# ON EXTREMES AND STREAMS OF UPCROSSINGS[1]

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We study relations between $\mathbf{P}\{\sup_{t\in[0,h]}\xi(t)>u\}$ and $h\,\underline{\lim}_{n\to\infty}2^n\mathbf{P}\{\xi(0)\le u<\xi(2^{-n})\}+\mathbf{P}\{\xi(0)>u\}$ for a stationary process $\xi(t)$. Applications include Markov jump processes, $\alpha$-stable processes, and quadratic functionals of Gaussian processes.

**1. Introduction.** Let $\{\xi(t)\}_{t\in[0,h]}$ be a real valued stationary stochastic process defined on a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$, where $h>0$ is a constant. Suppose that $\xi(t)$ is separable and continuous in probability ($\mathbf{P}$-continuous).

Let the support of $\xi(0)$ have endpoints $\underline{u}\equiv\inf\{u\in\mathbb{R}:\mathbf{P}\{\xi(0)\le u\}>0\}$ and

$$\overline{u}\;\equiv\;\sup\{u\in\mathbb{R}:\mathbf{P}\{\xi(0)\le u\}<1\}\qquad\text{where}\qquad\mathbf{P}\{\xi(0)=\overline{u}\}=0,\tag{1.1}$$

so that $\underline{u}<\overline{u}$. Set $M(h)\equiv\sup_{t\in[0,h]}\xi(t)$ and $J(u)\equiv\underline{\lim}_{n\to\infty}J(2^{-n},u)$ where

$$J(q,u)\;\equiv\;q^{-1}\,\mathbf{P}\{\xi(0)>u\ge\xi(q)\}\;=\;q^{-1}\,\mathbf{P}\{\xi(q)>u\ge\xi(0)\}\,.$$

If $\xi(t)$ is continuous a.s. with $\xi(0)$ continuously distributed, then $J(u)$ coincides with the upcrossing intensity $\mu(u)$ of the level $u$ by $\xi(t)$. Further we have

$$\mathbf{P}\{M(h)>u\}\;\le\;\mathbf{P}\{\xi(0)>u\}\;+\;h\,\mu(u):$$

This relation and work on formulae for $\mu(u)$ date back to Rice (1945). See also e.g., Leadbetter, Lindgren and Rootzén (1983, Chapters 7 and 13) and Albin (1992).

However, it seems natural to use $J(u)$ instead of $\mu(u)$, since we show that

$$\mathbf{P}\{M(h)>u\}\;\le\;\mathbf{P}\{\xi(0)>u\}+h\,J(u)\;\equiv\;\Psi_h(u)\qquad\text{for}\qquad u\in(\underline{u},\overline{u})\tag{1.2}$$

assuming $\mathbf{P}$-continuity only. Further, making use of the quantity

$$J_s(q,u)\;\equiv\;\frac{1}{q}\,\mathbf{P}\Big\{\xi(0)>u\ge\xi(q),\;\bigcup_{\ell=2}^{[s/q]}\{\xi(\ell q)>u\}\Big\}\qquad\text{for}\quad s\in(0,h]$$

[1]Research supported by NFR Grants M-AA/MA 09207-311 and R-RA 09207-312.
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AMS 1991 subject classifications. Primary 60G70. Secondary 60G10, 60G15, 60J27, 60J75.
*Key words and phrases.* Extrema, upcrossing, jump process, stable process, Gaussian process.

[so that $J_0(q,u)=J(q,u)$], we give characterizations of when any of the converses

$$\lim_{u\uparrow\overline{u}} \mathbf{P}\{M(h)>u\}/\Psi_h(u) = 1 \tag{1.3}$$

$$\underline{\lim}_{u\uparrow\overline{u}} \mathbf{P}\{M(h)>u\}/\Psi_h(u) > 0 \tag{1.4}$$

to (1.2) apply: Of course, in order for (1.3) or (1.4) to hold it is necessary that

$$\text{there exists a} \quad u_0\in(\underline{u},\overline{u}) \quad \text{such that} \quad J(u)<\infty \quad \text{for} \quad u\in[u_0,\overline{u}). \tag{1.5}$$

In Theorem 1 we do discrete approximation with a grid that works for all processes $\xi(t)$. This gives general characterizations of (1.3) and (1.4). In a typical application one need not compute $J(u)$ but only prove (1.5), which often is feasible.

We use a more traditional approach with a grid adapted to $\xi(t)$ in Theorem 2. This requires an additional often hard-to-verify technical condition, but gives virtually weaker and potentially easier to verify necessary and sufficient conditions for (1.3) and (1.4). In Theorem 3 we show how the technical condition can be modified and thus sometimes more easily dealt with.

In Examples 1 and 2 we show how Theorems 1 and 2 connect to contemporary research on the argmax process of Brownian motion minus parabolic drift and on $\alpha$-stable processes. In Sections 4-7 we use Theorems 1 and 3 to prove new results for $\alpha$-stable processes, for pure-jump Markov processes, and for quadratic functionals (squared norms) of Gaussian processes.

**2. Relations between extremes and streams of upcrossings I.** First we prove (1.2) and give general characterizations of (1.3) and (1.4) in terms of

$$\varDelta_{s,h} \equiv \overline{\lim}_{u\uparrow\overline{u}}\, \underline{\lim}_{n\to\infty}\, h J_s(2^{-n},u)/\Psi_h(u).$$

**Theorem 1.** *If $\{\xi(t)\}_{t\in[0,h]}$ is a separable and $\mathbf{P}$-continuous stationary process satisfying (1.1), then (1.2) holds. If in addition (1.5) holds, then we have*

$$(1.3)\ \textit{holds} \quad \Leftrightarrow \quad \varDelta_{s,h}=0 \quad \textit{for each} \quad s\in(0,h), \tag{2.1}$$

$$(1.4)\ \textit{holds} \quad \Leftrightarrow \quad \varDelta_{s,h}<1 \quad \textit{for some} \quad s\in(0,h). \tag{2.2}$$

*Proof.* By $\mathbf{P}$-continuity the dyadic numbers is a separant for $\xi(t)$. Hence we have

$$\begin{aligned} \mathbf{P}\{M(h)>u\} &= \mathbf{P}\left\{\bigcup_{n=0}^{\infty}\bigcup_{k=0}^{[2^n h]}\{\xi(2^{-n}k)>u\}\right\} \\ &= \lim_{n\to\infty}\mathbf{P}\left\{\bigcup_{k=0}^{[2^n h]}\{\xi(2^{-n}k)>u\}\right\} \end{aligned} \tag{2.3}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{[2^n h]} \mathbf{P}\left\{\xi(k) > u,\ \bigcap_{\ell=k+1}^{[2^n h]} \{\xi(2^{-n}\ell) \le u\}\right\}$$

$$= \mathbf{P}\{\xi(0) > u\} + \lim_{n\to\infty} \sum_{k=1}^{[2^n h]} \mathbf{P}\left\{\xi(0) > u,\ \bigcap_{\ell=1}^{k} \{\xi(2^{-n}\ell) \le u\}\right\}$$

(by stationatity). Here the right-hand side is bounded by $\Psi_h(u)$, which gives (1.2). Further an inspection of (2.3) shows that

$$\begin{aligned}
(2.4)\quad \varliminf_{u\uparrow\overline{u}} \frac{\mathbf{P}\{M(h)>u\}}{\Psi_h(u)} &= \lim_{u\uparrow\overline{u}} \lim_{n\to\infty} \frac{1}{\Psi_h(u)} \Bigg(\mathbf{P}\{\xi(0)>u\} + [2^n h]\mathbf{P}\{\xi(0)>u\ge\xi(2^{-n})\} \\
&\qquad - \sum_{k=2}^{[2^n h]} \mathbf{P}\left\{\xi(0) > u \ge \xi(2^{-n}),\ \bigcup_{\ell=2}^{k} \{\xi(2^{-n}\ell) > u\}\right\}\Bigg) \\
&= \lim_{u\uparrow\overline{u}} \lim_{n\to\infty} \frac{\mathbf{P}\{\xi(0)>u\} + 2^{-n}[2^n h]\, J(2^{-n},u) - \overline{J}_h(2^{-n},u)}{\Psi_h(u)} \\
&\le \varlimsup_{u\uparrow\overline{u}} \varliminf_{n\to\infty} \frac{\mathbf{P}\{\xi(0)>u\} + 2^{-n}[2^n h]\, J(2^{-n},u)}{\Psi_h(u)} - \delta_h \\
&\ge \varliminf_{u\uparrow\overline{u}} \varliminf_{n\to\infty} \frac{\mathbf{P}\{\xi(0)>u\} + 2^{-n}[2^n h]\, J(2^{-n},u)}{\Psi_h(u)} - \delta_h \\
&= 1 - \delta_h,
\end{aligned}$$

where

$$\delta_h = \varlimsup_{u\uparrow\overline{u}} \varliminf_{n\to\infty} \frac{\overline{J}_h(2^{-n},u)}{\Psi_h(u)} \qquad \text{and} \qquad \overline{J}_h(2^{-n},u) \equiv \sum_{k=2}^{[h/2^{-n}]} \frac{J_{2^{-n}k}(2^{-n},u)}{2^n}.$$

The implications to the right in (2.1) and (2.2) now follow from (1.2) using that

$$\delta_h \ge \varlimsup_{u\uparrow\overline{u}} \varliminf_{n\to\infty} (h-s)\, J_s(2^{-n},u)/\Psi_h(u) \ge (1-s/h)\,\Delta_{s,h} \qquad \text{for} \quad s<h.$$

Conversely, since $\Psi_s(u)/\Psi_h(u) \ge s/h$, the implications to the left follows from

$$\frac{h}{s} \varliminf_{u\uparrow\overline{u}} \frac{\mathbf{P}\{M(h)>u\}}{\Psi_h(u)} \ge \varliminf_{u\uparrow\overline{u}} \frac{\mathbf{P}\{M(s)>u\}}{\Psi_s(u)} \ge 1-\delta_s \ge 1-\Delta_{s,s} \ge 1-\Delta_{s,h}. \quad \square$$

Clearly (1.4) holds when $J(u) = O\big(\mathbf{P}\{\xi(0)>u\}\big)$ as $u\uparrow\overline{u}$, since we then have

$$(2.5)\qquad \varliminf_{u\uparrow\overline{u}} \mathbf{P}\{\xi(0)>u\}/\Psi_h(u) = \varliminf_{u\uparrow\overline{u}} \big(1 + h\,J(u)/\mathbf{P}\{\xi(0)>u\}\big)^{-1} > 0.$$

**Proposition 1.** *Let* $\{\xi(t)\}_{t\in[0,h]}$ *be* $\mathbf{P}$*-differentiable from the right at* $t=0$ *with*

$$(2.6)\qquad \lim_{t\downarrow 0} \frac{1}{t} \mathbf{P}\left\{\left|\frac{\xi(t)-\xi(0)-t\xi'(0)}{t}\right| > \lambda\right\} = 0 \qquad \textit{for each} \quad \lambda > 0.$$

*Let the distribution function $F_{\xi(0)}(x)$ have a right derivative at $x=u$. We have*

$$(2.7)\qquad \varliminf_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)>u\ge\xi(0)+t\xi'(0)\}}{t}\le J(u)\le\varlimsup_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)>u\ge\xi(0)+t\xi'(0)\}}{t},$$

$$(2.8)\qquad \varliminf_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)+t\xi'(0)>u\ge\xi(0)\}}{t}\le J(u)\le\varlimsup_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)+t\xi'(0)>u\ge\xi(0)\}}{t},$$

*and*

$$(2.9)\qquad \Delta_{s,h}\le\lim_{\nu\uparrow 2}\varlimsup_{u\uparrow\overline{u}}\varlimsup_{\varepsilon\downarrow 0}\frac{h}{\varepsilon\,\Psi_h(u)}\mathbf{P}\Big\{\xi(0)>u\ge\xi(0)+\varepsilon\xi'(0),\ \sup_{t\in[\nu\varepsilon,s]}\xi(t)>u\Big\}$$

$$\le\lim_{\nu\uparrow 1}\varlimsup_{u\uparrow\overline{u}}\varlimsup_{\varepsilon\downarrow 0}\frac{2\,h}{\varepsilon\,\Psi_h(u)}\mathbf{P}\Big\{\sup_{t\in[\nu\varepsilon,s]}\frac{\xi(t)-\xi(0)-t\xi'(0)}{t}\ge\frac{\xi(0)-u}{\varepsilon},\ \xi(0)>u\Big\}.$$

*If in addition $\xi(t)$ is continuous a.s. and $F_{\xi(0)}(u^-)=F_{\xi(0)}(u)$, then we have*

$$(2.10)\qquad J(u)=\lim_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)>u\ge\xi(0)+t\xi'(0)\}}{t}=\lim_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)+t\xi'(0)>u\ge\xi(0)\}}{t}.$$

*Proof.* For a non-increasing family of events $\{A_t\}_{t>0}\subseteq\mathcal{F}$ we have

$$\frac{\mathbf{P}\{\xi(0)>u\ge\xi(t),\,A_t\}}{t}\le\frac{1}{t}\mathbf{P}\Big\{\xi'(0)>-(1-\lambda)\frac{\xi(0)-u}{t},\ \xi(0)>u+\delta t,\ \xi(t)\le u\Big\}$$

$$+\frac{1}{t}\mathbf{P}\Big\{\xi'(0)\le-(1-\lambda)\frac{\xi(0)-u}{t},\ \xi(0)>u,\ A_t\Big\}$$

$$+\frac{\mathbf{P}\{u<\xi(0)\le u+\delta t\}}{t}$$

$$\le\frac{1}{t}\mathbf{P}\Big\{\frac{\xi(0)-\xi(t)+t\xi'(0)}{t}>\lambda\frac{\xi(0)-u}{t}>\lambda\delta\Big\}$$

$$+\frac{1}{1-\lambda}\,\frac{1-\lambda}{t}\mathbf{P}\Big\{\xi(0)>u\ge\xi(0)+\frac{t}{1-\lambda}\xi'(0),\ A_t\Big\}$$

$$+\delta\left(F'_{\xi(0)}(u^+)+o(t)\right)\qquad\text{for}\quad\lambda,\delta,t>0\ \ \text{(small)}.$$

Sending $t\downarrow 0$ and $\lambda,\delta\downarrow 0$ (in that order), (2.6) shows that (first taking $A_t=\Omega$)

$$(2.11)\qquad \varliminf_{t\downarrow 0}J(t,u)\le\varliminf_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)>u\ge\xi(0)+t\xi'(0)\}}{t},$$

$$(2.12)\qquad \varlimsup_{t\downarrow 0}\frac{\mathbf{P}\{\xi(0)>u\ge\xi(t),\,A_t\}}{t}\le\varlimsup_{t\downarrow 0}\frac{1}{t}\mathbf{P}\Big\{\xi(0)>u\ge\xi(0)+t\xi'(0),\,A_{(1-\lambda)t}\Big\}$$

for $\lambda\in(0,1)$. In an analogous way we obtain the estimates downwards

$$J(t,u)\ge-\frac{1}{t}\mathbf{P}\Big\{\xi'(0)\le-(1+\lambda)\frac{\xi(0)-u}{t},\ \xi(0)>u+\delta t,\ \xi(t)>u\Big\}$$

$$
\begin{aligned}
&\quad + \frac{1}{t}\mathbf{P}\bigg\{\xi'(0) \le -(1+\lambda)\frac{\xi(0)-u}{t},\ \xi(0)>u\bigg\} \\
&\quad - \frac{\mathbf{P}\{u<\xi(0)\le u+\delta t\}}{t} \\
&\ge -\frac{1}{t}\mathbf{P}\bigg\{\frac{\xi(0)-\xi(t)+t\xi'(0)}{t} < -\lambda\frac{\xi(0)-u}{t} < -\lambda\delta\bigg\} \\
&\quad + \frac{1}{1+\lambda}\,\frac{1+\lambda}{t}\mathbf{P}\bigg\{\xi(0)>u\ge \xi(0)+\frac{t}{1+\lambda}\xi'(0)\bigg\} \\
&\quad - \delta\big(F'_{\xi(0)}(u^+)+o(t)\big).
\end{aligned}
$$

Sending $t\downarrow 0$ and $\lambda,\delta\downarrow 0$, and using (2.6) and the differentiability of $F_{\xi(0)}$, we get

$$
\underline{\lim}_{t\downarrow 0} J(t,u) \ge \underline{\lim}_{t\downarrow 0} \frac{\mathbf{P}\{\xi(0)>u\ge\xi(0)+t\xi'(0)\}}{t} \tag{2.13}
$$

$$
\overline{\lim}_{t\downarrow 0} J(t,u) \ge \overline{\lim}_{t\downarrow 0} \frac{\mathbf{P}\{\xi(0)>u\ge\xi(0)+t\xi'(0)\}}{t}. \tag{2.14}
$$

Using (2.12), first together with (2.13) and for $A_t=\Omega$, and then for $A_t = \sup_{r\in[2t,s]}\xi(r)$, we get (2.7) and the first inequality in (2.9).

To prove the second inequality in (2.9), we note that the first inequality gives

$$
\varDelta_{s,h} \le \lim_{\nu\uparrow 2}\overline{\lim_{u\uparrow\overline{u}}}\,\overline{\lim_{\varepsilon\downarrow 0}}\,\frac{h}{\varepsilon\,\varPsi_h(u)}\mathbf{P}\bigg\{\sup_{t\in[\nu\varepsilon,s]}\frac{\xi(t)-\xi(0)-t\xi'(0)}{t} > \frac{\xi(0)-u}{\varepsilon t/(t-\varepsilon)},\ \xi(0)>u\bigg\},
$$

which in turn obviously is bounded by the right-hand side of (2.9).

In the proof of (2.7) we computed $\underline{\lim}_{t\downarrow 0}$ and $\overline{\lim}_{t\downarrow 0}$ for $J(t,u)=t^{-1}\mathbf{P}\{\xi(0)> u\ge\xi(t)\}$. Sending $t\downarrow 0$ in $J(t,u)=t^{-1}\mathbf{P}\{\xi(t)>u\ge\xi(0)\}$ instead, we get (2.8).

The left equality in (2.10) follows from (2.11), (2.14) and that (under the assumptions specified) $\lim_{t\downarrow 0} J_t(u)$ exists [and is the upcrossing intensity of the level $u$]. The right equality follows sending $t\downarrow 0$ in $J(t,u)=t^{-1}\mathbf{P}\{\xi(t)>u\ge\xi(0)\}$. □

**Example 1.** (The Argmax Process of Brownian Motion Minus Parabolic Drift.) Let $V(t)\equiv \mathrm{argmax}\{s\in\mathbb{R} : W(s)-(s-t)^2\}$ where $\{W(s)\}_{s\in\mathbb{R}}$ is standard Brownian motion. Hooghiemstra and Lopuhaä (1998) studied local extremes of $\xi(t)\equiv V(t)-t$ by quite difficult analytic methods. We shall recover that result here quite easily. As they did, we use some facts from the Rollo Davidson awarded work by Groeneboom (1989):

The process $V(t)$ is non-decreasing pure jump Markov with transition kernel

$$
\frac{P_x(dy)}{dy} \equiv \lim_{t\downarrow 0}\frac{\mathbf{P}\{V(t)\in dy\,|\,V(0)=x\}}{t\,dy} = \frac{2\,g(y)\,p(y-x)}{g(x)} \qquad \text{for} \quad y>x. \tag{2.15}
$$

Here $g$ and $p$ are continuous with $g$ locally bounded away from zero. Further

$$(2.16)\qquad p(x)\sim\begin{cases}4x\,\mathrm{e}^{-\kappa x} & \text{as } x\to\infty\\ \sqrt{2/(\pi x)} & \text{as } x\downarrow 0\end{cases}\qquad\text{and}\qquad g(x)\sim\begin{cases}4x\,\mathrm{e}^{-\frac{2}{3}x^3} & \text{as } x\to\infty\\ \lambda\mathrm{e}^{\kappa x} & \text{as } x\to-\infty\end{cases},$$

where $\lambda,\kappa>0$ are constants. The process $\xi(t)$ is stationary Markov and

$$(2.17)\qquad \xi(0)\quad\text{has density function}\quad f(x)\equiv f_{\xi(0)}(x)=f_{V(0)}(x)=\tfrac{1}{2}g(x)g(-x).$$

*Proof of* (1.3) *for* $\xi(t)$. Since $\xi(t)\geq\xi(0)-t$ (1.5) holds with [cf. (2.17)]

$$(2.18)\qquad J(u)\leq\underline{\lim}_{n\to\infty}2^n\,\mathbf{P}\big\{\xi(0)>u\geq\xi(0)-2^{-n}\big\}=f(u).$$

The time to a jump $\tau\equiv\inf\{t>0:V(t)\neq V(0)\}$ is exponentially distributed

$$\frac{\mathbf{P}\{\tau\in dt\,|\,V(0)=x\}}{dt}=c(x)\,\mathrm{e}^{-c(x)\,t}\qquad\text{with}\qquad\frac{P_x(dy)}{c(x)}=\mathbf{P}\{V(\tau)\in dy\,|\,V(0)=x\}.$$

By (2.15)-(2.17) and routine calulations we get [note that $g(u+u^{-2}x)\sim\mathrm{e}^{-2x}g(u)$]

$$(2.19)\qquad c(u)=\int_0^\infty\frac{g(y)p(y-u)\,dy}{g(u)}\sim\int_0^\infty\frac{g(u+u^{-2}x)\,dx}{ug(u)\sqrt{\pi x/2}}\sim\frac{1}{u}\int_0^\infty\frac{\mathrm{e}^{-2x}\,dx}{\sqrt{\pi x/2}}=\frac{1}{u}$$

as $u\to\infty$. By conditional independence of the past and the future we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\;2^n\,\mathbf{P}\Big\{\xi(0)>u\geq\xi(2^{-n}),\ \sup_{t\in(2^{-n},s]}\xi(t)>u\Big\}\\
&\leq\overline{\lim_{n\to\infty}}\;2^n\int_u^{u+2^{-n}}\mathbf{P}\Big\{V(0)>u\geq V(2^{-n})-2^{-n}\,\Big|\,V(2^{-n})=x\Big\}\\
&\qquad\qquad\times\mathbf{P}\Big\{\sup_{t\in(2^{-n},s]}V(t)>u+2^{-n}\,\Big|\,V(2^{-n})=x\Big\}\,f_{V(2^{-n})}(x)\,dx\\
&\leq\overline{\lim_{n\to\infty}}\;2^n\,\mathbf{P}\big\{\xi(0)>u\geq\xi(2^{-n})\big\}\ \sup_{x\in(u,u+2^{-n}]}\mathbf{P}\{\tau\leq s\,|\,V(0)=x\}\\
&=J(u)\ \overline{\lim_{n\to\infty}}\ \sup_{x\in(u,u+2^{-n}]}(1-\mathrm{e}^{-c(x)\,s})\qquad\text{for}\quad s>0,
\end{aligned}$$

which is $o(J(u))$ as $u\to\infty$ by (2.19). Hence $\Delta_{s,h}=0$ and (2.1) yields (1.3). □

**Remark 1.** Since $f(u)\sim 4\lambda u\,\mathrm{e}^{-\kappa u-\frac{2}{3}u^3}$ as $u\to\infty$ by (2.16) and (2.17), we have

$$\mathbf{P}\{\xi(0)>u\}=\textstyle\int_u^\infty f(x)\,dx\sim(2u^2)^{-1}f(u)=o(f(u))\qquad\text{as}\quad u\to\infty.$$

Further $J(u)\sim f(u)$ as $u\to\infty$, since $J(u)\leq f(u)$ [by (2.18)] and

$$J(u)\geq\underline{\lim}_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_u^{u+\varepsilon}\mathbf{P}\{\tau>s\,|\,V(0)=x\}f(x)\,dx\geq\underline{\lim}_{\varepsilon\downarrow 0}\inf_{x\in[u,u+\varepsilon]}\mathrm{e}^{-c(x)\varepsilon}f(u)\sim f(u)$$

by (2.19). Hence (1.3) reduces to $\mathbf{P}\{M(h)>u\}\sim h\,f(u)$ as $u\to\infty$.

**3. Relations between extremes and streams of upcrossings II.** Let $\Psi_h(q,u) \equiv \mathbf{P}\{\xi(0)>u\}+hJ(q,u)$. Choose functions $\{q_a:(\underline{u},\overline{u})\to(0,\infty)\}_{a>0}$ that satisfy

$$\lim_{a\downarrow 0}\overline{\lim}_{u\uparrow\overline{u}}\, q_a(u) = 0. \tag{3.1}$$

[Usually $q_a(u)=a\,q(u)$.] The grid $\{k\,q_a(u)\in[0,h]:k\in\mathbb{N}\}$ is dense enough when

$$\lim_{a\downarrow 0}\overline{\lim_{u\uparrow\overline{u}}}\,\frac{1}{\Psi_h(q_a(u),u)}\,\mathbf{P}\bigg\{M(h)>u,\ \bigcap_{\ell=0}^{[h/q_a(u)]}\{\xi(\ell q_a(u))\le u\}\bigg\} = 0: \tag{3.2}$$

This is the sparsest grid to which the proof of Theorem 1 carries over. It gives virtually weaker versions of the criteria $\Delta_{s,h}=0$ and $\Delta_{s,h}<1$ expressed in terms of

$$\mathfrak{D}_{s,h} \equiv \overline{\lim_{a\downarrow 0}}\,\overline{\lim_{u\uparrow\overline{u}}}\,\frac{hJ_s(q_a(u),u)}{\Psi_h(q_a(u),u)} \qquad \text{and} \qquad D_{s,h} \equiv \overline{\lim_{a\downarrow 0}}\,\overline{\lim_{u\uparrow\overline{u}}}\,\frac{hJ_s(q_a(u),u)}{\Psi_h(u)}.$$

In the presence of (3.2), the "natural" bounds for $\mathbf{P}\{M(h)>u\}$ that correspond to (1.2)-(1.4) are

$$\overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}\,\mathbf{P}\{M(h)>u\}\,/\,\Psi_h(q_a(u),u) \le 1, \tag{3.3}$$

$$\underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\mathbf{P}\{M(h)>u\}\,/\,\Psi_h(q_a(u),u) \ge 1, \tag{3.4}$$

$$\underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\mathbf{P}\{M(h)>u\}\,/\,\Psi_h(q_a(u),u) > 0. \tag{3.5}$$

As an alternative to (3.2), we will also use the requirement

$$\underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\Psi_h(q_a(u),u)/\Psi_h(u) \ge 1. \tag{3.6}$$

By Theorem 2 below, this requirement implies the following two relations

$$\lim_{a\downarrow 0}\overline{\lim}_{u\uparrow\overline{u}}\,\mathbf{P}\big\{\xi(0)\le u,\ M(q_a(u))>u,\ \xi(q_a(u))\le u\big\}\,/\,\big(q_a(u)\,\Psi_h(u)\big) = 0, \tag{3.7}$$

$$\lim_{a\downarrow 0}\underline{\lim}_{u\uparrow\overline{u}}\,\Psi_h(q_a(u),u)/\Psi_h(u) \;=\; \lim_{a\downarrow 0}\overline{\lim}_{u\uparrow\overline{u}}\,\Psi_h(q_a(u),u)/\Psi_h(u) \;=\; 1. \tag{3.8}$$

From (3.7) and (3.8) in turn one immediately gets (3.2), so that (3.6) implies (3.2). Further, under (3.8), (3.4) and (3.5) are equivalent to (1.3) and (1.4), respectively.

Note that (3.5) holds when $\overline{\lim}_{a\downarrow 0}\overline{\lim}_{u\uparrow\overline{u}}\,\Psi_h(q_a(u),u)/\mathbf{P}\{\xi(0)>u\}<\infty$ [cf. (2.5)].

If $J(u)>0$ for $u\in[u_0,\overline{u})$ for some $u_0\in(\underline{u},\overline{u})$, then (3.6) holds if e.g.,

$$\underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,J(q_a(u),u)/J(u) \ge 1. \tag{3.9}$$

This follows from the easily established inequality

$$\Psi_h(q_a(u),u)/\Psi_h(u) \;\ge\; 1+\min\big\{J(q_a(u),u)/J(u)-1,\,0\big\}.$$

The condition (3.9) in turn generalizes to the $\mathbf{P}$-continuous setting the requirement

$$\text{(3.10)} \qquad \underline{\lim}_{a\downarrow 0}\, \underline{\lim}_{u\uparrow\overline{u}}\, J(q_a(u),u)/\mu(u) \geq 1$$

of Leadbetter and Rootzén (1982): Assuming that $\xi(t)$ has a.s. continuous sample paths with $\xi(0)$ continuously distributed, and that the upcrossing intensity $\mu(u)$ [which in this case coincides with $J(u)$] is finite, they proved that (3.10) implies

$$\text{(3.11)} \qquad \lim_{a\downarrow 0}\, \overline{\lim}_{u\uparrow\overline{u}}\, \mathbf{P}\{\xi(0)\leq u,\, M(q_a(u))>u,\, \xi(q_a(u))\leq u\} \big/ \big(q_a(u)\,\mu(u)\big) = 0$$

and

$$\text{(3.12)} \qquad \lim_{a\downarrow 0}\, \underline{\lim}_{u\uparrow\overline{u}}\, J(q_a(u),u)/\mu(u) = \lim_{a\downarrow 0}\, \overline{\lim}_{u\uparrow\overline{u}}\, J(q_a(u),u)/\mu(u) = 1.$$

It has been established above (more or less) that the following implications hold

(3.2) $\Leftarrow$ (3.7)&(3.8) $\Leftarrow$ (3.6) $\Leftarrow$ (3.9) $\Leftarrow$ (3.11)&(3.12)&continuity $\Leftarrow$ (3.10)&continuity.

It is also quite clear that no converses to any of these implications hold in general.

To see that (3.6) $\not\Rightarrow$ (3.9), for example, let $J(u) = o\big(\mathbf{P}\{\xi(0)>u\}\big)$ as $u\uparrow\overline{u}$. Now (3.9) may or may not hold, while on the other hand (3.6) holds trivially since

$$\Psi_h(q_a(u),u)/\Psi_h(u) \;\geq\; 1 - h\,J(u)/\mathbf{P}\{\xi(0)>u\}.$$

It is (3.2) or (3.7), rather than (3.11), that is needed in extremes. In view of the above discussion, in order to ensure sufficient accuracy of the discrete approximation (denseness of the grid), it is both natural and beneficial to use condition (3.2) or (3.6) [that implies (3.7)], rather than (3.9) or indeed (3.10) [that implies (3.11)].

**Remark 2.** Hooghiemstra and Lopuhaä (1998) noted that (3.10) implies (3.11) for the jump process in Example 1. This was an important influence for me.

**Theorem 2.** *Let* $\{\xi(t)\}_{t\in[0,h]}$ *be a separable and* $\mathbf{P}$*-continuous stationary process satisfying* (1.1)*, and choose functions* $\{q_a(\cdot)\}_{a>0}$ *that satisfy* (3.1)*.*

**(i)** *If* (3.2) *holds, then* (3.3) *holds. Moreover we have*

$$\text{(3.13)} \qquad \text{(3.4) } \textit{holds} \quad\Leftrightarrow\quad \mathfrak{D}_{s,h}=0 \quad \textit{for each} \quad s\in(0,h),$$

$$\text{(3.14)} \qquad \text{(3.5) } \textit{holds} \quad\Leftrightarrow\quad \mathfrak{D}_{s,h}<1 \quad \textit{for some} \quad s\in(0,h).$$

**(ii)** *If* (3.6) *holds, then* (3.2), (3.7) *and* (3.8) *hold. Moreover we have*

$$\text{(3.15)} \qquad \text{(3.4) } \textit{holds} \;\Leftrightarrow\; \text{(1.3) } \textit{holds} \;\Leftrightarrow\; D_{s,h}=0 \quad \textit{for each} \quad s\in(0,h),$$

$$\text{(3.16)} \qquad \text{(3.5) } \textit{holds} \;\Leftrightarrow\; \text{(1.4) } \textit{holds} \;\Leftrightarrow\; D_{s,h}<1 \quad \textit{for some} \quad s\in(0,h).$$

*Proof of (i).* Using (3.1) together with (3.2), we get [cf. (2.3) and (2.4)]

$$
\begin{aligned}
(3.17)\qquad & \underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\mathbf{P}\{M(h)>u\}\,/\,\Psi_h(q_a(u),u) \\
&= \underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\frac{1}{\Psi_h(q_a(u),u)}\left(\mathbf{P}\Bigg\{\bigcup_{k=0}^{[h/q_a]}\{\xi(kq_a)>u\}\Bigg\}+\mathbf{P}\Bigg\{M(h)>u,\,\bigcap_{k=0}^{[h/q_a]}\{\xi(kq_a)\le u\}\Bigg\}\right) \\
&= \underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\frac{1}{\Psi_h(q_a(u),u)}\sum_{k=0}^{[h/q_a]}\mathbf{P}\Bigg\{\xi(kq_a)>u,\,\bigcap_{\ell=k+1}^{[h/q_a]}\{\xi(\ell q_a)\le u\}\Bigg\} \\
&= \underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\frac{1}{\Psi_h(q_a(u),u)}\left(\mathbf{P}\{\xi(0)>u\}+\sum_{k=1}^{[h/q_a]}\mathbf{P}\Bigg\{\xi(0)>u,\,\bigcap_{\ell=1}^{k}\{\xi(\ell q_a)\le u\}\Bigg\}\right) \\
&= \underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\frac{1}{\Psi_h(q_a(u),u)}\Bigg(\mathbf{P}\{\xi(0)>u\}+[h/q_a]\,\mathbf{P}\{\xi(0)>u\ge\xi(q_a)\} \\
&\qquad\qquad -\sum_{k=2}^{[h/q_a]}\mathbf{P}\Bigg\{\xi(0)>u\ge\xi(q_a),\,\bigcup_{\ell=2}^{k}\{\xi(\ell q_a)>u\}\Bigg\}\Bigg) \\
&= \underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\frac{\mathbf{P}\{\xi(0)>u\}+q_a[h/q_a]\,J(q_a(u),u)-\overline{J}_h(q_a(u),u)}{\Psi_h(q_a(u),u)} \\
&= 1-\mathfrak{d}_h,
\end{aligned}
$$

where

$$
\mathfrak{d}_h=\overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}\,\frac{\overline{J}_h(q_a(u),u)}{\Psi_h(q_a(u),u)}\equiv\overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}\,\frac{q_a(u)}{\Psi_h(q_a(u),u)}\sum_{k=2}^{[h/q_a(u)]}J_{q_a(u)k}(q_a(u),u).
$$

Here we have, for each choice of $s\in(0,h)$,

$$
\mathfrak{d}_h\ge\overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}\,(h-s)\,J_s(q_a(u),u)/\Psi_h(q_a(u),u)=(1-s/h)\,\mathfrak{D}_{s,h}
$$

and

$$
\begin{aligned}
\mathfrak{d}_h &\le \overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}\,s\,J_s(q_a(u),u)/\Psi_h(q_a(u),u)+\overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}(h-s)\,J(q_a(u),u)/\Psi_h(q_a(u),u) \\
&\le (s/h)\,\mathfrak{D}_{s,h}+(1-s/h).
\end{aligned}
$$

Inserting this in (3.17) we obtain

$$
\underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\,\mathbf{P}\{M(h)>u\}\,/\,\Psi_h(q_a(u),u)\;\begin{matrix}\le 1-(1-s/h)\,\mathfrak{D}_{s,h}\\ \ge (s/h)\,(1-\mathfrak{D}_{s,h})\end{matrix}\,,
$$

which readily give (3.13) and (3.14). Further, using (3.1) and (3.2) exactly as in (3.17), we get the upper estimate (3.3) in the following way:

$$
\overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}\,\frac{\mathbf{P}\{M(h)>u\}}{\Psi_h(q_a,u)}=\overline{\lim}_{a\downarrow 0}\,\overline{\lim}_{u\uparrow\overline{u}}\,\frac{\mathbf{P}\{\xi(0)>u\}+q_a[h/q_a]\,J(q_a,u)-\overline{J}_h(q_a,u)}{\Psi_h(q_a,u)}
$$

$$
\begin{aligned}
&\le \overline{\lim_{a\downarrow 0}}\,\overline{\lim_{u\uparrow\overline{u}}}\,\frac{\mathbf{P}\{\xi(0)>u\}+q_a[h/q_a]\,J(q_a,u)}{\Psi_h(q_a,u)}\\
&=1.
\end{aligned}
$$

*Proof of (ii).* Using (1.2) and (3.6), we get (3.7) and (3.8) from the estimate

$$
\begin{aligned}
\frac{\mathbf{P}\{\xi(0)\le u,\ M(q_a)>u,\ \xi(q_a)\le u\}}{q_a(u)\,\Psi_h(u)} &= \frac{\mathbf{P}\{\xi(0)\le u,\ M(q_a)>u\}-q_a(u)J(q_a,u)}{q_a(u)\,\Psi_h(u)}\\
&\le \big(\Psi_h(u)-\Psi_h(q_a,u)\big)\,/\,\big(h\,\Psi_h(u)\big).
\end{aligned}
$$

The first equivalencies in (3.15) and (3.16) follows directly from (3.8). The two remaining ones reduce to (3.13) and (3.14), respectively, when (3.8) holds. □

**Corollary 1.** *If* $\{\xi(t)\}_{t\in[0,h]}$ *is a separable and* $\mathbf{P}$*-continuous stationary process satisfying* (1.1) *and* (3.6), *then* (1.3) *holds if the following two conditions hold*

$$
\lim_{N\to\infty}\overline{\lim_{u\uparrow\overline{u}}}\sum_{k=N}^{[h/q_a(u)]}\frac{\mathbf{P}\{\xi(0)>u,\ \xi(kq_a(u))>u\}}{q_a(u)\,\Psi_h(u)}=0,
$$

$$
\lim_{u\uparrow\overline{u}}\frac{\mathbf{P}\big\{\xi(0)\le u,\xi(q_a(u))>u,\xi(\ell q_a(u))>u,\xi((\ell+1)q_a(u))\le u\big\}}{q_a(u)\,\Psi_h(u)}=0 \qquad \textit{for}\ \ 1\le\ell\in\mathbb{N}.
$$

*Proof.* Using the elementary fact that

$$
\begin{aligned}
\bigg(\Big(\bigcup_{k=2}^{N-1}\{\xi(kq_a)>u\}\Big)\cap\{\xi(Nq_a)\le u\}\bigg)^c\cap\Big(\bigcup_{k=N}^{[h/q_a]}\{\xi(kq_a)>u\}\Big)^c &= \bigcap_{k=2}^{[h/q_a]}\{\xi(kq_a)\le u\}\\
&=\Big(\bigcup_{k=2}^{[h/q_a]}\{\xi(kq_a)>u\}\Big)^c,
\end{aligned}
$$

the corollary follows from Theorem 2 (ii) together with the estimates

$$
\begin{aligned}
(3.18)\qquad & J_h(q_a(u),u)\,/\,\Psi_h(u)\\
&=\frac{1}{q_a\,\Psi_h(u)}\,\mathbf{P}\Big\{\xi(0)>u\ge\xi(q_a),\ \bigcup_{k=2}^{[h/q_a]}\{\xi(kq_a)>u\}\Big\}\\
&\le\frac{1}{q_a\,\Psi_h(u)}\,\mathbf{P}\Big\{\xi(q_a)\le u,\ \bigcup_{k=2}^{N-1}\{\xi(kq_a)>u\},\ \xi(Nq_a)\le u\Big\}\\
&\qquad+\frac{1}{q_a\,\Psi_h(u)}\,\mathbf{P}\Big\{\xi(0)>u,\ \bigcup_{k=N}^{[h/q_a]}\{\xi(kq_a)>u\}\Big\}\\
&\le\sum_{k=2}^{N-1}\sum_{\ell=k}^{N-1}\frac{\mathbf{P}\big\{\xi((k-1)q_a)\le u,\ \xi(kq_a)>u,\ \xi(\ell q_a)>u,\ \xi((\ell+1)q_a)\le u\big\}}{q_a\,\Psi_h(u)}\\
&\qquad+\sum_{k=N}^{[h/q_a]}\frac{\mathbf{P}\{\xi(0)>u,\ \xi(kq_a)>u\}}{q_a\,\Psi_h(u)}. \qquad\square
\end{aligned}
$$

Now assume that the distribution of $\xi(0)$ belongs to a domain of attraction of extremes. This means that there exist a constant $\hat{x}\in[-\infty,0)$ together with continuous functions $w:(\underline{u},\overline{u})\to(0,\infty)$ and $F:(\hat{x},\infty)\to(-\infty,1)$ such that

$$(3.19)\qquad \lim_{u\uparrow\overline{u}}\mathbf{P}\big\{\xi(0)>u+xw(u)\big\}\,/\,\mathbf{P}\{\xi(0)>u\} = 1-F(x) \quad \text{for} \quad x\in(\hat{x},\infty).$$

Set $q_a(u)=a\,q(u)$ for some function $q:(\underline{u},\overline{u})\to(0,\infty)$ that satisfies

$$(3.20)\qquad \lim_{u\uparrow\overline{u}} q(u)<\infty \qquad \text{and} \qquad \lim_{x\downarrow 0}\lim_{u\uparrow\overline{u}} q(u)/q(u-xw(u))<\infty.$$

Assume that, to each choice of $\delta>0$, there exists a constant $\hat{t}=\hat{t}(\delta)>0$ such that

$$(3.21)\qquad \lim_{N\to\infty}\overline{\lim_{u\uparrow\overline{u}}}\sum_{k=N}^{[h/(q(u)t)]}\frac{\mathbf{P}\{\xi(0)>u,\,\xi(kq(u)t)>u\}}{\mathbf{P}\{\xi(0)>u\}}=0,$$

(3.22)

$$\lim_{u\uparrow\overline{u}}\frac{\mathbf{P}\big\{\xi(0)\le u,\,\xi(q(u)t)>u+\delta w(u)t,\,\xi(\ell q(u)t)>u+\delta w(u)t,\,\xi((\ell+1)q(u)t)\le u\big\}}{\mathbf{P}\{\xi(0)>u\}}=0,$$

for $1\le\ell\in\mathbb{N}$ and $t\in(0,\hat{t}]$. Further assume that, to each choice of $\delta>0$, there exist constants $C,\hat{\nu}>0$, $\hat{u}\in(\underline{u},\overline{u})$ and $\varrho>1$ such that

$$(3.23)\qquad \mathbf{P}\big\{\xi(0)\le u+\nu w(u),\,\xi(q(u)t)>u+(\nu+\delta t)w(u),\,\xi(2q(u)t)\le u+\nu w(u)\big\}$$
$$\le C\,t^{\varrho}\,\mathbf{P}\{\xi(0)>u\} \qquad \text{for} \quad u\in[\hat{u},\overline{u}),\quad t\ge 0 \quad \text{and} \quad \nu\in[0,\hat{\nu}].$$

**Theorem 3.** *Let* $\{\xi(t)\}_{t\in[0,h]}$ *be a separable and* $\mathbf{P}$*-continuous stationary process satisfying* (1.1) *and* (3.19)-(3.23)*. Equations* (3.3) *and* (3.4) *hold with*

$$(3.24)\qquad 0<\underline{\lim_{a\downarrow 0}}\,\underline{\lim_{u\uparrow\overline{u}}}\frac{\Psi_h(aq(u),u)}{\mathbf{P}\{\xi(0)>u\}/q(u)}\le\overline{\lim_{a\downarrow 0}}\,\overline{\lim_{u\uparrow\overline{u}}}\frac{\Psi_h(aq(u),u)}{\mathbf{P}\{\xi(0)>u\}/q(u)}<\infty.$$

**Remark 3.** Even when (3.6) holds, one needs $\mathfrak{D}_{s,h}=0$ to get (3.4), which typically means proving (3.21) and (3.22) (cf. Corollary 1). Theorem 3 is useful because, instead of (3.6), what has to be shown in addition to (3.21) and (3.22) is (3.23), which often follows from the same calculations that gave (3.22) (e.g., Theorem 4 and Section 7 below). [Usually $q$ is non-increasing, so that (3.20) holds trivially.]

*Proof of Theorem 3.* By Albin (2000, Theorem 1), (3.19), (3.20) and (3.23) give

$$(3.25)\qquad \lim_{a\downarrow 0}\overline{\lim_{u\uparrow\overline{u}}}\frac{q}{\mathbf{P}\{\xi(0)>u\}}\,\mathbf{P}\Bigg\{M(h)>u,\,\bigcap_{\ell=0}^{[h/(aq)]}\{\xi(\ell q)\le u\}\Bigg\}=0$$

[where from now on $q\equiv q(u)$]. Using Boole's inequality and stationarity, this gives

$$(3.26)\qquad \overline{\lim_{u\uparrow\overline{u}}}\,q\,\mathbf{P}\{M(h)>u\}\,/\,\mathbf{P}\{\xi(0)>u\}$$

$$\leq \overline{\lim_{u\uparrow\overline{u}}} \frac{q}{\mathbf{P}\{\xi(0)>u\}} \left(\mathbf{P}\left\{M(h)>u, \bigcap_{\ell=0}^{[h/(aq)]} \{\xi(\ell q)\leq u\}\right\} + \mathbf{P}\left\{\bigcup_{\ell=0}^{[h/(aq)]} \{\xi(\ell q)>u\}\right\}\right)$$
$$\leq \overline{\lim_{u\uparrow\overline{u}}} \frac{q}{\mathbf{P}\{\xi(0)>u\}} \mathbf{P}\left\{M(h)>u, \bigcap_{\ell=0}^{[h/(aq)]} \{\xi(\ell q)\leq u\}\right\} + \frac{h}{a}$$
$$< \infty \qquad \text{for } a \text{ sufficiently small.}$$

On the other hand, Albin (1990, Theorem 2.b) together with (3.21) imply that

$$\underline{\lim}_{u\uparrow\overline{u}}\, q\,\mathbf{P}\{M(h)>u\} \,/\, \mathbf{P}\{\xi(0)>u\} > 0. \tag{3.27}$$

By an obvious modification of (3.17), using (3.25) instead of (3.2), we get

$$\lim_{u\uparrow\overline{u}} \frac{\mathbf{P}\{M(h)>u\}}{\mathbf{P}\{\xi(0)>u\}/q} = \lim_{a\downarrow 0}\lim_{u\uparrow\overline{u}} \frac{\Psi_h(aq,u) - \overline{J}_h(aq,u)}{\mathbf{P}\{\xi(0)>u\}/q}. \tag{3.28}$$

Arguing as for (3.18), (3.19) together with (3.21) and (3.22) show that

$$\text{(3.29)} \qquad q\,\overline{J}_h(aq,u) \,/\, \mathbf{P}\{\xi(0)>u\}$$
$$\leq \frac{1}{a\,\mathbf{P}\{\xi(0)>u\}}\,\mathbf{P}\left\{\xi(0)>u\geq\xi(aq), \bigcup_{k=2}^{[h/(aq)]} \{\xi(kaq)>u\}\right\}$$
$$\leq \sum_{k=2}^{N-1}\sum_{\ell=k}^{N-1} \frac{\mathbf{P}\{\xi((k-1)aq)\leq u,\, \xi(kaq)>u+\delta aw,\, \xi(\ell aq)>u+\delta aw,\, \xi((\ell+1)aq)\leq u\}}{a\,\mathbf{P}\{\xi(0)>u\}}$$
$$+ \sum_{k=2}^{N-1}\sum_{\ell=k}^{N-1} \frac{\mathbf{P}\{\{u<\xi(kaq)\leq u+\delta aw\}\cup\{u<\xi(\ell aq)\leq u+\delta aw\}\}}{a\,\mathbf{P}\{\xi(0)>u\}}$$
$$+ \sum_{k=N}^{[h/(aq)]} \frac{\mathbf{P}\{\xi(0)>u,\, \xi(kaq)>u\}}{a\,\mathbf{P}\{\xi(0)>u\}}$$
$$\to 0 + 2\,\delta\,(N-2)^2 F'(0) + 0 \qquad \text{as} \quad u\uparrow\overline{u} \quad \text{and} \quad a\downarrow 0 \qquad \text{(in that order).}$$

Using (3.27) and (3.28) together with (3.29) we thus obtain

$$\underline{\lim}_{a\downarrow 0}\,\underline{\lim}_{u\uparrow\overline{u}}\, q\,\Psi_h(aq,u) \,/\, \mathbf{P}\{\xi(0)>u\} > 0. \tag{3.30}$$

Hence (3.25) implies (3.2), so we get (3.3). Sending $\delta\downarrow 0$ in (3.29) and using (3.30) we get $\mathfrak{D}_{s,h}=0$, so that (3.22) gives (3.4). Further (3.4) and (3.26) show that

$$\overline{\lim_{a\downarrow 0}}\,\overline{\lim_{u\uparrow\overline{u}}} \frac{\Psi_h(aq,u)}{\mathbf{P}\{\xi(0)>u\}/q} \leq \overline{\lim_{a\downarrow 0}}\,\overline{\lim_{u\uparrow\overline{u}}} \frac{\Psi_h(aq,u)}{\mathbf{P}\{M(h)>u\}} \frac{\mathbf{P}\{M(h)>u\}}{\mathbf{P}\{\xi(0)>u\}/q} < \infty. \quad \square$$

**4. P-differentiable $\alpha$-stable processes.** A strictly $\alpha$-stable random variable $Z$ with scale $\sigma=\sigma_Z\geq 0$, skewness $\beta=\beta_Z\in[-1,1]$ and $\alpha\in(0,2)\setminus\{1\}$ satisfies

$$\mathbf{E}\{\exp[i\theta Z]\} = \exp\{-|\theta|^\alpha\sigma^\alpha[1+i\beta\tan(\tfrac{\pi(2-\alpha)}{2})\,\mathrm{sign}(\theta)]\} \qquad \text{for} \quad \theta\in\mathbb{R}.$$

Let $\{\xi(t)\}_{t\in\mathbb{R}}$ be a separable and **P**-differentiable strictly $\alpha$-stable stationary process and $\{\eta(t)\}_{t\in\mathbb{R}}$ an $\alpha$-stable Lévy process with $\sigma_{\eta(t)}=|t|^{1/\alpha}$. We have

$$\text{(4.1)} \quad \text{the finite dimensional distributions of} \quad \xi(t) \;=_d\; \text{those of} \;\; \int_{-\infty}^{\infty} g_t(s)\,d\eta(s)$$

for a suitable choice of $\{g_t(\cdot)\}_{t\in\mathbb{R}}\subseteq\mathbb{L}^\alpha(\mathbb{R})$. Here stationarity for $\xi(t)$ means that

$$\text{(4.2)} \qquad \Big\|\sum_{i=1}^n \theta_i g_{t_i+\tau}\Big\|_\alpha \qquad \text{and} \qquad \Big\langle \sum_{i=1}^n \theta_i g_{t_i+\tau}\Big\rangle_\alpha \qquad \text{do not depend on} \quad \tau,$$

where we use the notation $\langle g\rangle \equiv \int_{\mathbb{R}} g(x)\,dx$, $\langle g\rangle_\alpha \equiv \langle |g|^\alpha \mathrm{sign}(g)\rangle$ and $\|g\|_\alpha \equiv \langle |g|^\alpha\rangle^{1/\alpha}$. Further **P**-differentiability is equivalent with

$$\text{(4.3)} \qquad t^{-1}[g_t-g_0]\to g_0' \quad \text{in} \;\; \mathbb{L}^\alpha(\mathbb{R}) \quad \text{as} \quad t\to 0 \quad \text{for some} \;\; g_0'(\cdot)\subseteq\mathbb{L}^\alpha(\mathbb{R}).$$

See Samorodnitsky and Taqqu (1994) on properties of stable stochastic processes.

**Proposition 2.** *Let $\xi(t)$ be a separable $\alpha$-stable process given by* (4.1) *and* (4.2) *with $\alpha>1$ and $\|g_0^-\|_\alpha>0$. If* (4.3) *holds, then* (1.3) *holds if and only if*

$$\lim_{a\downarrow 0}\frac{1}{a}\Big[\big\langle g_0^-\vee g_a^-\big\rangle_\alpha - \langle g_0^-\rangle_\alpha - \big\langle g_0^-\vee\ldots\vee g_{[s/a]a}^-\big\rangle_\alpha + \big\langle g_0^-\vee\ldots\vee g_{[s/a]a-a}^-\big\rangle_\alpha\Big] = 0$$

*for $s\in(0,h)$. [Here we use the notation $g^-=\max\{-g,0\}$.] Further,* (1.4) *holds.*

*Proof.* With $C_\alpha^{-1}=\int_0^\infty \sin(x)\,dx/x^\alpha$, Samorodnitsky (1988, Theorem 3.1) gives

$$\text{(4.4)} \qquad \begin{array}{l} \mathbf{P}\{\bigcap_{\ell=0}^n\{\xi(\ell a)>u\}\} \;\sim\; C_\alpha\big\langle g_0^-\wedge\ldots\wedge g_{na}^-\big\rangle_\alpha u^{-\alpha} \\ \mathbf{P}\{\bigcup_{\ell=0}^n\{\xi(\ell a)>u\}\} \;\sim\; C_\alpha\big\langle g_0^-\vee\ldots\vee g_{na}^-\big\rangle_\alpha u^{-\alpha} \end{array} \qquad \text{as} \quad u\to\infty.$$

Moreover Albin and Leadbetter (1999, Theorem 5 and Corollary 5) show that

$$\text{(4.5)} \qquad J(u) \;=\; \mu(u) \;\sim\; \alpha\, C_\alpha\big\langle (g_0')^-(g_0^-)^{\alpha-1}\big\rangle\, u^{-\alpha} \qquad \text{as} \quad u\to\infty.$$

From this we readily conclude that (3.9) holds with $q_a(u)=a$, since

$$\begin{aligned}\lim_{u\to\infty}\frac{\Psi_h(a,u)}{\Psi_h(u)} &= \frac{(1+h/a)\,\langle g_0^-\rangle_\alpha - (h/a)\,\langle g_0^-\wedge g_{-a}^-\rangle_\alpha}{\langle g_0^-\rangle_\alpha + h\alpha\,\langle (g_0')^-(g_0^-)^{\alpha-1}\rangle} \\ &\sim \frac{(1+h/a)\,\langle g_0^-\rangle_\alpha - (h/a)\,\langle g_0^-\wedge[g_0(1-ag_0'/g_0)]^-\rangle_\alpha}{\langle g_0^-\rangle_\alpha + h\alpha\,\langle (g_0')^-(g_0^-)^{\alpha-1}\rangle} \\ &\to 1 \qquad \text{as} \quad a\downarrow 0.\end{aligned}$$

Since $J(u) = O\big(\mathbf{P}\{\xi(0)>u\}\big)$ by (4.4) and (4.5), (1.4) holds [cf. (2.5)]. Further (3.15) shows that (1.3) holds if and only if

$$\varlimsup_{u\to\infty} \frac{u^\alpha}{a}\,\mathbf{P}\Big\{\xi(0)>u,\ \xi(a)\le u,\ \bigcup_{\ell=2}^{[s/a]}\{\xi(\ell a)>u\}\Big\}$$

$$=\varlimsup_{u\to\infty} \frac{u^\alpha}{a}\Big[\mathbf{P}\Big\{\bigcup_{\ell=0}^{1}\{\xi(\ell a)>u\}\Big\}-\mathbf{P}\Big\{\bigcup_{\ell=0}^{0}\ldots\Big\}-\mathbf{P}\Big\{\bigcup_{\ell=0}^{[s/a]}\ldots\Big\}+\mathbf{P}\Big\{\bigcup_{\ell=0}^{[s/a]-1}\ldots\Big\}\Big]$$

$$=C_\alpha\,\frac{1}{a}\Big[\big\langle g_0^-\vee g_a^-\big\rangle_\alpha-\langle g_0^-\rangle_\alpha-\big\langle g_0^-\vee\ldots\vee g_{[s/a]a}^-\big\rangle_\alpha+\big\langle g_0^-\vee\ldots\vee g_{[s/a]a-a}^-\big\rangle_\alpha\Big]$$

$$\to 0 \qquad \text{as} \quad a\downarrow 0, \quad \text{for} \quad s\in(0,h). \quad \square$$

**Example 2.** ($\alpha$-STABLE MOVING AVERAGE PROCESSES.) Take $\alpha>1$ and $g_t^-(x) = f(x-t)$ where $\|f\|_\alpha>0$. This holds when $\xi(t)$ is a moving average process.

When $f$ is unimodal, $f$ is non-decreasing on $(-\infty,\hat{t})$ and non-increasing on $(\hat{t},\infty)$ for some $\hat{t}\in\mathbb{R}$. Assume that $f$ is left- or right-continuous at $\hat{t}$, and set

$$\hat{\lambda} = \hat{\lambda}_a \equiv \operatorname{argmax}\big\{\textstyle\int_{\hat{t}-(1-\lambda)a}^{\hat{t}+\lambda a} f(x)^\alpha\,dx : \lambda\in[0,1]\big\} \qquad \text{for} \quad a>0.$$

It is a straightforward matter to see that, for $a>0$ sufficiently small,

$$\big\langle g_0^-\vee\ldots\vee g_{na}^-\big\rangle_\alpha=\int_{\mathbb{R}}\max_{0\le k\le n} f(x-ka)^\alpha\,dx=\int_{\mathbb{R}} f(x)^\alpha\,dx+n\int_{\hat{t}-(1-\hat{\lambda})a}^{\hat{t}+\hat{\lambda}a} f(x)^\alpha\,dx.$$

Hence the limit in Proposition 2 is zero (already before $a\downarrow 0$) so that (1.3) holds.

When $f$ have multiple modes, $f$ is non-decreasing on $(\hat{r}-\varepsilon,\hat{r})$ and $(\hat{s},\hat{t})$ and non-increasing on $(\hat{r},\hat{s})$ and $(\hat{t},\hat{t}+\varepsilon)$ for some $\hat{r}<\hat{s}<\hat{t}$ and $\varepsilon>0$, with $f(\hat{r}), f(\hat{t}) > f(\hat{s})$. Assume that $f$ is left- or right-continuous at $\hat{r}, \hat{s}$ and $\hat{t}$, and set

$$\lambda(t) = \lambda_a(t) \equiv \operatorname{argmax}\big\{\textstyle\int_{t-(1-\lambda)a}^{t+\lambda a} f(x)^\alpha\,dx : \lambda\in[0,1]\big\} \qquad \text{for} \quad a>0 \quad \text{and} \quad t=\hat{r},\hat{t},$$

$$\lambda(t) = \lambda_a(t) \equiv \operatorname{argmin}\big\{\textstyle\int_{t-(1-\lambda)a}^{t+\lambda a} f(x)^\alpha\,dx : \lambda\in[0,1]\big\} \qquad \text{for} \quad a>0 \quad \text{and} \quad t=\hat{s}.$$

Now straightforward reasoning reveal that (for $a>0$ sufficiently small)

$$\int_{\hat{r}-\varepsilon}^{\hat{t}+\varepsilon}\big((g_0^-\vee g_a^-)(x)-g_a^-(x)\big)\,dx=\Big[\int_{\hat{r}-(1-\lambda(\hat{r}))a}^{\hat{r}+\lambda(\hat{r})a}-\int_{\hat{s}-(1-\lambda(\hat{s}))a}^{\hat{s}+\lambda(\hat{s})a}+\int_{\hat{t}-(1-\lambda(\hat{t}))a}^{\hat{t}+\lambda(\hat{t})a}\Big]f(x)^\alpha\,dx,$$

and

$$\int_{\hat{r}-\varepsilon}^{\hat{t}+\varepsilon}\Big(\big(g_0^-\vee\ldots\vee g_{[s/a]a}^-\big)(x)-\big(g_a^-\vee\ldots\vee g_{[s/a]a}^-\big)(x)\Big)\,dx=0 \qquad \text{for} \quad s>\hat{t}-\hat{r}+2\varepsilon.$$

Hence the limit in Proposition 2 is bounded from below by

$$\lim_{a\downarrow 0}\frac{1}{a}\Big[\int_{\hat{r}-(1-\lambda(\hat{r}))a}^{\hat{r}+\lambda(\hat{r})a}-\int_{\hat{s}-(1-\lambda(\hat{s}))a}^{\hat{s}+\lambda(\hat{s})a}+\int_{\hat{t}-(1-\lambda(\hat{t}))a}^{\hat{t}+\lambda(\hat{t})a}\Big]f(x)^\alpha\,dx=f(\hat{r})^\alpha+f(\hat{t})^\alpha-f(\hat{s})^\alpha>0$$

(since $g_0^-\vee\ldots\vee g_{[s/a]a}^- - g_a^-\vee\ldots\vee g_{[s/a]a}^- \le g_0^-\vee g_a^- - g_a^-$), so that (1.3) does not hold.

**Remark 4.** Local extremes as $u\to\infty$ of stable $\xi(t)$ with $\beta_{\xi(t)}>-1$ are studied in de Acosta (1977) and Samorodnitsky (1988). But relations between extremes and upcrossings have not been investigated, and our characterization of (1.3) is new.

Now take $\alpha<1$ and skewness $\beta_{\xi(t)}=-1$, so that the scale becomes $\sigma=\sigma_{\xi(t)}=\|g_0^+\|_\alpha$ [with the notation $g^+=\max\{g,0\}$]. We have $\overline{u}=0$ with

$$\mathbf{P}\{\xi(t)>u\}\sim A_\alpha\,(-u/\sigma)^{\alpha/[2(1-\alpha)]}\exp\bigl\{-B_\alpha\,(-\sigma/u)^{\alpha/(1-\alpha)}\bigr\}\qquad\text{as}\quad u\uparrow 0 \tag{4.6}$$

for some constants $A_\alpha, B_\alpha>0$ [e.g., Samorodnitsky and Taqqu (1994, p. 17)].

By Minkowski's inequality, we have $\|g_0{+}g_t\|_\alpha\ge\|2g_0\|_\alpha$ with equality iff. $g_0=g_t$ a.e., i.e., $\xi(0)=\xi(t)$ a.s. So when $\|g_0+g_t\|_\alpha=\|2g_0\|_\alpha$ for arbitrarily small $t>0$ stationarity show that $\xi(t)$ is periodic with arbitrarily small period. By separability and $\mathbf{P}$-continuity this gives $\xi(t)=\xi(0)$ for all $t$ a.s. Since this is uninteresting, two times differentaibility of $\|g_0+g_t\|_\alpha$ makes natural the requirement

$$\|g_0+g_t\|_\alpha\ge\|2g_0\|_\alpha+Ct^2\qquad\text{for}\quad t\in[0,t_1],\quad\text{some constants}\quad C,t_1>0. \tag{4.7}$$

[The first derivative is zero since $\|g_0+g_t\|_\alpha\ge\|2g_0\|_\alpha$.] To ensure "total skewness" $\beta=-1$ as well as sufficient accuracy of certain Taylor expansions of $\xi(t)$ we assume that to each $\ell\in\mathbb{N}$ there is a constant $t_2=t_2(\ell)>0$ such that

$$g_t+\lambda\frac{(g_t-g_0)-(g_{(\ell+1)t}-g_{\ell t})}{t^2}\ge 0\quad\text{a.e.}\qquad\text{for}\quad\lambda,t\in[0,t_2], \tag{4.8}$$

$$\inf_{t\in[0,t_2]}\left\|g_t+\lambda\frac{(g_t-g_0)-(g_{(\ell+1)t}-g_{\ell t})}{t^2}\right\|_\alpha\ge\|g_0\|_\alpha+O(\lambda)\qquad\text{as}\quad\lambda\downarrow 0. \tag{4.9}$$

**Theorem 4.** *Let $\xi(t)$ be a separable $\alpha$-stable process given by (4.1) and (4.2) with $\alpha<1$ and $\|g_0\|_\alpha=\|g_0^+\|_\alpha>0$. If (4.7)-(4.9) hold, then (3.3)-(3.4) and (3.19)-(3.24) hold with $q(u)=(-u)^{\alpha/[2(1-\alpha)]}$ and $w(u)=(-u)^{1/(1-\alpha)}$ for $u<0$.*

*Proof.* Set $\hat{t}=t_1\wedge h$. Since $q$ is non-increasing (3.20) holds, while (4.6) gives (3.19) with $F(x)=1-\exp\bigl\{-B_\alpha\frac{\alpha}{1-\alpha}\|g_0\|_\alpha^{\alpha/(1-\alpha)}x\bigr\}$. Taking $D>0$ such that $\bigl(1+\frac{1}{2}\|g_0\|_\alpha^{-1}Ct^2\bigr)^{\alpha/(1-\alpha)}\ge 1+Dt^2$ for $t\in[0,\hat{t}]$, (4.6) further shows that

$$\begin{aligned}\mathbf{P}\{\xi(0)>u,\,\xi(t)>u\}&\le\mathbf{P}\{\xi(0)+\xi(t)>2u\}\\&\le 2A_\alpha\left(\frac{-2u}{\|2g_0\|_\alpha}\right)^{\alpha/[2(1-\alpha)]}\exp\left\{-B_\alpha\left(\frac{\|2g_0\|_\alpha+Ct^2}{-2u}\right)^{\alpha/(1-\alpha)}\right\}\end{aligned}$$

$$\leq 2\,A_\alpha\biggl(\frac{-u}{\|g_0\|_\alpha}\biggr)^{\alpha/[2(1-\alpha)]}\exp\biggl\{-B_\alpha\biggl(\frac{\|g_0\|_\alpha}{-u}\biggr)^{\alpha/(1-\alpha)}\bigl(1+Dt^2\bigr)\biggr\}$$

$$\leq 4\,\mathbf{P}\{\xi(0)>u\}\,\exp\Bigl\{-B_\alpha D\bigl(-\|g_0\|_\alpha/u\bigr)^{\alpha/(1-\alpha)}t^2\Bigr\}$$

for $t\in[0,\hat{t}]$ and $u$ close to zero (negative). From this we readily obtain (3.21).

Noting that $w/(q(-u))=q$ and $q/(-u)^{\alpha/(1-\alpha)}=1/q$, (4.8) and (4.9) yield

$$\text{(4.10)}\quad \mathbf{P}\biggl\{\xi(qt)+q\,\frac{[\xi(qt)-\xi(0)]-[\xi((\ell+1)qt)-\xi(\ell qt)]}{(qt)^2}>u+\frac{2\,\delta\,w}{qt}\biggr\}$$

$$\leq 2\,A_\alpha\biggl(\frac{-u}{\|g_0\|_\alpha+O(q)}\biggr)^{\alpha/[2(1-\alpha)]}\exp\biggl\{-B_\alpha\biggl(\frac{\|g_0\|_\alpha+O(q)}{-u-2\delta w/(qt)}\biggr)^{\alpha/(1-\alpha)}\biggr\}$$

$$\leq 3\,A_\alpha\biggl(\frac{-u}{\|g_0\|_\alpha}\biggr)^{\alpha/[2(1-\alpha)]}\exp\biggl\{-B_\alpha\biggl(\frac{\|g_0\|_\alpha}{-u}\biggr)^{\alpha/(1-\alpha)}\biggl[1+\frac{\alpha}{1-\alpha}\,\frac{\delta q}{t}\biggr]\biggr\}$$

$$\leq 4\,\mathbf{P}\{\xi(0)>u\}\,\exp\biggl\{-B_\alpha\,\|g_0\|_\alpha^{\alpha/(1-\alpha)}\,\frac{\alpha}{1-\alpha}\,\frac{\delta}{qt}\biggr\}$$

for $t>0$ small compared with $\delta>0$, and $u$ close to zero. From this we conclude that (3.22) and (3.23) hold. Now Theorem 3 gives (3.3), (3.4) and (3.24). □

**Remark 5.** When $\beta_{\xi(t)}=-1$ and $\alpha>1$ we have $\overline{u}=\infty$ with a very light tail for $\mathbf{P}\{\xi(t)>u\}$ as $u\to\infty$: This case is studied in Albin (1999, 2000).

**5. Markov jump processes.** Let $\{\xi(t)\}_{t\in\mathbb{R}}$ be a stationary pure jump Markov process with transition probabilities $P_x(t,dy)=\mathbf{P}\{\xi(t)\in dy\,|\,\xi(0)=x\}$ such that

$$J(u)=\varliminf_{n\to\infty}\int_{x\leq u}2^nP_x(2^{-n},(u,\overline{u}))\,F_{\xi(0)}(dx)=\varliminf_{n\to\infty}\int_{x>u}2^nP_x(2^{-n},(u,\overline{u})^c)\,F_{\xi(0)}(dx)$$

is finite [i.e., (1.5) holds]. By the Markov property we have

$$\text{(5.1)}\qquad \varliminf_{n\to\infty}2^n\,\mathbf{P}\biggl\{\xi(0)>u\geq\xi(2^{-n}),\,\bigcup_{\ell=2}^{[2^ns]}\{\xi(2^{-n}\ell)>u\}\biggr\}$$

$$\leq\varliminf_{n\to\infty}2^n\sum_{\ell=2}^{[2^ns]}\mathbf{P}\Bigl\{\xi(0)>u\geq\xi(2^{-n}),\,\xi(2^{-n}(\ell-1))\leq u<\xi(2^{-n}\ell)\Bigr\}$$

$$=\varliminf_{n\to\infty}2^n\sum_{\ell=2}^{[2^ns]}\int_{x>u,\,y\leq u,\,z\leq u}P_x(2^{-n},dy)\,P_y(2^{-n}(\ell-2),dz)\,P_z(2^{-n},(u,\overline{u}))\,F_{\xi(0)}(dx).$$

Introducing the transition kernel $P_x(dy)\equiv\lim_{t\downarrow0}t^{-1}[P_x(t,dy)-\delta_x(\{y\})]$, we expect the right-hand side of (5.1) to be equal to

$$\text{(5.2)}\qquad \int_{0<t<s,\,x>u,\,y\leq u,\,z\leq u}P_x(dy)\,P_y(t,dz)\,P_z((u,\overline{u}))\;F_{\xi(0)}(dx):$$

If (5.1) really is bounded by (5.2), then Theorem 1 and Fatou's lemma show that

(5.3) $\quad$ (1.3) [(1.4)] holds if $\quad \overline{\lim}_{u\uparrow\overline{u}} \sup_{z\le u} P_z((u,\overline{u})) = 0 \ [<1]$.

Note that (1.3) [(1.4)] and (1.5) follow directly from (5.1) (and Theorem 1) when

$$\overline{\lim}_{u\uparrow\overline{u}}\, \overline{\lim}_{n\to\infty} \sup_{z\le u} 2^n P_z(2^{-n},(u,\overline{u})) = 0 \ [<1].$$

Now assume that $\xi(t)$ is a stationary Markov chain in $\mathbb{Z}$ that satisfies

(5.4) $\quad \lim_{t\downarrow 0} P_x(t,\{y\}) = \delta_x(\{y\}), \quad |P| \equiv \sup_{y\in\mathbb{Z}} |P_y(\{y\})| < \infty \quad$ and $\quad \sum_{y\in\mathbb{Z}} P_x(\{y\}) = 0$

for each $x\in\mathbb{Z}$. By e.g., Chung (1967, Corollary p. 137), we have

(5.5) $\quad 2^n P_x(2^{-n},A) \to P_x(A) \quad$ with $\quad 2^n P_x(2^{-n},A) \le |P| \quad$ for $\quad A\subseteq \mathbb{Z}\setminus\{x\}$.

Using (5.5) together with the Dominated Convergence Theorem we see that

$$J(u) = \underline{\lim}_{n\to\infty} \sum_{y\,\le\,[u]\,<\,x} 2^n P_x(2^{-n},\{y\})\,\mathbf{P}\{\xi(0)=x\} = \sum_{y\,\le\,[u]\,<\,x} P_x(\{y\})\,\mathbf{P}\{\xi(0)=x\}$$

is bounded by $|P|$ and thus finite. Since $P_y(t,\{z\})$ is continuous in $t$ we have

$$\frac{1}{[2^n s]-1} \sum_{\ell=2}^{[2^n s]} P_y(2^{-n}(\ell-2),\{z\}) \to \frac{1}{s}\int_0^s P_y(t,\{z\})\,dt \qquad \text{as} \quad n\to\infty.$$

Since the functions on both sides are densities, (5.5) and Scheffe's Theorem give

$$\sum_{\ell=2}^{[2^n s]} \int_{z\le u} P_y(2^{-n}(\ell-2),dz)\,P_z(2^{-n},(u,\infty)) \to \sum_{z\le[u]} \int_0^s P_y(t,\{z\}) \sum_{v>[u]} P_z(\{v\})\,dt,$$

where the left-hand side is bounded by $s\,|P|$. Moreover (5.5) shows that

$$2^n \sum_{y\le[u]} P_x(2^{-n},\{y\}) \to \sum_{y\le[u]} P_x(\{y\}) \qquad \text{and} \qquad 2^n \sum_{y\le[u]} P_x(2^{-n},\{y\}) \le |P|.$$

Using the Dominated Convergence Theorem we conclude that the right-hand side in (5.1) is equal to the expression in (5.2). Hence (5.3) yields the following result:

**Theorem 5.** *For a separable and stationary Markov chain* $\xi(t)$ *in* $\mathbb{Z}$ *that satisfies* (5.4), *we have that*

$$\text{(1.3) [(1.4)]} \quad \textit{holds if} \quad \lim_{u\to\infty} \sup_{x\le[u]} \textstyle\sum_{y>[u]} P_x(\{y\}) = 0 \ [<1].$$

**6. Differentiable $n$-dimensional Gaussian processes.** Let $\{X(t)\}_{t\in\mathbb{R}}$ be a centered and a.s. continuously differentiable stationary Gaussian process with values in $\mathbb{R}_{n|1}$ [the set of $n\times 1$-matrices with real elements]. Since differentiability a.s. implies that in mean-square for Gaussian processes, we have

$$(6.1) \qquad R(t) \equiv \mathbf{E}\{X(s)X(s+t)^T\} = R(0)+r't+\tfrac{1}{2}r''t^2+o(t^2) \qquad \text{as} \quad t\to 0.$$

We can assume that $R(0)$ is diagonal since rotations do not affect (6.3) below. Writing $\mathcal{P}$ for the projection on the span of the eigen-vectors of $R(0)$ with maximal eigen-value, the probability that a "high" extrema of $X(t)$ is not generated by $\mathcal{P}X(t)$ is asymptotically negligible [e.g., Albin (1992, Section 5)], so we can assume that all eigen-values of $R(0)$ are equal. Taking unit eigen-values we get $R(0)=I$. We shall study extremes of the process $\xi(t) = \|X(t)\| = (X(t)^TX(t))^{1/2}$. To exclude periodic components we require that, writing $S_n \equiv \{z\in\mathbb{R}_{n|1} : z^Tz=1\}$,

$$(6.2) \qquad \inf_{x\in S_n} x^T[I-R(t)R(t)^T]x > 0 \qquad \text{for each choice of} \quad t\in(0,h].$$

The first works on extremes for Gaussian processes in $\mathbb{R}^n$ are Sharpe (1978) and Lindgren (1980). See Albin (2000, Section 5) for more historic details.

In Albin (2000, Theorem 4) we treat two-times differentiable processes. However, in some applications one has differentiability only once [e.g., Jarušková (2000)].

Let $\kappa^n(\cdot)$ denote the $(n-1)$-dimensional Hausdorff measure over $\mathbb{R}_{n|1}$.

**Theorem 6.** *Let $\{X(t)\}_{t\in\mathbb{R}}$ be $\mathbb{R}_{n|1}$-valued, centered and a.s. continuously differentiable stationary Gaussian with $R(0)=I$ in (6.1). If (6.2) holds, then we have*

$$(6.3) \qquad \lim_{u\to\infty} \frac{\mathbf{P}\{\sup_{t\in[0,h]}\|X(t)\|>u\}}{u\,\mathbf{P}\{\|X(0)\|>u\}} = h\int_{y\in S_n} \sqrt{y^T[r'r'-r'']y}\;\frac{d\kappa^n(y)}{\sqrt{2\pi}\,\kappa^n(S_n)}.$$

*Proof.* The process $X_\varepsilon(t) \equiv X(t)-R(t+\varepsilon)^TX(-\varepsilon)$ is independent of $X(-\varepsilon)$ and

$$(6.4) \qquad \xi(t)^2-\xi(-\varepsilon)^2$$
$$= \|X_\varepsilon(t)\|^2 + 2\,X(-\varepsilon)^TR(t+\varepsilon)\,X_\varepsilon(t) - X(-\varepsilon)^T[I-R(t+\varepsilon)R(t+\varepsilon)^T]\,X(-\varepsilon).$$

Clearly $X_0(t) = X(t)-R(t)^TX(0)$ is continuously differentiable a.s., with $X_0'(0) =_d \mathrm{N}(0, r'r'-r'')$ [by (6.1) together with symmetry of $r''$ and skew-symmetry of $r'$]. Since $X_0(0)=0$, this in turn gives (with obvious notation and a.s. convergence)

$$(6.5) \qquad X_0'(0) = \lim_{\varepsilon\downarrow 0}\frac{X_0(\varepsilon)}{\varepsilon}$$

$$= \lim_{\varepsilon\downarrow 0}\bigg(\frac{X(\varepsilon)-R(\varepsilon)^T X(0)-X(0)+R(\varepsilon)^T X(-\varepsilon)}{\varepsilon}+\frac{X_\varepsilon(0)}{\varepsilon}\bigg)$$

$$= \lim_{\varepsilon\downarrow 0}\frac{X_\varepsilon(0)}{\varepsilon} =_d \mathrm{N}\big(0, r'r'-r''\big).$$

Using (6.1) again it follows that, given constants $u>0$ and $y\in\mathbb{R}_{n|1}$, we have

$$X_\varepsilon^{u,y} \equiv \frac{\|X_\varepsilon(0)\|^2+2uy^T R(\varepsilon)X_\varepsilon(0)-u^2y^T[I-R(\varepsilon)R(\varepsilon)^T]y}{2\varepsilon u} \rightarrow y^T X_0'(0) \tag{6.6}$$

a.s. as $\varepsilon\downarrow 0$. Writing $X(-\varepsilon)\equiv\xi(-\varepsilon)\,\hat{X}(-\varepsilon)$ with $\hat{X}(-\varepsilon)\in S_n$, we further have

$$dF_{\xi(-\varepsilon),\hat{X}(-\varepsilon)}(x,y) = f_{\xi(-\varepsilon)}(x)\,\kappa^n(S_n)^{-1}\,dx\,d\kappa^n(y) \qquad \text{for} \quad (x,y)\in(0,\infty)\times S_n.$$

Taking $0\le\delta<\varDelta\le u/\varepsilon$, and using (6.4)-(6.6), we obtain

$$\frac{1}{\varepsilon}\,\mathbf{P}\Big\{u-\varDelta\varepsilon<\xi(0)\le u-\delta\varepsilon,\ u<\xi(-\varepsilon)\le u+\varDelta\varepsilon\Big\} \tag{6.7}$$

$$= \int_{x\in(0,\varDelta),\,y\in S_n}\mathbf{P}\bigg\{x+\delta+\frac{(x^2-\delta^2)\varepsilon}{2u}<\frac{\xi(-\varepsilon)^2-\xi(0)^2}{2\varepsilon u}\le x+\varDelta+\frac{(x^2-\varDelta^2)\varepsilon}{2u}$$

$$\bigg|\,\xi(-\varepsilon)=u+x\varepsilon,\ \hat{X}(-\varepsilon)=y\bigg\}\,\frac{f_{\xi(-\varepsilon)}(u+x\varepsilon)\,dx\,d\kappa^n(y)}{\kappa^n(S_n)}$$

$$\rightarrow \int_0^\varDelta\bigg[\int_{y\in S_n}\mathbf{P}\Big\{x+\delta<-y^T X_0'(0)\le x+\varDelta\Big\}\,\frac{d\kappa^n(y)}{\kappa^n(S_n)}\bigg]\,dx\ f_{\xi(0)}(u) \qquad \text{as} \quad \varepsilon\downarrow 0$$

for $0\le\delta<\varDelta\le\infty$. Since $\mathbf{E}\{[y^T X_0'(0)]^+\} = \sqrt{y^T[r'r'-r'']y}\,/\sqrt{2\pi}$ by (6.5) and $f_{\xi(0)}(u)\sim u\,\mathbf{P}\{\|X(0)\|>u\}$ as $u\rightarrow\infty$ (which is elementary), it follows that

$$\lim_{u\rightarrow\infty}\frac{J(u)}{u\,\mathbf{P}\{\|X(0)\|>u\}} = \int_{y\in S_n}\sqrt{y^T[r'r'-r'']y}\;\frac{d\kappa^n(y)}{\sqrt{2\pi}\,\kappa^n(S_n)}\,. \tag{6.8}$$

When the integral in (6.8) is zero (1.2) gives (6.3). When the integral is non-zero, (2.1) together with (6.7) and (6.8) show that (1.3) and (6.3) hold if

$$\varlimsup_{\varepsilon\downarrow 0}\frac{1}{\varepsilon\,f_{\xi(0)}(u)}\,\mathbf{P}\Big\{u-\varDelta\varepsilon<\xi(0)\le u-\delta\varepsilon,\ u<\xi(-\varepsilon)\le u+\varDelta\varepsilon,\ M(h)>u\Big\}\rightarrow 0 \tag{6.9}$$

as $u\rightarrow\infty$ for $0<\delta<\varDelta<\infty$: Using (6.4) as for (6.7), the limit in (6.9) becomes

$$\varlimsup_{\varepsilon\downarrow 0}\int_{x\in(0,\varDelta),\,y\in S_n}\mathbf{P}\bigg\{x+\delta+\frac{(x^2-\delta^2)\varepsilon}{2u}<-X_\varepsilon^{u,y}\le x+\varDelta+\frac{(x^2-\varDelta^2)\varepsilon}{2u},\ \sup_{t\in[0,h]}\Big(\|X_\varepsilon(t)\|^2$$

$$+2uy^T R(t+\varepsilon)X_\varepsilon(t)+u^2y^T R(t+\varepsilon)R(t+\varepsilon)^T y\Big)>u^2\bigg\}\,\frac{f_{\xi(-\varepsilon)}(u+x\varepsilon)\,dx\,d\kappa^n(y)}{f_{\xi(0)}(u)\,\kappa^n(S_n)}.$$

Using that $-X_\varepsilon^{u,y} \geq x+\delta+\frac{(x^2-\delta^2)\varepsilon}{2u}$ and rearranging, this can be bounded by

$$\overline{\lim_{\varepsilon\downarrow 0}} \int_{x\in(0,\Delta),\, y\in S_n} \mathbf{P}\bigg\{\bigcup_{t\in[0,h]}\Big\{\|X_\varepsilon(t)\|^2+2uy^TR(t+\varepsilon)X_\varepsilon(t)-2utX_\varepsilon^{u,y}$$
$$> u^2-u^2y^TR(t+\varepsilon)R(t+\varepsilon)^Ty+t\big[2u(x+\delta)+(x^2-\delta^2)\varepsilon\big]\Big\}\bigg\}\frac{dx\,d\kappa^n(y)}{\kappa^n(S_n)}$$

Sending $\varepsilon\downarrow 0$ and using (6.6), this becomes

$$\int_{x\in(0,\Delta),\, y\in S_n} \mathbf{P}\bigg\{\bigcup_{t\in[0,h]}\Big\{\frac{u^{-1}\|X_0(t)\|^2+2y^TR(t)X_0(t)}{t}-2y^TX_0'(0)$$
$$> \frac{uy^T[I-R(t)R(t)^T]y}{t}+2(x+\delta)\Big\}\bigg\}\frac{dx\,d\kappa^n(y)}{\kappa^n(S_n)}.$$

Hence (6.9) follows from (6.2) and the fact that [by (6.5)] the process

$$\zeta(t)\equiv\frac{\|X_0(t)\|^2+2y^TR(t)X_0(t)}{t}-2y^TX_0'(0) \quad \text{is continuous a.s. with } \zeta(0)=0. \ \square$$

**7. Moving $\mathbb{L}^2$-norms of differentiable Gaussian processes.** Let $\{X(t)\}_{t\in\mathbb{R}}$ be a standardized, stationary, separable and mean-square differentiable Gaussian process. The covariance function $R$ of such a process $X$ satisfies

$$\text{(7.1)} \qquad R(t)\equiv\mathbf{Cov}\{X(s),X(s+t)\}=1+\tfrac{1}{2}R''(0)t^2+o(t^2) \quad \text{as } t\to 0.$$

We shall use Theorem 3 to prove the relations (3.3), (3.4) and (3.24), by means of a verification of the conditions (3.19)-(3.23), for the moving $\mathbb{L}^2$-norm process

$$\text{(7.2)} \qquad \xi(t)\equiv\|I_{[t,1+t]}X\|^2_{\mathbb{L}^2(\mathbb{R})}=\|I_{[0,1]}(\cdot)X(\cdot+t)\|^2_{\mathbb{L}^2(\mathbb{R})}=\int_t^{1+t}X(s)^2\,ds, \quad t\in\mathbb{R}.$$

To that end we have to make two additional requirements: The first one is that

$$\text{(7.3)} \qquad \int_0^1\int_0^1 f(r)f(s)\,R''(s-r)\,dsdr<0 \qquad \text{for}$$
$$f=\operatorname{argmax}\bigg\{\int_0^1\int_0^1 f(r)f(s)\,R(s-r)\,dsdr : f\in\mathbb{L}^2([0,1]),\ \|f\|_{\mathbb{L}^2([0,1])}=1\bigg\}.$$

[We have $\int_0^1\int_0^1 f(r)f(s)\,R''(s-r)\,dsdr\leq 0$ since $-R''$ is a covariance function.] In Examples 3 and 5 below we show that (7.3) holds when the derivative process $X'(t)$ has a spectral density, but also for example when the spectrum is discrete with only one frequency-component (i.e., when $X$ is a cosine-process).

The second additional requirement we have to impose is that

$$\text{(7.4)} \qquad \int_0^1\int_0^1 f(r)f(s)\,[R(s-r)-R(s-r-t)]\,dsdr>0 \qquad \text{for} \quad t\in(0,h] \quad \text{and}$$

$$f = \operatorname{argmax}\biggl\{ \int_0^1\int_0^1 f(r)f(s)\,R(s-r)\,dsdr : f\in\mathbb{L}^2([0,1]),\ \|f\|_{\mathbb{L}^2([0,1])}=1\biggr\}.$$

[The integral on the left-hand side is equal to $\frac{1}{2}\mathbf{Var}\{\int_0^1 f(s)\,[X(s)-X(s-t)]\,ds\}$, and is therefore non-negative.] In Examples 4 and 5 below we show that (7.4) holds in the case when $\lim_{t\to\infty} R(t)=0$, but also for example for the (periodic) cosine-process. In addition, it follows from the latter part of the proof of Theorem 7 below, that (7.3) implies that (7.4) holds for some (sufficiently small) $h>0$.

Put $\langle f\,|\,g\rangle = \int_{s\in\mathbb{R}} f(s)g(s)\,ds$ for $f,g\in\mathbb{L}^2([0,1])$. We can view $\mathfrak{X}\equiv I_{[0,1]}X$ as an $\mathbb{L}^2([0,1])$-valued zero-mean Gaussian random element, because

$$\sum_{i=1}^n \langle f_i\,|\,\mathfrak{X}\rangle = \sum_{i=1}^n \int_0^1 f_i(s)X(s)\,ds \qquad \text{is a zero-mean Gaussian random variable in } \mathbb{R}$$

for any choice of $n\in\mathbb{N}$ and $f_1,\ldots,f_n\in\mathbb{L}^2([0,1])$. The covariance operator $\mathfrak{R}: \mathbb{L}^2([0,1])\times\mathbb{L}^2([0,1])\to\mathbb{R}$ of $\mathfrak{X}$ is given by

$$\langle\mathfrak{R}f\,|\,g\rangle = \mathbf{E}\bigl\{\langle f\,|\,\mathfrak{X}\rangle\,\langle g\,|\,\mathfrak{X}\rangle\bigr\} = \int_0^1\int_0^1 R(s-r)\,f(r)g(s)\,drds \quad \text{for } f,g\in\mathbb{L}^2([0,1]).$$

There exists a complete ortonormal system $\{e_n\}_{n=1}^{\infty}$ in $\mathbb{L}^2([0,1])$ of eigen-functions of $\mathfrak{R}$ with eigen-values $\lambda_1=\ldots=\lambda_N>\lambda_{N+1}\geq\ldots\geq 0$ that satisfy $\sum_{n=1}^{\infty}\lambda_n<\infty$. Since the process $X$ is continuous a.s., we have the Karhunen-Loève expansion $X(s)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\,\eta_n\,e_n(s)$ (with uniform convergence) for all $s\in[0,1]$ with probability 1, where $\{\eta_n\}_{n=1}^{\infty}$ are independent N(0,1)-distributed random variables. [See for example Adler (1990, Sections 3.2 and 3.3).] This gives

$$\text{(7.5)}\qquad \xi(0)=\int_0^1 X(s)^2\,ds=\langle X\,|\,X\rangle=\sum_{n=1}^{\infty}\lambda_n\,\eta_n^2 \qquad \text{a.s. and in mean-square.}$$

Here we have $\sum_{n=1}^{\infty}\lambda_n=\mathbf{E}\{\xi(0)\}=1$, and the largest eigen-value $\lambda_1$ is given by

$$\text{(7.6)}\qquad \lambda_1=\sup\biggl\{\int_0^1\int_0^1 f(s)f(r)\,R(s-r)\,dsdr : f\in\mathbb{L}^2([0,1]),\ \|f\|_{\mathbb{L}^2([0,1])}=1\biggr\}.$$

The asymptotic behaviour of the upper tail $\mathbf{P}\{\|Z\|>u\}$ as $u\to\infty$ of a zero-mean Gaussian random element $Z$ in a separable Hilbert space were determined by Zolotarev (1961). The next lemma completes Zolotarev's result with upper bounds on the tail-probability where $Z$ may depend on an "external parameter" (e.g., $u$):

**Lemma 1.** *Let $\{\eta_n\}_{n=1}^{\infty}$ be independent $N(0,1)$-distributed random variables and $\lambda_1=\ldots=\lambda_N>\lambda_{N+1}\geq\ldots\geq 0$ $(N\geq 1)$ constants with $\sum_{n=1}^{\infty}\lambda_n\leq 1$. Put $Z\equiv\sum_{n=1}^{\infty}\lambda_n\eta_n^2$, and let $g_N$ be the density function of the random variable*

*$\lambda_1 \sum_{n=1}^{N} \eta_n^2$. There exists a (universal) constant $K \geq 0$ such that*

$$\mathbf{P}\{Z > u\} \leq K\, 2\lambda_1\, g_N(u)\, \exp\left\{ \frac{1 + \lambda_1/\lambda_{N+1}}{2\,(\lambda_1 - \lambda_{N+1})} \right\} \quad \textit{for} \quad u \geq 4\lambda_1\,(N/2-1)^+. \tag{7.7}$$

*Moreover, we have, by Zolotarev (1961),*

$$\mathbf{P}\{Z > u\} \sim 2\lambda_1\, g_N(u) \prod_{n=N+1}^{\infty} \frac{1}{\sqrt{1 - \lambda_n/\lambda_1}} \qquad \textit{as} \quad u \to \infty. \tag{7.8}$$

*Proof.* We only prove the inequality (7.7) when $\lambda_{N+1} > 0$, because it reduces to an elementary statement concerning the $\chi^2(N)$-distribution when $\lambda_{N+1} = 0$.

Take $N \geq 2$ and let $\hat{g}$ be the density function of $\sum_{n=N+1}^{\infty} \lambda_n \eta_n^2$. Since

$$g_N(x) = (\Gamma(\tfrac{N}{2}))^{-1}\, (2\lambda_1)^{-N/2}\, x^{N/2-1}\, \mathrm{e}^{-x/(2\lambda_1)} \quad \text{for} \quad x > 0, \tag{7.9}$$

elementary considerations reveal that

$$\mathbf{P}\{Z > u\} = g_N(u) \int_{x=0}^{x=\infty} \int_{y=0}^{y=x+u} \left(1 + \frac{x-y}{u}\right)^{N/2-1} \mathrm{e}^{-(x-y)/(2\lambda_1)}\, \hat{g}(y)\, dy dx.$$

Since $(1+x/u)^{N/2-1} \exp\{-x/(4\lambda_1)\}$ is a non-increasing function of $x \geq 0$ provided that $u \geq 4\lambda_1(N/2-1)$, it follows that (for such values of $u$)

$$\begin{aligned} &\mathbf{P}\{Z > u\} \\ &\quad \leq g_N(u) \int_{x=0}^{x=\infty} \int_{y=0}^{y=\infty} \mathrm{e}^{-x/(4\lambda_1) + y/(2\lambda_1)}\, \hat{g}(y)\, dy dx = 4\lambda_1\, g_N(u) \prod_{n=N+1}^{\infty} \frac{1}{\sqrt{1 - \lambda_n/\lambda_1}}. \end{aligned}$$

Since $-\ln(1-x) \leq \lambda_1 (\lambda_1 - \lambda_{N+1})^{-1} x$ for $x \in [0, \lambda_{N+1}/\lambda_1]$, we have

$$\begin{aligned} \prod_{n=N+1}^{\infty} \frac{1}{\sqrt{1 - \lambda_n/\lambda_1}} &= \exp\left\{ \frac{1}{2} \sum_{n=N+1}^{\infty} \ln\left(1 + \frac{\lambda_n}{\lambda_1 - \lambda_n}\right) \right\} \leq \exp\left\{ \frac{1}{2} \sum_{n=N+1}^{\infty} \frac{\lambda_n}{\lambda_1 - \lambda_{N+1}} \right\} \\ &\leq \exp\left\{ \frac{1}{2\,(\lambda_1 - \lambda_{N+1})} \right\}. \end{aligned} \tag{7.10}$$

Here we used that $\sum_{n=1}^{\infty} \lambda_n \leq 1$ by assumption. This gives (7.7) for $N \geq 2$.

Now take $N = 1$. By basic properties of the modified Bessel function $I_0(x) = \sum_{k=0}^{\infty} (x/2)^{2k}/(k!)^2$, the random variable $\lambda_1 \eta_1^2 + \lambda_2 \eta_2^2$ has density function

$$\begin{aligned} f_{\lambda_1 \eta_1^2 + \lambda_2 \eta_2^2}(z) &= \int_0^z \frac{1}{\sqrt{x(z-x)}} \exp\left\{ -\left( \frac{z-x}{2\lambda_1} + \frac{x}{2\lambda_2} \right) \right\} \frac{dx}{2\pi \sqrt{\lambda_1 \lambda_2}} \\ &= \frac{1}{2\sqrt{\lambda_1 \lambda_2}} \exp\left\{ -\frac{z}{4}\left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \right\} I_0\left( \frac{z}{4}\left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \right) \quad \text{for} \quad z > 0 \end{aligned}$$

[recall (7.9)]. Since $I_0$ is continuous with $I_0(x) \sim (2\pi x)^{-1/2}\mathrm{e}^x$ as $x\to\infty$, we have $K_1 = \sup_{x\ge 0}\sqrt{2\pi x}\,\mathrm{e}^{-x} I_0(x) \in [1,\infty)$. In the case when $\lambda_3 = 0$ we get

$$\begin{aligned}\mathbf{P}\{Z>u\} &= \int_0^\infty \exp\Big\{-\frac{u+x}{4}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big\} I_0\Big(\frac{u+x}{4}\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)\Big)\frac{dx}{2\sqrt{\lambda_1\lambda_2}}\\ &\le K_1\,g_1(u)\int_0^\infty \frac{\sqrt{u}}{\sqrt{\lambda_2}}\Big((u+x)\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)\Big)^{-1/2}\mathrm{e}^{-x/(2\lambda_1)}\,dx\\ &\le K_1\,2\lambda_1\,g_1(u)\sqrt{\frac{\lambda_1}{\lambda_1-\lambda_2}} \qquad \text{for} \quad u\ge 0.\end{aligned}$$

This gives (7.7) in case $\lambda_3=0$ since $\sqrt{x}\le \mathrm{e}^{x/2}$.

Finally consider the case when $N=1$ and $\lambda_3>0$. Write $\tilde g$ for the density function of $\sum_{n=3}^\infty \lambda_n\eta_n^2$ and put $K_2 = \sup_{x\ge 0}\sqrt{\pi/2}\,\mathrm{e}^{-x} I_0(x)$. The fact that (7.7) holds also in this case follows from the following sequence of estimates [cf. (7.10)]

$$\begin{aligned}&\mathbf{P}\{Z>u\}\\ &= \int_{x=0}^{x=\infty}\int_{y=0}^{y=x+u} \exp\Big\{-\frac{u+x-y}{4}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big\} I_0\Big(\frac{u+x-y}{4}\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)\Big)\tilde g(y)\,\frac{dydx}{2\sqrt{\lambda_1\lambda_2}}\\ &\le K_1\,g_1(u)\int_{x=0}^{x=\infty}\int_{y=0}^{y=u/2}\frac{\sqrt{u}}{\sqrt{\lambda_2}}\Big((u+x-y)\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)\Big)^{-1/2}\mathrm{e}^{-(x-y)/(2\lambda_1)}\tilde g(y)\,dydx\\ &\quad + K_2\,g_1(u)\int_{x=0}^{x=\infty}\int_{y=u/2}^{y=\infty}\frac{\sqrt{u}}{\sqrt{\lambda_2}}\,\mathrm{e}^{-(x-y)/(2\lambda_1)}\tilde g(y)\,dydx\\ &\le K_1\,2\lambda_1\,g_1(u)\sqrt{\frac{2\lambda_1}{\lambda_1-\lambda_2}}\int_0^\infty \mathrm{e}^{y/(2\lambda_1)}\tilde g(y)\,dy\\ &\quad + K_2\,2\lambda_1\,g_1(u)\Big[\sup_{y>u/2}\frac{\sqrt{u}}{\sqrt{\lambda_2}}\exp\Big\{\frac{y}{4}\Big(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\Big)\Big\}\Big]\int_0^\infty \exp\Big\{\frac{y}{4}\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\Big)\Big\}\tilde g(y)\,dy\\ &\le K_1\,2\lambda_1\,g_1(u)\sqrt{\frac{2\lambda_1}{\lambda_1-\lambda_2}}\prod_{n=3}^\infty\frac{1}{\sqrt{1-\lambda_n/\lambda_1}}\\ &\quad + K_2\,2\lambda_1\,g_1(u)\Big[\sup_{z>0}\frac{\sqrt{z}}{\sqrt{\lambda_2}}\exp\Big\{\frac{z}{8}\Big(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\Big)\Big\}\Big]\prod_{n=3}^\infty\frac{1}{\sqrt{1-\frac12\lambda_n(\lambda_1^{-1}+\lambda_2^{-1})}}\\ &\le \sqrt{2}\,K_1\,2\lambda_1\,g_1(u)\,\exp\Big\{\frac12\,\frac{\lambda_1}{\lambda_1-\lambda_2}\Big\}\,\exp\Big\{\frac12\sum_{n=3}^\infty \ln\Big(1+\frac{\lambda_n}{\lambda_1-\lambda_n}\Big)\Big\}\\ &\quad + K_2\,2\lambda_1\,g_1(u)\sqrt{\frac{4\lambda_1/\mathrm{e}}{\lambda_1-\lambda_2}}\,\exp\Big\{\frac12\sum_{n=3}^\infty \ln\Big(1+\frac{\frac12\lambda_n(\lambda_1^{-1}+\lambda_2^{-1})}{1-\frac12\lambda_n(\lambda_1^{-1}+\lambda_2^{-1})}\Big)\Big\}\\ &\le \sqrt{2}\,K_1\,2\lambda_1\,g_1(u)\,\exp\Big\{\frac12\,\frac{\lambda_1}{\lambda_1-\lambda_2}\Big\}\,\exp\Big\{\frac12\sum_{n=3}^\infty\frac{\lambda_n}{\lambda_1-\lambda_2}\Big\}\\ &\quad + 2K_2\,2\lambda_1\,g_1(u)\,\exp\Big\{\frac12\,\frac{\lambda_1}{\lambda_1-\lambda_2}-\frac12\Big\}\,\exp\Big\{\frac12\sum_{n=3}^\infty\frac{\frac12\lambda_n(\lambda_1^{-1}+\lambda_2^{-1})}{1-\frac12\lambda_2(\lambda_1^{-1}+\lambda_2^{-1})}\Big\}\end{aligned}$$

$$\leq \sqrt{2}\, K_1\, 2\lambda_1\, g_1(u)\ \exp\bigg\{\frac{1}{2(\lambda_1-\lambda_2)}\sum_{n=1}^{\infty}\lambda_n\bigg\}$$

$$+\, 2\, K_2\, 2\lambda_1\, g_1(u)\ \exp\bigg\{\frac{1}{2}\,\frac{\lambda_1}{\lambda_1-\lambda_2}\bigg\}\ \exp\bigg\{\frac{1}{2}\sum_{n=3}^{\infty}\frac{\lambda_n(1+\lambda_1/\lambda_2)}{\lambda_1-\lambda_2}\bigg\}$$

$$\leq (\sqrt{2}\, K_1 + 2\, K_2)\, 2\lambda_1\, g_1(u)\ \exp\bigg\{\frac{1+\lambda_1/\lambda_2}{2\,(\lambda_1-\lambda_2)}\bigg\} \qquad \text{for} \quad u\geq 0.$$

Here we used the inequality $\sqrt{x}\leq \mathrm{e}^{x/2}$ and the assumption $\sum_{n=1}^{\infty}\lambda_n\leq 1$. □

**Theorem 7.** *Let* $\{\xi(t)\}_{t\in\mathbb{R}}$ *be given by* (7.2) *where* $\{X(t)\}_{t\in\mathbb{R}}$ *is a standardized, stationary and separable Gaussian process such that conditions* (7.1)*,* (7.3) *and* (7.4) *hold. Equations* (3.19)*-*(3.23) *hold with* $q(u)=(1\vee u)^{-1/2}$ *and* $w(u)=1$*.*

*Proof.* Using (7.5) together with (7.8) and (7.9) it follows that (3.19) holds with $F(x) = 1-\mathrm{e}^{-x/(2\lambda_1)}$. Further (3.20) holds since $q$ is decreasing.

By a classical result due to Landau and Shepp (1970) and Marcus and Shepp (1972), given an $\varepsilon>0$ there exists a constant $u_0=u_0(\varepsilon)>1$ such that

$$\mathbf{P}\Big\{\sup_{s\in[0,t]} X(s)>u\Big\} \leq \mathbf{P}\Big\{\sup_{s\in[0,1]} X(s)>u\Big\} \leq \exp\bigg\{\frac{-u^2}{2\,(1+\varepsilon)}\bigg\} \quad \text{for } u\geq u_0 \text{ and } t\in(0,1].$$

Moreover, there exists a constant $C_1=C_1(\ell)>0$ such that

$$V(t) = \mathbf{Var}\bigg\{\int_0^t [X(\ell t+s)-X(s)]\, ds\bigg\} = 2\,\bigg(\int_0^t\!\!\int_0^t - \int_{\ell t}^{(\ell+1)t}\!\!\int_0^t\bigg) R(s-r)\, ds dr \leq C_1\, t^4$$

for $t\in[0,1]$: This follows from noting that $V(0)=0$ and

$$V'(t) = 4\textstyle\int_0^t R(s)\, ds - 2\,(\ell+1)\int_{\ell t}^{(\ell+1)t} R(s)\, ds + 2\,(\ell-1)\int_{(\ell-1)t}^{\ell t} R(s)\, ds,$$

so that $V'(0)=0$ and, by (7.1),

$$\tfrac{1}{2}V''(t) = 2R(t) - (\ell{+}1)^2 R((\ell{+}1)t) + 2\ell^2 R(\ell t) - (\ell{-}1)^2 R((\ell{-}1)t) = -6R''(0)t^2 + o(t^2)$$

as $t\to 0$. Taking $C_1\equiv \sup_{t\in(0,1]} V''(t)/t^2$, a Taylor expansion therefore gives

$$V(t) = \int_0^t (t-s)\, V''(s)\, ds \leq C_1\, t^4 \qquad \text{for} \quad t\in[0,1].$$

By application of the estimates obtained above we conclude that

$$\mathbf{P}\Big\{\xi(0)\leq u+\nu,\ \xi(qt)>u+\nu+\delta t,\ \xi(\ell qt)>u+\nu+\delta t,\ \xi((\ell+1)qt)\leq u+\nu\Big\}$$

$$\leq \mathbf{P}\bigg\{\bigg(\int_1^{1+qt} - \int_0^{qt}\bigg) X(s)^2\,ds > \delta t,\ \bigg(\int_{1+\ell qt}^{1+(\ell+1)qt} - \int_{\ell qt}^{(\ell+1)qt}\bigg) X(s)^2\,ds < -\delta t\bigg\}$$

$$\leq \mathbf{P}\bigg\{\bigg(\int_{1+\ell qt}^{1+(\ell+1)qt} - \int_{\ell qt}^{(\ell+1)qt}\bigg) X(s)^2\,ds - \bigg(\int_1^{1+qt} - \int_0^{qt}\bigg) X(s)^2\,ds < -2\delta t\bigg\}$$

$$\leq \mathbf{P}\bigg\{\bigg\{\bigg(\int_{1+\ell qt}^{1+(\ell+1)qt} - \int_1^{1+qt}\bigg) X(s)^2\,ds < -\delta t\bigg\} \cup \bigg\{\bigg(\int_{\ell qt}^{(\ell+1)qt} - \int_0^{qt}\bigg) X(s)^2\,ds > \delta t\bigg\}\bigg\}$$

$$= 2\,\mathbf{P}\bigg\{\int_0^{qt} \big[X(\ell qt+s)^2 - X(s)^2\big]\,ds > \delta t\bigg\}$$

$$\leq 4\,\mathbf{P}\bigg\{\sup_{s\in[0,qt]} X(s) > \frac{\sqrt{u}}{2\sqrt{t}}\bigg\} + 2\,\mathbf{P}\bigg\{\int_0^{qt} \big[X(\ell qt+s) - X(s)\big]\,ds > \frac{\delta\, t^{3/2}}{\sqrt{u}}\bigg\}$$

$$\leq 4\,\exp\bigg\{\frac{-u}{8\,(1+\varepsilon)\,t}\bigg\} + \mathbf{P}\bigg\{\mathrm{N}(0,1) > \frac{\delta\, t^{3/2}}{\sqrt{u}\sqrt{C_1}\,(qt)^2}\bigg\}$$

for $t\in(0,1]$, $\ell\geq 1$ and $u\geq u_0$. Hence (7.5) and (7.8) give (3.22) and (3.23).

Let $\mathbb{L}^2([0,1],\mathbb{R}^2)$ denote the Hilbert space of functions $(f_1,f_2):[0,1]\to\mathbb{R}^2$ such that $f_1,f_2\in\mathbb{L}^2([0,1])$, with inner product $\langle (f_1,f_2)\,|\,(g_1,g_2)\rangle = \langle f_1|g_1\rangle + \langle f_2|g_2\rangle$ and norm $\|(f_1,f_2)\|^2 = \|f_1\|^2_{\mathbb{L}^2([0,1])} + \|f_2\|^2_{\mathbb{L}^2([0,1])}$. We can view $\mathfrak{Y}_t \equiv \sqrt{1/2}\,(I_{[0,1]}X, I_{[0,1]}X(\cdot+t))$ as an $\mathbb{L}^2([0,1],\mathbb{R}^2)$-valued zero-mean Gaussian random element. The arguments used to establish (7.5) carry over to show that

$$\text{(7.11)}\qquad \tfrac{1}{2}[\xi(0)+\xi(t)] = \tfrac{1}{2}\big[\big(\textstyle\int_0^1 + \int_t^{1+t}\big) X(s)^2\,ds\big] = \|\mathfrak{Y}_t\|^2 = \displaystyle\sum_{n=1}^{\infty} \lambda_n(t)\,\eta_n^2$$

(a.s. and in mean-square), where $\lambda_1(t) = \ldots = \lambda_N(t) > \lambda_{N+1}(t) \geq \ldots \geq 0$ satisfy $\sum_{n=1}^{\infty} \lambda_n(t) < \infty$ and $\{\eta_n\}_{n=1}^{\infty}$ are independent $\mathrm{N}(0,1)$-distributed random variables. Since the process $\{\|\mathfrak{Y}_t\|^2\}_{t\in\mathbb{R}}$ is (two times) mean-square differentiable,

$$\lambda_k(t) = \mathbf{E}\big\{\tfrac{1}{2}(\eta_k^2-1)\,\|\mathfrak{Y}_t\|^2\big\} \quad \text{is two (four) times differentiable.}$$

The condition (3.21) holds if the largest eigen-value $\lambda_1(t)$ in (7.11) satisfies

$$\text{(7.12)}\qquad \lambda_1(t) \leq \lambda_1 - K_3 t^2 \quad \text{for } t\in(0,h], \text{ for some constant } K_3>0,$$

where $\lambda_1=\lambda_1(0)$ is given by (7.5): Because by Lemma 1 together with continuity of the eigen-values $\lambda_n(t)$ (as functions of $t$), (7.12) readily yields

$$\frac{\mathbf{P}\{\xi(0)>u,\ \xi(kqt)>u\}}{\mathbf{P}\{\xi(0)>u\}} \leq \frac{\mathbf{P}\{\frac{1}{2}[\xi(0)+\xi(kqt)]>u\}}{\mathbf{P}\{\xi(0)>u\}} \leq K_4\,\exp\{-K_3(kt)^2\}$$

for $u\geq 1$ and $kqt\leq h$, for some constant $K_4>0$. This in turn easily gives (3.21).

In analogy with (7.6), the largest eigen-value $\lambda_1(t)$ in (7.11) is given by

$$
\begin{aligned}
&\lambda_1(t)\\
&= \sup_{\|(f_1,f_2)\|=1} \mathbf{E}\big\{\langle(f_1,f_2)|\mathfrak{Y}_t\rangle\,\langle(f_1,f_2)|\mathfrak{Y}_t\rangle\big\}\\
&= \sup_{\|(f_1,f_2)\|=1} \frac{1}{2}\int_0^1\!\!\int_0^1\Big([f_1(r)f_1(s)+f_2(r)f_2(s)]\,R(s-r)\\
&\qquad\qquad + f_1(r)f_2(s)\,R(s-r+t) + f_2(r)f_1(s)\,R(s-r-t)\Big)\,dsdr\\
&= \sup_{\|(f_1,f_2)\|=1} \frac{1}{2}\int_0^1\!\!\int_0^1\Big([f_1(r)+f_2(r)][f_1(s)+f_2(s)]\,R(s-r)\\
&\qquad\qquad + \tfrac{1}{2}t^2[f_1(r)f_2(s)+f_2(r)f_1(s)]\,R''(s-r)\Big)\,dsdr + o(t^2)\\
&= \sup_{\|(\frac{f_1+f_2}{\sqrt{2}},\frac{f_1-f_2}{\sqrt{2}})\|=1} \int_0^1\!\!\int_0^1\bigg(\frac{f_1(r)+f_2(r)}{\sqrt{2}}\,\frac{f_1(s)+f_2(s)}{\sqrt{2}}\,R(s-r)\\
&\qquad\qquad + \frac{1}{4}\,t^2\Big(\frac{f_1(r)+f_2(r)}{\sqrt{2}}\,\frac{f_1(s)+f_2(s)}{\sqrt{2}}\\
&\qquad\qquad - \frac{f_1(r)-f_2(r)}{\sqrt{2}}\,\frac{f_1(s)-f_2(s)}{\sqrt{2}}\Big)\,R''(s-r)\bigg)\,dsdr + o(t^2)\\
&= \sup_{\|(f,g)\|=1} \int_0^1\!\!\int_0^1\Big(f(r)f(s)\,R(s-r)\\
&\qquad\qquad + \tfrac{1}{4}t^2[f(r)f(s) - g(r)g(s)]R''(s-r)\Big)\,dsdr + o(t^2)\\
&= \sup_{\|f\|_{\mathbb{L}^2([0,1])}=1} \int_0^1\!\!\int_0^1\Big(f(r)f(s)\,R(s-r) + \tfrac{1}{4}t^2 f(r)f(s)\,R''(s-r)\Big)\,dsdr + o(t^2)
\end{aligned}
$$

for $t\geq 0$ sufficiently small. It follows that $\lambda_1(t) = \lambda_1 - K_5t^2 + o(t^2)$ as $t\to 0$, for some constant $K_5>0$ when (7.3) holds. In order to establish (7.12) it is therefore, by continuity, sufficient to prove that $\lambda_1(t)<\lambda_1$ for each $t\in(0,h]$: We have

$$
\begin{aligned}
\lambda_1(t) &= \sup_{\|(f_1,f_2)\|=1} \frac{1}{2}\,\mathbf{Var}\bigg\{\int_0^1 f_1(r)X(r)\,dr + \int_0^1 f_2(s)X(s+t)\,ds\bigg\}\\
&\leq \sup_{\|(f_1,f_2)\|=1} \bigg(\mathbf{Var}\bigg\{\int_0^1 f_1(r)X(r)\,dr\bigg\} + \mathbf{Var}\bigg\{\int_0^1 f_2(s)X(s+t)\,ds\bigg\}\bigg)\\
&= \sup_{\|f\|_{\mathbb{L}^2([0,1])}=1} \mathbf{Var}\bigg\{\int_0^1 f(s)X(s)\,ds\bigg\}\\
&= \lambda_1.
\end{aligned}
$$

Here equality takes place for a $t\in(0,h]$ if and only if

$$
\mathbf{Var}\bigg\{\int_0^1 f(r)X(r)\,dr + \int_0^1 f(s)X(s+t)\,ds\bigg\}
$$

$$= 2\,\mathbf{Var}\bigg\{\int_0^1 f(r)X(r)\,dr\bigg\} + 2\,\mathbf{Var}\bigg\{\int_0^1 f(s)X(s{+}t)\,ds\bigg\},$$

with $f$ choosen as in (7.4). This in turn holds if and only if $\mathbf{Var}\{\int_0^1 f(s)X(s)\,dr - \int_0^1 f(s)X(s{+}t)\,ds\} = 0$, which contradicts (7.4). $\square$

**Example 3.** (The Case with Continuous Spectra.) Assume that the covariance functions $R$ and $-R''$ have spectralrepresentations

$$R(\tau) = \int_{-\infty}^{\infty} \mathrm{e}^{i2\pi\nu\tau}\,\phi(\nu)\,d\nu \qquad \text{and} \qquad -R''(\tau) = \int_{-\infty}^{\infty} \mathrm{e}^{i2\pi\nu\tau}\,(2\pi\nu)^2\phi(\nu)\,d\nu.$$

Let $\hat{f}(\nu) =_{\mathbb{L}^2} \int_0^1 \mathrm{e}^{i2\pi\nu\tau} f(s)\,ds$, where $f$ is the function in (7.3). We have

$$\begin{aligned} \int_0^1\int_0^1 f(s)f(r)\,R(s-r)\,dsdr &= \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2\,d\nu \\ \int_0^1\int_0^1 f(s)f(r)\,R''(s-r)\,dsdr &= -\int_{-\infty}^{\infty} (2\pi\nu)^2|\hat{f}(\nu)|^2\,d\nu \end{aligned},$$

where the upper two integrals are strictly positive with common value $\lambda_1$. It follows that the lower integrals are strictly negative, so that (7.3) holds.

**Example 4.** (The Case when $\lim_{t\to\infty} R(t) = 0$.) Let $Y(t) = \int_0^1 f(s)X(s{+}t)\,ds$ for $t\in\mathbb{R}$, where $f$ is the function in (7.4). The process $Y$ has covariance function $\rho(\tau) = \mathbf{Cov}\{Y(t), Y(t{+}\tau)\} = \int_0^1\int_0^1 f(r)f(s)\,R(s-r-\tau)\,dsdr$, so that the integral in (7.4) is equal to $\rho(0)-\rho(t)$. Now an elementary argument shows that (7.4) holds if and only if $\rho$ is not periodic with period at most $h$. It follows that (7.4) holds when $\lim_{t\to\infty} R(t) = 0$, since $\rho(t)$ cannot be periodic in that case.

**Example 5.** (The Cosine Process.) Let $X(t) = \eta\cos(\omega t) + \zeta\sin(\omega t)$ for $t\in\mathbb{R}$, where $\eta$ and $\zeta$ are independent $\mathrm{N}(0,1)$-distributed random variables and $\omega\neq 0$ a constant. We use the programme `Mathematica` to compute the eigen-value $\lambda_1$ in (7.5) together with its multiplicity $N$. (The actual calculations are quite elementary, but turn out to be very long indeed. This is the reason we find it fitting to do them by means of a computer.)

First we calculate $\xi(0) = \int_0^1 X(s)^2\,ds$:

```
In[1]:= Integrate[(eta*Cos[omg*s]+zeta*Sin[omg*s])^2,{s,0,1}]
```

`Out[1]=` $\dfrac{\text{eta zeta}}{2\,\text{omg}} + \dfrac{\text{eta}^2+\text{zeta}^2}{2} - \dfrac{\text{eta zeta Cos[2 omg]}}{2\,\text{omg}} + \dfrac{(\text{eta}^2-\text{zeta}^2)\,\text{Sin[2 omg]}}{4\,\text{omg}}$

We can write $\xi(0) = (A\eta{+}B\zeta)^2 + (C\eta{+}D\zeta)^2$ where $A\eta{+}B\zeta$ and $C\eta{+}D\zeta$ are independent, so that $\lambda_1 = \max\{A^2+B^2, C^2+D^2\}$, $\lambda_2 = \min\{A^2+B^2, C^2+D^2\}$ and $\lambda_3=\lambda_4=\ldots=0$ in (7.5). Now we calculate $\lambda_1$ and $\lambda_2$:

```
In[2]:= Solve[{A*C+B*D==0,A^2+C^2==1/2+Sin[2*omg]/(4*omg),B^2+D^2==
          1/2-Sin[2*omg]/(4*omg),2*A*B+2*C*D==(1-Cos[2*omg])/(2*omg)},
                                                            {A,B,C,D}]
In[3]:= Simplify[A^2+B^2/.%]
```

Out[3]= $\left\{\frac{\text{omg}-\sqrt{\text{Sin[omg]}^2}}{2\,\text{omg}}, \frac{\text{omg}+\sqrt{\text{Sin[omg]}^2}}{2\,\text{omg}}\right\}$

```
In[4]:= Simplify[C^2+D^2/.%%]
```

Out[4]= $\left\{\frac{\text{omg}+\sqrt{\text{Sin[omg]}^2}}{2\,\text{omg}}, \frac{\text{omg}-\sqrt{\text{Sin[omg]}^2}}{2\,\text{omg}}\right\}$

This gives $\lambda_1 = \frac{1}{2}[1+|\text{sinc}(\omega)|]$ and $\lambda_2 = \frac{1}{2}[1-|\text{sinc}(\omega)|]$. Since $R(t) = \cos(\omega t)$ and $R''(t) = -\omega^2 R(t)$, (7.3) hold with $\int_0^1\int_0^1 f(r)f(s)\,R''(s-r)\,dsdr = -\omega^2\lambda_1$.

In order to check (7.4) we note that

$$\begin{aligned}
&\int_0^1\int_0^1 f(r)f(s)\,[R(s-r)-R(s-r-t)]\,dsdr \\
&\quad= \int_0^1\int_0^1 f(r)f(s)\,\frac{1}{2}\Big([R(s-r)-R(s-r+t)]+[R(s-r)-R(s-r-t)]\Big)\,dsdr \\
&\quad= \int_0^1\int_0^1 f(r)f(s)\,\sin(\tfrac{1}{2}\omega t)\,\Big(\sin\big(\omega(s-r+\tfrac{1}{2}t)\big)-\sin\big(\omega(s-r-\tfrac{1}{2}t)\big)\Big)\,dsdr \\
&\quad= 2\int_0^1\int_0^1 f(r)f(s)\,\sin^2(\tfrac{1}{2}\omega t)\,\cos(\omega(s-r))\,dsdr \\
&\quad= 2\sin^2(\tfrac{1}{2}\omega t)\,\lambda_1
\end{aligned}$$

when $f$ is the function in (7.4). It follows that (7.4) holds if and only if $h<2\pi/|\omega|$.

**Acknowledgement.** I am grateful to *Christer Borell* and to *Ross Leadbetter* for valuable discussions and advice. An *anonymous referee* did an unusually careful reading, and provided many detailed and constructive comments that all were very useful. This work is hereby gratefully acknowledged. My special thanks go to *Dr. Daniela Jarušková* who suggested to me the problem dealt with in Section 6.

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