SPECTRAL PROPERTIES AT A THRESHOLD FOR TWO-CHANNEL HAMILTONIANS. II. APPLICATIONS TO SCATTERING THEORY

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ABSTRACT. Spectral properties and scattering theory in the low-energy limit are investigated for two-channel Hamiltonians with Schrödinger operators as component Hamiltonians. In various, mostly fairly 'singular' settings asymptotic expansions of the resolvent are deduced as the spectral parameter tends to the threshold zero. Furthermore scattering theory for pairs of two-channel Hamiltonians are established. As an application of the expansions of the resolvent, asymptotic expansions of the scattering matrix are derived as the energy parameter tends to the threshold zero.

1. INTRODUCTION

In the paper [16] we investigated spectral properties at thresholds of two-channel Hamiltonians of the form $\boldsymbol{H} = \boldsymbol{H}_{diag} + \boldsymbol{V}_{off}$, where

$$\boldsymbol{H}_{diag} = \begin{pmatrix} H_a & 0 \\ 0 & H_b \end{pmatrix}, \quad \boldsymbol{V}_{off} = \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix} \tag{1.1}$$

act on the Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$. We assume that H_a and H_b are self-adjoint operators in $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. Moreover we assume that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{H}_a)$, the space of bounded operators, and require $V_{ba} = V_{ab}^*$.

Due to the diagonal structure of $\boldsymbol{H}_{diag}$ its spectrum can be an arbitrary combination of those of H_a and H_b. There are several possible situations, and we treated only some of them in detail. A case of particular interest is the following. We assume $\sigma(H_a) = \sigma_{\rm ac}(H_a) = [\lambda, \infty)$ and $\sigma_{\rm ac}(H_b) = [\lambda_1, \infty)$ for some $\lambda_1 > \lambda$. Furthermore, we assume that λ is an isolated eigenvalue of H_b with eigenprojection P_b. Thus $\boldsymbol{H}_{diag}$ has an eigenvalue embedded at the threshold λ. In [16] we deduced an asymptotic expansion of the resolvent $\boldsymbol{R}(\zeta) = (\boldsymbol{H} - \zeta)^{-1}$ of $\boldsymbol{H}$ as the spectral parameter ζ tends to the threshold λ. To obtain this result we required some a priori information on the threshold of H_a. More precisely, let $\mathcal{K}_a$ be a Hilbert space, which is densely and continuously

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embedded in $\mathcal{H}_a$. Motivated by known results for Schrödinger operators [8, 5, 1], we assumed the existence of an expansion, valid in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$,

$$R_a(\zeta) = G_0 + i(\zeta - \lambda)^{1/2} G_1 - (\zeta - \lambda) G_2 - i(\zeta - \lambda)^{3/2} G_3 + \mathrm{O}(|\zeta - \lambda|^2) \tag{1.2}$$

as $\zeta \to \lambda$, $\zeta \in \mathbf{C} \setminus [\lambda, \infty)$.

Additional assumptions on the potentials V_{ab} and V_{ba} are necessary. Suppose that $V_{ab} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{K}_a)$, the compact operators, and that the operator $P_b V_{ba} G_0 V_{ab} P_b$ is strictly positive and invertible in $\mathcal{B}(P_b \mathcal{H}_b)$.

Under the above-mentioned assumptions the following result holds, see Theorem 3.4 in [16]. As $|\zeta - \lambda| \to 0$, $\zeta \in \mathbf{C} \setminus [\lambda, \infty)$, the resolvent of $\boldsymbol{H}$ has an asymptotic expansion

$$\boldsymbol{R}(\zeta) = \boldsymbol{R}_0 + i(\zeta - \lambda)^{1/2} \boldsymbol{R}_1 + \mathrm{O}(|\zeta - \lambda|), \tag{1.3}$$

valid in the norm topology of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$. This result holds generically. In [16] we gave a number of similar results in various, mostly fairly 'singular' settings.

The result (1.3) is obtained by using the asymptotic expansion (1.2) in combination with the Feshbach formula and a technique based on factoring out the identity plus a finite rank operator.

Despite the singular nature of the problem, (1.3) reveals that, generically, the singularities cancel. In particular, the resolvent has a well-defined limit $\boldsymbol{R}_0$ at the threshold point in the norm topology of $\mathcal{B}(\mathcal{K}_a \oplus \mathcal{H}_b, \mathcal{K}_a^* \oplus \mathcal{H}_b)$.

This companion paper is devoted to scattering theory for pairs of two-channel Hamiltonians with Schrödinger operators as component Hamiltonians. Generally the Hamiltonians are of the form

$$H = H_0 + V(x) = \begin{pmatrix} p^2 & 0 \\ 0 & p^2 + 1 \end{pmatrix} + \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix}, \tag{1.4}$$

acting in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^d) \oplus L^2(\mathbf{R}^d)$. Here p is the momentum operator and the $V_{ij}(x)$, $i, j = 1, 2$, are short range potentials. The paper is organized as follows. The notation is fixed in Section 2. In Section 3 we study two-channel Hamiltonians of the form (1.4) with one-dimensional Schrödinger operators as component Hamiltonians, i.e. $p = id/dx$. We give a detailed account in the case where V_{ij} are short range (local) potentials decaying as $\mathrm{O}(|x|^{-\beta})$ at infinity for some $\beta > 2$. In Section 3.3 we give asymptotic expansions of the resolvent of H as the spectral parameter tends to the threshold zero (the bottom of the continuous spectrum of H); this is the low-energy limit. In the proper framework these results follow immediately from the abstract theory developed in [16] in conjunction with well-known asymptotic expansions of the resolvents of the component Hamiltonians. We stress that the assumption $V_{ab} \in \mathcal{B}_\infty(\mathcal{H}_b, \mathcal{H}_a)$ in [16] is not necessary. Sufficient decay of $V_{ij}(x)$ at infinity is enough to carry over the results. In Section 3.4 we establish scattering theory for the pair (H, H_0) by the abstract short range theory developed by Jensen, Mourre and Perry [10]. In Section 3.5, as an application of the expansions for the resolvent we derive asymptotic expansions of the scattering matrix in the low-energy limit, i.e. as the energy parameter tends to the threshold zero. Theorems 3.19 and 3.20 are the main results on scattering theory in this paper.

In Sections 4 and 5 we discuss briefly how similar results can be obtained when the component Hamiltonians are higher-dimensional Schrödinger operators. In the three-dimensional case we also discuss how to treat the problem in the presence of a constant magnetic field and an axisymmetrical electric potential. Complete results will be given elsewhere.

Related work on N-channel scattering in one dimension is found in [11]. The authors consider the case where the threshold energies are equal and they are mainly interested in developing a formulation of Levinson's Theorem. Consequently, no comparison can be made to the present work.

2. Preliminaries

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T)$. The spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$, respectively. We use standard terminology for the various parts of the spectrum, see for example [18]. The resolvent is $R(\zeta) = (T-\zeta)^{-1}$. The spectral family associated to T is denoted by $E_T(\lambda)$, $\lambda \in \mathbf{R}$. For an interval $\Delta \subset \mathbf{R}$, $F(\lambda \in \Delta)$ stands for the smoothed out characteristic function of Δ:

$$F(\lambda \in \Delta) = \begin{cases} 1 & \text{if } \lambda \in \Delta \text{ and } \operatorname{dist}(\lambda, \partial\Delta) \geq \delta, \;\; \delta \ll |\Delta|, \\ 0 & \text{if } \lambda \notin \Delta. \end{cases}$$

Given a selfadjoint operator T, $\tilde{E}_T(\Delta) = F(T \in \Delta)$ will denote the smoothed-out spectral projection of T on the interval Δ.

For a complex number $z \in \mathbf{C} \setminus [0, \infty)$ we denote by $z^{1/2}$ the branch of the square root with positive imaginary part.

Let $\mathbf{R}^d$ be the d-dimensional Euclidean space, denote points of $\mathbf{R}^d$ by $x = (x_1, \dots, x_d)$ and let $|x| = (\sum_{j=1}^{d} x_j^2)^{1/2}$. For $1 \leq p \leq \infty$ let $L^p(\mathbf{R}^d)$ be the space of (equivalence classes of) complex-valued functions ψ which are measurable and satisfy $\int |\psi(x)|^p dx < \infty$ if $p < \infty$ and $\|\psi\|_{L^\infty(\mathbf{R}^d)} = \operatorname{ess\,sup} |\psi| < \infty$ if $p = \infty$. The measure dx is the Lebesgue measure. For any p the $L^p(\mathbf{R}^d)$ space is a Banach space with norm $\|\cdot\|_{L^p(\mathbf{R}^d)} = (\int_{\mathbf{R}^d} |\cdot|^p dx)^{1/p}$. In the case $p = 2$, $L^2(\mathbf{R}^d)$ is a complex and separable Hilbert space with scalar product $(\phi, \psi)_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} \overline{\phi}\psi$ and corresponding norm $\|\psi\|_{L^2(\mathbf{R}^d)} =$

$(\psi,\psi)^{1/2}_{L^2(\mathbf{R}^d)}$. Let $\langle x\rangle = (1+|x|^2)^{1/2}$. We use the weighted L^2 spaces $L^{2,s}(\mathbf{R}^d) = \{\psi \mid \langle x\rangle^s\psi \in L^2(\mathbf{R}^d)\}$, $s \in \mathbf{R}$. Similarly $L^2(\mathbf{R}^d)^m$, the m-fold cartesian product of $L^2(\mathbf{R}^d)$, is equipped with the scalar product $\langle \phi,\psi\rangle = \sum_{j=1}^m (\phi_j,\psi_j)_{L^2(\mathbf{R}^d)}$ and the norm $\|\psi\| = (\psi,\psi)^{1/2}$. For $m=2$ we need the weighted spaces $\mathcal{L}^s(\mathbf{R}^d) = L^{2,s}(\mathbf{R}^d)^2$, $s \in \mathbf{R}$.

The space of infinite differentiable complex-valued functions with compact support will be denoted by $C_0^\infty(\mathbf{R}^d)$ or $\mathcal{D}(\mathbf{R}^d)$, the space of test functions. The adjoint space of $\mathcal{D}(\mathbf{R}^d)$, $\mathcal{D}'(\mathbf{R}^d)$, is the space of distributions on $\mathcal{D}(\mathbf{R}^d)$. The Schwarz space of rapidly decreasing functions and its adjoint space of tempered distributions are denoted by $\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}'(\mathbf{R}^d)$, respectively.

Let p denote the momentum operator $-i\nabla$ and let $\langle p\rangle = (1+p^2)^{1/2}$. We use the weighted Sobolev space $H^{m,s}(\mathbf{R}^d)$ given by

$$H^{m,s}(\mathbf{R}^d) = \{\psi \in \mathcal{S}'(\mathbf{R}^d) \mid ||\psi||_{m,s} = ||\langle x\rangle^s\langle p\rangle^m\psi||_{L^2} < \infty\}.$$

We use $\langle\cdot,\cdot\rangle$ to denote the inner product on $L^2(\mathbf{R}^d)$ and also the natural duality between $H^{m,s}(\mathbf{R}^d)$ and $H^{-m,-s}(\mathbf{R}^d)$. $\mathcal{B}(H^{m,s}(\mathbf{R}^d), H^{m',s'}(\mathbf{R}^d))$ denotes the space of bounded operators from $H^{m,s}$ to $H^{m',s'}$ with the operator norm. The Fourier transform is given by

$$(\mathcal{F}\psi)(\xi) = \widehat{\psi}(\xi) = (2\pi)^{-1/2}\int_{\mathbf{R}^d} e^{-ix\cdot\xi}\psi(x)dx$$

and is a bounded map from $H^{m,s}(\mathbf{R}^d)$ to $H^{s,m}(\mathbf{R}^d)$.

3. 1D Schrödinger operators as component Hamiltonians

In this section the abstract results in [16] are applied to scattering theory in the low-energy limit for two-channel Hamiltonians with one-dimensional Schrödinger operators as component Hamiltonians.

3.1. **Two-channel Hamiltonians.** Let $p = -id/dx$ be the momentum operator. The free two-channel Hamiltonian H_0 in (1.4) is a self-adjoint operator in $\mathcal{H} = L^2(\mathbf{R}) \oplus L^2(\mathbf{R})$ with domain $\mathcal{D}(H_0) = H^2(\mathbf{R}) \oplus H^2(\mathbf{R})$, where $H^2(\mathbf{R})$ denotes the Sobolev space $W^{2,2}(\mathbf{R})$ of order 2. Its spectrum $\sigma(H_0) = \sigma_{ess}(H_0) = [0,\infty)$ is the union of two semilines starting at zero and one, respectively. This motivates the definition of the threshold set $\mathsf{T} = \{0,1\}$. Throughout this section the component potentials are subject to the following assumption.

Assumption 3.1. Let V_{ij}, $i,j = 1,2$, be bounded, real-valued functions such that for some constants $c > 0$ and $\beta > 2$

$$|V_{ij}(x)| \le c(1+|x|)^{-\beta}, \quad i,j = 1,2, \text{ and } V_{12} = V_{21}.$$

We refer to β as the decay parameter. Under Assumption 3.1 the potential $V(x)$ in (1.4) is H_0-compact. Hence, the two-channel Hamiltonian $H = H_0 + V$ is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ and spectrum $\sigma(H) = \sigma_d(H) \cup \sigma_{ess}(H)$, where $\sigma_{ess}(H) = \sigma_{ess}(H_0)$.

3.2. **Auxiliary results.** It is well-known that zero cannot be an eigenvalue of the one-dimensional Schrödinger operator $H_1 = p^2 + V_{11}(x)$ in $L^2(\mathbf{R})$ when the (local) potential $V_{11}(x)$ decays like $V_{11}(x) = \mathrm{O}(|x|^{-\beta})$ at infinity for some $\beta > 2$. Hence, there are only two possible zero-energy properties of H_1: Case I) Zero is a regular point of H_1, i.e. zero is not an eigenvalue nor a zero resonance of H_1. Case II) Zero is an exceptional point of H_1, i.e. zero is not an eigenvalue but zero is a resonance. In the latter case, the equation $H_1\psi = 0$ has a unique (up to multiplicative constants) solution in $L^{2,-s}(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $5/2 < s < \beta - 5/2$, but not in $L^2(\mathbf{R})$ (see [14]). We have the following results for the resolvent $R_1(\zeta)$ of H_1 as $\zeta \to 0$.

Theorem 3.2. *Suppose zero is a regular point of H_1. Assume $\beta > 9$ and let s satisfy $9/2 < s < \beta - 9/2$. For some $\delta > 0$ we have in the norm of $\mathcal{B}(H^{-1,s}(\mathbf{R}), H^{1,-s}(\mathbf{R}))$ the asymptotic expansion*

$$R_1(\zeta) = B_0^{(0)} + i\zeta^{1/2} B_1^{(0)} - \zeta B_2^{(0)} + O(\zeta^{3/2}) \tag{3.1}$$

for $|\zeta| < \delta$, $\operatorname{Im}\zeta^{1/2} > 0$, where $(B_j^{(0)})^* = B_j^{(0)}$, $j = 0, 1, 2$, as operators in $\mathcal{B}(H^{-1,s}(\mathbf{R}), H^{1,-s}(\mathbf{R}))$.

Theorem 3.3. *Suppose zero is an exceptional point of H_1. Assume $\beta > 11$ and let s satisfy $11/2 < s < \beta - 11/2$. For some $\delta > 0$ we have in the norm of $\mathcal{B}(H^{-1,s}(\mathbf{R}), H^{1,-s}(\mathbf{R}))$ the asymptotic expansion*

$$R_1(\zeta) = -i\zeta^{-1/2} B_{-1}^{(1)} + B_0^{(1)} + i\zeta^{1/2} B_1^{(1)} + O(\zeta) \tag{3.2}$$

for $|\zeta| < \delta$, $\operatorname{Im}\zeta^{1/2} > 0$, where $(B_j^{(1)})^* = B_j^{(1)}$, $j = -1, 0, 1$, as operators in $\mathcal{B}(H^{-1,s}(\mathbf{R}), H^{1,-s}(\mathbf{R}))$.

Theorems 3.2 and 3.3 are found in [14] under the assumption that $\int V_{11}(x)dx \neq 0$. Expansions with a similar structure can be derived if $\int V_{11}(x)dx = 0$ but in this case the coefficients are different (see [15] for details). By stating the theorems as above, we do not differentiate between the two situations. The proofs of Theorems 3.1 and 3.2 are based on a combination of the methods in [8] and [2, 4]. If the potential V_{11} decays exponentially at infinity, (3.1) and (3.2) are norm-convergent Taylor, respectively Laurent series [2, 4].

3.3. **Asymptotic expansions of the resolvent in the low-energy limit.** In this subsection we carry over the abstract results in [16] to the concrete Hamiltonian H. We show how to apply one of the abstract theorems in [16]. The remaining theorems in [16] are carried over in a similar way. We state the remaining theorems without further explanation.

Let Assumption 3.1 be satisfied throughout. The component Hamiltonians $H_1 = p^2 + V_{11}(x)$ and $H_2 = p^2 + 1 + V_{22}(x)$ play the roles of H_a and H_b in [16, Section 3]. We consider the situation, where Assumption 3.1 in [16] hold for H_1 at the threshold zero. In the present context Assumption 3.1[(i) and (iii)] in [16] can be formulated as follows.

Assumption 3.4. Let $0 \in \sigma(H_1)$.
(i) Suppose that zero is a regular point of H_1.
(ii) Assume that zero is a simple isolated eigenvalue of H_2, with normalized eigenfunction χ_0. Its reduced resolvent is denoted by C_2.

Assumption 3.4(i) implies that the asymptotic expansion (3.1) holds when $\beta > 9$ and s satisfy $9/2 < s < \beta - 9/2$. The expansion (3.1) corresponds to the expansion under Assumption 3.1(i) in [16].

We use the notation $P_2 = \langle \cdot,\, \chi_0\rangle \chi_0$ for the eigenprojection. Let $5/2 < s < \beta - 5/2$. The following real numbers are needed to state the results.

$$\alpha_0 = \langle V_{21} B_0^{(0)} V_{12} \chi_0,\, \chi_0 \rangle, \tag{3.3}$$

$$\gamma_0 = \langle V_{21} B_0^{(0)} V_{12} C_2 V_{21} B_0^{(0)} V_{12} \chi_0,\, \chi_0 \rangle. \tag{3.4}$$

Define, formally, the operators

$$E_{-2} = -V_{21} B_0^{(0)} V_{12} P_2, \quad E_{-1} = -P_2 V_{21} B_1^{(0)} V_{12},$$

and, for $j \geq 0$,

$$E_j = \begin{cases} -V_{21} B_{j+2}^{(0)} V_{12} P_2 + \sum_{k=1}^{j-\frac{j-2}{2}} (-1)^k V_{21} B_{j+2-2k}^{(0)} V_{12} C_2^k, & j = 0, 2, 4, \ldots \\ -V_{21} B_{j+2}^{(0)} V_{12} P_2 + \sum_{k=1}^{j-\frac{j-1}{2}} (-1)^k C_2^k V_{21} B_{j+2-2k}^{(0)} V_{12} C_2^k, & j = 1, 3, 5, \ldots \end{cases}$$

We have the elementary lemma.

Lemma 3.5. *Let Assumption 3.1 and Assumption 3.4 be satisfied. Then the operators E_j, $j = -2, -1$, are rank one operators in $L^{2,s}(\mathbf{R})$ for $\beta > 2j + 9$ and $j + 9/2 < s < \beta - j - 9/2$ and the operators E_j, $j = 0, 1, 2, \ldots$, are compact operators in $L^{2,s}(\mathbf{R})$ for $\beta > 2j + 9$ and $j + 9/2 < s < \beta - j - 9/2$.*

Proof. The projection P_2 extends to a bounded operator from $L^{2,s}(\mathbf{R})$ to $L^{2,-s}(\mathbf{R})$, $s > 0$ and under Assumption 3.1 with $\beta > 2$ we have that V_{ij} is a compact map from $H^{1,0}(\mathbf{R})$ to $H^{-1,\beta'}(\mathbf{R})$ for all $2 < \beta' < \beta$. The assertions follow from these observations in conjunction with the mapping properties of $B_j^{(0)}$; the latter imposes the requirements on β and s. $\square$

Assume that $\alpha_0 \neq 0$. Define the projections

$$P_1 = \alpha_0^{-1} \langle \cdot, \chi_0 \rangle V_{21} B_0^{(0)} V_{12} \chi_0, \quad P_0 = I - P_1,$$

in $L^{2,s}(\mathbf{R})$, $5/2 < s < \beta - 5/2$, where $\langle \cdot, \cdot \rangle$ is understood in the dual sense (between $L^{2,s}$ and $L^{2,-s}$). From Lemma 3.5 and [16, Lemma 3.2 and Theorem 3.2] we obtain immediately the following result.

Theorem 3.6. *Let Assumption 3.1 with $\beta > 11$ hold and let $11/2 < s < \beta - 11/2$. Let Assumption 3.4 hold. Assume that $\alpha_0 \neq 0$. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{L}^s(\mathbf{R}), \mathcal{L}^{-s}(\mathbf{R}))$ the asymptotic expansion*

$$R(\zeta) = \begin{pmatrix} B_0^{(0)} + B_0^{(0)} V_{12} b_0 V_{21} B_0^{(0)} & -B_0^{(0)} V_{12} b_0 \\ -b_0 V_{21} B_0^{(0)} & b_0 \end{pmatrix} + i(\zeta - \lambda)^{1/2}$$
$$\times \begin{pmatrix} B_1^{(0)} + B_0^{(0)} V_{12} b_0 V_{21} B_1^{(0)} + B_1^{(0)} V_{12} b_0 V_{21} B_0^{(0)} + B_0^{(0)} V_{12} b_1 V_{21} B_0^{(0)} \\ -b_0 V_{21} B_1^{(0)} - b_1 V_{21} B_0^{(0)} \end{pmatrix.$$
$$\begin{matrix} -B_0^{(0)} V_{12} b_1 - B_1^{(0)} V_{12} b_0 \\ b_1 \end{pmatrix} + O(|\zeta - \lambda|) \tag{3.5}$$

as $|\zeta| \to 0$, $\operatorname{Im} \zeta > 0$, where the coefficients b_0 and b_1 are given by

$$b_0 = (1 - P_0 C_2 V_{21} B_0^{(0)} V_{12})^{-1} P_0 C_2, \tag{3.6}$$

and

$$b_1 = \alpha_0^{-1} (1 - P_0 C_2 V_{21} B_0^{(0)} V_{12})^{-1} \times (\alpha_0 P_0 E_1 - E_{-1})(1 - P_0 C_2 V_{21} B_0^{(0)} V_{12})^{-1} P_0 C_2. \tag{3.7}$$

Likewise we obtain the following theorem corresponding to [16, Theorem 3.1].

Theorem 3.7. *Let Assumption 3.1 with $\beta > 9$ hold and let $9/2 < s < \beta - 9/2$. Let Assumption 3.4*(i) *be fulfilled. Assume that $0 \in \rho(H_2)$. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{L}^s(\mathbf{R}), \mathcal{L}^{-s}(\mathbf{R}))$ the asymptotic expansion*

$$R(\zeta) = \begin{pmatrix} B_0^{(0)} + B_0^{(0)} V_{12} a_0 V_{21} B_0^{(0)} & -B_0^{(0)} V_{12} a_0 \\ -a_0 V_{21} B_0^{(0)} & a_0 \end{pmatrix} + i(\zeta - \lambda)^{1/2}$$
$$\times \begin{pmatrix} B_1^{(0)} + B_0^{(0)} V_{12} a_0 V_{21} B_1^{(0)} + B_1^{(0)} V_{12} a_0 V_{21} B_0^{(0)} + B_0^{(0)} V_{12} a_1 V_{21} B_0^{(0)} \\ -a_1 V_{12} B_0^{(0)} - a_0 V_{21} B_1^{(0)} \end{matrix}$$
$$\begin{matrix} -B_0^{(0)} V_{12} a_1 - B_1^{(0)} V_{12} a_0 \\ a_1 \end{pmatrix} + O(|\zeta - \lambda|) \tag{3.8}$$

as $|\zeta| \to 0$, $\operatorname{Im} \zeta > 0$, where

$$a_0 = L_0, \quad a_1 = L_0 V_{21} B_1^{(0)} V_{12} R_2(0) L_0, \tag{3.9}$$

$$a_2 = L_0 V_{21} B_2^{(0)} V_{12} R_2(0) L_0 - L_0 V_{21} B_0^{(0)} V_{12} R_2(0)^2 L_0 + L_0 (V_{21} B_1^{(0)} V_{12} R_2(0) L_0)^2 \tag{3.10}$$

and $L_0 = (I - V_{21} B_0^{(0)} V_{12} R_2(0))^{-1}$.

Returning to Assumption 3.4, we now consider the case $\alpha_0 = 0$. Assuming that $\gamma_0 \neq 0$ and $5/2 < s < \beta - 5/2$ we introduce the operators

$$J_1 = \langle \cdot, \chi_0 \rangle V_{21} B_0^{(0)} V_{12} \chi_0, \qquad J_0 = I - J_1,$$

and the projections

$$\tilde{J}_1 = \gamma_0^{-1} \langle \cdot, \chi_0 \rangle V_{21} B_0^{(0)} V_{12} C_2 V_{12} B_0^{(0)} V_{12} \chi_0, \qquad \tilde{J}_0 = I - \tilde{J}_1.$$

Then we obtain the following result from [16, Theorem 3.3].

Theorem 3.8. *Let Assumption 3.1 with* $\beta > 15$ *hold and let* $15/2 < s < \beta - 15/2$. *Let Assumption 3.4 hold. Assume that* $\alpha_0 = 0$ *and* $\gamma_0 \neq 0$. *Then, generically, we have in* $\mathcal{B}(\mathcal{L}^s(\mathbf{R}), \mathcal{L}^{-s}(\mathbf{R}))$ *the asymptotic expansion*

$$R(\zeta) = \begin{pmatrix} B_0^{(0)} + B_0^{(0)} V_{12} c_0 V_{21} B_0^{(0)} & -B_0^{(0)} V_{12} c_0 \\ -c_0 V_{21} B_0^{(0)} & c_0 \end{pmatrix} + i(\zeta - \lambda)^{1/2}$$
$$\times \begin{pmatrix} B_1^{(0)} + B_0^{(0)} V_{12} c_0 V_{21} B_1^{(0)} + B_1^{(0)} V_{12} c_0 V_{21} B_0^{(0)} + B_0^{(0)} V_{12} c_1 V_{21} B_0^{(0)} & -B_0^{(0)} V_{12} c_1 - B_1^{(0)} V_{12} c_0 \\ -c_0 V_{21} B_1^{(0)} - c_1 V_{21} B_0^{(0)} & c_1 \end{pmatrix} + O(|\zeta - \lambda|) \tag{3.11}$$

as $|\zeta| \to 0$, $\operatorname{Im} \zeta > 0$, *where the coefficients* c_0 *and* c_1 *are given by*

$$c_0 = C_2 \tilde{J}_0 (I - V_{21} B_0^{(0)} V_{12} C_2 \tilde{J}_0)^{-1}, \tag{3.12}$$

$$\begin{aligned} c_1 = {} & \gamma_0^{-1} C_2 \tilde{J}_0 (I - V_{21} B_0^{(0)} V_{12} C_2 \tilde{J}_0)^{-1} \\ & \times \Big(V_{21} B_1^{(0)} V_{12} P_2 + V_{21} B_1^{(0)} V_{12} C_2 J_1 \tilde{J}_1 \\ & - \gamma_0 V_{21} B_1^{(0)} V_{12} C_2 \tilde{J}_0 \Big) \times (I - V_{21} B_0^{(0)} V_{12} C_2 \tilde{J}_0)^{-1}. \end{aligned} \tag{3.13}$$

Remark 3.9.
(a) If 0 is an eigenvalue of H_2 then 0 is simple [19]. Hence, the situation described in [16, Theorem 3.4] never occurs under the above-mentioned assumptions.
(b) The setting considered in [16, Theorem 3.5] does not occur for the Hamiltonian H because 0 cannot be an eigenvalue of H_1 under Assumption 3.1 for $\beta > 2$.

Finally we consider the case when H_1 has a half-bound state (or resonance) at 0. In the present context Assumption 3.5 in [16] can be formulated as follows.

Assumption 3.10. Let $0 \in \sigma(H_1)$.
(i) Suppose that zero is an exceptional point of H_1 (1st kind), i.e. there exists a solution ϕ to the equation $H_1 \phi = 0$, where $\phi \in L^{2,-s}(\mathbf{R}) \cap$

$L^\infty(\mathbf{R})$, $5/2 < s < \beta - 5/2$, but $\phi \notin L^2(\mathbf{R})$.
(ii) Assume that $0 \in \rho(H_2)$.

Let $5/2 < s < \beta - 5/2$ and introduce the real constant

$$\theta_0 = \langle V_{12} R_2(0) V_{21}\phi, \phi\rangle.$$

Assuming that $\theta_0 \neq 0$ we may introduce the projections

$$\mathcal{P}_1 = \theta_0^{-1}\langle \cdot, \phi\rangle V_{12} R_2(0) V_{21}\phi, \quad \mathcal{P}_0 = I - \mathcal{P}_1.$$

in $L^{2,-s}(\mathbf{R})$. Moreover, it is convenient to introduce the operator $E = V_{12}R_2(0)V_{21}B_0^{(1)}\mathcal{P}_0$. It is well-defined when $9/2 < s < \beta - 9/2$. Then we obtain the following result from [16, Theorem 3.6].

Theorem 3.11. *Let Assumption 3.1 with $\beta > 13$ hold and let $13/2 < s < \beta - 13/2$. Let Assumption 3.10 be fulfilled. Assume that $\theta_0 \neq 0$. Then, generically, we have in $\mathcal{B}(\mathcal{L}^s(\mathbf{R}), \mathcal{L}^{-s}(\mathbf{R}))$ the asymptotic expansion*

$$\begin{aligned} R(\zeta) = &\begin{pmatrix} f_0 & -f_0 V_{12} R_2(0) \\ -R_2(0) V_{21} f_0 & R_2(0) + R_2(0) V_{21} f_0 V_{12} R_2(0) \end{pmatrix} \\ &+ i(\zeta - \lambda)^{1/2} \begin{pmatrix} f_1 & -f_1 V_{12} R_2(0) \\ -R_2(0) V_{12} f_1 & R_2(0) V_{21} f_1 V_{12} R_2(0) \end{pmatrix} + O(|\zeta - \lambda|) \end{aligned} \tag{3.14}$$

as $|\zeta| \to 0$, $\operatorname{Im}\zeta > 0$, where the coefficients f_0 and f_1 are given by

$$f_0 = B_0^{(1)}\mathcal{P}_0(I - E)^{-1} + \theta_0^{-1} B_{-1}^{(1)}(I - E)^{-1}, \tag{3.15}$$

$$\begin{aligned} f_1 = &\theta_0^{-1} B_0^{(1)}\mathcal{P}_0(I - E)^{-1} V_{12} R_2(0) V_{21} B_0^{(1)}(I - E)^{-1} \\ &+ B_0^{(1)}\mathcal{P}_0(I - E)^{-1} V_{12} R_2(0) V_{21} B_1^{(1)}\mathcal{P}_0(I - E)^{-1} + \theta_0^{-1} B_0^{(1)}(I - E)^{-1} \\ &- \theta_0^{-1} B_0^{(1)}\mathcal{P}_0(I - E)^{-2} + B_1^{(1)}\mathcal{P}_0(I - E)^{-1}. \end{aligned} \tag{3.16}$$

3.4. **Scattering theory for the pair (H, H_0).** We establish the scattering theory for the pair (H, H_0) of two-channel Hamiltonians by means of the abstract short range scattering theory developed by Jensen, Mourre and Perry [10] (See also [7]) which is based on the following two definitions:

Definition 3.12. Let I_0 be an open interval. Let A be a self-adjoint operator in $\mathcal{H}$. We say that H_0 satisfies propagation estimates with respect to A on I_0 if there exist real numbers $s > s' > 1$ such that for all $g \in C_0^\infty(I_0)$ the following two estimates hold:

$$\|(1 + A^2)^{-s/2} e^{-itH_0} g(H_0)(1 + A^2)^{-s/2}\| \leq c(1 + |t|)^{-s'} \text{ for all } t \in \mathbf{R}, \tag{3.17}$$

$$\|(1 + A^2)^{-s/2} e^{-itH_0} g(H_0) P_A^{\pm}\| \leq c(1 + |t|)^{-s'} \text{ for all } \pm t > 0. \tag{3.18}$$

Here $P_A^+ = E_A((0, \infty))$ and $P_A^- = 1 - P_A^+$.

Definition 3.13. Let A be a self-adjoint operator on $\mathcal{H}$. The potential V is said to be a short range perturbation of H_0 with respect to A, if

$$(H+i)^{-1} - (H_0+i)^{-1} \text{ is a compact operator on } \mathcal{H}, \tag{3.19}$$

and if there exist a real number $\mu > 1$ and integers $j, k \geq 0$ such that the operator

$$(H+i)^{-j} V (H_0+i)^{-k} (1+A^2)^{\mu/2} \tag{3.20}$$

extends to a bounded operator on $\mathcal{H}$.

The main theorem is:

Theorem 3.14 (Jensen-Mourre-Perry). *Let H_0, V and H be as above. Assume that there exists a self-adjoint operator A such that H_0 satisfies the propagation estimates with respect to A and such that the potential V is a short range perturbation of H_0 with respect to A. Then the wave operators $W_\pm(H, H_0; I_0)$ exist and are strongly asymptotically complete. Furthermore, $\sigma_s(H) \cap I_0$ is discrete in I_0.*

To complete the abstract theory one can give other conditions which are simpler to verify than the propagation estimates in Definition 3.12. One such method is the Mourre theory. We state the result (see [10, 7]) in the following form:

Theorem 3.15 (Jensen-Mourre-Perry). *Let H_0 and A be self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\lambda_0 \in \mathbf{R}$. Suppose:*
(a) *$\mathcal{D}(A) \cap \mathcal{D}(H_0)$ is a core of H_0.*
(b) *$e^{i\gamma A}\,\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ and for each $\psi \in \mathcal{D}(H_0)$ we have that $\sup_{|\gamma|\leq 1} \|H_0 e^{i\gamma A}\| < \infty$.*
(c) *The commutator $[H_0, iA]$, defined as a form on $\mathcal{D}(A) \cap \mathcal{D}(H_0)$, is bounded below and closable. The self-adjoint operator associated with its closure is denoted iB_1. Assume $\mathcal{D}(H_0) \subset \mathcal{D}(B_1)$. Assume inductively for $j = 2, 3, \ldots$ that the form $i[iB_{j-1}, A]$ is bounded below and closable. The associated operator is denoted iB_j. Assume $\mathcal{D}(H_0) \subset \mathcal{D}(B_j)$.*
(d) *There exist $\alpha > 0$, $\delta > 0$, and a compact operator K such that with $J = (\lambda_0 - \delta, \lambda_0 + \delta)$ the Mourre estimate*

$$E_{H_0}(J) iB_1 E_{H_0} \geq \alpha E_{H_0}(J) + K \tag{3.21}$$

holds.
Then we have:
(i) *$\sigma_s(H_0) \cap J$ is discrete in J.*
(ii) *The propagation estimates in (3.17) and (3.18) hold with $I_0 = J \backslash \sigma_s(H_0)$ for all $s > s' > 0$.*

We refer to the papers for the proofs [10, 7]. To establish the scattering theory for the pair of Hamiltonians (H, H_0) we need the following preparations. Let $I_1 = (0, 1)$ and $I_2 = (1, \infty)$. We consider the scattering theory for the pair (H, H_0) localized to the intervals I_1 and I_2.

We define the self-adjoint operator $A_x = (1/2)(xp + px)$ called the (one-dimensional) generator of dilations. The operator A_x is defined as the infinitesimal generator of the unitary group U_t defined by $U_t\phi(x) = e^{-t/2}\phi(e^{-t}x)$ for any $\phi \in L^2(\mathbf{R})$. Then we may define the self-adjoint operator $A = A_x\Pi$, where Π is the 2×2 identity operator in $\mathcal{H}$. The operator A plays the role of the conjugate operator in the Mourre theory for H_0. Furthermore, the scale of spaces is $\mathcal{H}_m := H^m(\mathbf{R}) \oplus H^m(\mathbf{R})$, $m = -2, -1, 0, 1, 2$, where $H^m(\mathbf{R})$ denotes the Sobolev space $W^{m,2}(\mathbf{R})$ of order m.

We are ready to verify the conditions in Theorem 3.15.

Lemma 3.16. *The self-adjoint operators H_0 and A satisfy conditions* (a)-(c) *in Theorem 3.15. Moreover, for any $\lambda_0 \in \mathbf{R}\backslash\mathsf{T}$, condition* (d) *in Theorem 3.15 is satisfied.*

Proof. We verify the hypotheses in Theorem 3.15.
(a) H_0 and A are defined on $\mathcal{S} = \mathcal{S}(\mathbf{R}) \oplus \mathcal{S}(\mathbf{R})$ and $\mathcal{S}$ is a common core of H_0 and A.
(b) The explicit formula

$$e^{i\gamma A}(H_0 + i)^{-1} = \begin{pmatrix} (e^{-2\gamma}p^2 + i)^{-1}e^{iA_x\gamma} & 0 \\ 0 & (e^{-2\gamma}p^2 + 1 + i)^{-1}e^{iA_x\gamma} \end{pmatrix}$$

shows that $e^{i\gamma A}$ leaves $\mathcal{D}(H_0)$ invariant.
(c) We apply Proposition II.1 in [17]. For this purpose, we verify several conditions. By (a)-(b), the set $\mathcal{S} \subset \mathcal{D}(A) \cap \mathcal{D}(H_0)$ is a core of both A and H_0 and $e^{i\gamma A}\mathcal{S} \subset \mathcal{S}$. In addition, the form $[H_0, iA]$ on $\mathcal{S}$ satisfies $[H_0, iA] = 2p^2 I =: 2H_0'$ in the sense of forms. Here I denotes the 2×2 identity matrix. Clearly, the form H_0' defined on $\mathcal{S}$ is bounded below and closable in $\mathcal{H}$; the closed form generates a unique self-adjoint operator $iB_{1,\mathcal{S}}$ with domain $\mathcal{D}(B_{1,\mathcal{S}}) = H^2(\mathbf{R}) \oplus H^2(\mathbf{R})$; the operator $iB_{1,\mathcal{S}}$ is the self-adjoint realization, also denoted by H_0', of the diagonal matrix H_0' with elements p^2 and p^2 and domain $H^2(\mathbf{R}) \oplus H^2(\mathbf{R})$. In particular, $\mathcal{D}(B_{1,\mathcal{S}}) = \mathcal{D}(H_0)$. Therefore, Proposition II.1 in [17] asserts that the form $[H_0, iA]$ defined on $\mathcal{D}(A) \cap \mathcal{D}(H_0)$ is bounded below and closable, and the associated self-adjoint operator, denoted by iB_1, satisfies $B_1 = B_{1,\mathcal{S}} = H_0'$. The multiple commutators are treated in a similar way.
(d) Assume $\lambda_0 \in (0, 1)$. Define $d_1(\lambda_0) = \min\{\lambda_0, 1 - \lambda_0\}$. Let $\delta_1 = d_1(\lambda_0)/2$ and let $J = (\lambda_0 - \delta_1, \lambda_0 + \delta_1)$. It follows easily that (3.21) is satisfied with $\alpha = 2(\lambda_0 - \delta_1)$ and $K = 0$. Assume that $\lambda_0 \in (1, \infty)$. Define $d_2(\lambda_0) = \lambda_0 - 1$. Let $\delta = \delta_2(\lambda_0)/2$ and let $J = (\lambda_0 - \delta_2, \lambda_0 + \delta_2)$. Now, $\tilde{E}_{p^2+1}(J) = F(\lambda + 1 \in J) = F(\lambda \in J - 1) = \tilde{E}_{p^2}(J - 1)$. Then,

for any $\phi \in \mathcal{H}$, we have that

$$
\begin{aligned}
&\langle \phi, \tilde{E}_{H_0}(J) B_1 \tilde{E}_{H_0}(J)\phi\rangle_{\mathcal{H}} = \\
&\quad = 2\langle \phi_1, \tilde{E}_{p^2}(J) p^2 \tilde{E}_{p^2}(J)\phi_1\rangle_{L^2(\mathbf{R})} \\
&\quad +2\langle \phi_2, \tilde{E}_{p^2+1}(J) p^2 \tilde{E}_{p^2+1}(J)\phi_2\rangle_{L^2(\mathbf{R})} \\
&\quad = 2\langle \phi_1, \tilde{E}_{p^2}(J) p^2 \tilde{E}_{p^2}(J)\phi_1\rangle_{L^2(\mathbf{R})} \\
&\quad +2\langle \phi_2, \tilde{E}_{p^2}(J-1) p^2 \tilde{E}_{p^2}(J-1)\phi_2\rangle_{L^2(\mathbf{R})} \\
&\quad = 2\int_J \lambda \langle \phi_1, \tilde{E}_{p^2}(J) E_{p^2}(\lambda) \tilde{E}_{p^2}(J)\phi_1\rangle_{L^2(\mathbf{R})} \\
&\quad +2\int_{J-1} \lambda \langle \phi_2, \tilde{E}_{p^2}(J-1) E_{p^2}(\lambda) \tilde{E}_{p^2}(J-1)\phi_2\rangle_{L^2(\mathbf{R})} \\
&\quad \geq 2(\lambda_0-\delta_2)\langle \phi_1, E_{p^2}(J)\phi_1\rangle_{L^2(\mathbf{R})} + \\
&\quad +2(\lambda_0-1-\delta_2)\langle \phi_2, E_{p^2+1}(J)\phi_2\rangle_{L^2(\mathbf{R})} \\
&\quad \geq \alpha\langle \phi, E_{H_0}(J)\phi\rangle_{\mathcal{H}},
\end{aligned}
$$

where $\alpha = 2(\lambda_0 - 1 - \delta_2)$. This establishes (3.21) in the case when $\lambda_0 \in (1,\infty)$. □

Since $\sigma(H_0) = \sigma_{ac}(H_0)$ implies that $\sigma_s(H_0) = \emptyset$, Lemma 3.16 in combination with Theorem 3.15 imply that the propagation estimates (3.17) and (3.18) are valid on both intervals I_1 and I_2 for all $s > s' > 0$. In order to establish Theorem 3.14, it remains to verify that V is a short range perturbation of H_0 with respect to A.

Proposition 3.17. *Let Assumption 3.1 hold with $\beta > 1$. Then V is a short range perturbation of H_0 with respect to A.*

Proof. In order to apply Theorem 3.14, we need to verify (3.19) and (3.20) for H_0, H and A. It follows immediately from the second resolvent equation and the H_0-compactness of V that the condition (3.19) is satisfied.

For any $\beta > 1$ we show that (3.20) is satisfied with $\mu > 1$, $j = 1$ and $k = 2$. First, we show that for any $0 \leq \mu \leq 2$,

$$T(\mu) := (1+x^2)^{-\mu/2}(H_0+i)^{-2}(1+A^2)^{\mu/2}$$

extends to a bounded operator on $\mathcal{H}$. (Here $(1+x^2)^{-\mu/2}$ is short hand notation for the 2×2 diagonal operator with $(1+x^2)^{-\mu/2}$ along the diagonal). For $\mu = 0 + i\tau$, $\tau \in \mathbf{R}$, $T(0+i\tau)$ clearly is a bounded operator since $(H_0+i)^{-2}$ is bounded in $\mathcal{H}$ and, moreover, the operators $(1+x^2)^{-i\tau/2}$ and $(1+a^2)^{i\tau/2}$ are both unitary on $\mathcal{H}$. For $\mu = 2+i\tau$, $\tau \in \mathbf{R}$, we first extend $(1+x^2)^{-(2+i\tau)/2}(p^2+i)^{-2}(1+A_x^2)^{(2+i\tau)/2}$ to a bounded operator on $L^2(\mathbf{R})$. Then $T(2+i\tau)$ extends to a bounded operator on $\mathcal{H}$. For any $0 \leq \mu \leq 2$ we only have to apply Hadamard's three line theorem to the operator $T(z) = (1+x^2)^{-z/2}(H_0+i)^{-2}(1+A^2)^{z/2}$. To complete the proof, taking $j = 1$, we observe that $(1+$

$\overline{x^2)^{\mu/2}V(H+i)^{-1}} \in \mathcal{B}(\mathcal{H})$. It follows that $[(1+x^2)^{\mu/2}V(H+i)^{-1}]^* = \overline{(H+i)^{-1}V(1+x^2)^{\mu/2}}$ is bounded and therefore, for $\mu > 1$, $j = 1$ and $k = 2$, $\overline{(H+i)^{-1}V(1+x^2)^{\mu/2}}T(\mu)$ extends to a bounded operator on $\mathcal{H}$. This proves (3.20). □

Consequently, we have established Theorem 3.14 for the pair (H, H_0) and their conjugate operator A.

3.5. **The scattering matrix in the low-energy limit.** It follows from Theorem 3.14 that the local scattering operator S_j, defined by

$$S_j = W_+^*(H, H_0; I_j)W_-(H, H_0; I_j), \;\; j = 1, 2,$$

is a unitary operator on $E_{I_j}(H_0)P_{ac}(H_0)\mathcal{H}$. Let $\tilde{S}_1$ denote the unitary representation of S_1 in $L^2(I_1; \mathbf{C}^2)$. There is a general theorem asserting that $\tilde{S}_1$ admits a diagonal representation $(\tilde{S}_1\psi)(\lambda) = \tilde{S}_1(\lambda)\psi(\lambda)$ (see, e.g., [12, Theorem 6.2]). In the present context the scattering matrix $S(\lambda)$ (henceforth we suppress the lower index and the tilde character) can be represented on the interval I_1 as follows for any $\lambda \in I_1 \backslash \sigma_{pp}(H)$:

$$S(\lambda) = \\ 1 - \pi i \lambda^{-1/2}\gamma_0(\lambda^{1/2})\mathcal{F}V(1 - R(\lambda + i0)V)\mathcal{F}^*\gamma_0(\lambda^{1/2})^*, \quad (3.22)$$

Here $\mathcal{F}$ is the Fourier transform and $\gamma_0(\mu)$ is the trace operator given by

$$\gamma_0(\mu)f = \begin{pmatrix} f(-\mu^{1/2}) \\ f(\mu^{1/2}) \end{pmatrix}, \;\; f \in H^{s,0}(\mathbf{R}_\xi), \;\; s > 1/2, \;\; \mu = \xi^2.$$

(A proof of (3.22) can be found in [13, Section 10.4]). The scattering matrix $S(\lambda)$ on I_1 is a unitary operator in $\mathcal{B}(\mathbf{C}^2)$. (The spectral multiplicity changes from two to four at the threshold 1). Via (3.22) and the asymptotic expansions of the resolvent we derive asymptotic expansions for the scattering matrix $S(\lambda)$ for the (H, H_0) as $\lambda \downarrow 0$. For this purpose we need expansions for the operators $\gamma_0(\lambda^{1/2})\mathcal{F}$ and $\mathcal{F}^*\gamma_0(\lambda^{1/2})^*$. Formally, we have

$$\gamma_0(\lambda^{1/2})\mathcal{F} = \sum_{j=0}^{\infty}(i\lambda^{1/2})^j\Gamma_j, \quad (3.23)$$

where

$$\Gamma_j \; : \; (2\pi)^{-1/2}(j!)^{-1}\begin{pmatrix} (-x)^j \\ x^j \end{pmatrix}. \quad (3.24)$$

This follows from a formal expansion of

$$\gamma_0(\lambda^{1/2})\mathcal{F} \; : \; (2\pi)^{-1/2}\begin{pmatrix} \exp(-i\lambda^{1/2}x) \\ \exp(i\lambda^{1/2}x) \end{pmatrix}.$$

We see that $\Gamma_j \in \mathcal{B}(L^{2,s}(\mathbf{R}), \mathbf{C}^2)$, $s > j + 1/2$. Expansion (3.23) is valid as $\lambda \downarrow 0$ in the sense that if $\gamma_0(\lambda^{1/2})\mathcal{F}$ is approximated by a finite series up to $j = k$, k being the largest integer satisfying $s > k + 1/2$, then the remainder is $\mathrm{o}(\lambda^{k/2})$ in the norm of $\mathcal{B}(L^{2,s}(\mathbf{R}), \mathbf{C}^2)$.

Theorem 3.18. *Let Assumption 3.1 hold with $\beta > 9$. Let Assumption 3.4(i) hold. Assume that $0 \in \rho(H_2)$. Then, generically, we have in the norm of $\mathcal{B}(\mathbf{C}^2)$ the asymptotic expansion*

$$S(\lambda) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + o(1) \tag{3.25}$$

as $\lambda \downarrow 0$.

Proof. Let s satisfy $9/2 < s < \beta - 9/2$ and let R_j, $j = 0, 1$, denote the coefficients in (3.8). From (3.22), (3.23) and Theorem 3.7 we have the expansion $S(\lambda) = -i\lambda^{-1/2}S_{-1} + S_0 + \mathrm{o}(1)$ in $\mathcal{B}(\mathbf{C}^2)$, where

$$\begin{aligned} S_{-1} &= \pi\Gamma_0(V - VR_0V)\Gamma_0^*, \\ S_0 &= 1_\lambda + \pi\Gamma_1(V - VR_0V)\Gamma_0^* - \pi\Gamma_0 VR_1V\Gamma_0^* - \pi\Gamma_0(V - VR_0V)\Gamma_1^*. \end{aligned}$$

Using $1 = S(\lambda)S(\lambda)^*$ and the simple fact that $T^2 = 0$ implies that $T = 0$ for any self-adjoint operator T, we obtain that $\Gamma_0(V - VR_0V)\Gamma_0^* = 0$. Thus $S_{-1} = 0$. As for S_0 we begin by rewriting the term $\pi\Gamma_0VR_1V\Gamma_0^*$ via the expression for Γ_0 in (3.24) and the expression for R_1 given in Theorem 3.7. For any $(z_1, z_2) \in \mathbf{C}^2$, the operator acts as

$$\pi\Gamma_0VR_1V\Gamma_0^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = c \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where

$$\begin{aligned} c = \frac{1}{2}\langle & V_{11}B_1^{(0)}V_{11}1 + V_{11}B_0^{(0)}V_{12}a_0V_{21}B_1^{(0)}V_{11}1 \\ & + V_{11}B_0^{(0)}V_{12}a_0V_{21}B_0^{(0)}V_{11}1 + V_{11}B_0^{(0)}V_{12}a_1V_{21}B_0^{(0)}V_{11}1 \\ & - V_{11}B_0^{(0)}V_{12}a_1V_{21}1 - V_{11}B_1^{(0)}V_{12}a_0V_{21}1 - V_{12}a_1V_{21}B_0^{(0)}V_{11}1 \\ & - V_{12}a_0V_{21}B_1^{(0)}V_{11}1 + V_{12}a_1V_{21}1 + V_{11}B_0^{(0)}V_{12}a_0V_{21}B_1^{(0)}V_{12}1 \\ & + V_{11}B_0^{(0)}V_{12}a_0V_{21}B_0^{(0)}V_{12}1 + V_{11}B_0^{(0)}V_{12}a_1V_{21}B_0^{(0)}V_{21}1 \\ & - V_{11}B_0^{(0)}V_{12}a_1V_{22}1 - V_{11}B_1^{(0)}V_{12}a_0V_{22}1 - V_{12}a_1V_{21}B_0^{(0)}V_{12}1 \\ & - V_{12}a_0V_{21}B_1^{(0)}V_{12}1 + V_{12}a_1V_{22}1, 1\rangle. \end{aligned} \tag{3.26}$$

The operator $\Gamma_1(V - VR_0V)\Gamma_0^*$ can be written as a matrix with real elements. Therefore, for some real number a we find that

$$\{\pi\Gamma_1(V - VR_0V)\Gamma_0^* - \pi\Gamma_0(V - VR_0V)\Gamma_1^*\} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

since the terms on the left-hand side are each other adjoints. Hence,

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - c \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}.$$

By the unitarity of S_0, we infer that $a = 0$ and either $c = 0$ or $c = 1$. We show that $c = 1$. First, we observe that c depends continuously on V_{11}, V_{12}, V_{21} and V_{22}, hence it suffices to consider the case where $V_{12} = V_{21} = 0$. Moreover, only the first term on the right-hand side of (3.26) remains. Thus, in order to compute c we need the expression for $B_1^{(0)}$ which is given in [14, Theorem 4]. Using this expression we find that $\langle V_{11} B_1^{(0)} V_{11}, 1\rangle = 2$. Hence $c = 1$ as desired. □

In a similar way we establish the following theorems which are the main theorems on scattering theory herein.

Theorem 3.19. *Let Assumption 3.1 hold with $\beta > 11$. Let Assumption 3.4 be satisfied. Assume that $\alpha_0 \neq 0$. Then, generically, we have in the norm of $\mathcal{B}(\mathbf{C}^2)$ the asymptotic expansion (3.25) as $\lambda \downarrow 0$.*

Theorem 3.20. *Let Assumption 3.1 hold with $\beta > 15$. Let Assumption 3.4 be satisfied. Assume that $\alpha_0 = 0$ and $\gamma_0 \neq 0$. Then, generically, we have in the norm of $\mathcal{B}(\mathbf{C}^2)$ the asymptotic expansion (3.25) as $\lambda \downarrow 0$.*

Remark 3.21. In principle, the method allows us to derive an asymptotic expansion of the scattering matrix under Assumption 3.10. However, we would need to derive an expansion of the resolvent of the one-dimensional Schrödinger operator H_1 up to an error term of order $\mathrm{O}(|\zeta|)$ via the method used in [14]. In practice this turns out to be extremely tedious and complicated to do and, as a consequence, we have not succeeded in doing so.

4. Higher-dimensional Schrödinger operators as component Hamiltonians

Consider the two-channel Hamiltonian

$$H = H_0 + V(x) = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + 1 \end{pmatrix} + \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix} \tag{4.1}$$

acting in $\mathcal{H} = L^2(\mathbf{R}^d) \oplus L^2(\mathbf{R}^d)$, $d > 1$, where $-\Delta$ is the d-dimensional Laplacian and the $V_{ij}(x)$, $x \in \mathbf{R}^d$, $i, j = 1, 2$, are real-valued, electric potentials decaying as $\mathrm{O}(|x|^{-\beta})$ at infinity for some $\beta > 2$. For $d \geq 3$ odd Jensen-Kato [8] and Jensen [5] (see also [1]) have deduced asymptotic expansions of the resolvent of d-dimensional Schrödinger operators $-\Delta + V(x)$. For $\operatorname{Im}\zeta \geq 0$, $\operatorname{Im} \geq 0$ and $|\zeta| \to 0$, the expansions take the form

$$R(\zeta) = -\zeta^{-1} B_{-2} - i\zeta^{-1/2} B_{-1} + B_0 + i\zeta^{1/2} B_1 + \cdots, \tag{4.2}$$

valid in the norm topology of $\mathcal{B}(H^{-1,s}(\mathbf{R}^d), H^{1,-s}(\mathbf{R}^d))$, provided $V(x)$ decays sufficiently rapidly and s is large enough. The assumptions in the abstract theory in [16] is modelled on such expansions. Consequently, the methods in [16] and this companion paper in conjunction with the expansions (4.2) allows one to derive asymptotic expansions of the resolvent of H in (4.1) as well as the scattering matrix associated with the pair (H, H_0) as the spectral and energy parameters tend to zero. Furthermore, it is easy to extend the results to cover the even-dimensional case. From Jensen [5] one has for such Schrödinger operators in dimensions $d \geq 6$ an expansion of the form

$$R(\zeta) = -\zeta^{-1}P_0 - \ln\zeta B_{-1}^0 + B_0^0 + \zeta(\ln\zeta)^2 B_1^2 + \zeta\ln\zeta B_1^1 + \zeta B_1^0 + o(\zeta)$$

as $\zeta \to 0$. We always have $B_{-1}^0 P_0 = B_{-1}^0$, and generically $P_0 = 0$, i.e. zero is not an eigenvalue. In dimensions $d \geq 5$ there is no zero resonance (half-bound state). Similar expansions hold in dimensions $d = 2, 4$, but here additionally the zero resonance may occur [6, 3]. The abstract arguments in [16] can clearly be adapted to cover this type of expansion.

5. 3D Schrödinger operator with a constant magnetic field

The results obtained in Section 3 can be carried over to a Schrödinger operator in $L^2(\mathbf{R}^3)$ with a constant magnetic field and an axisymmetrical electric potential. Under these assumptions the operator can be represented in a multi-channel framework. For the lowest Landau level we can fit the problem into the two-channel framework considered here. It requires considerable preparation to apply our results. Preliminary results on this interesting application are contained in [13]. Complete results will be published elsewhere.

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