SPECTRAL PROPERTIES AT A THRESHOLD FOR TWO-CHANNEL HAMILTONIANS. I. ABSTRACT THEORY

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ABSTRACT. Spectral properties at thresholds are investigated for two-channel Hamiltonians in various, mostly fairly 'singular' settings. In an abstract framework we deduce asymptotic expansions of the resolvent as the spectral parameter tends to a threshold. The results are based on given asymptotic expansions of the component Hamiltonians. Applications to scattering theory are given in a companion paper.

1. INTRODUCTION

In the first of a series of papers we study spectral properties at thresholds of two-channel Hamiltonians of the form $\boldsymbol{H} = \boldsymbol{H}_{diag} + \boldsymbol{V}_{off}$, where

$$\boldsymbol{H}_{diag} = \begin{pmatrix} H_a & 0 \\ 0 & H_b \end{pmatrix}, \quad \boldsymbol{V}_{off} = \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix} \tag{1.1}$$

act on the Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$. We assume that H_a and H_b are self-adjoint operators in $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. Moreover we assume that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{H}_a)$, the space of bounded operators, and require $V_{ba} = V_{ab}^*$.

Due to the diagonal structure of $\boldsymbol{H}_{diag}$ its spectrum can be an arbitrary combination of those of H_a and H_b. There are several possible situations, and we have only treated some of them in detail. A case of particular interest is the following. We assume $\sigma(H_a) = \sigma_{\mathrm{ac}}(H_a) = [\lambda, \infty)$ and $\sigma_{\mathrm{ac}}(H_b) = [\lambda_1, \infty)$ for some $\lambda_1 > \lambda$. Furthermore, we assume that λ is an isolated eigenvalue of H_b with eigenprojection P_b. Thus $\boldsymbol{H}_{diag}$ has an eigenvalue embedded at the threshold λ. Our aim is to derive an asymptotic expansion of the resolvent $\boldsymbol{R}(\zeta) = (\boldsymbol{H} - \zeta)^{-1}$ of $\boldsymbol{H}$ as the spectral parameter ζ tends to the threshold λ. To obtain results we require some a priori information on the threshold of H_a. More precisely, let $\mathcal{K}_a$ be a Hilbert space, which is densely and continuously

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embedded in $\mathcal{H}_a$. Motivated by known results for Schrödinger operators [6, 4, 15, 1], we assume the existence of an expansion, valid in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$,

$$R_a(\zeta) = G_0 + i(\zeta-\lambda)^{1/2}G_1 - (\zeta-\lambda)G_2 - i(\zeta-\lambda)^{3/2}G_3 + \mathrm{O}(|\zeta-\lambda|^2) \tag{1.2}$$

as $\zeta \to \lambda$, $\zeta \in \mathbf{C}\setminus[\lambda,\infty)$. This type of asymptotic expansion is known to hold generically for a Schrödinger operator $-\Delta + V(x)$ on $L^2(\mathbf{R}^d)$ for d odd, provided $V(x)$ decays sufficiently rapidly.

Additional assumptions on the potentials V_{ab} and V_{ba} are necessary. Suppose that $V_{ab} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{K}_a)$, the compact operators, and that the operator $P_bV_{ba}G_0V_{ab}P_b$ is strictly positive and invertible in $\mathcal{B}(P_b\mathcal{H}_b)$.

Under the above-mentioned assumptions the following result holds, see Theorem 3.10. As $|\zeta-\lambda| \to 0$, $\zeta \in \mathbf{C}\setminus[\lambda,\infty)$, the resolvent of $\boldsymbol{H}$ has an asymptotic expansion

$$\boldsymbol{R}(\zeta) = \boldsymbol{R}_0 + i(\zeta-\lambda)^{1/2}\boldsymbol{R}_1 + \mathrm{O}(|\zeta-\lambda|), \tag{1.3}$$

valid in the norm topology of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$. This result holds generically (see Remark 3.18(b)). In Section 3 we give a number of similar results in various settings. The coefficients depend on the concrete setting. Explicit expressions are given for the first few coefficients.

The result (1.3) is obtained by using the asymptotic expansion (1.2) in combination with the Feshbach formula and a technique based on factoring out the identity plus a finite rank operator. The latter technique was pioneered by Vaĭnberg [19] and later used by Murata [15] in a context resembling ours.

We emphasize that although most of the settings have a 'singular' nature, generically the singular terms (negative powers in $(\zeta-\lambda)^{1/2}$) cancel. Consequently, no singularities appear in the expansion (1.3). In particular, the resolvent has a well-defined limit $\boldsymbol{R}_0$ at the threshold point in the norm topology of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$. Recently, the latter feature has been subject to a thorough analysis in the paper [7] in which it is shown that under small off-diagonal perturbations the embedded eigenvalue λ of $\boldsymbol{H}_{diag}$ never moves into the continuous spectrum. Applications to the Friedrichs model and Schrödinger operators with confined channels are given. We refer to the paper for details.

The objective is different here. Indeed, the companion paper [13] is devoted to scattering theory for pairs of two-channel Hamiltonians with Schrödinger operators as component Hamiltonians. As an application of the expansions for the resolvent deduced in the present paper we derive asymptotic expansions of the S-matrix as the energy parameter tends to a threshold.

There is a vast literature on 2×2 operator-valued matrices, e.g. in system theory (see e.g. [2]) and in semigroup theory (see e.g. [3]). Most notably in this context is the substantial number of questions of a general nature which have been answered on spectral theory recently, see e.g. the survey by Tretter [18]. However, the methods therein are not related to ours although some of the questions addressed clearly are, e.g. the appearance of resonances discussed by Mennicken and Motovilov [14]. In this paper we are not imposing assumptions on the Hamiltonians, which make it possible to give a reasonable definition of a resonance, hence we delay the discussion of resonances to a future work.

2. Preliminaries

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T)$. The spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$, respectively. We use standard terminology for the various parts of the spectrum, see for example [16]. The resolvent is $R(\zeta) = (T - \zeta)^{-1}$.

The spaces of bounded and compact operators from a Hilbert $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$ are denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$, respectively. If $\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H}$ we use the notation $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively.

If λ is an isolated eigenvalue of T with associated eigenprojection P, then the reduced resolvent is given as

$$C = \lim_{\zeta \to \lambda} (I - P) R(\zeta), \tag{2.1}$$

and we have the norm convergent expansion

$$R(\zeta) = -\frac{P}{\zeta - \lambda} + \sum_{n=0}^{\infty} (\zeta - \lambda)^n C^{n+1}. \tag{2.2}$$

The expansion is valid for $0 < |\zeta - \lambda| < \delta$ for some small $\delta > 0$. See for example [9, 17].

The Feshbach formula gives a convenient explicit representation of the resolvent $\boldsymbol{R}(\zeta)$ of $\boldsymbol{H}$. There are two variants. We give only one of them. The other version is just an interchange of indices. Define

$$R_a(\zeta) = (H_a - \zeta)^{-1}, \tag{2.3}$$

$$T_b(\zeta) = H_b - \zeta - V_{ba} R_a(\zeta) V_{ab}. \tag{2.4}$$

Then for $\operatorname{Im} \zeta \neq 0$ we have

$$\boldsymbol{R}(\zeta) = \begin{pmatrix} R_a(\zeta) + R_a(\zeta) V_{ab} T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & -R_a(\zeta) V_{ab} T_b(\zeta)^{-1} \\ -T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & T_b(\zeta)^{-1} \end{pmatrix} \tag{2.5}$$

For a complex number $z \in \mathbf{C} \setminus [0, \infty)$ we denote by $z^{1/2}$ the branch of the square root with positive imaginary part.

3. Asymptotic expansions of the resolvent at a threshold

In several different settings asymptotic expansions of the resolvent $\boldsymbol{R}(\zeta)$ are deduced as the spectral parameter ζ tends to a threshold λ.

Assumption 3.1. Let $\lambda \in \sigma(H_a)$.
(i) Assume that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, such that for some $\delta > 0$ we have an asymptotic expansion in the norm of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$, viz.

$$R_a(\zeta) = G_0 + i(\zeta - \lambda)^{1/2} G_1 - (\zeta - \lambda) G_2 - i(\zeta - \lambda)^{3/2} G_3 + \mathrm{O}(|\zeta - \lambda|^2) \tag{3.1}$$

for $|\zeta - \lambda| < \delta$, $\operatorname{Im}\zeta > 0$. Assume furthermore that $G_j = G_j^*$, $j = 0, 1, 2, 3$, as operators in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$.
(ii) Assume that $V_{ab} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{K}_a)$.
(iii) Assume that λ is a simple isolated eigenvalue of H_b, with normalized eigenfunction ψ. Its reduced resolvent is denoted by C_b.

We use the notation $P_b = \langle \cdot, \psi \rangle \psi$ for the eigenprojection. The following real numbers are needed to state the results.

$$\alpha_0 = \langle V_{ba} G_0 V_{ab} \psi, \psi \rangle, \tag{3.2}$$
$$\gamma_0 = \langle V_{ba} G_0 V_{ab} C_b V_{ba} G_0 V_{ab} \psi, \psi \rangle. \tag{3.3}$$

Before treating the case where λ is an eigenvalue of H_b we briefly consider the situation where $\lambda \in \rho(H_b)$. Under this assumption we have that

$$R_b(\zeta) = \sum_{n=0}^{\infty} (\zeta - \lambda)^n C_n \tag{3.4}$$

for $|\zeta - \lambda|$ sufficiently small. The series converges in $\mathcal{B}(\mathcal{H}_b)$, and we have $C_n = R_b(\lambda)^{n+1}$. We have the following lemma.

Lemma 3.2. *Let Assumption 3.1*(i) *and* (ii) *hold for H_a at $\lambda \in \mathbf{R}$ with the expansion (3.1) replaced by*

$$R_a(\zeta) = G_0 + i(\zeta - \lambda)^{1/2} G_1 - (\zeta - \lambda) G_2 + O(|\zeta - \lambda|^{3/2}). \tag{3.5}$$

Assume that $\lambda \in \rho(H_b)$. Then, generically, we have in $\mathcal{B}(\mathcal{H}_b)$ the following asymptotic expansion

$$T_b(\zeta)^{-1} = a_0 + i(\zeta - \lambda)^{1/2} a_1 - (\zeta - \lambda) a_2 + O(|\zeta - \lambda|^{3/2}) \tag{3.6}$$

as $|\zeta - \lambda| \to 0$, $\operatorname{Im}\zeta > 0$, where the coefficients are given by

$$a_0 = L_0, \quad a_1 = L_0 V_{ba} G_1 V_{ab} C_0 L_0, \tag{3.7}$$

$$\begin{aligned} a_2 = {} & L_0 V_{ba} G_2 V_{ab} C_0 L_0 \\ & - L_0 V_{ba} G_0 V_{ab} C_1 L_0 + L_0 (V_{ba} G_1 V_{ab} C_0 L_0)^2. \end{aligned} \tag{3.8}$$

Here $L_0 = (I_b - V_{ba} G_0 V_{ab} C_0)^{-1}$.

Proof. The strategy of the proof is to factor the operator $T_b(\zeta)$ in order to show that the inverse of $T_b(\zeta)$ exists and admits an asymptotic expansion in the norm topology of $\mathcal{B}(\mathcal{H}_b)$ for $|\zeta - \lambda|$ small enough.

In the sequel we always assume at least $|\zeta - \lambda| < \delta$ (with the δ from Assumption 3.1) and $\operatorname{Im}\zeta > 0$. We use the factorization

$$T_b(\zeta) = (I_b - V_{ba}R_a(\zeta)V_{ab}R_b(\zeta))\,(H_b - \zeta). \tag{3.9}$$

The assumption gives the following asymptotic expansion in $\mathcal{B}(\mathcal{H}_b)$.

$$\begin{aligned} &I_b - V_{ba}R_a(\zeta)V_{ab}R_b(\zeta) \\ &\quad = I_b - V_{ba}G_0V_{ab}C_0 - i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_0 \\ &\quad +(\zeta-\lambda)V_{ba}G_2V_{ab}C_0 - (\zeta-\lambda)V_{ba}G_0V_{ab}C_1 \\ &\quad +\mathrm{O}(|\zeta-\lambda|^{3/2}). \end{aligned} \tag{3.10}$$

The operator $V_{ba}G_0V_{ba}C_0$ is compact. Therefore, generically the operator $I_b - V_{ba}G_0V_{ab}C_0$ is invertible. Let L_0 denote its bounded inverse in $\mathcal{K}_b$. Then we may factorize as follows

$$\begin{aligned} &I_b - V_{ba}R_a(\zeta)V_{ab}R_b(\zeta) = \\ &\quad = \big[\big(-i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_0 \\ &\quad +(\zeta-\lambda)V_{ba}G_2V_{ab}C_0 - (\zeta-\lambda)V_{ba}G_0V_{ab}C_1 \\ &\quad +\mathrm{O}(|\zeta-\lambda|^{3/2})\big)\,L_0 + I_b\big]\,(I_b - V_{ba}G_0V_{ab}C_0). \end{aligned} \tag{3.11}$$

In the latter an expansion of the inverse of $[\cdots]$ is obtained by the Neumann series. The result follows immediately from this Neumann series, the factorizations (3.11), (3.9) and the expansion of $R_b(\zeta)$. $\square$

From Lemma 3.2 and the Feshbach formula we immediately obtain the following theorem.

Theorem 3.3. *Let the assumptions in Lemma 3.2 be fulfilled. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$ the asymptotic expansion*

$$\begin{aligned} \boldsymbol{R}(\zeta) = &\begin{pmatrix} G_0 + G_0V_{ab}a_0V_{ba}G_0 & -G_0V_{ab}a_0 \\ -a_0V_{ba}G_0 & a_0 \end{pmatrix} + \\ &+i(\zeta-\lambda)^{1/2}\begin{pmatrix} G_1 + G_0V_{ab}a_0V_{ba}G_1 + G_1V_{ab}a_0V_{ba}G_0 + G_0V_{ab}a_1V_{ba}G_0 & -G_0V_{ab}a_1 - G_1V_{ab}a_0 \\ -a_1V_{ba}G_0 - a_0V_{ba}G_1 & a_1 \end{pmatrix} + O(|\zeta-\lambda|) \end{aligned} \tag{3.12}$$

as $|\zeta - \lambda| \to 0$, $\operatorname{Im}\zeta > 0$.

We now consider the case where λ is an eigenvalue of H_b. Let us first consider the case $\alpha_0 \neq 0$. It is convenient to introduce the following projections in $\mathcal{H}_b$:

$$P_1 = \alpha_0^{-1}\langle\cdot,\psi\rangle V_{ba}G_0V_{ab}\psi, \quad P_0 = I_b - P_1. \tag{3.13}$$

First we show that the transfer function $T_b(\zeta)$ is invertible in a neighborhood of λ and that its inverse admits an asymptotic expansion in this neighborhood.

Lemma 3.4. *Let Assumption 3.1 hold at $\lambda \in \mathbf{R}$. Assume that $\alpha_0 \neq 0$. Then, generically, we have in $\mathcal{B}(\mathcal{H}_b)$ the following asymptotic expansion*

$$T_b(\zeta)^{-1} = b_0 + i(\zeta-\lambda)^{1/2} b_1 + O(|\zeta-\lambda|) \tag{3.14}$$

as $|\zeta-\lambda| \to 0$, $\operatorname{Im}\zeta > 0$, where the coefficients b_0 and b_1 are given by

$$b_0 = C_b P_0 (I_b - V_{ba} G_0 V_{ab} C_b P_0)^{-1}, \tag{3.15}$$

$$\begin{aligned} b_1 = \alpha_0^{-1} & C_b P_0 (I_b - V_{ba} G_0 V_{ab} C_b P_0)^{-1} \\ & \times (\alpha_0 V_{ba} G_1 V_{ab} C_b P_0 + V_{ba} G_1 V_{ab} P_b) \\ & \times (I_b - V_{ba} G_0 V_{ab} C_b P_0)^{-1}. \end{aligned} \tag{3.16}$$

Proof. We follow the strategy in the proof of Lemma 3.2. We begin by making the factorization (3.9). The assumption gives the following asymptotic expansion in $\mathcal{B}(\mathcal{H}_b)$.

$$\begin{aligned} I_b & - V_{ba} R_a(\zeta) V_{ab} R_b(\zeta) \\ & = I_b + \frac{1}{\zeta-\lambda}\alpha_0 P_1 + \frac{i}{(\zeta-\lambda)^{1/2}} V_{ba} G_1 V_{ab} P_b \\ & - (V_{ba} G_0 V_{ab} C_b + V_{ba} G_2 V_{ab} P_b) \\ & - i(\zeta-\lambda)^{1/2} (V_{ba} G_3 V_{ab} P_b + V_{ba} G_1 V_{ab} C_b) \\ & + \mathrm{O}(|\zeta-\lambda|). \end{aligned} \tag{3.17}$$

We have expressed the second term on the right-hand side in terms of the projection P_1. Since

$$\left(I_b + \frac{\alpha_0}{\zeta-\lambda} P_1\right)^{-1} = P_0 + \frac{\zeta-\lambda}{\alpha_0+\zeta-\lambda} P_1,$$

we make the following factorization

$$\begin{aligned} I_b & - V_{ba} R_a(\zeta) V_{ab} R_b(\zeta) \\ & = \Bigg[\Bigg(\frac{i}{(\zeta-\lambda)^{1/2}} V_{ba} G_1 V_{ab} P_b - V_{ba} G_0 V_{ab} C_b \\ & - V_{ba} G_2 V_{ab} P_b - i(\zeta-\lambda)^{1/2} V_{ba} G_3 V_{ab} P_b \\ & - i(\zeta-\lambda)^{1/2} V_{ba} G_1 V_{ab} C_b + (\zeta-\lambda) V_{ba} G_0 V_{ab} C_b^2 \\ & + \mathrm{O}(|\zeta-\lambda|)\Bigg) \left(P_0 + \frac{\zeta-\lambda}{\alpha_0+\zeta-\lambda} P_1\right) + I_b\Bigg] \\ & \times \left(I_b + \frac{\alpha_0}{\zeta-\lambda} P_1\right). \end{aligned} \tag{3.18}$$

Consider $[\cdots]$ in (3.18). Using $P_b P_0 = 0$ we find that

$$\begin{aligned} [\cdots](\alpha_0+\zeta-\lambda) & = \alpha_0 (I_b - V_{ba} G_0 V_{ab} C_b P_0) \\ & + i(\zeta-\lambda)^{1/2} (V_{ba} G_1 V_{ab} P_b - \alpha_0 V_{ba} G_1 V_{ab} C_b P_0) + \mathrm{O}(|\zeta-\lambda|) \end{aligned} \tag{3.19}$$

From Assumption 3.1(ii) it follows that the operator $V_{ba}G_0V_{ab}C_bP_0$ is a compact operator in $\mathcal{H}_b$, hence generically $I_b - V_{ba}G_0V_{ab}C_bP_0$ is invertible and the inverse is bounded in $\mathcal{H}_b$. As a consequence we are able to factor as follows.

$$\begin{aligned}
&\text{Right-hand side of (3.19)}\\
&\quad = \big[I_b + \{i(\zeta-\lambda)^{1/2}\,(\alpha_0 V_{ba}G_1V_{ab}C_bP_0 + V_{ba}G_1V_{ab}P_b)\\
&\quad + \mathrm{O}(|\zeta-\lambda|)\}\,\alpha_0^{-1}(I_b - V_{ba}G_0V_{ab}C_bP_0)^{-1}\big]\\
&\quad \times\alpha_0(I_b - V_{ba}G_0V_{ab}C_bP_0)^{-1}.
\end{aligned}\tag{3.20}$$

For $|\zeta-\lambda|$ small enough, the Neumann series implies that the inverse of the right-hand side of (3.20) has the following expansion:

$$\begin{aligned}
&\text{Inverse of right-hand side of (3.20)}\\
&\quad = \alpha_0^{-1}(I_b - V_{ba}G_0V_{ab}C_bP_0)^{-1} +\\
&\quad +i(\zeta-\lambda)^{1/2}\alpha_0^{-2}(I_b - V_{ba}G_0V_{ab}C_bP_0)^{-1}\\
&\quad \times(\alpha_0V_{ba}G_1V_{ab}C_bP_0 + V_{ba}G_1V_{ab}P_b)\\
&\quad \times(I_b - V_{ba}G_0V_{ab}C_bP_0)^{-1} + \mathrm{O}(|\zeta-\lambda|).
\end{aligned}\tag{3.21}$$

In combination with (3.18), we obtain from (3.21) that

$$\begin{aligned}
&(I_b - V_{ba}R_a(\zeta)V_{ab}R_b(\zeta))^{-1} = P_0(I_b - V_{ba}G_0V_{ab}C_bP_0)^{-1}\\
&\quad +i(\zeta-\lambda)^{1/2}\alpha_0^{-1}P_0(I_b - V_{ba}G_0V_{ab}C_bP_0)^{-1}\,(\alpha_0V_{ba}G_1V_{ab}C_bP_0\\
&\quad +V_{ba}G_1V_{ab}P_b)\,(I_b - V_{ba}G_0V_{ab}C_bP_0^{-1} + \mathrm{O}(|\zeta-\lambda|).
\end{aligned}\tag{3.22}$$

Finally, we obtain the desired expansion (3.14) by using the factorization (3.9), the expansion for $R_b(\zeta)$, (3.22) and the relation $P_bP_0 = 0$. □

Theorem 3.5. *Let Assumption 3.1 hold at $\lambda \in \mathbf{R}$. Assume that $\alpha_0 \neq 0$. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$ the asymptotic expansion*

$$\begin{aligned}
\boldsymbol{R}(\zeta) &= \begin{pmatrix} G_0 + G_0V_{ab}b_0V_{ba}G_0 & -G_0V_{ab}b_0\\ -b_0V_{ba}G_0 & b_0\end{pmatrix} +\\
&+i(\zeta-\lambda)^{1/2}\begin{pmatrix} G_1 + G_0V_{ab}b_0V_{ba}G_1 + G_1V_{ab}b_0V_{ba}G_0 + G_0V_{ab}b_1V_{ba}G_0 & -G_0V_{ab}b_1 - G_1V_{ab}b_0\\ -b_0V_{ba}G_1 - b_1V_{ba}G_0 & b_1\end{pmatrix} + O(|\zeta-\lambda|)
\end{aligned}\tag{3.23}$$

as $|\zeta-\lambda| \to 0$, $\operatorname{Im}\zeta > 0$.

Proof. The result follows immediately from the Feshbach formula (2.5) and Lemma 3.4. □

We now consider the case $\alpha_0 = 0$. Assuming that $\gamma_0 \neq 0$ we introduce the operators

$$J_1 = \langle\cdot,\psi\rangle V_{ba}G_0V_{ab}\psi, \qquad J_0 = I_b - J_1,$$

and the projections

$$\tilde{J}_1 = \gamma_0^{-1}\langle\cdot,\psi\rangle V_{ba}G_0V_{ab}C_bV_{ab}G_0V_{ab}\psi, \qquad \tilde{J}_0 = I_b - \tilde{J}_1.$$

Then we have the following result.

Lemma 3.6. *Let Assumption 3.1 hold at $\lambda \in \mathbf{R}$. Assume that Assumption 3.1(i) is fulfilled with (3.1) replaced by*

$$\begin{aligned} R_a(\zeta) = G_0 + i(\zeta-\lambda)^{1/2}G_1 - (\zeta-\lambda)G_2 - i(\zeta-\lambda)^{3/2}G_3 + \\ +(\zeta-\lambda)^2G_4 + i(\zeta-\lambda)^{5/2}G_5 + O(|\zeta-\lambda|^3) \end{aligned} \tag{3.24}$$

Assume that $\alpha_0 = 0$ and $\gamma_0 \neq 0$. Then, generically, we have in $\mathcal{B}(\mathcal{H}_b)$ the following asymptotic expansion

$$T_b(\zeta)^{-1} = c_0 + i(\zeta-\lambda)^{1/2}c_1 + O(|\zeta-\lambda|) \tag{3.25}$$

as $|\zeta-\lambda| \to 0$, $\operatorname{Im}\zeta > 0$, where the coefficients c_0 and c_1 are given by

$$c_0 = C_b\tilde{J}_0(I_b - V_{ba}G_0V_{ab}C_b\tilde{J}_0)^{-1}, \tag{3.26}$$

$$\begin{aligned} c_1 = \gamma_0^{-1}&C_b\tilde{J}_0(I_b - V_{ba}G_0V_{ab}C_b\tilde{J}_0)^{-1} \\ &\times\Big(V_{ba}G_1V_{ab}P_b + V_{ba}G_1V_{ab}C_bJ_1\tilde{J}_1 \\ &-\gamma_0V_{ba}G_1V_{ab}C_b\tilde{J}_0\Big) \times (I_b - V_{ba}G_0V_{ab}C_b\tilde{J}_0)^{-1}. \end{aligned} \tag{3.27}$$

Proof. The proof follows the same lines as the proof of Lemma 3.4 but due to the assumption $\alpha_0 = 0$ it is more elaborate. Again we start from the factorization (3.9). The assumption gives the following asymptotic expansion in $\mathcal{B}(\mathcal{H}_b)$.

$$\begin{aligned} I_b &- V_{ba}R_a(\zeta)V_{ab}R_b(\zeta) \\ &= I_b + \frac{1}{\zeta-\lambda}J_1 + \frac{i}{(\zeta-\lambda)^{1/2}}V_{ba}G_1V_{ab}P_b \\ &- (V_{ba}G_0V_{ab}C_b + V_{ba}G_2V_{ab}P_b) \\ &-i(\zeta-\lambda)^{1/2}\left(V_{ba}G_3V_{ab}P_b + V_{ba}G_1V_{ab}C_b\right) \\ &+(\zeta-\lambda)\left(V_{ba}G_0V_{ab}C_b^2 + V_{ba}G_2V_{ab}C_b\right. \\ &\left.+V_{ba}G_4V_{ab}P_b\right) + i(\zeta-\lambda)^{3/2}V_{ba}G_5V_{ab}P_b \\ &-i(\zeta-\lambda)^{3/2}V_{ba}G_1V_{ab}C_b^2 + i(\zeta-\lambda)^{3/2}V_{ba}G_3V_{ab}C_b \\ &+\mathrm{O}(|\zeta-\lambda|^2). \end{aligned} \tag{3.28}$$

Since

$$\left(I_b + \frac{1}{\zeta-\lambda}J_1\right)^{-1} = I_b - \frac{1}{\zeta-\lambda}J_1 \tag{3.29}$$

we make the factorization

$$\begin{aligned}
&I_b - V_{ba}R_a(\zeta)V_{ab}R_b(\zeta)\\
&\quad = \Bigg[I_b + \Bigg\{\frac{i}{(\zeta-\lambda)^{1/2}}V_{ba}G_1V_{ab}P_b - V_{ba}G_0V_{ab}C_b\\
&\quad -V_{ba}G_2V_{ab}P_b - i(\zeta-\lambda)^{1/2}V_{ba}G_3V_{ab}P_b\\
&\quad -i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_b + (\zeta-\lambda)V_{ba}G_0V_{ab}C_b^2\\
&\quad +(\zeta-\lambda)V_{ba}G_2V_{ab}C_b + (\zeta-\lambda)V_{ba}G_4V_{ab}P_b\\
&\quad +i(\zeta-\lambda)^{3/2}V_{ba}G_5V_{ab}P_b - i(\zeta-\lambda)^{3/2}V_{ba}G_1V_{ab}C_b^2\\
&\quad +i(\zeta-\lambda)^{3/2}V_{ba}G_3V_{ab}C_b + \mathrm{O}(|\zeta-\lambda|^2)\Bigg\}\left(I_b - \frac{1}{\zeta-\lambda}J_1\right)\Bigg]\\
&\quad \times\left(I_b + \frac{1}{\zeta-\lambda}J_1\right). \qquad (3.30)
\end{aligned}$$

Using $P_bJ_1 = 0$ and the projection $\tilde{J}_1$ we find that $[\cdots]$ on the right-hand side of (3.30) can be written as follows.

$$\begin{aligned}
&[\cdots] \text{ on right-hand side of (3.30)}\\
&\quad = I_b + \frac{\gamma_0}{\zeta-\lambda}\tilde{J}_1 + \frac{i}{(\zeta-\lambda)^{1/2}}V_{ba}G_1V_{ab}P_b + \frac{i}{(\zeta-\lambda)^{1/2}}V_{ba}G_1V_{ab}C_bJ_1\\
&\quad -V_{ba}G_0V_{ab}C_b - V_{ba}G_2V_{ab}P_b - V_{ba}G_0V_{ab}C_b^2J_1 - V_{ba}G_2V_{ab}C_bJ_1\\
&\quad -i(\zeta-\lambda)^{1/2}V_{ba}G_3V_{ab}P_b - i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_b\\
&\quad +i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_b^2J_1 - i(\zeta-\lambda)^{1/2}V_{ba}G_3V_{ab}C_bJ_1\\
&\quad +\mathrm{O}(|\zeta-\lambda|). \qquad (3.31)
\end{aligned}$$

Now, using

$$\left(I_b + \frac{\gamma_0}{\zeta-\lambda}\tilde{J}_1\right)^{-1} = \tilde{J}_0 + \frac{\zeta-\lambda}{\gamma_0+\zeta-\lambda}\tilde{J}_1,$$

the expression in (3.31) can be factorized as follows.

$$\begin{aligned}
&\text{Right-hand side of (3.31)}\\
&\quad = \Bigg[I_b + \Bigg(\frac{i}{(\zeta-\lambda)^{1/2}}V_{ba}G_1V_{ab}P_b + \frac{i}{(\zeta-\lambda)^{1/2}}V_{ba}G_1V_{ab}C_bJ_1\\
&\quad -V_{ba}G_0V_{ab}C_b - V_{ba}G_2V_{ab}P_b - V_{ba}G_0V_{ab}C_b^2J_1 - V_{ba}G_2V_{ab}C_bJ_1\\
&\quad -i(\zeta-\lambda)^{1/2}V_{ba}G_3V_{ab}P_b - i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_b\\
&\quad +i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_b^2J_1 - i(\zeta-\lambda)^{1/2}V_{ba}G_3V_{ab}C_bJ_1\\
&\quad +\mathrm{O}(|\zeta-\lambda|)\Bigg)\times\\
&\quad \times\left(\tilde{J}_0 + \frac{\zeta-\lambda}{\gamma_0+\zeta-\lambda}\tilde{J}_1\right)\Bigg] \times \left(I_b + \frac{\gamma_0}{\zeta-\lambda}\tilde{J}_1\right). \qquad (3.32)
\end{aligned}$$

Consider $[\cdots]$ on the right-hand side of (3.32). Using that $J_1\tilde{J}_0 = 0$ and $P_b\tilde{J}_0 = 0$ we find that

$$\begin{aligned}
&(\gamma_0 + \zeta - \lambda) \times [\cdots] \text{ on right-hand side of (3.32)} \\
&\quad = \gamma_0(I_b - V_{ba}G_0V_{ab}C_b\tilde{J}_0) \\
&\quad +i(\zeta-\lambda)^{1/2}\Big(V_{ba}G_1V_{ab}P_b\tilde{J}_1 + V_{ba}G_1V_{ab}C_bJ_1\tilde{J}_1 \\
&\quad -\gamma_0V_{ba}G_1V_{ab}C_b\tilde{J}_0\Big) + \mathrm{O}(|\zeta-\lambda|). \qquad (3.33)
\end{aligned}$$

The latter expression has the same structure as the expression in (3.19) in the proof of Lemma 3.4. Therefore, we can continue in a similar way as in the proof of Lemma 3.4. We use the relations $J_1\tilde{J}_0 = 0$ and $P_b\tilde{J}_0 = 0$ several times. The details are omitted. □

Theorem 3.7. *Let Assumption 3.1 hold at $\lambda \in \mathbf{R}$ with (3.1) replaced by (3.24). Assume $\alpha_0 = 0$ and $\gamma_0 \neq 0$. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$ the asymptotic expansion*

$$\begin{aligned}
\boldsymbol{R}(\zeta) &= \begin{pmatrix} G_0 + G_0V_{ab}c_0V_{ba}G_0 & -G_0V_{ab}c_0 \\ -c_0V_{ba}G_0 & c_0 \end{pmatrix} + \\
&+i(\zeta-\lambda)^{1/2}\begin{pmatrix} G_1 + G_0V_{ab}c_0V_{ba}G_1 + G_1V_{ab}c_0V_{ba}G_0 + G_0V_{ab}c_1V_{ba}G_0 & -G_0V_{ab}c_1 - G_1V_{ab}c_0 \\ -c_0V_{ba}G_1 - c_1V_{ba}G_0 & c_1 \end{pmatrix} + O(|\zeta-\lambda|) \qquad (3.34)
\end{aligned}$$

as $|\zeta - \lambda| \to 0$, $\operatorname{Im}\zeta > 0$.

Proof. The result follows immediately from the Feshbach formula (2.5) and Lemma 3.6. □

We now consider the case when λ is an isolated eigenvalue of H_b of arbitrary multiplicity. We limit ourselves to discussing the simplest case.

Assumption 3.8. Let parts (i) and (ii) of Assumption 3.1 hold at $\lambda \in \mathbf{R}$ with (3.1) replaced by (3.24). Assume that λ is an isolated eigenvalue of H_b with eigenprojection P_b such that the operator $P_bV_{ba}G_0V_{ab}P_b$ is strictly positive and invertible in $\mathcal{B}(P_b\mathcal{H}_b)$.

Under Assumption 3.8 we define the operators $L_1 = (P_bV_{ba}G_0V_{ab}P_b)^{-1}$, $K_1 = V_{ba}G_1V_{ab}P_b$ and

$$\begin{aligned}
M_1 &= V_{ba}G_0V_{ab}C_b - V_{ba}G_2V_{ab}P_b + V_{ba}G_0V_{ab}C_bV_{ba}G_0V_{ab}P_bL_1P_b \\
&\quad +V_{ba}G_2V_{ab}P_bV_{ba}G_0V_{ab}P_bL_1P_b. \qquad (3.35)
\end{aligned}$$

Then we obtain the following result.

Lemma 3.9. *Let Assumption 3.8 hold at $\lambda \in \mathbf{R}$. Then, generically, we have in $\mathcal{B}(\mathcal{H}_b)$ the following asymptotic expansion*

$$T_b(\zeta)^{-1} = d_0 + i(\zeta-\lambda)^{1/2}d_1 + O(|\zeta-\lambda|) \qquad (3.36)$$

as $|\zeta-\lambda|\to 0$, $\operatorname{Im}\zeta>0$, *where the coefficients* d_0 *and* d_1 *are given by*

$$d_0=(C_b-P_bL_1P_b)(I_b-M_1)^{-1}-C_bV_{ba}G_0V_{ab}P_bL_1P_b(I_b-M_1)^{-1}, \tag{3.37}$$

$$\begin{aligned}
d_1 &= P_bL_1P_b(I_b-M_1)^{-1}V_{ba}G_0V_{ab}C_bV_{ba}G_1V_{ab}P_bL_1P_b(I_b-M_1)^{-1}\\
&-P_bL_1P_b(I_b-M_1)^{-1}V_{ba}G_0V_{ab}C_bV_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_b(I_b-M_1)^{-1}\\
&+C_b(V_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_b-V_{ba}G_1V_{ab}P_bL_1P_b)(I_b-M_1)^{-1}\\
&+C_b(I_b-V_{ba}G_0V_{ab}P_bL_1P_b)(I_b-M_1)^{-1}\\
&\times\{V_{ba}G_1V_{ab}C_b+V_{ba}G_0V_{ab}C_bV_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_b\\
&-V_{ba}G_0V_{ab}C_bV_{ba}G_1V_{ab}P_bL_1P_b-V_{ba}G_1V_{ab}C_bV_{ba}G_0V_{ab}P_bL_1P_b\}\\
&\times(I_b-M_1)^{-1}+P_bL_1P_bK_1P_bL_1P_b(I_b-M_1)^{-1}.
\end{aligned}\tag{3.38}$$

Proof. We start from (3.28) with the modification that we do not introduce J_1. Thus, the second term on the right-hand side of (3.28) has the coefficient $V_{ba}G_0V_{ab}P_b$. Define

$$Y(\zeta)=V_{ba}G_0V_{ab}+i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}$$

and

$$S(\zeta)=I_b+\frac{1}{\zeta-\lambda}Y(\zeta)P_b. \tag{3.39}$$

Hence, $S(\zeta)$ consists of the identity plus the singular terms in (3.28). Introduce also

$$Z(\zeta)=(\zeta-\lambda)P_b+P_bV_{ba}G_0V_{ab}P_b+i(\zeta-\lambda)^{1/2}P_bV_{ba}G_1V_{ab}P_b. \tag{3.40}$$

We use the following abstract result. Let P be a projection in $\mathcal{B}(\mathcal{H})$ and let $X\in\mathcal{B}(\mathcal{H})$. Assume that the operator $P+PXP$ is invertible in $\mathcal{B}(P\mathcal{H})$. Then the operator $I+XP$ is invertible, and we have, with an obvious notation, $(I+XP)^{-1}=I-XP(P+PXP)^{-1}P$. If, in the present situation, we assume that for some ζ the operator $Z(\zeta)$ is invertible in $\mathcal{B}(P_b\mathcal{H}_b)$ then $S(\zeta)$ is invertible, and the inverse is given by

$$S(\zeta)^{-1}=I_b-Y(\zeta)P_bZ(\zeta)^{-1}P_b. \tag{3.41}$$

Consider $Z(\zeta)$ first and bear in mind that $L_1=(P_bV_{ba}G_0V_{ab}P_b)^{-1}$ and $K_1=P_bV_{ba}G_1V_{ab}P_b$. Under Assumption 3.8 and for $|\zeta-\lambda|$ small enough, the Neumann series yields that

$$\begin{aligned}
Z(\zeta)^{-1} &= P_bL_1-i(\zeta-\lambda)^{1/2}P_bL_1P_bK_1P_bL_1\\
&-(\zeta-\lambda)P_bL_1P_bL_1-(\zeta-\lambda)P_bL_1P_bK_1P_bL_1P_bK_1P_bL_1\\
&+i(\zeta-\lambda)^{3/2}L_1P_bL_1P_bK_1P_bL_1+i(\zeta-\lambda)^{3/2}(P_bL_1K_1)^3P_bL_1\\
&+i(\zeta-\lambda)^{3/2}P_bL_1P_bK_1P_bL_1P_bL_1+\mathrm{O}(|\zeta-\lambda|^2).
\end{aligned}\tag{3.42}$$

Next, we use (3.41) to obtain the following expansion for $S(\zeta)^{-1}$:

$$\begin{aligned} S(\zeta)^{-1} &= I_b - V_{ba}G_0V_{ab}P_bL_1P_b \\ &+i(\zeta-\lambda)^{1/2}V_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_b \\ &-i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}P_bL_1P_b + (\zeta-\lambda)V_{ba}G_0V_{ab}P_bL_1P_bL_1P_b \\ &+(\zeta-\lambda)V_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_bK_1P_bL_1P_b \\ &-(\zeta-\lambda)V_{ba}G_1V_{ab}P_bL_1P_bK_1P_bL_1P_b \\ &-i(\zeta-\lambda)^{3/2}V_{ba}G_0V_{ab}P_bL_1P_bL_1P_bK_1P_bL_1P_b \\ &-i(\zeta-\lambda)^{3/2}V_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_bK_1P_bK_1P_bL_1P_b \\ &-i(\zeta-\lambda)^{3/2}V_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_bL_1P_b \\ &+i(\zeta-\lambda)^{3/2}V_{ba}G_1V_{ab}P_bL_1P_bK_1P_bL_1P_bK_1P_bL_1P_b \\ &+i(\zeta-\lambda)^{3/2}V_{ba}G_1V_{ab}P_bL_1P_bL_1P_b + \mathrm{O}(|\zeta-\lambda|^2). \end{aligned} \tag{3.43}$$

Next, we consider $U(\zeta)$ defined by

$$U(\zeta) = I_b - \left(V_{ba}G_0V_{ab}C_b + V_{ba}G_2V_{ab}P_b + \cdots + \mathrm{O}(|\zeta-\lambda|^2\right) S(\zeta)^{-1}.$$

Using the definition of M_1 in (3.35) we find the following expression for $U(\zeta)$ up to an error term:

$$\begin{aligned} U(\zeta) &= I_b - M_1 - i(\zeta-\lambda)^{1/2}V_{ba}G_3V_{ab}P_b \\ &-i(\zeta-\lambda)^{1/2}V_{ba}G_1V_{ab}C_b - i(\zeta-\lambda)^{1/2}\times \\ &\times\left(V_{ba}G_0V_{ab}C_b + V_{ba}G_2V_{ab}P_b\right)V_{ba}G_0V_{ab}P_bL_1P_bK_1P_b \\ &+i(\zeta-\lambda)^{1/2}(V_{ba}G_0V_{ab}C_b + V_{ba}G_2V_{ab}P_b)V_{ba}G_1V_{ab}P_bL_1P_b \\ &+i(\zeta-\lambda)^{1/2}(V_{ba}G_3V_{ab}P_b + V_{ba}G_1V_{ab}C_b)V_{ba}G_0V_{ab}P_bL_1P_b \\ &+(\zeta-\lambda)(V_{ba}G_0V_{ab}C_b^2 + V_{ba}G_2V_{ab}C_b + V_{ba}G_4V_{ab}P_b) \\ &-(\zeta-\lambda)(V_{ba}G_0V_{ab}C_b + V_{ba}G_2V_{ab}P_b)V_{ba}G_0V_{ab}P_bL_1P_bL_1P_b \\ &-(\zeta-\lambda)(V_{ba}G_0V_{ab}C_b + V_{ba}G_2V_{ab}P_b)V_{ba}G_0V_{ab}P_bL_1P_b \\ &\times P_bK_1P_bL_1P_bK_1P_bL_1P_b \\ &+(\zeta-\lambda)(V_{ba}G_0V_{ab}C_b + V_{ba}G_2V_{ab}P_b)V_{ba}G_1V_{ab}P_bL_1P_b \\ &\times P_bK_1P_bL_1P_b - (\zeta-\lambda)(V_{ba}G_0V_{ab}C_b^2 + V_{ba}G_2V_{ab}C_b + V_{ba}G_4V_{ab}P_b) \\ &\times V_{ba}G_0V_{ab}P_bL_1P_b + (\zeta-\lambda)(V_{ba}G_3V_{ab}P_b + V_{ba}G_1V_{ab}C_b) \\ &\times\left[V_{ba}G_0V_{ab}P_bL_1P_bK_1P_bL_1P_b - V_{ba}G_1V_{ab}P_bL_1P_b\right] \\ &+\mathrm{O}(|\zeta-\lambda|^{3/2}). \end{aligned} \tag{3.44}$$

The operator M_1 is compact. Hence, generically, the operator $I_b - M_1$ is invertible. Therefore, we factorize $U(\zeta)$ in the following way:

$$\begin{aligned} U(\zeta) = (I_b - M_1)&\left\{I_b + (I_b-M_1)^{-1}\times\right. \\ &\left.\left(i(\zeta-\lambda)V_{ba}G_3V_{ab}P_b + \cdots + \mathrm{O}(|\zeta-\lambda|^{3/2})\right)\right\}. \end{aligned} \tag{3.45}$$

From (3.44), (3.45) and the Neumann series we obtain an expansion for the inverse of $U(\zeta)$ up to $\mathrm{O}(|\zeta-\lambda|^{3/2})$. Finally, the expansion (3.36) is obtained via the factorization $T_b(\zeta)^{-1} = R_b(\zeta)S(\zeta)^{-1}U(\zeta)^{-1}$

in conjunction with the expansions (2.2), (3.43) and the expansion of the inverse of $U(\zeta)$. □

In view of Lemma 3.9 and the Feshbach formula we obtain the following theorem.

Theorem 3.10. *Let Assumption 3.8 hold at $\lambda \in \mathbf{R}$. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$ the asymptotic expansion*

$$\boldsymbol{R}(\zeta) = \begin{pmatrix} G_0 + G_0 V_{ab} d_0 V_{ba} G_0 & -G_0 V_{ab} d_0 \\ -d_0 V_{ba} G_0 & d_0 \end{pmatrix} +$$
$$+ i(\zeta-\lambda)^{1/2} \begin{pmatrix} G_1 + G_0 V_{ab} d_0 V_{ba} G_1 + G_1 V_{ab} d_0 V_{ba} G_0 + G_0 V_{ab} d_1 V_{ba} G_0 & -G_0 V_{ab} d_1 - G_1 V_{ab} d_0 \\ -d_0 V_{ba} G_1 - d_1 V_{ba} G_0 & d_1 \end{pmatrix} + O(|\zeta-\lambda|) \tag{3.46}$$

as $|\zeta-\lambda| \to 0$, $\operatorname{Im}\zeta > 0$.

We now turn to the case where $\lambda \in \rho(H_b)$ and λ is a threshold eigenvalue of H_a. We assume that the asymptotic expansion of $R_a(\zeta)$ around λ has a particular structure which we know occurs for Schrödinger-type operators, see [6, 4, 15].

Assumption 3.11. Let λ be an eigenvalue of H_a with associated eigenprojection P_a.
(i) Assume that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, such that for some $\delta > 0$ we have an asymptotic expansion

$$R_a(\zeta) = -\frac{1}{\zeta-\lambda} P_a - \frac{i}{(\zeta-\lambda)^{1/2}} G_{-1} + G_0 + $$
$$+ i(\zeta-\lambda)^{1/2} G_1 - (\zeta-\lambda) G_2 + \mathrm{O}(|\zeta-\lambda|^{3/2}). \tag{3.47}$$

for $|\zeta-\lambda| < \delta$, $\operatorname{Im}\zeta > 0$, in norm in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume that $G_j = G_j^*$ for $j = -1, 0, 1, 2$, in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume $P_a \in \mathcal{B}(\mathcal{K}_a)$ and furthermore $G_{-1} P_a = G_{-1}$.
(ii) Assume that $V_{ab} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{K}_a)$.
(iii) Assume that $\lambda \in \rho(H_b)$.

We bear in mind that for $\lambda \in \rho(H_b)$, the expansion (3.4) is valid.

Assumption 3.12. Let Assumption 3.11 hold. Assume that the operator $P_a V_{ab} C_0 V_{ba} P_a$ is strictly positive and invertible in $\mathcal{B}(\mathcal{K}_a)$.

Introduce the following operators:

$$L_2 = (P_a V_{ab} C_0 V_{ba} P_a)^{-1}, \quad K_2 = P_a V_{ab} C_0 V_{ba} G_{-1},$$

$$M_2 = V_{ab} C_0 V_{ba} G_0 - V_{ab} C_1 V_{ba} P_a - (V_{ab} C_0 V_{ba} G_0 - V_{ab} C_1 V_{ba} P_a) V_{ab} C_0 V_{ba} P_a L P_a.$$

Then we obtain the following result.

Lemma 3.13. *Let Assumption 3.12 hold at* $\lambda \in \mathbf{R}$. *Then, generically, we have in* $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$ *the following asymptotic expansion*

$$T_a(\zeta)^{-1} = e_0 + i(\zeta-\lambda)^{1/2} e_1 + O(|\zeta-\lambda|) \tag{3.48}$$

as $|\zeta-\lambda| \to 0$, $\operatorname{Im}\zeta > 0$, *where the coefficients* e_0 *and* e_1 *are given by*

$$e_0 = G_0(I_a - M_2)^{-1} - G_0 V_{ab} C_0 V_{ba} P_a L_2 P_a (I_a - M_2)^{-1}, \tag{3.49}$$

$$\begin{aligned}
e_1 = {} & G_0 V_{ab} C_0 V_{ba} P_a L_2 P_a K_2 P_a L_2 P_a (I_a - M_2)^{-1} \\
& - G_0 V_{ab} C_0 V_{ba} G_{-1} L_2 P_a (I_a - M_2)^{-1} + G_1 (I_a - M_2)^{-1} \\
& - G_1 V_{ab} C_0 V_{ba} P_a L_2 P_a (I_a - M_2)^{-1} \\
& - (P_a L_2 P_a L_2 + P_a L_2 P_a K_2 P_a L_2 P_a K_2 P_a L_2 P_a) \\
& - P_a K_2 P_a L_2 P_a K_2 P_a L_2 P_a L_2 P_a) \\
& \times (I_a - M_2)^{-1} \{ V_{ab} C_0 V_{ba} G_1 - V_{ab} C_1 V_{ba} G_{-1} \\
& - V_{ab} C_0 V_{ba} G_1 V_{ab} C_0 V_{ba} P_a L_2 P_a + V_{ab} C_1 V_{ba} G_{-1} V_{ab} C_0 V_{ba} P_a L_2 P_a \\
& + V_{ab} C_0 V_{ba} G_0 V_{ab} C_0 V_{ba} P_a L_2 P_a K_2 P_a L_2 P_a \\
& - V_{ab} C_0 V_{ba} G_0 V_{ab} C_0 V_{ba} G_{-1} L_2 P_a \\
& - V_{ab} C_1 V_{ba} P_a V_{ab} C_0 V_{ba} P_a L_2 P_a K_2 P_a L_2 P_a \\
& + V_{ab} C_1 V_{ba} P_a V_{ab} C_0 V_{ba} G_{-1} L_2 P_a \} (I_a - M_2)^{-1} \\
& - G_{-1} P_a L_2 P_a (I_a - M_2)^{-1}.
\end{aligned} \tag{3.50}$$

Proof. We follow the strategy of the proof of Lemma 3.9. To facilitate comparison with this proof we use analogous notation for some of the operator-valued functions. However, this time we interchange the roles of a and b in the Feshbach formula, since now $R_b(\zeta)$ is regular at $\zeta = \lambda$.

The first step is again to factor $T_a(\zeta)$.

$$T_a(\zeta) = (I_a - V_{ab} R_b(\zeta) V_{ba} R_a(\zeta)) (H_a - \zeta).$$

Inserting the two expansions we find the following asymptotic expansion in $\mathcal{B}(\mathcal{K}_a)$.

$$\begin{aligned}
& I_a - V_{ab} R_b(\zeta) V_{ba} R_a(\zeta) \\
& = \frac{1}{\zeta-\lambda} V_{ab} C_0 V_{ba} P_a + \frac{i}{(\zeta-\lambda)^{1/2}} V_{ab} C_0 V_{ba} G_{-1} \\
& + I_a - V_{ab} C_0 V_{ba} G_0 + V_{ab} C_1 V_{ba} P_a \\
& - i(\zeta-\lambda)^{1/2} V_{ab} C_0 V_{ba} G_1 + i(\zeta-\lambda)^{1/2} V_{ab} C_1 V_{ba} G_{-1} \\
& + (\zeta-\lambda) V_{ab} C_0 V_{ba} G_2 + (\zeta-\lambda) V_{ab} C_2 V_{ba} P_a \\
& - (\zeta-\lambda) V_{ab} C_1 V_{ba} G_0 + \mathrm{O}(|\zeta-\lambda|^{3/2}).
\end{aligned} \tag{3.51}$$

We observe that the singular part is contained in

$$S(\zeta) = I_a + \frac{1}{\zeta-\lambda} Y(\zeta) P_a,$$

where we have introduced

$$Y(\zeta) = V_{ab} C_0 V_{ba} + i(\zeta-\lambda)^{1/2} V_{ab} C_0 V_{ba} G_{-1}.$$

The operator $S(\zeta)$ is invertible, if

$$Z(\zeta) = (\zeta - \lambda)P_a + i(\zeta - \lambda)^{1/2} P_a V_{ab} C_0 V_{ba} G_{-1} P_a + P_a V_{ab} C_0 V_{ba} P_a$$

is invertible in the space $\mathcal{B}(\mathcal{K}_a)$, and we have again

$$S(\zeta)^{-1} = I_a - Y(\zeta) P_a Z(\zeta)^{-1} P_a,$$

see (3.41). The remainder of the proof is analogous to the proof of Lemma 3.9, and is omitted. □

The latter result and the Feshbach formula yield the following theorem.

Theorem 3.14. *Let Assumption 3.11 hold at $\lambda \in \mathbf{R}$. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$ the asymptotic expansion*

$$\begin{aligned} \boldsymbol{R}(\zeta) &= \begin{pmatrix} e_0 & -e_0 V_{ab} C_0 \\ -C_0 V_{ba} e_0 & C_0 + C_0 V_{ba} e_0 V_{ab} C_0 \end{pmatrix} \\ &+ i(\zeta - \lambda)^{1/2} \begin{pmatrix} e_1 & -e_1 V_{ab} C_0 \\ -C_0 V_{ba} e_1 & C_0 V_{ba} e_1 V_{ab} C_0 \end{pmatrix} + O(|\zeta - \lambda|) \end{aligned} \tag{3.52}$$

as $|\zeta - \lambda| \to 0$, $\operatorname{Im} \zeta > 0$.

Finally we consider the case when H_a has a so-called half-bound state (or resonance) at λ, i.e. there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, and a solution ϕ to $H_a \phi = \lambda \phi$, where $\phi \in \mathcal{K}_a^*$ but $\phi \notin \mathcal{H}_a$. Motivated by the known results for Schrödinger operators in dimensions one and three, see e. g. [1, 6, 10], we assume a particular form of the singularity of the resolvent.

Assumption 3.15. Let λ be a half-bound state of H_a.
(i) Assume that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, such that for some $\delta > 0$ we have an asymptotic expansion

$$R_a(\zeta) = \frac{i}{(\zeta - \lambda)^{1/2}} Q_a + G_0 + i(\zeta - \lambda)^{1/2} G_1 + \mathrm{O}(|\zeta - \lambda|) \tag{3.53}$$

for $|\zeta - \lambda| < \delta$, $\operatorname{Im} \zeta > 0$, in norm in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume that $G_0 = G_0^*$ for $G_0 \in \mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume $Q_a = \langle \cdot, \varphi \rangle \varphi$ for some $\varphi \in \mathcal{K}_a^*$.
(ii) Assume that $V_{ab} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{K}_a)$.
(iii) Assume that $\lambda \in \rho(H_b)$.

Under this assumption (3.4) holds. We introduce the real constant $\theta_0 = \langle V_{ab} C_0 V_{ba} \varphi, \varphi \rangle$. Assuming that $\theta_0 \neq 0$, we may introduce the projections

$$\mathcal{P}_1 = \theta_0^{-1} \langle \cdot, \phi \rangle V_{ab} C_0 V_{ba} \phi, \quad \mathcal{P}_0 = I_a - \mathcal{P}_1.$$

Moreover, it is convenient to introduce the operator $E = V_{ab} C_0 V_{ba} G_0 \mathcal{P}_0$. Then we obtain the following result.

Lemma 3.16. *Let Assumption 3.15 hold at* $\lambda \in \mathbf{R}$. *Assume that* $\theta_0 \neq 0$. *Then, generically, we have in* $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$ *the following asymptotic expansion*

$$T_a(\zeta)^{-1} = f_0 + i(\zeta - \lambda)^{1/2} f_1 + O(|\zeta - \lambda|) \tag{3.54}$$

as $|\zeta - \lambda| \to 0$, $\operatorname{Im} \zeta > 0$, *where the coefficients* f_0 *and* f_1 *are given by*

$$f_0 = G_0 \mathcal{P}_0 (I_a - E)^{-1} - \theta_0^{-1} Q_a (I_a - E)^{-1}, \tag{3.55}$$

$$\begin{aligned} f_1 = \theta_0^{-1} G_0 \mathcal{P}_0 (I_a - E)^{-1} V_{ab} C_0 V_{ba} G_0 (I_a - E)^{-1} + \\ + G_0 \mathcal{P}_0 (I_a - E)^{-1} V_{ab} C_0 V_{ba} G_1 \mathcal{P}_0 (I_a - E)^{-1} + \theta_0^{-1} G_0 (I_a - E)^{-1} \\ - \theta_0^{-1} G_0 \mathcal{P}_0 (I_a - E)^{-2} + G_1 \mathcal{P}_0 (I_a - E)^{-1}. \end{aligned} \tag{3.56}$$

Proof. Having introduced the projections $\mathcal{P}_0$, $\mathcal{P}_1$, the proof follows the pattern of the proof of Lemma 3.4. The details are omitted. □

As usual we immediately obtain an expansion of the resolvent $R(\zeta)$ via the Feshbach formula.

Theorem 3.17. *Let Assumption 3.15 hold at* $\lambda \in \mathbf{R}$. *Then, generically, we have in the norm of* $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$ *the asymptotic expansion*

$$\begin{aligned} \boldsymbol{R}(\zeta) = \begin{pmatrix} f_0 & -f_0 V_{ab} C_0 \\ -C_0 V_{ba} f_0 & C_0 + C_0 V_{ba} f_0 V_{ab} C_0 \end{pmatrix} \\ + i(\zeta - \lambda)^{1/2} \begin{pmatrix} f_1 & -f_1 V_{ab} C_0 \\ -C_0 V_{ba} f_1 & C_0 V_{ba} f_1 V_{ab} C_0 \end{pmatrix} + O(|\zeta - \lambda|) \end{aligned} \tag{3.57}$$

as $|\zeta - \lambda| \to 0$, $\operatorname{Im} \zeta > 0$.

Remark 3.18.
(a) There are several other cases which could be considered. It is possible to have both an eigenvalue and a resonance at a threshold for H_a, and furthermore, an eigenvalue of H_b could also occur at λ. It seems that the present technique is difficult to adapt to these problems. One will have to go through several stages of decomposition.
(b) Throughout this section we have used the word *generic* whenever we assume that an operator is invertible, e.g. the operator $I_b - V_{ba} G_0 V_{ab} C_0$ in the proof of Lemma 3.2 or the operator $I_b - V_{ba} G_0 V_{ab} C_b P_0$ in the proof of Lemma 3.4. It would be interesting, if the non-generic cases could be treated as well. Some preliminary work on this issue can be found in [10], where two-channel Hamiltonians with one-dimensional Schrödinger operators as component Hamiltonians are studied. Some non-generic cases are treated by introducing an auxiliary one-dimensional Schrödinger operator with a non-local potential. However, no unified treatment (as the present one) has been developed.

4. Applications

The main motivation for deriving asymptotic expansions of the resolvent as the spectral parameter tends to a threshold is to study scattering theory near thresholds for pairs of concrete two-channel Hamiltonians $(\boldsymbol{H}, \boldsymbol{H}_0)$ on the form

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{V} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} + \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

The companion paper [13] is devoted to scattering theory for pairs of such two-channel Hamiltonians with Schrödinger operators as component Hamiltonians. We give a detailed account in the case of one-dimensional Schrödinger operators with short range (local) potentials decaying as $\mathrm{O}(|x|^{-\beta})$ at infinity for some $\beta > 2$. First we establish scattering theory by the abstract short range theory developed by Jensen, Mourre and Perry [8]. Secondly, as an application of the results in this paper we derive asymptotic expansions of the S-matrix in the low-energy limit, i.e. as the energy parameter tends to the threshold zero. Moreover, we discuss how similar results can be obtained when the component Hamiltonians are d-dimensional Schrödinger operators, $3 \le d$ odd. In the three-dimensional case we also discuss how to treat the problem in the presence of a constant magnetic field.

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