A strict inequality for the random triangle model

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November 10, 2000

Abstract

The random triangle model on a graph G, is a random graph model where the usual i.i.d. measure is perturbed by a factor $q^{t(\omega)}$, where $q \geq 1$ is a constant, and $t(\omega)$ is the number of triangles in the random subgraph ω. Here we consider the case where G is the usual two-dimensional triangular lattice, for which there exists a percolation threshold $p_c(q)$ such that the probability of getting an infinite connected component of retained edges is 0 for $p < p_c(q)$, and 1 for $p > p_c(q)$. It has previously been shown that $p_c(q)$ is a decreasing function of q. Here we strengthen this by showing that $p_c(q)$ is *strictly* decreasing. This confirms a conjecture by Häggström and Jonasson.

1 Introduction

In the standard random graph model, each edge of a graph $G = (V, E)$ is removed independently with the same probability $1 - p$, for some $p \in (0, 1)$. Motivated by the transitivity phenomenon in social networks (friends of friends are often friends as well), Jonasson [7] generalized this to obtain the so-called **random triangle model**. This model arises by biasing the product measure in the i.i.d. model by a factor $q^{t(\omega)}$, where $q \geq 1$ is a constant, and $t(\omega)$ is the number of triangles in the subgraph ω of G. The focus in [7] is on the case where G is the complete graph. Häggström and Jonasson [5] instead considered the case where G is the two-dimensional triangular lattice (denoted $\mathbf{T}$) which is also the case we consider in this paper.

Because the graph we are dealing with is infinite, the definition of the random triangle model becomes less straightforward than in the finite case. The definition in [5] uses the so-called DLR (Dobrushin–Lanford–Ruelle) approach to infinite-volume Gibbs measures. The details of this are deferred to Section 2.

As in [5], we are interested in the percolation-theoretic question of whether the random triangle model on $\mathbf{T}$ produces an infinite connected component of retained edges. The answer turns out to depend on the parameters p and q in a manner described in the following result from [5].

Theorem 1.1 (Häggström and Jonasson) *There exists a function $p_c : [1, \infty) \to [0, 1]$ such that any Gibbs measure for the random triangle model on $\mathbf{T}$ with parameters*

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$p \in (0,1)$ and $q \geq 1$ produces an infinite connected component with probability

$$\begin{cases} 0 & \text{if } p < p_c(q) \\ 1 & \text{if } p > p_c(q)\,. \end{cases}$$

The function $p_c(q)$ satisfies

$$p_c(q) = \begin{cases} 2\sin(\frac{\pi}{18}) & \text{for } q = 1 \\ (q-1)^{-2/3} & \text{for } q \geq 27 + 15\sqrt{3}\,. \end{cases}$$

Furthermore, the function is continuous and decreasing throughout $[1, \infty)$.

(The fact that $p_c(1) = 2\sin(\frac{\pi}{18})$ goes back to Wierman [9].) Regarding the behavior of $p_c(q)$ on $(1, 27+15\sqrt{3})$, nothing else beyond continuity and decreasingness was obtained in [5]. Note that Theorem 1.2 precludes the possibility that the formula $p_c(q) = (q-1)^{-2/3}$ extends throughout that interval. It was conjectured in [5] that $p_c(q)$ is *strictly* decreasing, and the purpose of this note is to prove that conjecture.

Theorem 1.2 *The function $p_c(q)$, as defined in Theorem 1.1, is strictly decreasing throughout the interval $[1, 27 + 15\sqrt{3}]$.*

Some related strict inequalities for other models have been obtained (using other, less elementary, methods), e.g., by Aizenman and Grimmett [1] and Bezuidenhout, Grimmett and Kesten [2].

In the next section, we give a more precise description of the model. After some preliminaries on stochastic domination in Section 3, we prove our main result (Theorem 1.2) in Section 4. Finally, in Section 5, we mention some extensions of the result.

2 The model

The triangular lattice $\mathbf{T} = (V, E)$ is defined as the graph with vertex set

$$V = \left\{(x,y) \in \mathbf{R}^2 : \frac{x}{\sqrt{3}} \in \mathbf{Z}, y \in \mathbf{Z}\right\} \cup \left\{(x,y) \in \mathbf{R}^2 : \frac{x}{\sqrt{3}} - \frac{1}{2} \in \mathbf{Z}, y - \frac{1}{2} \in \mathbf{Z}\right\},$$

and edge set E consisting of pairs $\langle u, v\rangle$ of vertices at Euclidean distance 1 from each other. A subset of E is identified in the natural way with an element of $\{0,1\}^E$.

We use the DLR approach (see, e.g., [3] for a general introduction) to defining the random triangle model on $\mathbf{T}$. That is, we consider a probability measure on $\{0,1\}^E$ to be a Gibbs measure for the random triangle model if it satisfies a certain set of desired conditional distributions on finite subsets of E. For $p \in (0,1)$, $q \geq 1$, a finite subset S of E, and an edge configuration $\eta \in \{0,1\}^{E \setminus S}$, we define $\mu^{p,q}_{S,\eta}$ as the probability measure on $\{0,1\}^S$ which to each $\xi \in \{0,1\}^S$ assigns probability

$$\mu^{p,q}_{S,\eta}(\xi) = \frac{1}{Z^{p,q}_{S,\eta}} q^{t_S(\xi \vee \eta)} \prod_{e \in S} p^{\xi(e)}(1-p)^{1-\xi(e)} \tag{1}$$

Here $(\xi \vee \eta) \in \{0,1\}^E$ is the configuration which agrees with ξ on S and with η on $S \setminus E$, $t_S(\xi \vee \eta)$ is the number of triangles in $(\xi \vee \eta)$ that have at least one edge in S, and $Z^{p,q}_{S,\eta}$ is a normalizing constant.

Definition 2.1 *A probability measure μ on $\{0,1\}^E$ is said to be a Gibbs measure for the random triangle model on $\mathbf{T}$ with parameters $p \in (0,1)$ and $q \geq 1$ if it admits conditional probabilities such that a $\{0,1\}^E$-valued random object Y with distribution μ has the following property: For every finite $S \subset E$ and every $\eta \in \{0,1\}^{E \setminus S}$, the conditional distribution of $Y(S)$, given the event $\{Y(E \setminus S) = \eta\}$, equals $\mu^{p,q}_{S,\eta}$.*

For the set of parameter values we are interested in, i.e., $q \in [1, 27+15\sqrt{3}]$, it was shown in [5] that the random triangle model on $\mathbf{T}$ with parameters p and q has a unique Gibbs measure; we denote this Gibbs measure by $\mu^{p,q}_{\mathbf{T}}$. (For $q > 27+15\sqrt{3}$, it was shown that there is a unique Gibbs measure for $p \neq (q-1)^{-2/3}$, and multiple Gibbs measures for $p = (q-1)^{-2/3}$.)

Note that the random triangle model exhibits the following Markov random field property. For a finite $S \subset E$, define ∂S to be the set of edges in $E \setminus S$ that share some triangle with some edge in E. We then have, for any $\{0,1\}^E$-valued random object X distributed according to a Gibbs measure for the random triangle model, any $\xi \in \{0,1\}^S$ and any $\eta \in \{0,1\}^{E \setminus S}$, that

$$\mathbf{P}(X(S) = \xi \mid X(E \setminus S) = \eta) = \mathbf{P}(X(S) = \xi \mid X(\partial S) = \eta(\partial S)) .$$

This is immediate from (1) and Definition 2.1.

For a vertex $x \in V$, define the edge set T_x as

$$T_x = \left\{ \left\langle x, x+(1,0) \right\rangle, \left\langle x, x+(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \right\rangle, \left\langle x+(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}), x+(1,0) \right\rangle \right\}, \qquad (2)$$

and note that $\{T_x\}_{x \in V}$ is a partitioning of E into triangles. We will need an increasing sequence of finite edge sets $\Lambda_1 \subset \Lambda_2 \subset \ldots$ converging to E in the sense that each edge $e \in E$ is in all but finitely many of the Λ_i's. For concreteness, we take (somewhat arbitrarily)

$$\Lambda_n = \bigcup_{\substack{x \in V \\ |x| \leq n}} T_x$$

where $|x|$ is Euclidean norm. The fact that $\mu^{p,q}_{\mathbf{T}}$ is the unique Gibbs measure for the random triangle model with given parameter values, implies that $\mu^{p,q}_{\Lambda_n,\eta_n}$ converges in distribution to $\mu^{p,q}_{\mathbf{T}}$, for any sequence of boundary conditions η_n.

3 Stochastic domination

The concept of stochastic domination will play a leading role in our proof of Theorem 1.2. We begin with some basic definitions. Let S be a finite or countable set. For two configurations $\xi, \xi' \in \{0,1\}^S$, we write $\xi \preceq \xi'$ if $\xi(s) \leq \xi'(s)$ for all $s \in S$. A function $f : \{0,1\}^S \to \mathbf{R}$ is said to be increasing if $f(\xi) \leq f(\xi')$ whenever $\xi \preceq \xi'$. An event $A \subseteq \{0,1\}^S$ is said to be increasing if its indicator function is increasing. We say that A is nontrivial if $A \notin \{\emptyset, \{0,1\}^S\}$. Write $\mathcal{A}_S$ for the class of increasing events in $\{0,1\}^S$, and write $\mathcal{A}^*_S$ for the class of nontrivial increasing events in $\{0,1\}^S$.

For two probability measures μ and μ' on $\{0,1\}^S$, we say that μ' **stochastically dominates** μ, writing $\mu \preceq_d \mu'$, if

$$\int_{\{0,1\}^S} f \, d\mu \leq \int_{\{0,1\}^S} f \, d\mu' \qquad (3)$$

for all bounded increasing functions $f : \{0,1\}^S \to \mathbf{R}$. By a well-known theorem of Strassen (see, e.g., [8]), this is equivalent to the existence of two $\{0,1\}^S$-valued random objects X and X' defined jointly on the same probability space, with the properties that X has distribution μ, X' has distribution μ', and

$$\mathbf{P}(X \preceq X') = 1\,.$$

A common tool for proving stochastic domination and monotonicity is Holley's Theorem (see [3] for an extensive discussion):

Theorem 3.1 (Holley) *Let S be finite, and let μ and μ' be probability measures on $\{0,1\}^S$ that assign positive probability to all elements of $\{0,1\}^S$. Suppose that for all $s \in S$ and all $\xi, \xi' \in \{0,1\}^{S\setminus\{s\}}$ such that $\xi \preceq \xi'$, we have*

$$\mu(X(s) = 1 \,|\, X(S \setminus \{s\}) = \xi) \le \mu'(X'(s) = 1 \,|\, X(S \setminus \{s\}) = \xi')\,.$$

Then $\mu \preceq_d \mu'$.

This theorem was applied in [5], to prove, among others, the following result for the random triangle model. Recall the definition of $\mu^{p,q}_{S,\eta}$ in (1).

Proposition 3.2 (Häggström and Jonasson) *Let $S \subset E$ be finite, and let $\eta, \eta' \in \{0,1\}^{E\setminus S}$ be configurations satisfying $\eta \preceq \eta'$. Then, for any $p \in (0,1)$ and $q \ge 1$, we have*

$$\mu^{p,q}_{S,\eta} \preceq_d \mu^{p,q}_{S,\eta'}\,.$$

More generally, if $p \le p'$ and $q \le q'$, we have

$$\mu^{p,q}_{S,\eta} \preceq_d \mu^{p',q'}_{S,\eta'}\,. \tag{4}$$

By taking $p = p'$, $q \le q'$, $S = \Lambda_n$ and taking limits as $n \to \infty$, the decreasingness statement concerning $p_c(q)$ in Theorem 1.1 is easily obtained. However, to prove *strict* decreasingness of $p_c(q)$, one needs to prove (4) for values of p, p', q and q' that are not covered by Proposition 3.2; see Proposition 4.3 in the next section. In fact, Proposition 4.3 cannot be obtained by direct application of Holley's Theorem. Instead, we shall use the following generalization of Holley's Theorem, where the single-edge conditional distribution assumption is replaced by an analogous assumption for triangles.

Proposition 3.3 *Let $T = T_x$ be any triangle as in (2). Suppose that $p, p' \in (0,1)$ and $q, q' \ge 1$ are chosen in such a way that for any $\xi \in \{0,1\}^{E\setminus T}$ we have*

$$\mu^{p,q}_{T,\xi} \preceq_d \mu^{p',q'}_{T,\xi}\,. \tag{5}$$

Then

$$\mu^{p,q}_{\Lambda_n,\eta} \preceq_d \mu^{p',q'}_{\Lambda_n,\eta} \tag{6}$$

for any n and any $\eta \in \{0,1\}^{E\setminus\Lambda_n}$.

Proof: We shall exploit a similar Markov chain coupling idea as in the standard dynamical proof of Holley's Theorem (see [3]). Let p, p', q, q', n and η be as in the proposition.

We can define a $\{0,1\}^{\Lambda_n}$-valued Markov chain $\{Y_k\}_{k=0}^{\infty}$ as follows. First pick the initial configuration $Y_0 \in \{0,1\}^{\Lambda_n}$ according to $\mu^{p,q}_{\Lambda_n,\eta}$. Then, for each k, Y_{k+1} is obtained from Y_k by picking a triangle (as in (2)) $T^k \subset \Lambda_n$ at random (according to uniform distribution and independent for different values of k), setting $Y_{k+1}(e) = Y_k(e)$ for all $e \in \Lambda_n \backslash T^k$, and picking $Y_{k+1}(T^k)$ according to the conditional distribution $\mu^{p,q}_{T^k,(Y_k(\Lambda_n \backslash T^k)\vee\eta)}$. It is easy to see that this transition rule (which is a special case of the well-known Gibbs sampler) preserves the initial distribution $\mu^{p,q}_{\Lambda_n,\eta}$.

Analogously, define the Markov chain $\{Y'_k\}_{k=0}^{\infty}$ as follows. First pick $Y'_0 \in \{0,1\}^{\Lambda_n}$ according to $\mu^{p',q'}_{\Lambda_n,\eta}$. Then Y'_{k+1} is obtained from Y'_k by picking a triangle $T^k \subset \Lambda_n$ at random (again uniformly and independent for different values of k), setting $Y'_{k+1}(e) = Y'_k(e)$ for all $e \in \Lambda_n \setminus T^k$, and picking $Y'_{k+1}(T^k)$ according to $\mu^{p',q'}_{T^k,(Y'_k(\Lambda_n \backslash T^k)\vee\eta)}$. This transition mechanism preserves the initial distribution $\mu^{p',q'}_{\Lambda_n,\eta}$.

Our next aim is to describe how the chains $\{Y_k\}_{k=0}^{\infty}$ and $\{Y'_k\}_{k=0}^{\infty}$ should be run jointly (coupled) on the same probability space. For any $T = T_x \subset \Lambda_n$ and any $\xi, \xi' \in \{0,1\}^{\Lambda_n \backslash E_x}$, we define a probability measure $P_{T,\xi,\xi'}$ on $\{0,1\}^T \times \{0,1\}^T$ satisfying the following properties:

(i) The first and second marginals of $P_{T,\xi,\xi'}$ equal $\mu^{p,q}_{T,(\xi\vee\eta)}$ and $\mu^{p',q'}_{T,(\xi'\vee\eta)}$, respectively.

(ii) For all ξ and ξ' such that $\xi \preceq \xi'$, the measure $P_{T,\xi,\xi'}$ puts all of its probability mass on the set
$$\{(\zeta, \zeta') \in \{0,1\}^T \times \{0,1\}^T : \zeta \preceq \zeta'\}.$$

Property (i) is what makes $P_{T,\xi,\xi'}$ a coupling of $\mu^{p,q}_{T,(\xi\vee\eta)}$ and $\mu^{p',q'}_{T,(\xi'\vee\eta)}$. Property (ii) is our key use of the assumption (5). To see that a coupling satisfying (i) and (ii) exists, note that for $\xi \preceq \xi'$ we have
$$\mu^{p',q'}_{T,\xi} \preceq_d \mu^{p',q'}_{T,\xi'}$$
as a special case of Proposition 3.2. This, in combination with the assumption (5), implies that
$$\mu^{p,q}_{T,\xi} \preceq_d \mu^{p',q'}_{T,\xi'}$$
so that, by Strassen's Theorem, the desired coupling exists.

The chains $\{Y_k\}_{k=0}^{\infty}$ and $\{Y'_k\}_{k=0}^{\infty}$ can now be run jointly as follows. First take the initial values Y_0 and Y'_0 to be independent. Then synchronize the evolution of the two chains by letting, at each time k, the two chains pick the same triangle T^k to update. The pair $(Y_{k+1}(T^k), Y'_{k+1}(T^k))$ is chosen according to the probability measure $P_{T^k,(Y_k(\Lambda_n \backslash T^k)\vee\eta),(Y'_k(\Lambda_n \backslash T^k)\vee\eta)}$. It is easy to see that this gives the correct marginal behavior for each of the two chains. By property (ii), we also have that if $Y_K \preceq Y'_K$ for some K, then this ordering is preserved, i.e., $Y_k \preceq Y'_k$ for all $k \geq K$. If we now take K to be the smallest k for which Y'_k takes its maximal value (i.e., $Y'_k(e) = 1$ for all $e \in \Lambda_n$), then K is finite a.s., due to irreducibility of the chain $\{Y'_k\}_{k=0}^{\infty}$. Clearly, $Y_K \preceq Y'_K$ holds for that choice of K, and we can conclude that
$$\lim_{k\to\infty} \mathbf{P}(Y_k \preceq Y'_k) = 1\,.$$

Hence, the limiting distribution as $k \to \infty$ of the pair (Y_k, Y'_k) is a coupling which (by Strassen's Theorem) establishes (6). $\square$

4 Proof of main result

The key lemma, which will provide us with (5) in Proposition 3.3, is the following.

Lemma 4.1 *Let $T = T_x$ be any triangle as in (2). For any $p \in (0,1)$, any $q \geq 1$ and any $\varepsilon > 0$, we can find a $\delta > 0$ such that the following holds. For any $\xi \in \{0,1\}^{E \setminus T}$, we have*

$$\mu_{T,\xi}^{p,q} \preceq_d \mu_{T,\xi}^{p-\delta,q+\varepsilon}. \tag{7}$$

Proof: Enumerate the edges of T as $\{e_1, e_2, e_3\}$, and the edges of ∂T as $\bigcup_{i=1}^{3}\{e_i^1, e_i^2\}$, where for each i the set $\{e_i, e_i^1, e_i^2\}$ forms a triangle. Also set $\Delta_i = \xi(e_i^1)\xi(e_i^2)$ for $i = 1, 2, 3$. By direct application of the definition (1), we have that the measure $\mu_{T,\xi}^{p,q}$ assigns probability

$$\mu_{T,\xi}^{p,q}(\zeta) = \frac{(1-p)^3}{Z_{T,\xi}^{p,q}} q^{\zeta(e_1)\zeta(e_2)\zeta(e_3)} \prod_{i=1}^{3} \left(\frac{pq^{\Delta_i}}{1-p}\right)^{\zeta(e_i)} \tag{8}$$

to each $\zeta \in \{0,1\}^T$.

We next define an auxiliary probability measure $\tilde{\mu}_{T,\xi}^{p,q,\tilde{q}}$ on $\{0,1\}^T$ with the extra parameter $\tilde{q} \geq 1$. For each $\zeta \in \{0,1\}^T$, set

$$\tilde{\mu}_{T,\xi}^{p,q,\tilde{q}}(\zeta) = \frac{(1-p)^3}{\tilde{Z}_{T,\xi}^{p,q,\tilde{q}}} \tilde{q}^{\zeta(e_1)\zeta(e_2)\zeta(e_3)} \prod_{i=1}^{3} \left(\frac{pq^{\Delta_i}}{1-p}\right)^{\zeta(e_i)}, \tag{9}$$

where of course $\tilde{Z}_{T,\xi}^{p,q,\tilde{q}}$ is another normalizing constant. Note that by setting $\tilde{q} = q$ in (9), we simply recover $\mu_{T,\xi}^{p,q}$. Note also that $\mu_{T,\xi}^{p,q}$ and $\tilde{\mu}_{T,\xi}^{p,q,\tilde{q}}$ depend on ξ only through Δ_1, Δ_2 and Δ_3.

Now take $\varepsilon > 0$ as in the lemma, and let $B \subset \{0,1\}^T$ be the event that all three edges in T are present. Our preliminary aim is to compare $\mu_{T,\xi}^{p,q}(B)$ and $\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(B)$. A direct calculation using (8) and (9) gives

$$\frac{\mu_{T,\xi}^{p,q}(B)\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(\neg B)}{\mu_{T,\xi}^{p,q}(\neg B)\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(B)} = \frac{q}{q+\varepsilon} < 1.$$

Hence

$$\frac{\mu_{T,\xi}^{p,q}(B)}{\mu_{T,\xi}^{p,q}(\neg B)} < \frac{\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(B)}{\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(\neg B)},$$

so that

$$\mu_{T,\xi}^{p,q}(B) < \tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(B). \tag{10}$$

Now let $A \in \mathcal{A}_T^*$ be a nontrivial increasing event in $\{0,1\}^T$, and note that $\neg A \subseteq \neg B$ for any such A. Another direct application of (8) and (9) gives that $\frac{\mu_{T,\xi}^{p,q}(\zeta)}{\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(\zeta)}$ is the same for all $\zeta \in \neg B$. This implies that

$$\frac{\mu_{T,\xi}^{p,q}(\neg A)}{\mu_{T,\xi}^{p,q}(\neg B)} = \frac{\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(\neg A)}{\tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(\neg B)}.$$

Using (10), we therefore get that

$$\mu_{T,\xi}^{p,q}(A) < \tilde{\mu}_{T,\xi}^{p,q,q+\varepsilon}(A) \tag{11}$$

for all ξ and all $A \in \mathcal{A}_T^*$. Now, the right hand side of (11) is of course continuous in p, so the inequality still holds with $p-\delta$ in place of p in the right hand side, for some sufficiently small $\delta > 0$. Since there are only finitely many $A \in \mathcal{A}_T^*$, and effectively only finitely many boundary conditions ξ (because ξ only influences the probabilities via $(\Delta_1, \Delta_2, \Delta_3)$), we can in fact find a $\delta > 0$ such that, uniformly in ξ and $A \in \mathcal{A}_T^*$, we have

$$\mu_{T,\xi}^{p,q}(A) \leq \tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon}(A)\,. \tag{12}$$

Trivially, the same holds for all ξ and all $A \in \mathcal{A}_T$. Now we can show that $\mu_{T,\xi}^{p,q}$ is stochastically dominated by $\tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon}$. Indeed, let f be any increasing function from $\{0,1\}^T$ to $\mathbf{R}$. Clearly, f takes only finitely many values, say $a_0 < a_1 < \cdots < a_m$. For $k = 1, \ldots, m$, define the event

$$A_k = \{\omega \in \{0,1\}^T : f(\omega) \geq a_k\}\,,$$

which allows us to rewrite $\int_{\{0,1\}^T} f \, d\mu_{T,\xi}^{p,q}$ as

$$\int_{\{0,1\}^T} f \, d\mu_{T,\xi}^{p,q} = a_0 + \sum_{k=1}^{m} (a_k - a_{k-1}) \mu_{T,\xi}^{p,q}(A_k)\,,$$

and similarly for $\int_{\{0,1\}^T} d\tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon}$. The events $A_1, \ldots, A_k$ are of course increasing, so that, using (12), we get

$$\begin{aligned}
&\int_{\{0,1\}^T} f \, d\tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon} - \int_{\{0,1\}^T} f \, d\mu_{T,\xi}^{p,q} \\
&= \sum_{k=1}^{m} (a_k - a_{k-1}) \left(\tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon}(A_k) - \mu_{T,\xi}^{p,q}(A_k) \right) \\
&\geq 0\,,
\end{aligned}$$

and (3) is verified with $\mu = \mu_{T,\xi}^{p,q}$ and $\mu' = \tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon}$. We have thus established that

$$\mu_{T,\xi}^{p,q} \preceq_d \tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon}\,. \tag{13}$$

By an application of Holley's Theorem (we omit the trivial calculations of single-edge conditional probabilities needed to check that the conditions in the theorem are fulfilled), we furthermore get

$$\tilde{\mu}_{T,\xi}^{p-\delta,q,q+\varepsilon} \preceq_d \mu_{T,\xi}^{p-\delta,q+\varepsilon}\,. \tag{14}$$

By combining (13) and (14), we get (7), as desired. □

Closely similar to Lemma 4.1 is the next result, where we decrease (rather than increase) q by ε.

Lemma 4.2 *Let $T = T_x$ be any triangle as in (2). For any $p \in (0,1)$, any $q > 1$ and any $\varepsilon > 0$, we can find a $\delta > 0$ such that the following holds. For any $\xi \in \{0,1\}^{E \setminus T}$, we have*

$$\mu_{T,\xi}^{p,q} \succeq_d \mu_{T,\xi}^{p+\delta,q-\varepsilon}\,.$$

Proof: Follows by a completely straightforward modification (basically just the reversal of a few of the inequalities) of the proof of Lemma 4.1. $\square$

We now move on from considering a single triangle T, to larger subgraphs.

Proposition 4.3 *For any n, $\eta \in \{0,1\}^{E\setminus\Lambda_n}$, $p \in (0,1)$ and $q > 1$ we have the following.*

(i) *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\mu^{p,q}_{\Lambda_n,\eta} \preceq_d \mu^{p-\delta,q+\varepsilon}_{\Lambda_n,\eta}$.*

(ii) *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\mu^{p,q}_{\Lambda_n,\eta} \succeq_d \mu^{p+\delta,q-\varepsilon}_{\Lambda_n,\eta}$.*

Proof: Part (i) follows by applying Proposition 3.3 with $p' = p - \delta$ and $q' = q + \varepsilon$, where ε and δ are as in Lemma 4.1. Similarly, part (ii) follows by combining Lemma 4.2 and Proposition 3.3. $\square$

In the next result, we let $\alpha(p,q)$ denote the $\mu^{p,q}_{\mathbf{T}}$-probability of having an infinite connected component of retained edges.

Corollary 4.4 *For any $p \in (0,1)$ and $q \in (1, 27 + 15\sqrt{3})$ we have the following.*

(i) *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\alpha(p,q) \le \alpha(p-\delta, q+\varepsilon)$.*

(ii) *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\alpha(p,q) \ge \alpha(p+\delta, q-\varepsilon)$.*

Proof: By taking the limit as $n \to \infty$ in Proposition 4.3 (i), we get that $\mu^{p,q}_{\mathbf{T}} \preceq_d \mu^{p-\delta,q+\varepsilon}_{\mathbf{T}}$, where ε and δ are as in Proposition 4.3 (i). Hence

$$\mu^{p,q}_{\mathbf{T}}(\exists \text{ an infinite connected component}) \le \mu^{p-\delta,q+\varepsilon}_{\mathbf{T}}(\exists \text{ an infinite connected component}),$$

because the existence of an infinite connected component is an increasing event. Part (i) of the corollary is therefore established. Part (ii) follows similarly. $\square$

Proof of Theorem 1.2: Suppose, for the sake of deriving a contradiction, that $p_c(q)$ fails to be strictly decreasing somewhere on $[1, 27 + 15\sqrt{3}]$. By Theorem 1.1, there then exists a closed interval $[q_1, q_2] \subset [1, 27 + 15\sqrt{3}]$, with $q_1 < q_2$, on which $p_c(q)$ is constant. Set $q^* = \frac{q_1+q_2}{2}$ and $p^* = p_c(q^*)$. Since $\mu^{p^*,q^*}_{\mathbf{T}}$ is the unique Gibbs measure for the random triangle model with the given parameter values, we have by general Gibbs theory that $\mu^{p^*,q^*}_{\mathbf{T}}$ is ergodic, so that in particular $\alpha(p^*, q^*) \in \{0,1\}$.

Suppose that $\alpha(p^*, q^*) = 1$. Then, by Corollary 4.4 (i), we have $\alpha(p^* - \delta, q_2) = 1$ for some $\delta > 0$. Hence, $p_c(q_2) \le p^* - \delta < p^* = p_c(q^*)$, which contradicts the constancy of $p_c(q)$ on $[q_1, q_2]$.

We can therefore conclude that $\alpha(p^*, q^*) = 0$. But Corollary 4.4 (ii) now gives us that $\alpha(p^* + \delta, q_1) = 0$ for some $\delta > 0$. This implies that $p_c(q_1) \ge p^* + \delta > p^* = p_c(q^*)$, and we again have a contradiction to the constancy of $p_c(q)$ on $[q_1, q_2]$. This completes the proof. $\square$

5 Some extensions

The main idea in our proof was to use a Holley-type inequality whose monotonicity assumption concerns not the conditional distribution of a single variable, but that of a larger (but still managable) set of variables. We hope that the usefulness of this idea is not limited to the particular setting of the present paper. Here we mention a couple of modest extensions of Theorem 1.2, for which the proof can immediately be adapted.

1. Rigidity percolation. Very few properties of the event

$$A_{conn} = \{\exists \text{ an infinite connected component}\}$$

were actually used in our proof. The key properties were that it is increasing, and that it has a nontrivial threshold $p_c(q)$. Another event of this type, that has received some attention recently, is

$$A_{rig} = \{\exists \text{ an infinite } \textit{rigid} \text{ component}\}\,.$$

See, e.g., Holroyd [6] or Häggström [4] for the definition of rigid components. For i.i.d. bond percolation on $\mathbf{T}$, it was shown in [6] that there is a critical value p_r, satisfying $p_c(1) < p_r < 1$, such that $\mathbf{P}(A_{rig})$ is 0 or 1 depending on whether p is below or above p_c. Using Holley's Theorem, it is easy to extend this to the random triangle model and show that the $\mu_{\mathbf{T}}^{p,q}$-probability of having an infinite rigid component is 0 or 1 depending on whether p is below or above $p_r(q)$, where $p_r(q)$ is a continuous and decreasing function of q. The methods of the present paper show that $p_r(q)$ is strictly decreasing.

2. The random square model. To consider the random triangle model on the square lattice $\mathbf{Z}^2$ (with edges connecting Euclidean nearest neighbors) is of course pointless, because $\mathbf{Z}^2$ contains no triangles. In [5], it was instead suggested to consider a random *square* model on $\mathbf{Z}^2$, where, in the defining formula analogous to (1), we take q raised to the number of squares rather than triangles. Again, arguments based on Holley's Theorem imply the existence of a continuous and decreasing function $p_c(q)$ such that the probability of having an infinite connected component is 0 if $p < p_c(q)$ and 1 if $p > p_c(q)$, and again our methods can easily be adapted to show that $p_c(q)$ is strictly decreasing.

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