödinger operator with a constant magnetic field and an axisymmetrical electric potential. Under these assumptions the operator can be represented in a multi-channel framework. For the lowest Landau level we can fit the problem into the afore-mentioned two-channel framework, apply the results in [12] and, consequently, obtain expansions of the resolvent and the scattering matrix near the lowest Landau threshold. It requires considerable preparation to apply the results in [12] and a crucial ingredient is asymptotic expansions of the resolvent of an auxiliary one-dimensional Schrödinger operator with a non-local potential satisfying the conditions in the present paper. Thus, the main theorems herein are used explicitly. Preliminary results are contained in [11]. Complete results will be published elsewhere.

Finally we note that we intend to give results under the (complementary) assumption $\langle 1, V1\rangle = 0$ in a forthcoming publication.

## 2. The free resolvent

The basic Hilbert space is $L^2(\mathbf{R})$. Let $\mathcal{S}'(\mathbf{R})$ denote the tempered distributions. Let $p$ denote the momentum operator $-id/dx$, $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\langle p\rangle = (1+p^2)^{1/2}$. We use the weighted Sobolev space $H^{m,s} = H^{m,s}(\mathbf{R})$ given by

$$H^{m,s}(\mathbf{R}) = \{u \in \mathcal{S}'(\mathbf{R})|\ \|u\|_{m,s} = \|\langle x\rangle^s \langle p\rangle^m u\|_{L^2} < \infty\}.$$

We use $\langle\cdot,\cdot\rangle$ to denote the inner product on $L^2$ and also the natural duality between $H^{m,s}$ and $H^{-m,-s}$. $B(m,s;m',s')$ denotes the space of bounded operators from $H^{m,s}$ to $H^{m',s'}$ with the operator norm. The Fourier transform is given by

$$(\mathcal{F}\psi)(\xi) = \widehat{\psi}(\xi) = (2\pi)^{-1/2}\int_{\mathbf{R}} e^{-ix\xi}\psi(x)dx$$

and is a bounded map from $H^{m,s}$ to $H^{s,m}$.

The free Hamiltonian $H_0 = -d^2/dx^2$ is a self-adjoint operator in $L^2(\mathbf{R})$ with domain $D(H_0) = H^2(\mathbf{R})$ and spectrum $\sigma(H_0) = \sigma_{ess}(H_0) = [0,\infty)$, where $\sigma_{ess}$ denotes the essential spectrum. Its resolvent $R_0(\zeta) = (H_0-\zeta)^{-1}$, $\zeta \in \rho(H_0) = \mathbf{C}\backslash[0,\infty)$, has kernel [16, Theorem 9.5.2]

$$R_0(x,y;\zeta) = \frac{i}{2\sqrt{\zeta}} e^{i\sqrt{\zeta}|x-y|}, \tag{2.1}$$

where, for $\zeta \in \mathbf{C}\backslash[0,\infty)$, $\sqrt{\zeta}$ is chosen such that $\mathrm{Im}\sqrt{\zeta} > 0$. The following properties of the free resolvent can be easily checked. For $s, s' > 1/2$, $R_0(\zeta)$ is a continuous map from the closure of $\mathbf{C}\backslash[0,\infty)$ to $B(-1,s;1,-s')$. Moreover, $R_0(\zeta) = -i\zeta^{-1/2}G_{-1} + \tilde{R}_0(\zeta)$, where $G_{-1} = -(1/2)\langle\cdot,1\rangle 1$ is a rank one operator and $\tilde{R}_0(\zeta)$ is a continuous map from $\overline{\mathbf{C}}$ to $B(-1,s;1,-s')$, $s,s'>3/2$. Formally, (2.1) gives the expansion

$$R_0(\zeta)=\sum_{n=-1}^{\infty}(i\zeta^{1/2})^n G_n,$$

where $G_n$ is an integral operator with kernel

$$G_n(x,y)=-\frac{1}{2(n+1)!}|x-y|^{n+1}.$$

For $s,s'>n+3/2$ we have that

$$G_n\in B(-1,s;1,-s'),\qquad n=-1,0,1,2,\dots \tag{2.2}$$

Let $N\in\mathbf{N}$, $s,s'>N+3/2$ and let $\theta$ satisfy $0<\theta<\min\,\{1,\min\{s,s'\}-N-3/2\}$. As $|\zeta|\to 0$, $\zeta\in\overline{\mathbf{C}}\backslash\{0\}$, $\operatorname{Im}\sqrt{\zeta}>0$, we have in $B(-1,s;1,-s')$ the expansion

$$R_0(\zeta)=\sum_{n=-1}^{N}(i\zeta^{1/2})^n G_n+\mathrm{O}(|\zeta|^{(N+\theta)/2}). \tag{2.3}$$

Henceforth we use the notation

$$H^{m,s+0}=\bigcup_{s'>s}H^{m,s'},\quad H^{m,s-0}=\bigcap_{s'<s}H^{m,s'}.$$

These are regarded simply as algebraic vector spaces; we do not introduce topologies in them.

The following properties of $G_0$ will be needed later. First, $H_0G_0\phi=\phi$ for any $\phi\in H^{-1,3/2+0}$. Secondly, using the notation $(\cdot)=x$, we have the following result.

**Lemma 2.1.**
*(i) Assume that* $\psi\in H^{0,\frac{3}{2}+0}$ *and* $\langle\psi,1\rangle=0$. *Then* $G_0\psi\in H^{2,-\frac{1}{2}-0}\cap L^\infty$.
*(ii) Assume that* $\psi\in H^{-1,s}$, $s>3/2$ *and* $\langle\psi,1\rangle=0$. *Then* $G_0\psi\in H^{1,-1/2-0}\cap L^\infty$.
*(iii) Assume that* $\psi\in H^{0,2+0}$, $\langle\psi,1\rangle=0$ *and* $\langle\psi,(\cdot)\rangle=0$. *Then* $G_0\psi\in H^{2,0+0}$.

*Proof.* We first prove that $\phi\in H^{0,s}$, $s>3/2$, and $\langle\phi,1\rangle=0$ imply that $G_0\phi\in H^{0,-1/2-0}$. Since

$$(G_0\phi)(x)=-\frac{1}{2}\int(|x-y|-|x|)\phi(y)dy,$$

we have that $G_0\phi=G_0'\phi$, where $G_0'$ has kernel $G_0'(x,y)=|x-y|-|x|$. From $||x-y|-|x||\le|y|$ the kernel of $G_0'$ is dominated by $|y|$, which is the kernel of an operator that maps $H^{0,s}$ to $H^{0,1-s}$ for $s>3/2$ and also $H^{0,s}$ to $L^\infty$. Thus, $G_0\phi\in H^{0,-1/2-0}\cap L^\infty$ and since $(1-d^2/dx^2)G_0\phi=G_0\phi+\phi\in H^{0,-1/2-0}$, we have that $G_0\phi\in H^{2,-1/2-0}$. This proves (i). To prove (ii), set $\phi=\langle p\rangle^{-1}\psi$. Then $\phi\in H^{0,s}$, $s>3/2$, and $\langle\phi,1\rangle=\langle\psi,1\rangle=0$ by assumption. Hence $G_0\phi\in H^{2,-1/2-0}\cap L^\infty$ by what was just proved. Then $G_0\psi \in H^{1,-1/2-0} \cap L^\infty$ as required. Similar techniques are used to prove (iii). $\square$

**Lemma 2.2.** *Assume that* $\psi, \phi \in H^{0,\frac{7}{2}+0}$. *If* $\langle \psi, 1\rangle = 0$, $\langle \phi, 1\rangle = 0$, $\langle \psi, (\cdot)\rangle = 0$ *and* $\langle \phi, (\cdot)\rangle = 0$ *then*

$$\langle G_2\psi, \phi\rangle = -\langle G_0\psi, G_0\phi\rangle. \tag{2.4}$$

*Proof.* First, we have to verify that both sides of (2.4) make sense. Now, $\psi \in H^{0,\frac{7}{2}+0}$ implies that $G_2\psi \in H^{1,-\frac{7}{2}-0}$ by (2.2), so that the left-hand side of (2.4) makes sense. Furthermore, the right-hand side makes sense because $G_0\psi, G_0\phi \in H^{2,0+0} \subset H^{0,0}$ by Lemma 2.1(iii). The inverse Fourier transform of $1/(|\xi|^2+\epsilon)$, $\epsilon > 0$, is given by $(1/2\sqrt{\epsilon})\exp(-\sqrt{\epsilon}|x|)$. Using Taylor's formula and taking the limit $\epsilon \downarrow 0$, we find that $\langle \frac{1}{\xi^2}\widehat{\psi}, \widehat{\psi}\rangle = \langle G_0\psi, \psi\rangle$, if $|x|^2\psi \in L^1$ and $\langle \psi, 1\rangle = 0$, which is clearly satisfied. Using this, in conjunction with $\langle G_{-1}\psi, \phi\rangle = 0$ and $\langle G_1\psi, \phi\rangle = 0$, (2.4) follows from

$$\begin{aligned}
\langle G_2\psi, \phi\rangle &= \lim_{\zeta\to 0}(-\zeta^{-1})\langle (R_0(\zeta) - G_0)\psi, \phi\rangle \\
&= \lim_{\zeta\to 0}(-\zeta^{-1}) \int \left[(|\xi|^2 - \zeta)^{-1} - |\xi|^{-2}\right] \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}d\xi \\
&= -\lim_{\zeta\to 0}\int |\xi|^{-2}(|\xi|^2 - \zeta)^{-1}\widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}d\xi \\
&= -\int |\xi|^{-4}\widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}d\xi \\
&= -\langle |\xi|^{-2}\widehat{\psi}, |\xi|^{-2}\widehat{\phi}\rangle \\
&= -\langle G_0\psi, G_0\phi\rangle.
\end{aligned}$$

In the third equality we applied dominated convergence as $\zeta \to 0$; note that $\widehat{\psi}$ and $\widehat{\phi}$ are in $C^1$ with $\widehat{\psi}(0) = \widehat{\phi}(0) = 0$. $\square$

## 3. The Schrödinger operator $H$

We consider $H = H_0 + V$ where $V$ is supposed to satisfy the following abstract short-range condition.

**Assumption 3.1.** Let $V$ be a symmetric operator in $L^2$. Assume that $V$ is a compact operator from $H^{1,0}$ to $H^{-1,\beta}$ for some $\beta > 1$, and $V$ extends to a compact operator from $H^{1,-\beta}$ to $H^{-1,0}$. Assume, in addition, that $\langle 1, V1\rangle \neq 0$.

We refer to $\beta$ above as the *decay parameter*. We have that $\langle p\rangle 1 = \mathcal{F}[(1+|\xi|^2)^{1/2}\mathcal{F}^{-1}1] = 1$ in the sense of distributions and $1 \in L^{2,-\beta/2}(\mathbf{R})$ for $\beta > 1$, hence $1 \in H^{1,-\beta/2}(\mathbf{R})$ for $\beta > 1$. Since $V \in B(1, -\beta/2; -1, \beta/2)$, we conclude that $V1 \in H^{-1,\beta/2}(\mathbf{R})$ and finally we note that $H^{1,-\beta/2}(\mathbf{R})$ is the dual space of $H^{-1,\beta/2}(\mathbf{R})$. This justifies the assumption $\langle 1, V1\rangle \neq$

0. We note that the potential $V$ is a compact map from $H^{1,s}$ to $H^{-1,\beta+s}$ for any $s \in \mathbf{R}$.

Since $V$ is a symmetric operator in $L^2(\mathbf{R})$ which by Assumption 3.1 is $H_0$-form compact, the KLMN theorem generates a self-adjoint operator $H = H_0 + V$ (quadratic form sense) in $L^2(\mathbf{R})$. Furthermore, $H$ is bounded from below and any domain of essential self-adjointness of $H_0$ is a form core of $H$. Due to the $H_0$-form compactness, we have that $\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty)$. Often $H$ is extended to a larger space. In the sequel we will work in a framework of spaces $H^{-1,s}$ and $H^{1,-s}$ in order to utilize the duality of these spaces.

We introduce the constant $\nu = 1/\langle 1, V1\rangle$ and define an operator $Q_1$ by $Q_1\phi = \nu\langle\phi, V1\rangle 1$. Then $Q_1$ is a projection in $H^{1,-s}(\mathbf{R})$ for $1/2 < s < \beta - 1/2$. We also introduce its complement $Q_0 = 1 - Q_1$ in the same space. We have the following elementary result.

**Lemma 3.2.** *Let $V$ satisfy Assumption 3.1 with $\beta > 1$. The following relations holds in $H^{1,-s}(\mathbf{R})$ for $1/2 < s < \beta - 1/2$.*

$$VQ_0 = Q_0^* V, \quad VQ_1 = Q_1^* V \tag{3.1}$$

$$G_{-1}VQ_0 = 0, \quad Q_0^* V G_{-1} = 0 \tag{3.2}$$

*Proof.* Writing explicitly $Q_0 \cdot = 1 - \nu\langle\cdot, V1\rangle 1$, $Q_0^* \cdot = 1 - \nu\langle\cdot, 1\rangle V1$ and $G_{-1}\cdot = -(1/2)\langle\cdot, 1\rangle 1$, we obtain the equalities by straightforward computations, which are omitted. □

## 4. The point zero in the spectrum of $H$

We analyze the point zero in the spectrum of $H$. We assume that Assumption 3.1 with $\beta > 3$ is satisfied. Hereafter, the parameter $s$ is subject to the restriction $3/2 < s < \beta - 3/2$. Define the spaces

$$\mathcal{M} = \{\phi \in H^{1,-s} \mid Q_0 G_0 V Q_0 \phi = -\phi\},$$

$$\mathcal{N} = \{\psi \in H^{-1,s} \mid Q_0^* V G_0 Q_0^* \psi = -\psi\}.$$

Notice that $Q_0 G_0 V G_0$ and $Q_0^* V G_0 Q_0^*$ are compact operators. A priori these spaces may depend on $s$, but obviously $\mathcal{M}$ is monotone increasing and $\mathcal{N}$ is monotone decreasing in $s$. By the compactness of $Q_0 G_0 V Q_0$ and $Q_0^* V G_0 Q_0^*$ and the duality of $H^{1,-s}$ and $H^{-1,s}$ we have $\dim \mathcal{M} = \dim \mathcal{N} < \infty$. Since $\mathcal{M}$ and $\mathcal{N}$ are monotone in $s$ in the opposite direction, they must be independent of $s \in (3/2, \beta - 3/2)$. Further,

$$\mathcal{M} \subset H^{1,-3/2-0}, \quad \mathcal{N} \subset H^{-1,\beta-3/2-0} \subset H^{-1,3/2+0}. \tag{4.1}$$

Consider a family $H_\mu = H_0 + \mu V$ with a real coupling parameter $\mu$. Let $\mathcal{M}(\mu)$ ($\mathcal{N}(\mu)$) denote the null space of $1 + \mu Q_0 G_0 V Q_0$ ($1 + \mu Q_0^* V G_0 Q_0^*$). Then $\mathcal{M}(\mu) = \mathcal{N}(\mu) = \{0\}$ except for a discrete set of $\mu$'s, because $V$ is compact. Thus, generically $\mathcal{M} = \mathcal{N} = \{0\}$.

**Lemma 4.1.** $(H_0+V)\mathcal{M} = \{0\}$ *and both* $H_0$ *and* $V$ *are injective from* $\mathcal{M}$ *onto* $\mathcal{N}$.

*Proof.* Lemma 3.2 yields that $V$ maps $\mathcal{M}$ into $\mathcal{N}$. Since $H_0G_0\phi = \phi$ for $\phi \in \mathcal{M}$, this also implies that $H_0$ maps $\mathcal{M}$ into $\mathcal{N}$. Let $\phi \in \mathcal{M}$ and assume $H_0\phi = 0$. Then $\phi(x) = \alpha + \gamma x$ for some constants $\alpha, \gamma$. Now, $\phi = -Q_0G_0VQ_0\phi = -Q_0G_0Q_0^*V\phi$ and $Q_0^*V\phi \in H^{-1,s}$ for some $s > 3/2$ and satisfies $\langle Q_0^*V\phi, 1\rangle = 0$. Hence we can use Lemma 2.1(ii) to conclude that $G_0Q_0^*V\phi \in H^{1,-r}$ for any $r > 1/2$. Since $Q_0$ maps $H^{1,-r}$ into itself for any $r > 1/2$, we get that $\phi \in H^{1,-r}$ for any $r > 1/2$. Thus $\gamma = 0$. Since $\phi = Q_0\phi$, we have $Q_1\phi = 0$. But in this case $Q_1\phi = \alpha$, so $\alpha = 0$ and we have proved $\phi = 0$. Thus $H_0$ is injective on $\mathcal{M}$. Since $\mathcal{M}$ and $\mathcal{N}$ have the same dimensions, it is also onto. The result for $V$ follows from $H_0\phi = -V\phi$ for all $\phi \in \mathcal{M}$. □

Contrary to the situation for local potentials, we need to introduce the following space $\tilde{\mathcal{M}}$.

**Lemma 4.2.** *Let*

$$\tilde{\mathcal{M}} = \{\phi \in \mathcal{M} \mid G_0V\phi = -\phi\}.$$

*(i) Then* $\tilde{\mathcal{M}} = \{\phi \in \mathcal{M} \mid \langle G_0V\phi, V1\rangle = 0\}$. *Furthermore,* $\dim \mathcal{M}/\tilde{\mathcal{M}} \leq 1$.
*(ii) Assume* $\mathcal{M}/\tilde{\mathcal{M}} \neq \{0\}$ *and let* $\phi \in \mathcal{M}$. *Then*

$$G_0V\phi = -\phi + \nu\langle G_0V\phi, V1\rangle 1. \tag{4.2}$$

*The last term vanishes for* $\phi \in \tilde{\mathcal{M}}$.

*Proof.* Let $\phi \in \tilde{\mathcal{M}}$. Then $\phi = Q_0\phi$, since $\phi \in \tilde{\mathcal{M}} \subset \mathcal{M}$ and, furthermore, $\phi = Q_0\phi$ implies that $Q_1\phi = 0$. The latter yields that $\langle \phi, V1\rangle = 0$. Since $\phi \in \tilde{\mathcal{M}}$ it can be rewritten as $\langle G_0V\phi, V1\rangle = 0$. Hence, $\tilde{\mathcal{M}} \subset \{\phi \in \mathcal{M} \mid \langle G_0V\phi, V1\rangle = 0\}$. Conversely, if $\phi \in \mathcal{M}$ and $\langle G_0V\phi, V1\rangle = 0$, then

$$G_0V\phi = Q_1G_0V\phi + Q_0G_0V\phi = \nu\langle G_0V\phi, V1\rangle 1 + Q_0G_0VQ_0\phi = -\phi$$

as desired. This proves that $\{\phi \in \mathcal{M} \mid \langle G_0V\phi, V1\rangle = 0\} \subset \tilde{\mathcal{M}}$. Hence we have shown both inclusions and this proves part (i). To prove (ii), suppose $\phi \in \mathcal{M}$, $\phi \neq 0$. Then, as claimed in (ii),

$$\phi = -G_0V\phi + Q_1G_0V\phi = -G_0V\phi + \nu\langle G_0V\phi, V1\rangle 1. \tag{4.3}$$

If $\phi \in \tilde{\mathcal{M}}$ then the last term vanishes, since $\langle G_0V\phi, V1\rangle = 0$ by (i). □

### 4.1. **Zero-energy behaviour of $H$.**

In this section we describe the zero-energy properties of $H$. We use the notation $\mathrm{Sign}(\cdot) = \cdot/|\cdot|$, where $(\cdot) = x$ (or $y$). We begin with two lemmas.

**Lemma 4.3.** *Let Assumption 3.1 with $\beta > 5$ be satisfied. Let $5/2 < s < \beta - 5/2$. Assume $\phi \in \mathcal{M}$. Then $\phi \in L^\infty(\mathbf{R})$ and $H\phi = 0$ in the sense of distributions. Moreover,*

$$\phi - Q_1 G_0 V\phi + \frac{1}{2}\mathrm{Sign}\,(\cdot)\langle V\phi, (\cdot)\rangle \in L^2(\mathbf{R}) \tag{4.4}$$

*and*

$$\phi(\pm\infty) = Q_1 G_0 V\phi \mp \frac{1}{2}\langle V\phi, (\cdot)\rangle. \tag{4.5}$$

*Proof.* First we show that $\phi \in L^\infty(\mathbf{R})$. By Lemma 4.2(ii) we have that

$$\phi = -G_0 V\phi + Q_1 G_0 V\phi. \tag{4.6}$$

Here $V\phi \in H^{-1,s}$ and, in addition, $\phi \in \mathcal{M}$ implies that $\langle \phi, V1\rangle = 0$. Therefore, Lemma 2.1(ii) asserts that $G_0 V\phi \in L^\infty$. Thus, $\phi \in L^\infty$ as desired. From Lemma 4.1 we have that $H\phi = 0$ in the sense of distributions. This proves the second assertion. Let us show (4.4). Define

$$\tilde{\phi}(x) = \phi(x) - Q_1 G_0 V\phi + \frac{1}{2}\mathrm{Sign}\,V(x)\langle V\phi, (\cdot)\rangle.$$

Using $\langle \phi, V1\rangle = 0$ we find that

$$\begin{aligned} \phi(x) &= Q_1 G_0 V\phi + \frac{1}{2}\int_{-\infty}^{\infty} |x-y|(V\phi)(y)dy \\ &= Q_1 G_0 V\phi - \frac{1}{2}\frac{x}{|x|}\langle V\phi, (\cdot)\rangle + \\ &\quad + \begin{cases} \int_{-\infty}^{x} (x-y)(V\phi)(y)dy, & x < 0, \\ \int_{x}^{\infty} (y-x)(V\phi)(y)dy, & x > 0, \end{cases} \end{aligned} \tag{4.7}$$

where, in the last equality, we have taken into account the sign of $x$. In the compact set $K = [-a, a]$, $a > 0$, $\tilde{\phi}$ is clearly bounded. Via (4.7), e.g. for $x > a$, we find that

$$\tilde{\phi}(x) = \int_{x}^{\infty} (y-x)(V\phi)(y)dy.$$

Using $|y - x| \leq 2|y|^2/|x|$ and $\phi \in L^\infty$, we obtain the inequality

$$|\tilde{\phi}(x)| \leq \frac{2\|\phi\|_\infty}{x}\int_{x}^{\infty} |y|^2|(V1)(y)|dy$$

and similar for $x < -a$. Since $1 \in H^{1,-s}$ for $s > 1/2$, we find that $V1 \in L^{2,s}$ for $1/2 < s < \beta - 1/2$. Hence, the latter integral can be estimated by Hölder's inequality. We find that

$$\int_{x}^{\infty} |y|^2|(V1)(y)|dy \leq \left(\int_{x}^{\infty} |y|^4(1+|y|^2)^{-s}dy\right)^{1/2} \|V1\|_{L^{2,s}}.$$

For $s > 5/2$ the integral on the right-hand side is finite. Finally, since $2/|x| \in L^2(\mathbf{R}\backslash K)$, it follows that $\tilde{\phi} \in L^2$. This completes the proof. ☐

The converse of Lemma 4.3 is contained in the next lemma.

**Lemma 4.4.** *Let Assumption 3.1 with $\beta > 5$ be satisfied. Let $5/2 < s < \beta - 5/2$. Assume that $\phi \in H^{1,-\frac{5}{2}-0} \cap L^\infty$ and $H\phi = 0$ in the sense of distributions. Then $\phi \in \mathcal{M}$, and the statements in Lemma 4.3 hold for this $\phi$.*

*Proof.* Define $\psi = Q_1 G_0 V\phi - G_0 V\phi$. In order to prove the lemma, we show that $\psi = \phi$ a.e. and $\phi \in \mathcal{M}$. The assumption $\phi \in H^{1,-\frac{5}{2}-0}$ implies that $V\phi \in H^{-1,\frac{3}{2}+0}$. Therefore, $H_0\psi = -V\phi$ since $Q_1 G_0 V\phi$ is a constant function. But $-V\phi = \phi''$ by assumption, so $\psi'' = \phi''$. Thus, for some constants $\alpha$ and $\gamma$ we have that

$$\psi(x) = \phi(x) + \alpha + \gamma x. \tag{4.8}$$

Since $\phi \in L^\infty$, we get that

$$\lim_{x\to\pm\infty} \frac{\psi(x)}{x} = \gamma. \tag{4.9}$$

On the other hand, we can rewrite $\psi$ as

$$\begin{aligned}\psi(x) = Q_1 G_0 V\phi + \frac{1}{2}x\langle\phi, V1\rangle + \frac{1}{2}\frac{x}{|x|}\langle\phi, V(\cdot)\rangle + \\ + \begin{cases} \int_{-\infty}^{x} (x-y)(V\phi)(y)dy, & x < 0, \\ \int_{x}^{\infty} (y-x)(V\phi)(y)dy, & x > 0. \end{cases}\end{aligned} \tag{4.10}$$

For $x > 0$, due to $|x - y| \le 2|y|$ and $\phi \in L^\infty$, we get the estimate

$$\left|\int_x^\infty (y-x)(V\phi)(y)dy\right| \le 2\|\phi\|_{L^\infty} \int_x^\infty |y||(V1)(y)|dy. \tag{4.11}$$

Now, $V1 \in L^{2,s}$ for $1/2 < s < \beta - 1/2$. Therefore, the latter integral can be estimated by Hölder's inequality. We find that

$$\int_x^\infty |y||(V1)(y)|dy \le \left(\int_x^\infty |y|^2(1+|y|^2)^{-s}dy\right) \|V1\|_{L^{2,s}}.$$

For $s > 3/2$ the integral on the right-hand side is finite. Hence, the right-hand side of (4.11) tends to zero as $x \to +\infty$. The case $x \to -\infty$ is handled analogously. Therefore, we obtain from (4.8) and (4.10) that $\gamma = \pm\frac{1}{2}\langle\phi, V1\rangle$. Hence $\gamma = 0$, which imlies that $Q_1\phi = 0$. Moreover, $Q_1\psi = 0$. Thus, by (4.8) $0 = Q_1(\psi - \phi) = \alpha$ and we have shown $\psi = \phi$. We conclude that $\phi + Q_0 G_0 V Q_0 \phi = 0$ as claimed. From here one can follow the proof of Lemma 4.3. □

The following result is crucial for our considerations. We emphasize that the proof differs from the analogous proofs in dimensions $3, 4, 5, \ldots$. Compared with [10], which treats local potentials, some modifications are necessary.

**Lemma 4.5.** *The geometric null space $\mathcal{M}$ coincides with the algebraic null space of the operator $1+Q_0G_0VQ_0$ in $Q_0H^{1,-s}$, $3/2 < s < \beta-3/2$. Thus there exists a projection operator $U$ and a linear operator $L$ in $H^{1,3/2-\beta+0}$, both in $B(H^{1,-s})$ such that*

$$U^2 = U, \quad UL = LU = 0, \tag{4.12}$$

$$(1 + Q_0G_0VQ_0)U = U(1 + Q_0G_0VQ_0) = 0, \tag{4.13}$$

$$(1 + Q_0G_0VQ_0)L = L(1 + Q_0G_0VQ_0) = 1 - U \tag{4.14}$$

*Similar results hold for the null space $\mathcal{N}$ of $1 + Q_0^*VG_0Q_0^*$ with $U$ and $L$ replaced by $U^*$ and $L^*$, respectively.*

*Proof.* It suffices to show that $(1 + Q_0G_0VQ_0)^2\phi = 0$, $\phi \in Q_0H^{1,-s}$, implies that $(1 + Q_0G_0VQ_0)\phi = 0$. Define $\psi = (1 + Q_0G_0VQ_0)\phi$ such that $(1 + Q_0G_0VQ_0)\psi = 0$. Let $\tilde{\phi} = V\phi$ and $\tilde{\psi} = (1 + Q_0^*VG_0Q_0^*)\tilde{\phi}$. Then $\langle\psi, \tilde{\psi}\rangle = \langle(1 + Q_0G_0VQ_0)^2\phi, \tilde{\phi}\rangle = 0$. On the other hand,

$$0 = \langle\psi, \tilde{\psi}\rangle = -\langle G_0Q_0^*V\psi, Q_0^*V\psi\rangle = \langle\frac{1}{\xi^2}\widehat{Q_0^*V\psi}, \widehat{Q_0^*V\psi}\rangle$$

if $|x|^2Q_0^*V\psi \in L^1$ and $\langle 1, Q_0^*V\psi\rangle = 0$. Now, $Q_0^*V\psi = V\psi - \nu\langle V\psi, 1\rangle V1$ and $V1 \in H^{0,\beta-1/2-\epsilon}$ since $1 \in H^{2,-1/2-\epsilon}$. Therefore $|x|^2V1 \in H^{0,\beta-5/2-\epsilon}$. By Cauchy-Schwarz' inequality $H^{0,s} \subset L^1$ for $s > 1/2$ and as a consequence $|x|^2V1 \in L^1$ for $\beta > 3$. From Lemma 4.3 we have that $\psi \in L^\infty$ and therefore we also have that $|x|^2V\psi \in L^1$. In addition, $\langle Q_0^*V\psi, 1\rangle = 0$. Hence, computing in momentum space, we get that $-\langle G_0Q_0^*V\psi, Q_0^*V\psi\rangle = -\|(1/|\xi|)(\widehat{Q_0^*V\psi})\|^2_{L^2}$, which implies that $(\widehat{Q_0^*V\psi})(\xi) = 0$ a.e. and therefore $Q_0^*V\psi = 0$ in $L^2$. Thus, $\psi = 0$. □

Next we study under which conditions $\phi \in \mathcal{M}$ is a zero-energy eigenfunction for $H$ or in other words, when $\phi \in \mathcal{M}$ belongs to $L^2$. Let $P_0$ denote the orthogonal projection in $L^2 = H^{0,0}$ onto the eigenspace for the eigenvalue zero for $H$. If zero is not an eigenvalue, we set $P_0 = 0$.

**Lemma 4.6.** *Let $\phi \in \mathcal{M}$. Then $\phi \in H^{0,0}$ if and only if $\psi = H_0\phi = -V\phi$ satisfies both $Q_1G_0\psi = 0$ and $\langle\psi, (\cdot)\rangle = 0$. In this case actually $\phi \in H^{1,-1/2-0}$. We have $P_0L^2(\mathbf{R}) = \{\phi \in \tilde{\mathcal{M}} \mid \langle V\phi, (\cdot)\rangle = 0\,\}$ and $\dim\,(\tilde{\mathcal{M}}/P_0L^2(\mathbf{R})) \le 1$. In particular, $P_0V1 = 0$, $P_0Vx = 0$ and $P_0G_0V1 = 0$.*

*Proof.* Suppose that $\phi \in \mathcal{M} \cap L^2$. From Lemma 4.3 we have that $Q_1G_0\psi = 0$ and $\langle\psi, (\cdot)\rangle = 0$. Conversely, suppose that $\phi \in \mathcal{M}$ and both $Q_1G_0\psi = 0$ and $\langle\psi, (\cdot)\rangle = 0$. As in the proof of Lemma 4.3 $\phi \in \mathcal{M}$ implies that $\phi$ can be written as in (4.7). Then $Q_1G_0\psi = 0$ and $\langle\psi, (\cdot)\rangle = 0$ imply that, e.g. for $x > 0$,

$$\phi(x) = \int_x^\infty (y - x)(V\phi)(y)dy.$$

Next one imitates the proof of (4.4) in Lemma 4.3 to conclude that $\phi \in L^2(\mathbf{R})$. Finally, let us show that $\phi \in \mathcal{M} \cap L^2$ implies that $\phi \in H^{1,-1/2-0}$. Now, if $\phi \in \mathcal{M}$ then $V\phi \in \mathcal{N}$ by Lemma 4.1. Thus, $V\phi \in \mathcal{N} \subset H^{-1,3/2+0}$ and from Lemma 2.1(ii) we infer that $G_0 V\phi \in H^{1,-1/2-0}$. Moreover, since $\phi \in \mathcal{M} \cap L^2$ we have from above that $Q_1 G_0 V\phi = 0$. Hence $\phi = -G_0 V\phi \in H^{1,-1/2-0}$ as desired. The remaining assertions are obvious. □

By Lemma 4.2 and Lemma 4.6 $P_0 L^2(\mathbf{R}) \subset \tilde{\mathcal{M}} \cap H^{1,-1/2-0}$. Let $\{\phi_n\}$ be an $L^2$-orthogonal basis for $P_0 L^2(\mathbf{R})$. Then $P_0 = \sum \langle \cdot, \phi_n \rangle \phi_n$, and thus $P_0$ extends to an operator from $H^{-1,1/2+0}$ to $H^{1,-1/2-0}$.

4.2. **Decomposition of $H^{1,-s}$.** We shall decompose the space $H^{1,-s}$ according to the description above. We have already decomposed $H^{1,-s}$ with respect to the projections $Q_0$ and $Q_1$, viz. $Q_0 + Q_1 = 1$. Furthermore, $\operatorname{Ran} U = \mathcal{M} \subset Q_0 H^{1,-s}$; in particular, $U = Q_0 U = U Q_0$. Hence we need not decompose $Q_1 H^{1,-s}$. By Lemma 4.5 we decompose $Q_0 H^{1,-s}$ with respect to $U$ and its complement $U_0 := Q_0 - U$; here $U_0 = Q_0 U_0 = U_0 Q_0$. We call any $\psi \in \mathcal{M} \backslash P_0 H^{0,0}$ a zero resonance function. Lemma 4.2 and Lemma 4.6 assert that $\dim(\mathcal{M} \backslash H^{0,0}) \leq 2$, so there is at most two (linear independent) zero resonance functions modulo $L^2$-functions. In comparison, in the three-dimensional case there is at most one zero resonance function [4]. Next, we introduce a projection onto $P_0 H^{0,0}$. To construct it, we need the following lemma.

**Lemma 4.7.**

$$-P_0 V G_2 V P_0 = P_0.$$

*Proof.* It suffices to prove that $-\langle G_2 V P_0 \psi, V P_0 \phi \rangle = \langle P_0 \psi, \phi \rangle$ for $\psi, \phi \in H^{0,0}$. From Lemma 4.6, we have that $\langle V P_0 \psi, 1 \rangle = 0$, $\langle V P_0 \phi, 1 \rangle = 0$, $\langle V P_0 \psi, (\cdot) \rangle = 0$ and $\langle V P_0 \phi, (\cdot) \rangle = 0$ and, by means of Lemma 2.1, we get

$$-\langle G_2 V P_0 \psi, V P_0 \phi \rangle = \langle G_0 V P_0 \psi, G_0 V P_0 \phi \rangle.$$

But $G_0 V P_0 \psi = -P_0 \psi$ since $P_0 \psi \in \tilde{\mathcal{M}}$. This yields the result. □

From (2.2) we have that $V G_2 V \in B(1, -s; -1, s)$ for any $s$ satisfying $7/2 < s < \beta - 7/2$. Lemma 4.7 implies that $-P_0 V G_2 V$ is a projection in $H^{1,-s}$. With the projection $U$ from Lemma 4.5 we have from Lemma 4.6 that $U P_0 = P_0$. Therefore, also $-P_0 V G_2 V U$ is a projection in $H^{1,-s}$, and in $\mathcal{M}$. Hence, we have projections

$$\begin{aligned} U_0 &= Q_0 - U, \\ U_1 &= (1 + P_0 V G_2 V) U, \\ U_2 &= -P_0 V G_2 V U \end{aligned}$$

with the following properties

$$U_j U_k = \delta_{jk} U_k, \quad , j, k = 0, 1, 2 \tag{4.15}$$

and the decomposition

$$U_0 + U_1 + U_2 + Q_1 = 1 \quad (\text{identity in } H^{1,-s}). \tag{4.16}$$

In view of Lemma 4.2 and Lemma 4.6 $U_1 H^{1,-s}$ is at most two-dimensional. If $U_1 \neq 0$ there exists functions $\phi_1$ and $\phi_2$ in $\mathcal{M}$ satisfying

$$\langle \phi_1, V G_0 V 1 \rangle \neq 0, \quad \langle \phi_1, V(\cdot) \rangle \neq 0, \tag{4.17}$$

$$\langle \phi_2, V G_0 V 1 \rangle = 0, \quad \langle \phi_2, V(\cdot) \rangle \neq 0, \tag{4.18}$$

The dual elements in $\mathcal{N}$ are $\tilde{\phi}_1 = V\phi_1$ and $\tilde{\phi}_2 = V\phi_2$. We choose the normalization

$$\langle \phi_j, \tilde{\phi}_k \rangle = \delta_{jk}, \quad j, k = 1, 2. \tag{4.19}$$

For $j = 1, 2$, introduce the constants

$$c_1^{(j)} = \nu \langle G_0 V \phi_j, V 1 \rangle, \quad c_2^{(j)} = \frac{1}{2} \langle V \phi_j, (\cdot) \rangle, \tag{4.20}$$

where the functions $\phi_j$ are described above. Then we make the following classification of the zero-energy properties of $H$: Zero is a *regular point for $H$* when $U_1 = U_2 = 0$. Zero is an *exceptional point of the 1st kind for $H$* when $U_1 \neq 0$ and $U_2 = 0$. Here we make the subclassification:

Type 1. $\dim \tilde{\mathcal{M}} = 0$,
Type 2. $\dim \tilde{\mathcal{M}} = 1$,
Type 3. $\dim \mathcal{M} = 2$.

Zero is an *exceptional point of the 2nd kind for $H$* when $U_1 = 0$ and $U_2 \neq 0$ and finally zero is an *exceptional point of the 3rd kind for $H$* when $U_1 \neq 0$ and $U_2 \neq 0$. Here we make the same subclassification as when zero is an exceptional point of the 1st kind. The classification has the following meaning. If zero is a regular point for $H$ there are no zero-energy resonances or zero-energy bound states for $H$; this represents the generic case. If zero is an exceptional point of the 1st kind for $H$ there are zero-energy resonances of $H$ but no zero-energy bound states. Zero is an exceptional point of the 2nd kind for $H$ if there are zero-energy bound states but no zero-energy resonances. Finally, zero is an exceptional point of the 3rd kind for $H$ if zero is both a resonance and an eigenvalue. In all exceptional cases we have $\langle \phi_j, V1 \rangle = 0$. It is convenient to collect the following relations in one place.

**Lemma 4.8.** *The following relations are valid.*

$$Q_0 Q_1 = Q_1 Q_0 = 0, \quad U U_0 = U_0 U = 0, \tag{4.21}$$

$$Q_1 U = U Q_1 = 0, \quad Q_0 U = U Q_0 = U, \tag{4.22}$$

$$Q_1 + U_1 + U_2 + U_0Q_0 = 1, \tag{4.23}$$

$$U_jU_k = \delta_{jk}U_k, \ \ j,k = 0,1,2, \tag{4.24}$$

$$Q_0U_j = U_jQ_0 = U_j, \ \ j = 1,2 \tag{4.25}$$

$$Q_0U_0 = U_0Q_0 = U_0 - Q_1 = Q_0 - U, \tag{4.26}$$

$$Q_1U_j = U_jQ_1 = 0, \ \ j = 1,2, \tag{4.27}$$

$$Q_1L = LQ_1 = Q_1, \ \ Q_0L = LQ_0 = L - Q_1, \ \ Q_1LQ_0 = 0, \tag{4.28}$$

$$H_0U = U^*H_0, \quad VU = U^*V. \tag{4.29}$$

*Proof.* Equalities (4.21)-(4.22) follow from the definitions of $Q_0$, $Q_1$, $U_0$ and $U$. (4.23)-(4.25) are just (4.15)-(4.16). Furthermore, (4.26) follows from the definition of $U_0$, for instance $U_0Q_0 = Q_0U_0 = Q_0(Q_0 - U) = Q_0 - U$. Equality (4.27) follows from (4.25)-(4.26) and (4.28) follows from Lemma 4.5. The operator $H_0$ maps $UH^{1,-s} = \mathcal{M}$ into $\mathcal{N}$ by Lemma 4.1, hence $H_0U = U^*H_0U$. By taking adjoints, the equality $H_0U = U^*H_0$ follows. The proof of the remaining equality in (4.29) is similar. □

## 5. Expansions of $R(\zeta)$ for small $\zeta$

In this section we obtain expansions for $R(\zeta)$ as $\zeta \to 0$. The order of the expansion is dependent on the decay parameter $\beta$ for the potential $V$. Generally, expansion to a high order requires a large $\beta$. The proof is based on the method introduced in [4] adapted to the current context. The crucial technique is contained in [4, Lemma 3.12]. For the sake of convenience we state it here.

**Lemma 5.1.** *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathbf{X}$ and $\mathbf{Y}$ be vector spaces. Let $A : \mathcal{X} \to \mathcal{Y}$, $B : \mathbf{X} \to \mathcal{X}$, $C : \mathcal{Y} \to \mathbf{Y}$ be linear operators. Define $\mathbf{A} = CAB$. If $\mathbf{A}^{-1}$ exists, then $A^{-1} = B\mathbf{A}^{-1}C$ provided $B$ is surjective and $C$ is injective.*

*Proof.* Let $D = B\mathbf{A}^{-1}C$. $AD = AB\mathbf{A}^{-1}C$, hence $CAD = CAB\mathbf{A}^{-1}C = C$ or $C(AD - 1) = 0$. Since $C$ is injective, we conclude $AD = 1$. Likewise, $DA = B\mathbf{A}^{-1}CA$ such that $DAB = B\mathbf{A}^{-1}CAB = B$ or $(DA - 1)B = 0$. The surjectivity of $B$ implies that $DA = 1$. □

We proceed to give the necessary definitions. Throughout this section we impose Assumption 3.1 with $\beta > 3$ on $V$, at least. Moreover, we impose at least the restriction $3/2 < s < \beta - 3/2$. We divide the analysis according to the zero-energy properties of $H$. In all cases we begin by obtaining an expansion for $(1 + R_0(\zeta)V)^{-1}$ around $\zeta = 0$ and then proceed to obtain the expansion for $R(\zeta)$ via the second resolvent equation $R(\zeta) = (1 + R_0(\zeta)V)^{-1}R_0(\zeta)$.

5.1. **The regular case.** We assume that zero is a regular point for $H$. Under this assumption $Q_0(1+G_0V)Q_0$ is invertible in $Q_0H^{1,-s}$ according to the analysis in Section 4.2. We denote by $K$ the inverse to $Q_0(1+G_0V)Q_0$ in $Q_0H^{1,-s}$ and extend $K$ to the whole of $H^{1,-s}$ by setting $KQ_1 = Q_1K = 0$. Thus the following relation hold:

$$KQ_0(1+G_0V)Q_0 = Q_0(1+G_0V)Q_0K = Q_0, \tag{5.1}$$

Furthermore, using the definitions we find that $-iQ_1^*VG_{-1}VQ_1 = \frac{i}{2}\langle\cdot, V1\rangle V1$. The inverse of this operator, which maps $Q_1H^{1,-s}$ into $Q_1^*H^{-1,s}$, is given by $-2i\nu^2\langle\cdot,1\rangle 1$. With these observations we are ready to state the following lemma.

**Lemma 5.2.** *Assume that zero is a regular point for $H$.*

**(i):** *Assume that $\beta > 5$ and let $s$ satisfy $5/2 < s < \beta - 5/2$. For $\zeta \to 0$ we have in $B(1,-s;1,-s)$ the expansion*

$$(1+R_0(\zeta)V)^{-1} = D_0^{(0)} + i\zeta^{1/2}D_1^{(0)} + O(\zeta) \tag{5.2}$$

*with*

$$D_0^{(0)} = K, \tag{5.3}$$

$$\begin{aligned} D_1^{(0)} &= -KG_1VK - 2\nu^2\langle\cdot, V1\rangle 1 + 2\nu^2\langle\cdot, V1\rangle KG_0V + \\ &+2\nu^2\langle\cdot, K^*(V+VG_0V)1\rangle 1 - 2\nu^2\langle\cdot, K^*(V+VG_0V)1\rangle KG_0V. \end{aligned} \tag{5.4}$$

**(ii):** *Assume that $\beta > 7$ and let $s$ satisfy $7/2 < s < \beta - 7/2$. For $\zeta \to 0$ we have in $B(1,-s;1,-s)$ the expansion*

$$(1+R_0(\zeta)V)^{-1} = D_0^{(0)} + i\zeta^{1/2}D_1^{(0)} - \zeta D_2^{(0)} + O(\zeta^{3/2}) \tag{5.5}$$

*with $D_0^{(0)}$ and $D_1^{(0)}$ given above and*

$$\begin{aligned} D_2^{(0)} &= -KG_2VK + KG_1VKG_1VK + 2\nu^2\langle VG_1VK\cdot, 1\rangle 1 + \\ &-4\nu^2\langle\cdot, V1\rangle\langle V(1+G_0V), 1\rangle 1 + 2\nu^2\langle\cdot, V1\rangle KG_1V + \\ &-2\nu^2\langle\cdot, V1\rangle KG_1VKG_0V - 2\nu^2\langle V(1+G_0V)KG_1VK\cdot, 1\rangle 1 + \\ &+4\nu^4\langle\cdot, V1\rangle\langle V(1+G_0V), 1\rangle KG_0V + \\ &+4\nu^4\langle\langle V(1+G_0V)K\cdot, 1\rangle\langle V(1+G_0V)1, 1\rangle 1 + \\ &+4\nu^4\langle\cdot, V1\rangle\langle V(1+G_0V)KG_0V, 1\rangle 1. \end{aligned} \tag{5.6}$$

*Proof.* We prove (i). The proof of (ii) is similar. Let $s$ satisfy $5/2 < s < \beta - 5/2$. In order to apply Lemma 5.1, we define the following spaces and operators:

$$\mathcal{X} = \mathcal{Y} = H^{1,-s},$$

$$\mathbf{X} = Q_0\mathcal{X} \oplus Q_1\mathcal{X}, \quad \mathbf{Y} = Q_0\mathcal{X} \oplus Q_1^*H^{-1,s},$$

$$A = 1 + R_0(\zeta)V, \quad B = (B_0, B_1) = (Q_0, \zeta^{1/4}Q_1),$$

$$C = \begin{pmatrix} C_0 \\ C_1 \end{pmatrix} = \begin{pmatrix} Q_0 \\ \zeta^{1/4} Q_1^* V \end{pmatrix},$$

Obviously, $B$ is surjective. Moreover, $C$ is injective. To prove this, assume that $\phi \in \mathcal{Y}$ and $C\phi = 0$. Then $Q_0\phi = 0$ such that $\phi = Q_1\phi = \nu\langle\phi, V1\rangle 1$. Moreover $Q_1^* V\phi = 0$, which means that $\nu\langle\phi, V1\rangle V1 = 0$. Since $V1 \neq 0$, we get $\langle\phi, V1\rangle = 0$ and hence $\phi = 0$. Thus, we have proved that $C$ is injective.

We define $\mathbf{A} = CAB$ and use matrix-notation for $\mathbf{A}$. Using the asymptotic expansions for $R_0(\zeta)$ from (2.3), we get

$$A = 1 + R_0(\zeta)V = 1 - i\zeta^{-1/2}G_{-1}V + G_0V + i\zeta^{1/2}G_1V - \zeta G_2 V + \mathrm{o}(\zeta),$$

under the assumptions $\beta > 5$ and $5/2 < s < \beta - 5/2$. Further,

$$\mathbf{A} = \begin{pmatrix} C_0 \\ C_1 \end{pmatrix} A(B_0, B_1) = \begin{pmatrix} C_0AB_0 & C_0AB_1 \\ C_1AB_0 & C_1AB_1 \end{pmatrix}.$$

Using Lemma 4.8 and $Q_0G_{-1}VQ_1 = 0$ we have

$$\begin{aligned}
A_{00} &= Q_0(1 + G_0V)Q_0 + i\zeta^{1/2}Q_0G_1VQ_0 + \mathrm{O}(\zeta),\\
A_{01} &= \zeta^{1/4}Q_0(1 + G_0V)Q_1 + i\zeta^{3/4}Q_0G_1VQ_1 + \mathrm{O}(\zeta^{5/4}),\\
A_{10} &= \zeta^{1/4}Q_1^*V(1 + G_0V)Q_0 + i\zeta^{3/4}Q_1^*VG_1VQ_0 + \mathrm{O}(\zeta^{5/4}),\\
A_{11} &= -iQ_1^*VG_{-1}VQ_1 + \zeta^{1/2}Q_1^*V(1 + G_0V)Q_1\\
&\quad + i\zeta Q_1^*VG_1VQ_1 + \mathrm{O}(\zeta^{3/2}).
\end{aligned}$$

Our assumption $5/2 < s < \beta - 5/2$ is necessary and sufficient to make this computation. Let us verify this for two of the terms. The remaining terms can be seen to make sense by similar arguments. Consider $Q_0G_1VQ_1$ first. If $u \in H^{1,-s}$, $Q_1u \in H^{1,-s}$ for $1/2 < s < \beta - 1/2$, so $VQ_1u \in H^{-1,\beta-s}$ and $G_1VQ_1u \in H^{1,-s}$ for $\beta - s > 5/2$ and $s > 5/2$, or $5/2 < s < \beta - 5/2$. $Q_0$ maps $H^{1,-s}$ into itself, thus $Q_0G_1VQ_1$ makes sense. Next, consider $Q_1^*VG_1VQ_1$ and show that this operator maps $Q_1H^{1,-s}$ into $Q_1^*H^{-1,s}$. If $u \in H^{1,-s}$, $Q_1u \in H^{1,-s}$ for $1/2 < s < \beta - 1/2$ and $VQ_1u \in H^{-1,\beta-s}$. Then $G_1VQ_1u \in H^{1,-s}$ for $5/2 < s < \beta - 5/2$ and thus $VG_1VQ_1u \in H^{-1,\beta-s} \subset H^{-1,s}$. Therefore, $Q_1^*VG_1VQ_1$ maps in the desired way.

We decompose $\mathbf{A} = \mathbf{D} + \mathbf{S}$ with

$$\mathbf{D} = \begin{pmatrix} Q_0(1 + G_0V)Q_0 & 0 \\ 0 & -iQ_1^*VG_{-1}VQ_1 \end{pmatrix}.$$

We also have, with an obvious notation, the order estimate

$$\mathbf{S} = \mathrm{O}\begin{pmatrix} \zeta^{1/2} & \zeta^{1/4} \\ \zeta^{1/4} & \zeta^{1/2} \end{pmatrix} \quad \text{as } \zeta \to 0.$$

From the observations previous to this lemma we know that $Q_0(1 + G_0V)Q_0$ has inverse $K$ in $Q_0H^{1,-s}$ and $iQ_1^*VG_{-1}VQ_1 = (i/2)\langle\cdot, V1\rangle V1$

has inverse $-2i\nu^2\langle\cdot,1\rangle 1$. Therefore, $\mathbf{D}$ has inverse

$$\mathbf{D}^{-1} = \begin{pmatrix} K & 0 \\ 0 & -2i\nu^2\langle\cdot,1\rangle 1 \end{pmatrix}.$$

This means that for sufficiently small $\zeta$, $\mathbf{A}$ is invertible and the inverse can be computed using the Neumann series:

$$\mathbf{A}^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{S}\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{S}\mathbf{D}^{-1}\mathbf{S}\mathbf{D}^{-1} - \cdots$$

Using the finite version of the expansion we obtain asymptotic expansions of $(1+R_0(\zeta)V)^{-1}$ to a certain order, in the current case up to terms of order $\zeta^{1/2}$ with an error of order $\mathrm{O}(\zeta)$. For this purpose the first three terms written above will be needed. Using the expansion for $\mathbf{A}^{-1}$, Lemma 5.1 then yields an expansion for $A^{-1}$. The calculations are tedious, but straightforward. This completes the proof of (i). □

We are ready to state the main result for the resolvent of $H$.

**Theorem 5.3.** *Assume that zero is a regular point for $H$. Suppose that $V$ satisfies Assumption 3.1 with $\beta > 7$. Let $s$ satisfy $7/2 < s < \beta - 7/2$. For $\zeta \to 0$ we have in $B(-1,s;1,-s)$ the expansion*

$$R(\zeta) = B_0^{(0)} + i\zeta^{1/2}B_1^{(0)} + O(\zeta)$$

*with*

$$B_0^{(0)} = Q_0 K G_0 Q_0^* + \nu\langle\cdot,1\rangle 1, \tag{5.7}$$

$$\begin{aligned} B_1^{(0)} = {} & KG_1 - KG_1VKG_0 + \nu\langle\cdot,1\rangle KG_1V + \nu\langle\cdot,\rangle KG_1VKG_0V + \\ & +2\nu^2\langle V(1+G_0V)KG_0\cdot,1\rangle 1 - 2\nu^2\langle V(1+G_0V)KG_0\cdot,1\rangle Q_0 \\ & +2\nu^2\langle VG_0\cdot,1\rangle KG_0V + 2\nu^3\langle\cdot,1\rangle\langle V(1+G_0V),1\rangle(1-KG_0V) \\ & -2\nu^2\langle VG_0\cdot,1\rangle 1 - 2\nu^3\langle\cdot,1\rangle\langle V(1+G_0V)KG_0V,1\rangle 1. \end{aligned} \tag{5.8}$$

*Proof.* It follows from Lemma 5.2, (2.3), $KG_{-1} = 0$ and the second resolvent equation. □

5.2. **Exceptional case of the first kind: Type 1.** By assumption $\dim\mathcal{M} = 1$ and $\dim\tilde{\mathcal{M}} = 0$. We define the projection $U_1'\cdot = \langle\cdot,\tilde{\phi}_1\rangle\phi_1$, where $\phi_1$ and $\tilde{\phi}_1$ satisfy (4.17) and (4.19). Hence the starting point of the analysis is the decomposition $Q_1 + U_1' + U_0Q_0 = 1$. It follows from the definitions of $U_0$, $U_1'$ and $Q_1$ and Lemma 4.8 that we decompose using three commuting projections. For convenience, define the operator $\tilde{U}_0 = U_0Q_0$. We shall use the following two lemmas.

**Lemma 5.4.** *The following relations hold.*

$$Q_1U_1' = 0, \quad (U_1')^*Q_1^* = 0, \quad (U_1')^*VG_{-1}U_1' = 0, \tag{5.9}$$

$$(U_1')^*V(1+G_0V)U_1' = 0, \quad U_1'G_0V\tilde{U}_0 = 0. \tag{5.10}$$

*Proof.* The equalities are straightforward consequences of the definitions of the various operators, (4.17) and Lemmas 4.8. □

Bear in mind that the operator $L$ was introduced in Lemma 4.5.

**Lemma 5.5.**
*(i) The inverse to* $\tilde{U}_0(1+G_0V)\tilde{U}_0$ *in* $B(\tilde{U}_0H^{1,-s})$ *is* $L$.
*(ii) The inverse to* $(i/2)\langle\cdot,V1\rangle V1$ *in* $B(Q_1H^{1,-s},Q_1^*H^{-1,s})$ *is* $-2i\nu^2\langle\cdot,1\rangle 1$.
*(iii) The inverse to* $\langle\cdot,\tilde{\phi}_1\rangle V1$ *in* $B(U_1'H^{1,-s},Q_1^*H^{-1,s})$ *is* $\nu\langle\cdot,1\rangle\phi_1$.
*(iv) The inverse to* $\langle\cdot,V1\rangle\tilde{\phi}_1$ *in* $B(Q_1H^{1,-s},(U_1')^*H^{-1,s})$ *is* $\nu\langle\cdot,\phi_1\rangle 1$.
*(v) The inverse to* $\langle\cdot,\tilde{\phi}_1\rangle\tilde{\phi}_1$ *in* $B(U_1'H^{1,-s},(U_1')^*H^{-1,s})$ *is* $\langle\cdot,\phi_1\rangle\phi_1$.

*Proof.* (i) follows immediately from (4.26) and (4.14); in this order. The proofs of (ii)-(v) are straightforward. □

Introduce the constant $d_1=-\nu^2/[|c_1^{(1)}|^2+|c_2^{(1)}|^2]$. Note that $d_1$ is always finite, according to the subclassification in the exceptional case of first kind.

**Lemma 5.6.** *Assume that zero is an exceptional point of the first kind for* $H$ *(Type 1). Let Assumption 3.1 with* $\beta>7$ *be satisfied. Let* $s$ *satisfy* $7/2<s<\beta-7/2$. *For* $\zeta\to 0$ *we have in* $B(1,-s;1,-s)$ *the expansion*

$$(1+R_0(\zeta)V)^{-1}=\zeta^{-1/2}\frac{i}{2}\frac{d_1}{\nu^2}\langle\cdot,\tilde{\phi}_1\rangle\phi_1+D_0^{(1,1)}+O(\zeta^{1/2}),$$

*where the constant term* $D_0^{(1,1)}$ *is given by*

$$\begin{aligned}
D_0^{(1,1)} &= \tilde{U}_0L\tilde{U}_0-\frac{c_1^{(1)}d_1}{\nu}\langle\cdot,\tilde{\phi}_1\rangle 1-\frac{(c_1^{(1)})^*d_1}{\nu}\langle\cdot,V1\rangle\phi_1-\\
&-\frac{c_1^{(1)}d_1}{\nu}\langle\cdot,\tilde{\phi}_1\rangle\tilde{U}_0L\tilde{U}_0(1+G_0V)1-\frac{d_1}{2\nu^2}\langle\cdot,\tilde{\phi}_1\rangle\tilde{U}_0L\tilde{U}_0G_1\tilde{\phi}_1\\
&-\frac{c_1^{(1)}d_1}{\nu}\langle(V+VG_0V)\tilde{U}_0L\tilde{U}_0\cdot,1\rangle\phi_1-\frac{d_1}{2\nu^2}\langle G_1V\tilde{U}_0L\tilde{U}_0\cdot,\tilde{\phi}_1\rangle\phi_1+\\
&+\Bigg\{\frac{|c_1^{(1)}|^2d_1^2}{\nu^2}\langle(V+VG_0V)1,1\rangle+\frac{(c_1^{(1)})^*d_1^2}{2\nu^3}\langle G_1V\phi_1,V1\rangle+\\
&+\frac{c_1^{(1)}d_1^2}{2\nu^3}\langle G_1V1,\tilde{\phi}_1\rangle+\frac{d_1^2}{4\nu^4}\langle G_2V\phi_1,\tilde{\phi}_1\rangle\Bigg\}\langle\cdot,\tilde{\phi}_1\rangle\phi_1+
\end{aligned}$$

$$+\left\{\frac{|c_1^{(1)}|^2 d_1^2}{\nu^2}\langle (V+VG_0V)\tilde{U}_0 L\tilde{U}_0(1+G_0V)1,1\rangle + \right.$$
$$+\frac{(c_1^{(1)})^* d_1^2}{2\nu^3}\langle (V+VG_0V)\tilde{U}_0 L\tilde{U}_0 G_1 V\phi_1,1\rangle +$$
$$+\frac{c_1^{(1)} d_1^2}{2\nu^3}\langle G_1 V\tilde{U}_0 L\tilde{U}_0(1+G_0V)1,\tilde{\phi}_1\rangle +$$
$$\left.+\frac{d_1^2}{4\nu^4}\langle G_1 V\tilde{U}_0 L\tilde{U}_0 G_1 V\phi_1,\tilde{\phi}_1\rangle\right\}\langle\cdot,\tilde{\phi}_1\rangle\phi_1.$$

*Proof.* The set-up needed to use Lemma 5.1 is as follows. The spaces are $\mathcal{X}=\mathcal{Y}=H^{1,-s}$,

$$\mathbf{X}=\tilde{U}_0\mathcal{X}\oplus Q_1\mathcal{X}\oplus U_1'\mathcal{X},\quad \mathbf{Y}=\tilde{U}_0\mathcal{X}\oplus Q_1^*H^{-1,s}\oplus (U_1')^*H^{-1,s}$$

and the operators are

$$A=1+R_0(\zeta)V,\quad B=(\tilde{U}_0,\zeta^{1/4}Q_1,\zeta^{-1/4}U_1'),$$

$$C=\begin{pmatrix}\tilde{U}_0\\ \zeta^{1/4}Q_1^*V\\ \zeta^{-1/4}(U_1')^*V\end{pmatrix}.$$

From the definition, $B$ is obviously surjective. Moreover, $C$ is injective as we will show now. Assume that $C\phi=0$, $\phi\in\mathcal{Y}$. This means

$$\tilde{U}_0\phi=0,\quad Q_1^*V\phi=0,\quad (U_1')^*V\phi=0.$$

Now, $(U_1')^*V\phi=VU_1'\phi$ and $V$ is injective on $\mathcal{M}$, so $U_1'\phi=0$. Due to $\phi=Q_0(1-U)\phi+U_1'\phi+Q_1\phi$, $Q_0(1-U)\phi=0$ and $U_1'\phi=0$ imply that $Q_1\phi=\phi$, or explicitly $\phi=Q_1\phi=\nu\langle\phi,V1\rangle 1$. Now, $Q_1^*V\phi=0$ yields $\nu\langle\phi,V1\rangle V1=0$ or $\langle\phi,V1\rangle=0$, hence $\phi=\nu\langle\phi,V1\rangle 1=0$. Thus, we have proved that $C$ is injective.

We define $\mathbf{A}=CAB$ and use matrix-notation for $\mathbf{A}$; explicitly

$$\mathbf{A}=CAB=\begin{pmatrix}C_0AB_0 & C_0AB_1 & C_0AB_2\\ C_1AB_0 & C_1AB_1 & C_1AB_2\\ C_2AB_0 & C_2AB_1 & C_2AB_2\end{pmatrix}.$$

From (2.3), requiring $7/2<s<\beta-7/2$, we obtain

$$\begin{aligned}A &= 1+R_0(\zeta)V\\ &= 1-i\zeta^{-1/2}G_{-1}V+G_0V+i\zeta^{1/2}G_1V-\zeta G_2V+\mathrm{O}(\zeta^{3/2}),\end{aligned}$$

and via Lemma 3.2 and Lemma 5.4 we get the following expansions for the individual terms in $\mathbf{A}$:

$$\begin{aligned}
A_{00} &= \tilde{U}_0(1+G_0V)\tilde{U}_0 + i\zeta^{1/2}\tilde{U}_0G_1V\tilde{U}_0 + \mathrm{O}(\zeta),\\
A_{10} &= \zeta^{1/4}Q_1^*(V+VG_0V)\tilde{U}_0 + i\zeta^{3/4}Q_1^*VG_1V\tilde{U}_0 + \mathrm{O}(\zeta^{5/4}),\\
A_{20} &= i\zeta^{1/4}(U_1')^*VG_1V\tilde{U}_0 + \mathrm{O}(\zeta^{3/4}),\\
A_{01} &= \zeta^{1/4}\tilde{U}_0(1+G_0V)Q_1 + i\zeta^{3/4}\tilde{U}_0G_1VQ_1 + \mathrm{O}(\zeta^{5/4}),\\
A_{11} &= \frac{i}{2\nu}VQ_1 + \zeta^{1/2}Q_1^*(V+VG_0V)Q_1 + \mathrm{O}(\zeta),\\
A_{21} &= VU_1'G_0VQ_1 + i\zeta^{1/2}(U_1')^*VG_1VQ_1 + \mathrm{O}(\zeta),\\
A_{02} &= i\zeta^{1/4}\tilde{U}_0G_1VU_1' + \mathrm{O}(\zeta^{3/4}),\\
A_{12} &= Q_1^*VG_0VU_1' + i\zeta^{1/2}Q_1^*VG_1VU_1' + \mathrm{O}(\zeta),\\
A_{22} &= i(U_1')^*VG_1VU_1' - \zeta^{1/2}(U_1')^*VG_2VU_1' + \mathrm{O}(\zeta).
\end{aligned}$$

Our assumption $7/2 < s < \beta - 7/2$ is necessary and sufficient to make this computation. Let us verify this for two of the terms. The remaining terms can be seen to make sense by similar arguments. Consider $\tilde{U}_0G_1VQ_1$ first. If $u \in H^{1,-s}$, $Q_1u \in H^{1,-s}$ for $1/2 < s < \beta - 1/2$, so $VQ_1u \in H^{-1,\beta-s}$ and $G_1VQ_1u \in H^{1,-s}$ for $\beta - s > 5/2$ and $s > 5/2$, or $5/2 < s < \beta - 5/2$. $P_0$, as well as $Q_0$, maps $H^{1,-s}$ into itself, thus $\tilde{U}_0G_1VQ_1u$ makes sense for $5/2 < s < \beta - 5/2$. Next, consider $(U_1')^*VG_2VU_1'$ and show that this operator maps $U_1'H^{1,-s}$ into $(U_1')^*H^{-1,s}$. If $u \in H^{1,-s}$, $U_1'u \in H^{1,-s}$ for $3/2 < s < \beta - 3/2$ and $VU_1'u \in H^{-1,\beta-s}$. Then $G_2VU_1'u \in H^{1,-s}$ for $7/2 < s < \beta - 7/2$ and therefore $VG_2VU_1'u \in H^{-1,\beta-s} \subset H^{-1,s}$. Thus, $(U_1')^*VG_2VP_1$ maps in the desired way.

As in the regular case the idea is to extract a leading invertible operator. We extract the following leading term operator

$$\mathbf{D} = \begin{pmatrix} \tilde{U}_0(1+G_0V)\tilde{U}_0 & 0 & 0\\ 0 & \frac{i}{2}\langle\cdot,V1\rangle V1 & c_1^{(1)}\langle\cdot,\tilde{\phi}_1\rangle V1\\ 0 & (c_1^{(1)})^*\langle\cdot,V1\rangle\tilde{\phi}_1 & 2i|c_2^{(1)}|^2\langle\cdot,\tilde{\phi}_1\rangle\tilde{\phi}_1 \end{pmatrix}.$$

Due to Lemma 5.5, a simple computation shows that this operator is invertible, and the inverse is given by

$$\mathbf{D}^{-1} = \begin{pmatrix} L & 0 & 0\\ 0 & 2id_1|c_2^{(1)}|^2\langle\cdot,1\rangle 1 & -\frac{c_1^{(1)}d_1}{\nu}\langle\cdot,\phi_1\rangle 1\\ 0 & -\frac{(c_1^{(1)})^*d_1}{\nu}\langle\cdot,1\rangle\phi_1 & \frac{i}{2}\frac{d_1}{\nu^2}\langle\cdot,\phi_1\rangle\phi_1 \end{pmatrix}.$$

At this point we have $\mathbf{A} = \mathbf{D} + \mathbf{S}$, where

$$\mathbf{S} = \mathrm{O}\begin{pmatrix} \zeta^{1/2} & \zeta^{1/4} & \zeta^{1/4}\\ \zeta^{1/4} & \zeta^{1/2} & \zeta^{1/2}\\ \zeta^{1/4} & \zeta^{1/2} & \zeta^{1/2} \end{pmatrix}$$

and

$$\mathbf{S}^2 = \mathrm{O}\begin{pmatrix} \zeta^{1/2} & \zeta^{3/4} & \zeta^{3/4} \\ \zeta^{3/4} & \zeta^{1/2} & \zeta^{1/2} \\ \zeta^{3/4} & \zeta^{1/2} & \zeta^{1/2} \end{pmatrix}.$$

Hence $\mathbf{A}$ is invertible and the inverse can be computed using the Neumann series. The expansion in the lemma requires the first three terms in

$$\mathbf{A}^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{S}\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{S}\mathbf{D}^{-1}\mathbf{S}\mathbf{D}^{-1} + \cdots$$

Lemma 5.1 then yields the desired expansion after some elementary computations, which are omitted. □

*Remark* 5.7. To obtain the leading order term in the asymptotic expansion of the $S$-matrix (see Section 6) when zero is an exceptional point of 1st kind for $H$ (Type 1), it is necessary to know explicitly the coefficient $D_1^{(1,1)}$ of the next term in the expansion above. It turns out that the expression for $D_1^{(1,1)}$ takes up a lot of space. As a consequence, we do not duplicate it here.

The main result for the resolvent of $H$ is:

**Theorem 5.8.** *Assume that zero is an exceptional point of first kind for $H$ (Type 1). Let Assumption 3.1 be satisfied. Assume $\beta > 7$ and let $s$ satisfy $7/2 < s < \beta - 7/2$. For $\zeta \to 0$ we have in $B(-1, s; 1, -s)$ the expansion*

$$R(\zeta) = -i\zeta^{-1/2}\frac{d_1}{2\nu^2}\langle\cdot, \phi_1\rangle\phi_1 + O(1).$$

*Proof.* It follows from Lemma 5.6, (2.3) and the second resolvent equation, in combination with the relations $\langle\phi_1, V1\rangle = 0$, $\tilde{U}_0 G_{-1} = 0$ and Lemma 4.2(ii) □

5.3. **Exceptional case of the first kind: Type 2.** By assumption $\dim\tilde{\mathcal{M}} = 1$. We may define the projection $U_1''\cdot = \langle\cdot, \tilde{\phi}_2\rangle\phi_2$, where $\phi_2$ and $\tilde{\phi}_2$ satisfy (4.18) and (4.19). Hence, the basic decomposition is $Q_1 + U_1'' + U_0 Q_0 = 1$. It follows from the definitions of $U_0$, $U_1''$ and $Q_1$ and Lemma 4.8 that we decompose using three commuting projections. For convenience, define the operator $\tilde{U}_0 = U_0 Q_0$. We shall use the following lemma.

**Lemma 5.9.** *The following relations are valid.*

$$Q_1 U_1'' = 0,\ \ (U_1'')^* Q_1^* = 0,\ \ (U_1'')^* V G_{-1} V U_1'' = 0 \tag{5.11}$$

$$(1 + G_0 V)U_1'' = 0,\ \ (U_1'')^* V(1 + G_0 V) = 0. \tag{5.12}$$

$$U_1'' G_0 V \tilde{U}_0 = 0 \tag{5.13}$$

$$VU_1''G_0VQ_1 = 0, \quad Q_1^*VG_0VU_1'' = 0, \tag{5.14}$$

$$(U_1'')^*VG_1VU_1'' = 2|c_2^{(2)}|^2\langle\cdot, \tilde{\phi}_2\rangle\tilde{\phi}_2. \tag{5.15}$$

*Proof.* The equalities are straightforward consequences of the definitions of the various operators, (4.17) and Lemmas 4.8. □

We introduce the constant $d_2 = -\nu^2/|c_2^{(2)}|^2$. Note that $d_2$ is always finite, according to the subclassification in the exceptional case of first kind, type 2.

**Lemma 5.10.** *Assume that zero is an exceptional point of first kind for $H$ (Type 2). Let Assumption 3.1 with $\beta > 7$ be satisfied. Let $s$ satisfy $7/2 < s < \beta - 7/2$. For $\zeta \to 0$ we have in $B(1,-s;1,-s)$ the expansion*

$$(1 + R_0(\zeta)V)^{-1} = \zeta^{-1/2}D_{-1}^{(1,2)} + D_0^{(1,2)} + O(\zeta^{1/2}),$$

*where the coefficients $D_{-1}^{(1,2)}$ and $D_0^{(1,2)}$ are given by*

$$D_{-1}^{(1,2)} = \frac{i}{2}\frac{d_2}{\nu^2}\langle\cdot, \tilde{\phi}_2\rangle\phi_2 \tag{5.16}$$

*and*

$$\begin{aligned} D_0^{(1,2)} &= \tilde{U}_0L\tilde{U}_0 + \frac{d_2}{2\nu^2}\langle\cdot, \tilde{\phi}_2\rangle\tilde{U}_0L\tilde{U}_0G_1\tilde{\phi}_2 + \frac{d_2}{2\nu^2}\langle G_1V\tilde{U}_0L\tilde{U}_0\cdot, \tilde{\phi}_2\rangle\phi_2 + \\ &\quad - \left(\frac{d_2}{2\nu^2}\right)^2 \langle\cdot, \tilde{\phi}_2\rangle\langle G_2\tilde{\phi}_2, \tilde{\phi}_2\rangle\phi_2 + \\ &\quad + \left(\frac{d_2}{2\nu^2}\right)^2 \langle\cdot, \tilde{\phi}_2\rangle\langle G_1V\tilde{U}_0L\tilde{U}_0G_1\tilde{\phi}_2, \tilde{\phi}_2\rangle\phi_2. \end{aligned} \tag{5.17}$$

*Proof.* With some obvious modifications the proof is similar to the proof of Lemma 5.6. □

The main result is:

**Theorem 5.11.** *Assume that zero is an exceptional point of first kind for $H$ (Type 2). Let Assumption 3.1 be satisfied. Assume $\beta > 7$ and let $s$ satisfy $7/2 < s < \beta - 7/2$. For $\zeta \to 0$ we have in $B(-1,s;1,-s)$ the expansion*

$$R(\zeta) = -i\zeta^{-1/2}\frac{d_2}{2\nu^2}\langle\cdot, \phi_2\rangle\phi_2 + O(1).$$

*Proof.* With some obvious modifications the proof is similar to the proof of Theorem 5.8. □

5.4. **Exceptional case of the first kind: Type 3.** By assumption, $\dim \mathcal{M} = 2$. In view of Lemma 4.2 and Lemma 4.6 we can choose a basis $\{\phi_1, \phi_2\}$ in $\mathcal{M}$ satisfying

$$\langle \phi_1, VG_0V1\rangle \neq 0, \quad \langle \phi_1, V(\cdot)\rangle = 0,$$

$$\langle \phi_2, VG_0V1\rangle = 0, \quad \langle \phi_2, V(\cdot)\rangle \neq 0,$$

Together with this basis we choose the dual basis in $\mathcal{N}$, $\{\tilde{\phi}_1, \tilde{\phi}_2\}$. Define the projections

$$U_1^{'} = \langle \cdot, \tilde{\phi}_1\rangle \phi_1, \qquad U_1^{''} = \langle \cdot, \tilde{\phi}_2\rangle \phi_2$$

Then the basic decomposition in $H^{1,-s}$ is $\tilde{U}_0 + U_1^{'} + U_1^{''} + Q_1 = 1$, where as usual we have introduced $\tilde{U}_0 = U_0 Q_0$. We shall use the following lemma.

**Lemma 5.12.** *The following relations are valid.*

$$VU_1^{'} = (U_1^{'})^* V, \;\; VU_1^{''} = (U_1^{''})^* V, \tag{5.18}$$

$$Q_1 U_1^{'} = (U_1^{'})^* Q_1^* = 0, \;\; Q_1 U_1^{''} = (U_1^{''})^* Q_1^* = 0, \tag{5.19}$$

$$Q_1^* V U_1^{'} = 0, \;\; (U_1^{'})^* V Q_1 = 0, \tag{5.20}$$

$$(U_1^{'})^* V(1 + G_0 V)\tilde{U}_0 = 0, \;\; \tilde{U}_0(1 + VG_0)U_1^{'} = 0, \tag{5.21}$$

$$(1 + G_0 V)U_1^{''} = 0, \;\; (U_1^{''})^* V(1 + G_0 V) = 0 \tag{5.22}$$

$$(U_1^{'})^* V(1 + G_0 V)U_1^{'} = 0. \tag{5.23}$$

$$(U_1^{'})^* V(1 + G_0 V)Q_1 = (c_1^{(1)})^* \langle \cdot, V1\rangle \tilde{\phi}_1, \tag{5.24}$$

$$Q_1^* V(1 + G_0 V)U_1^{'} = c_1^{(1)} \langle \cdot, \tilde{\phi}_1\rangle V1, \tag{5.25}$$

$$(U_1^{'})^* VG_1 VU_1^{'} \cdot = 0, \tag{5.26}$$

$$(U_1^{'})^* VG_1 VU_1^{''} \cdot = 0, \quad (U_1^{''})^* VG_1 VU_1^{'} \cdot = 0, \tag{5.27}$$

$$(U_1^{''})^* VG_1 VU_1^{''} \cdot = 2|c_2^{(2)}|^2 \langle \cdot, \tilde{\phi}_2\rangle \tilde{\phi}_2, \tag{5.28}$$

*Proof.* The relations follow from Lemma 3.2, the properties of $\phi_1$ and $\phi_2$, Lemma 4.8 and straightforward computations. □

**Lemma 5.13.** *Assume that zero is an exceptional point of the first kind for $H$ (Type 3). Let Assumption 3.1 with $\beta > 7$ be satisfied and let $s$ satisfy $7/2 < s < \beta - 7/2$. Let $e_1 = 1/c_1^{(1)}$ and $e_2 = 1/c_2^{(2)}$. For $\zeta \to 0$ we have in $B(1,-s;1,-s)$ the expansion*

$$(1 + R_0(\zeta)V)^{-1} = -i\zeta^{-1/2} D_{-1}^{(1,3)} + D_0^{(1,3)} + O(1),$$

*where*

$$D_{-1}^{(1,3)} = \frac{1}{2}|e_1|^2\langle\cdot,\tilde{\phi}_1\rangle\phi_1 + \frac{1}{2}|e_2|^2\langle\cdot,\tilde{\phi}_2\rangle\phi_2.$$

*and*

$$\begin{aligned}
D_0^{(1,3)} \quad = \quad & \tilde{U}_0 L \tilde{U}_0 - (1/2)|e_1|^2\langle\cdot,\tilde{\phi}_1\rangle\tilde{U}_0 L\tilde{U}_0 G_1\tilde{\phi}_1 \\
& -(1/2)|e_1|^2\langle\cdot,\tilde{\phi}_2\rangle\tilde{U}_0 L\tilde{U}_0 G_1\tilde{\phi}_2 \\
& -(1/4)|e_1|^2|e_2|^2\langle\cdot,\tilde{\phi}_2\rangle\langle VG_2V\phi_2,\phi_1\rangle\phi_1 \\
& -(1/4)|e_2|^4\langle\cdot,\tilde{\phi}_2\rangle\langle VG_2V\phi_2,\phi_2\rangle\phi_2 \\
& +(1/4)|e_1|^4\langle VG_1V\tilde{U}_0L\tilde{U}_0G_1V\phi_1,\phi_1\rangle\langle\cdot,\tilde{\phi}_1\rangle\phi_1 \\
& +(1/4)|e_1|^2|e_2|^2\langle VG_1V\tilde{U}_0L\tilde{U}_0G_1V\phi_2,\phi_1\rangle\langle\cdot,\tilde{\phi}_2\rangle\phi_1 \\
& +(1/4)|e_1|^2|e_2|^2\langle VG_1V\tilde{U}_0L\tilde{U}_0G_1V\phi_1,\phi_2\rangle\langle\cdot,\tilde{\phi}_1\rangle\phi_2 \\
& +(1/4)|e_2|^4\langle VG_1V\tilde{U}_0L\tilde{U}_0G_1V\phi_2,\phi_2\rangle\langle\cdot,\tilde{\phi}_2\rangle\phi_2 \\
& +\nu e_1^*\langle\cdot,\tilde{\phi}_1\rangle 1 + \nu e_1\langle\cdot,V1\rangle\phi_1 \\
& -\nu e_1^*\langle\cdot,\tilde{\phi}_1\rangle\tilde{U}_0L\tilde{U}_0(1+G_0V)1 \\
& -\nu e_1\langle V(1+G_0V)\tilde{U}_0L\tilde{U}_0\cdot,1\rangle\phi_1 \\
& -(\nu/2)|e_1|^2e_1\langle\cdot,\tilde{\phi}_1\rangle\langle VG_1V\phi_1,1\rangle\phi_1 \\
& -(\nu/2)|e_2|^2e_1\langle\cdot,\tilde{\phi}_2\rangle\langle VG_1V\phi_2,1\rangle\phi_1 \\
& -(\nu/2)|e_2|^2e_1^*\langle\cdot,\tilde{\phi}_1\rangle\langle VG_1V1,\phi_2\rangle\phi_2 \\
& -(\nu/4)|e_1|^2|e_2|^2\langle\cdot,\tilde{\phi}_1\rangle\langle VG_2V\phi_1,\phi_2\rangle\phi_2 \\
& +(\nu/2)|e_1|^2e_1\langle V(1+G_0V)\tilde{U}_0L\tilde{U}_0G_1V\phi_1,1\rangle\langle\cdot,\tilde{\phi}_1\rangle\phi_1 \\
& +(\nu/2)|e_1|^2e_1^*\langle VG_1V\tilde{U}_0L\tilde{U}_0(1+G_0V)1,\phi_1\rangle\langle\cdot,\tilde{\phi}_1\rangle\phi_1 \\
& +(\nu/2)|e_2|^2e_1\langle V(1+G_0V)\tilde{U}_0L\tilde{U}_0G_1V\phi_2,1\rangle\langle\cdot,\tilde{\phi}_2\rangle\phi_1 \\
& +(\nu/2)|e_2|^2e_1^*\langle VG_1V\tilde{U}_0L\tilde{U}_0(1+G_0V)1,\phi_2\rangle\langle\cdot,\tilde{\phi}_1\rangle\phi_2 \\
& -\nu^2|e_1|^2\langle V(1+G_0V)1,1\rangle\langle\cdot,\tilde{\phi}_1\rangle\phi_1 \\
& +\nu^2|e_1|^2\langle V(1+G_0V)\tilde{U}_0L\tilde{U}_0(1+G_0V)1,1\rangle\langle\cdot,\tilde{\phi}_1\rangle\phi_1.
\end{aligned}$$

*Proof.* The set-up needed to use Lemma 5.1 is as follows. The spaces are $\mathcal{X} = \mathcal{Y} = H^{1,-s}$,

$$\mathbf{X} = \tilde{U}_0\mathcal{X} \oplus Q_1\mathcal{X} \oplus U_1^{'}\mathcal{X} \oplus U_1^{''}\mathcal{X},$$

$$\mathbf{Y} = \tilde{U}_0\mathcal{X} \oplus Q_1^* H^{-1,s} \oplus (U_1^{'})^* H^{-1,s} \oplus (U_1^{''})^* H^{-1,s}$$

and the operators are

$$A = 1 + R_0(\zeta)V, \quad B = (\tilde{U}_0, \zeta^{1/4}Q_1, \zeta^{-1/4}U_1', \zeta^{-1/4}U_1''),$$

$$C = \begin{pmatrix} \tilde{U}_0 \\ \zeta^{1/4}Q_1^*V \\ \zeta^{-1/4}(U_1')^*V \\ \zeta^{-1/4}(U_1'')^*V \end{pmatrix}.$$

From here the lemma is proven in a similar way to Lemma 5.2; we use Lemma 5.12. We omit the tedious details. □

The main result for the full resolvent is given in the following theorem.

**Theorem 5.14.** *Assume that zero is an exceptional point of first kind for $H$ (Type 3). Let Assumption 3.1 be satisfied. Assume $\beta > 7$ and let $s$ satisfy $7/2 < s < \beta - 7/2$. For $\zeta \to 0$ we have in $B(-1,s;1,-s)$ the expansion*

$$R(\zeta) = i\zeta^{-1/2}\frac{1}{2|c_1^{(1)}|^2}\langle\cdot,\phi_1\rangle\phi_1 + i\zeta^{-1/2}\frac{1}{2|c_2^{(2)}|^2}\langle\cdot,\phi_2\rangle\phi_2 + O(1). \quad (5.29)$$

*Proof.* Except for a few modifications, the proof is similar to the proof of Theorem 5.8. The details are omitted. □

5.5. **Exceptional case of the second kind.** We assume that zero is an exceptional point of the second kind for $H$. By definition $U_1 = 0$ and $U_2 \neq 0$. Hence, the starting point of the analysis is the decomposition $Q_1 + U_2 + U_0Q_0 = 1$. It follows from the definitions of $U_0$, $U_2$, $Q_1$ and $Q_0$ and Lemma 4.8 that we decompose using three commuting projections. As usual, let $\tilde{U}_0 = U_0Q_0$.

**Lemma 5.15.** *The following relations hold.*

$$Q_1^*VG_{-1}VU_2 = 0, \quad U_2^*VG_{-1}VQ_1 = 0, \quad (5.30)$$

$$(1 + G_0V)U_2 = 0, \quad U_2^*V(1 + G_0V) = 0, \quad (5.31)$$

$$\tilde{U}_0VG_1VU_2 = 0, \quad U_2^*VG_1V\tilde{U}_0 = 0, \quad (5.32)$$

$$U_2^*VG_1VU_2 = 0. \quad (5.33)$$

*Proof.* The relations follow from straightforward computations using, in particular the definition of $U_2$ and Lemma 4.6. □

Furthermore, we need the following result.

**Lemma 5.16.** *We have that $U_2^*VG_2VU_2 = -U_2^*U_2$, and this operator is invertible in $B(U_2H^{1,-s}, U_2^*H^{-1,s})$. Moreover, $U_2(U_2^*U_2)^{-1}U_2^* = P_0$.*

*Proof.* Let $\{\psi_j\}$ be an $L^2$-orthonormal basis for $U_2 H^{1,-s} = P_0 H^{0,0}$. Let $\{\chi_j\}$ be the dual basis in $U_2^* H^{-1,s}$. Then $U_2 = \sum_j \langle \cdot, \chi_j \rangle \psi_j$, $U_2^* = \sum_j \langle \cdot, \psi_j \rangle \chi_j$ and $P_2 = \sum_j \langle \cdot, \psi_j \rangle \psi_j$. Note that $U_2$ actually maps in $H^{1,\frac{1}{2}-0}$ and that $U_2^*$ extends to $H^{-1,\frac{1}{2}+0}$. Thus $U_2^* U_2$ makes sense. Lemma 4.7 and $P_0 U_2 = U_2$ imply $U_2^* V G_2 V U_2 = -U_2^* U_2$. Furhermore, a computation shows that $U_2^* U_2 = \sum_j \langle \cdot, \chi_j \rangle \chi_j$. Hence, $U_2 (U_2^* U_2)^{-1} U_2^* = P_0$. □

Then we can establish the following result.

**Lemma 5.17.** *Assume that zero is an exceptional point of second kind for $H$. Let Assumption 3.1 be satisfied. Assume that $\beta > 9$ and let $s$ satisfy $9/2 < s < \beta - 9/2$. For $\zeta \to 0$ we have in $B(1,-s;1,-s)$ the expansion*

$$(1 + R_0(\zeta) V)^{-1} = -\zeta^{-1} D_{-2}^{(2)} - i\zeta^{-1/2} D_{-1}^{(2)} + O(1) \tag{5.34}$$

*with*

$$D_{-2}^{(2)} = -P_0 V, \tag{5.35}$$

$$D_{-1}^{(2)} = -P_0 V G_3 V P_0 V - 2\nu^2 \langle V G_3 V P_0 V \cdot, 1 \rangle P_0 V G_1 V 1. \tag{5.36}$$

*Proof.* We only give the basic set-up. To apply Lemma 5.1 we need the definitions

$$\mathcal{X} = \mathcal{Y} = H^{1,-s},$$

$$\mathbf{X} = \tilde{U}_0 \mathcal{X} \oplus Q_1 \mathcal{X} \oplus U_2 \mathcal{X}, \quad \mathbf{Y} = \tilde{U}_0 \mathcal{X} \oplus Q_1^* H^{-1,s} \oplus U_2^* H^{-1,s},$$

$$A = 1 + R_0(\zeta) V, \quad B = (\tilde{U}_0, \zeta^{1/4} Q_1, \zeta^{-1/2} U_2),$$

$$C = \begin{pmatrix} \tilde{U}_0 \\ \zeta^{1/4} Q_1^* V \\ \zeta^{-1/2} U_2^* V \end{pmatrix}.$$

From here the proof follows the pattern of the proof of Lemma 5.2. The actual computations are quite lengthy and tedious. Details are omitted. □

The main result for the resolvent of $H$ is given in the following theorem.

**Theorem 5.18.** *Assume that zero is an exceptional point of second kind for $H$. Let Assumption 3.1 be satisfied. Assume $\beta > 11$ and let $s$ satisfy $11 < s < \beta - 11/2$. For $\zeta \to 0$ we have in $B(-1,s;1,-s)$ the expansion*

$$R(\zeta) = -\zeta^{-1} B_{-2}^{(2)} - i\zeta^{-1/2} B_{-1}^{(2)} + O(1)$$

*with*

$$B_{-2}^{(2)} = P_0, \tag{5.37}$$

$$B_{-1}^{(2)} = P_0VG_3VP_0 + 2\nu^2\langle VG_3VP_0\cdot, 1\rangle P_0VG_1V1. \tag{5.38}$$

*Proof.* It follows from Lemma 5.17, (2.3) and the second resolvent equation, in combination with the relations $\tilde{U}_0 1 = 0$ and $P_0V1 = 0$. $\square$

5.6. **Exceptional point of the third kind.** We collect the main results in the subcases 1, 2 and 3. No proofs are given. For $(1 + R_0(\zeta)V)^{-1}$ we have the following three lemmas.

**Lemma 5.19.** *Assume that zero is an exceptional point of the third kind for $H$ (Type 1). Let Assumption 3.1 with $\beta > 9$ be satisfied. Let $s$ satisfy $9/2 < s < \beta - 9/2$. For $\zeta \to 0$ we have in $B(1, -s; 1, -s)$ the expansion*

$$(1 + R_0(\zeta)V)^{-1} = -\zeta^{-1}D_{-2}^{(3,1)} - i\zeta^{-1/2}D_{-1}^{(3,1)} + O(1),$$

*where*

$$B_{-2}^{(3,1)} = -P_0V,$$

$$\begin{aligned}
D_{-1}^{(3,1)} \quad = \quad & -\frac{d_1}{2\nu^2}\langle\cdot, \tilde{\phi}_1\rangle\phi_1 - \frac{(c_1^{(1)})^* d_1}{\nu}\langle VG_1VP_0V\cdot, 1\rangle\phi_1 \\
& -\frac{d_1}{2\nu^2}\langle G_2VP_0V\cdot, \tilde{\phi}_1\rangle\phi_1 - \frac{c_1^{(1)} d_1}{\nu}\langle\cdot, \tilde{\phi}_1\rangle P_0VG_1V1 \\
& -\frac{d_1}{2\nu^2}\langle\cdot, \tilde{\phi}_1\rangle P_0VG_2\tilde{\phi}_1 - P_0VG_3VP_0V \\
& +2d_1|c_2^{(1)}|^2\langle G_1VP_0V\cdot, V1\rangle P_0VG_1V1 \\
& -\frac{(c_1^{(1)})^* d_1}{\nu}\langle G_1VP_0V\cdot, V1\rangle P_0VG_2\tilde{\phi}_1 \\
& -\frac{c_1^{(1)} d_1}{\nu}\langle G_2VP_0V\cdot, \tilde{\phi}_1\rangle P_0VG_1V1 \\
& -\frac{d_1}{2\nu^2}\langle G_2VP_0V\cdot, \tilde{\phi}_1\rangle P_0VG_2\tilde{\phi}_1.
\end{aligned}$$

**Lemma 5.20.** *Assume that zero is an exceptional point of the third kind for $H$ (Type 2). Let Assumption 3.1 with $\beta > 9$ be satisfied. Let $s$ satisfy $7/2 < s < \beta - 7/2$. For $\zeta \to 0$ we have in $B(1, -s; 1, -s)$ the expansion*

$$(1 + R_0(\zeta)V)^{-1} = -\zeta^{-1}D_{-2}^{(3,2)} - i\zeta^{-1/2}D_{-1}^{(3,2)} + O(1),$$

*where*

$$B_{-2}^{(3,2)} = -P_0V,$$

$$\begin{aligned} D_{-1}^{(3,2)} &= -\frac{d_2}{2\nu^2}\langle\cdot,\tilde{\phi}_2\rangle\phi_2 - P_0VG_3VP_0V \\ &-\frac{d_2}{2\nu^2}\langle\cdot,\tilde{\phi}_2\rangle P_0VG_2\tilde{\phi}_2 \\ &-\frac{d_2}{2i\nu^2}\langle G_2VP_0V\cdot,\tilde{\phi}_2\rangle\phi_2 \\ &-2\nu^2\langle G_1VP_0V\cdot,V1\rangle P_0VG_1V1 \\ &-\frac{d_2}{2\nu^2}\langle G_2VP_0V\cdot,\tilde{\phi}_2\rangle P_0VG_2\tilde{\phi}_2. \end{aligned}$$

**Lemma 5.21.** *Assume that zero is an exceptional point of the third kind for $H$ (Type 3). Let Assumption 3.1 with $\beta > 9$ be satisfied. Let $s$ satisfy $9/2 < s < \beta - 9/2$, $e_1 = 1/c_1^{(1)}$ and $e_2 = 1/c_2^{(2)}$. For $\zeta \to 0$ we have in $B(1,-s;1,-s)$ the expansion*

$$(1 + R_0(\zeta)V)^{-1} = \zeta^{-1}P_0 - i\zeta^{-1/2}D_{-1}^{(3,3)} + O(1),$$

*where*

$$\begin{aligned} D_{-1}^{(3,3)} &= (1/2)|e_1|^2\langle\cdot,\tilde{\phi}_1\rangle\phi_1 + (1/2)|e_2|^2\langle\cdot,\tilde{\phi}_2\rangle\phi_2 \\ &+\nu e_1\langle VG_1VP_0V\cdot,1\rangle\phi_1 + (1/2)|e_1|^2\langle G_2VP_0V\cdot,\tilde{\phi}_1\rangle\phi_1 \\ &+(1/2)|e_2|^2\langle G_2VP_0V\cdot,\tilde{\phi}_2\rangle\phi_2 + \nu e_1^*\langle\cdot,\tilde{\phi}_1\rangle P_0VG_1V1 \\ &+(1/2)|e_1|^2\langle\cdot,\tilde{\phi}_1 P_0VG_2V\phi_1 + |e_2|^2\langle\cdot,\tilde{\phi}_2\rangle P_0VG_2\tilde{\phi}_2 \\ &-|e_2|^2P_0VG_3VP_0V + \nu e_1\langle VG_1VP_0V\cdot,1\rangle P_0VG_2\tilde{\phi}_1 \\ &+\nu e_1^*\langle G_2VP_0V\cdot,\tilde{\phi}_1\rangle P_0VG_1V1 \\ &+(1/2)|e_1|^2\langle G_2VP_0V\cdot,\tilde{\phi}_1\rangle P_0VG_2\tilde{\phi}_1 \\ &+|e_2|^2\langle G_2VP_0V\cdot,\tilde{\phi}_2\rangle P_0VG_2\tilde{\phi}_2. \end{aligned}$$

Due to the complexity of the computations, we only give the leading cofficient in the main result for the resolvent:

**Theorem 5.22.** *Assume that zero is an exceptional point of third kind for $H$ (Type 1,2 or 3). Let Assumption 3.1 be satisfied. Assume $\beta > 9$ and let $s$ satisfy $9/2 < s < \beta - 9/2$. For $\zeta \to 0$ we have in $B(-1,s;1,-s)$ the expansion*

$$R(\zeta) = \zeta^{-1}P_0 + O(\zeta^{-1/2}).$$

Thus, the leading coefficients for the three subcases are identical as expected. Obviously, if we include the next term in the expansion, the coefficients are different in the three subcases.

## 6. The on-shell scattering matrix at low energies

Under Assumption 3.1 with $\beta > 1$ the results of [9] hold for $H$. Hence the wave operators $W_\pm$ exist for the pair of operators $(H, H_0)$ and the wave operators are strongly complete, viz.

$$\mathrm{Ran}\,(W_+) = \mathrm{Ran}\,(W_-) = \mathcal{H}_{ac}(H), \qquad \sigma_{sc}(H) = \emptyset,$$

In particular, the scattering matrix $S(\lambda)$ has the following representation:

$$S(\lambda) = 1 - \pi i \lambda^{-1/2} \gamma_0(\lambda^{1/2}) \mathcal{F} V (1 + R_0(\lambda + i0)V)^{-1} \mathcal{F}^* \gamma_0(\lambda^{1/2})^*, \tag{6.1}$$

where $\mathcal{F}$ is the Fourier transform and $\gamma_0(\mu)$ is the trace operator defined by

$$\gamma_0(\mu) f = \begin{pmatrix} f(\mu^{1/2}) \\ f(-\mu^{1/2}) \end{pmatrix}, \ f \in H^{s,0}(\mathbf{R}_\xi),\ s > 1/2,\ \mu = \xi^2.$$

(A proof of (6.1) can be found in [11, Appendix A]). The scattering matrix is a unitary operator in $B(\mathbf{C}^2)$. In this section we derive asymptotic expansions for the scattering matrix $S(\lambda)$ for the pair of operators $(H, H_0)$.

To derive asymptotic expansions for $S(\lambda)$ we need expansions for the operators $\gamma_0(\lambda^{1/2})\mathcal{F}$ and $\mathcal{F}^*\gamma_0(\lambda^{1/2})^*$. Formally, we have that

$$\gamma_0(\lambda^{1/2})\mathcal{F} = \sum_{j=0}^{\infty} (i\lambda^{1/2})^j \Gamma_j, \tag{6.2}$$

where

$$\Gamma_j \ : \ (2\pi)^{-1/2} (j!)^{-1} \begin{pmatrix} (-x)^j \\ x^j \end{pmatrix}.$$

This follows from a formal expansion of

$$\gamma_0(\lambda^{1/2})\mathcal{F} \ : \ (2\pi)^{-1/2} \begin{pmatrix} \exp(-i\lambda^{1/2}x) \\ \exp(i\lambda^{1/2}x) \end{pmatrix}.$$

We see that

$$\Gamma_j \in B(L^{2,s}(\mathbf{R}), \mathbf{C}^2),\ \ s > j + 1/2.$$

The expansion in (6.2) is valid as $\lambda \downarrow 0$ in the sense that if $\gamma_0(\lambda^{1/2})\mathcal{F}$ is approximated by the finite series up to $j = k$, $k$ being the greatest integer satisfying $s > k + 1/2$, then the remainder is $\mathrm{o}(\lambda^{k/2})$ in the norm of $B(L^{2,s}(\mathbf{R}), \mathbf{C}^2)$. We are ready to give:

**Theorem 6.1.** *Assume that zero is a regular point for $H$. Assume $\beta > 5$. In the norm of $B(\mathbf{C}^2)$ we have as $\lambda \downarrow 0$ the expansion*

$$S(\lambda) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + o(1) \tag{6.3}$$

*Proof.* The assertion follows from Lemma 5.2, (6.1) and (6.2) in conjunction with the relations $K\Gamma_0^* = 0$ and $\Gamma_0 V K = 0$. □

*Remark* 6.2. For local potentials, the leading coefficient in Theorem 6.1 agrees with the results, obtained by other methods, in [2], [3] and [15]. In the limit $\lambda \downarrow 0$ we have total reflection.

**Theorem 6.3.** *Assume that zero is an exceptional point for $H$ of first kind (Type 1). Assume $\beta > 9$. In the norm of $B(\mathbf{C}^2)$ we have for $\lambda \downarrow 0$ the expansion*

$$S(\lambda) = S_0^{(1,1)} + o(1) \tag{6.4}$$

*with*

$$S_0^{(1,1)} = \begin{pmatrix} \frac{|c_1^{(1)}|^2 - |c_2^{(1)}|^2}{|c_1^{(1)}|^2 + |c_2^{(1)}|^2} & \frac{-2c_1^{(1)}(c_2^{(1)})^*}{|c_1^{(1)}|^2 + |c_2^{(1)}|^2} \\ \\ \frac{2c_1^{(1)}(c_2^{(1)})^*}{|c_1^{(1)}|^2 + |c_2^{(1)}|^2} & \frac{|c_1^{(1)}|^2 - |c_2^{(1)}|^2}{|c_1^{(1)}|^2 + |c_2^{(1)}|^2} \end{pmatrix}. \tag{6.5}$$

*Proof.* In view of (6.2), Lemma 5.6 and (6.1) we have that

$$S(\lambda) = \lambda^{-1} S_{-2}^{(1,1)} + \lambda^{-1/2} S_{-1}^{(1,1)} + S_0^{(1,1)} + \mathrm{O}(\lambda^{1/2}),$$

where the coefficients $S_{-2}^{(1,1)}$, $S_{-1}^{(1,1)}$ and $S_0^{(1,1)}$ are given by

$$S_{-2}^{(1,1)} = -i\pi \Gamma_0 V D_{-1}^{(1,1)} \Gamma_0^*,$$

$$S_{-1}^{(1,1)} = \pi \left( \Gamma_1 V D_{-1}^{(1,1)} \Gamma_0^* - i\Gamma_0 V D_0^{(1,1)} \Gamma_0^* - \Gamma_0 V D_{-1}^{(1,1)} \Gamma_1^* \right). \tag{6.6}$$

and

$$\begin{aligned} S_0^{(1,1)} &= 1 - \pi \Gamma_0 V D_0^{(1,1)} \Gamma_1^* - i\pi \Gamma_1 V D_{-1}^{(1,1)} \Gamma_1^* + \\ &+ \pi \Gamma_0 V D_1^{(1,1)} \Gamma_0^* + \pi \Gamma_1 V D_0^{(1,1)} \Gamma_0^* \\ &+ \pi i \Gamma_2 V D_{-1}^{(1,1)} \Gamma_0^* + i\pi \Gamma_0 V D_{-1}^{(1,1)} \Gamma_2^*. \end{aligned} \tag{6.7}$$

We show that $S_{-2}^{(1,1)} = S_{-1}^{(1,1)} = 0$. Since $\langle 1, \tilde{\phi}_1 \rangle = 0$ implies that $\Gamma_0 \tilde{\phi}_1 = 0$, we infer that $S_{-2}^{(1,1)} = 0$. Via the relations $\Gamma_0 \tilde{\phi}_1 = 0$ and $\tilde{U}_0 \Gamma_0^* = 0$ it follows that $S_{-1}^{(1,1)} = 0$. Finally, we turn to (6.7). Due to $\Gamma_0 \tilde{\phi}_1 = 0$, we find that $\Gamma_2 V D_{-1}^{(1,1)} \Gamma_0^* = 0$ and, likewise, $i\pi \Gamma_0 V D_{-1}^{(1,1)} \Gamma_2^* = 0$. Obviously,

$$-i\pi \Gamma_1 V D_{-1}^{(1,1)} \Gamma_1^* = \frac{\pi d_1}{2\nu^2} \langle \cdot, \Gamma_1 \tilde{\phi}_1 \rangle \Gamma_1 \tilde{\phi}_1. \tag{6.8}$$

Furthermore, we have that

$$\pi \Gamma_1 V D_0^{(1,1)} \Gamma_0^* = -\frac{\pi (c_1^{(1)})^* d_1}{\nu} \langle \cdot, \Gamma_0 V 1 \rangle \Gamma_1 \tilde{\phi}_1. \tag{6.9}$$

Via $\Gamma_0\tilde{\phi}_1 = 0$ and $\Gamma_0 V\tilde{U}_0 = \Gamma_0 V Q_0 U_0 = 0$, we find that

$$-\pi\Gamma_0 V D_0^{(1,1)}\Gamma_1^* = \frac{\pi c_1^{(1)} d_1}{\nu}\langle\cdot, \Gamma_1\tilde{\phi}_1\rangle\Gamma_0 V1. \tag{6.10}$$

Next, we rewrite the term $\pi\Gamma_0 V D_1^{(1,1)}\Gamma_0^*$ via the explicit expression for $D_1^{(1,1)}$ (see [11] for details). Using that $\Gamma_0\tilde{\phi}_1 = 0$, $\tilde{U}_0\Gamma_0^* = 0$ and $\Gamma_0 V\tilde{U}_0 = 0$, we find that

$$\pi\Gamma_0 V D_1^{(1,1)}\Gamma_0^* = 2\pi d_1|c_2^{(1)}|^2\langle\cdot, \Gamma_0 V1\rangle\Gamma_0 V1 \tag{6.11}$$

Adding this by (6.8), (6.9) and (6.10) yields

$$S_0^{(1,1)} = 1 + 2\pi d_1|c_2^{(1)}|^2\langle\cdot, \Gamma_0 V1\rangle\Gamma_0 V1 + \frac{\pi d_1}{2\nu^2}\langle\cdot, \Gamma_1\tilde{\phi}_1\rangle\Gamma_1\tilde{\phi}_1 +$$

$$+\frac{\pi c_1^{(1)} d_1}{\nu}\langle\cdot, \Gamma_1\tilde{\phi}_1\rangle\Gamma_0 V1 - \frac{\pi(c_1^{(1)})^* d_1}{\nu}\langle\cdot, \Gamma_0 V1\rangle\Gamma_1\tilde{\phi}_1. \tag{6.12}$$

Finally, we rewrite this to obtain (6.5). (Here we use that $(c_1^{(1)})^* c_2^{(1)} = c_1^{(1)}(c_2^{(1)})^*$, since $\phi_1$ is unique up to multiplicative constants). This completes the proof. □

*Remark* 6.4. For local potentials, the result in Theorem 6.3 agrees with [1, Eqn. (4.15)]; the result in [1] has been obtained by combining Jost functions techniques with Fredholm methods, hence the entire approach is quite different from the present. More importantly, the method in [1] does not allow one to treat the abstract short-range potential $V$.

In a similar way we obtain the following theorems.

**Theorem 6.5.** *Assume that zero is an exceptional point for $H$ of first kind (Type 2). Assume $\beta > 9$. In the norm of $B(\mathbf{C}^2)$ we have as $\lambda \downarrow 0$ the expansion*

$$S(\lambda) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + o(1). \tag{6.13}$$

**Theorem 6.6.** *Assume that zero is an exceptional point for $H$ of first kind (Type 3). Assume $\beta > 9$. In the norm of $B(\mathbf{C}^2)$ we have as $\lambda \downarrow 0$ the expansion*

$$S(\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + o(1). \tag{6.14}$$

**Theorem 6.7.** *Assume that zero is an exceptional point for $H$ of second kind. Assume $\beta > 13$. In the norm of $B(\mathbf{C}^2)$ we have as $\lambda \downarrow 0$ the expansion*

$$S(\lambda) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + o(1). \tag{6.15}$$

Theorems 6.1-6.7 treat the most significant cases. Results on the scattering matrix when zero is an exceptional point of the third kind for $H$ (three subcases) are omitted, since the computations are extremely tedious and lengthy.

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