ON THE ASYMPTOTIC DISTRIBUTION OF A STATISTIC FOR TEST OF APPEARANCE OF LINEAR TREND

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Consider the stochastic process $Y(t) = \big(W(t)/\sqrt{t}, (\int_0^t z\,dW(z) - \frac{1}{2}tW(t))/\sqrt{t^3/12}\big)$ for $t>0$, where $\{W(s)\}_{s\geq 0}$ is a standard Wiener process. The Euclidian norm $|Y(t)|^2$ of this process arises as the asymptotic (large sample) distribution of a test statistic for a change point detection problem of appearance of linear trend. We study the asymptotic behavior of $\mathbf{P}\big\{\sup_{t\in[\alpha,1]}|Y(t)|^2>u\big\}$ as $u\to\infty$ for a fixed $\alpha\in(0,1)$, and when $\alpha=\alpha(u)\downarrow 0$ at a certain rate as $u\to\infty$. Of course, the statistical interest in these asymptotics lie in the possibility to obtain approximate test levels for the mentioned statistical test.

1. Introduction. Let $\{W(t)\}_{t\in\mathbb{R}}$ be a standard Wiener process, and define

$$Y(t) = \left(\frac{W(t)}{\sqrt{t}}, \frac{\int_0^t s\,dW(s) - \frac{1}{2}tW(t)}{\sqrt{t^3/12}}\right) = \left(\frac{W(t)}{\sqrt{t}}, \frac{\frac{1}{2}tW(t) - \int_0^t W(s)\,ds}{\sqrt{t^3/12}}\right). \tag{1.1}$$

We study the asymptotic behaviour of $\mathbf{P}\big\{\sup_{t\in[\alpha,1]}|Y(t)|^2>u\big\}$ as $u\to\infty$ when $\alpha\in(0,1)$ is a constant, as well as when $\alpha=\alpha(u)\downarrow 0$ at a certain rate as $u\to\infty$.

Consider a change point detection problem of appearance of linear trend, where the null hypothesis "$H_0: X_i=e_i$ for $i=1,\ldots,n$" is tested against the alternative

$$\text{“}H_1 : X_i = \begin{cases} a_0+a_1\frac{i}{n}+e_i & \text{for } i=1,\ldots,k \\ e_i & \text{for } i=k+1,\ldots,n \end{cases}, \quad \text{for some } k\in\mathbb{N} \text{ and } a_0,a_1\in\mathbb{R}\text{”}.$$

Here $\{e_i\}_{i=1}^{\infty}$ is standardized discrete white noise. Under H_0, the test statistic

$$\max_{[\alpha n]\leq k\leq n} \Big(\sum_{i=1}^k X_i\Big)^2\Big/k + \Big(\sum_{i=1}^k\big(\tfrac{i}{n}-\tfrac{k+1}{2n}\big)X_i\Big)^2\Big/\Big(\sum_{i=1}^k\big(\tfrac{i}{n}-\tfrac{k+1}{2n}\big)^2\Big) \to_D \sup_{t\in[\alpha,1]}|Y(t)|^2$$

as $n\to\infty$. Hence it is important to study the distribution of $\sup_{t\in[\alpha,1]}|Y(t)|^2$.

The literature on extremes of the norm of vector-valued Gaussian processes is rich, and includes, for example, Sharpe (1978), Lindgren (1980, 1989), Albin (1990

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Section 4, 2000a Section 5), and Piterbarg (1994). However, the component processes Y_1 and Y_2 in (1.1) are dependent, while the literature on non-differentiable processes only deals with independent components, and thus does not apply to Y.

2. The case with a fixed $\alpha\in(0,1)$**.** Here we establish that

$$\lim_{u\to\infty} u^{-1}\,\mathrm{e}^{\frac{1}{2}u}\,\mathbf{P}\big\{\sup_{t\in[\alpha,1]}|Y(t)|^2>u\big\} = -\ln(\alpha) \qquad \text{for a fixed} \quad \alpha\in(0,1).$$

This result follows immediately from our first theorem:

Theorem 1. *For the process* $X(t)=Y(e^t)$, $t\in\mathbb{R}$, *with* Y *given by* (1.1), *we have*

$$\lim_{u\to\infty} u^{-1}\,\mathrm{e}^{\frac{1}{2}u}\,\mathbf{P}\big\{\sup_{t\in[0,h]}|X(t)|^2>u\big\} = h \qquad \textit{for} \quad h>0.$$

The proof uses Albin (1990, Theorem 1) and Albin (1992, Proposition 2), together with Albin (2000b, Lemma 1), which is stated here for easy reference.

Lemma 1. *Let* $Z\equiv\sum_{n=1}^{\infty}\lambda_n\eta_n^2$ *where* $\{\eta_n\}_{n=1}^{\infty}$ *are independent* $N(0,1)$*-distributed random variables and* $\lambda_1=\ldots=\lambda_N>\lambda_{N+1}\geq\ldots\geq 0$ *are constants. There exists a constant* $K=K(N)>0$ *(that depends on* N *only) such that*

$$\mathbf{P}\{Z>u\}\leq K\,\exp\left\{\frac{(\sum_{n=1}^{\infty}\lambda_n)\,(1+\lambda_1/\lambda_{N+1})}{2(\lambda_1-\lambda_{N+1})}\right\}(u/\lambda_1)^{\frac{1}{2}N-1}e^{-\frac{1}{2}u/\lambda_1} \qquad \textit{for } u>0.$$

Proof of Theorem 1. Since $|X(0)|^2$ is $\chi^2(2)$-distributed, we have

$$\mathbf{P}\{|X(0)|^2>u+2x\}\,/\,\mathbf{P}\{|X(0)|^2>u\} = \mathrm{e}^{-x} \qquad \text{for} \quad x\geq-\tfrac{1}{2}u. \tag{2.1}$$

Hence Albin (1990, Eq. 2.1) holds for $|X|^2$, with $w(u)\equiv 2$ and $F(x)\equiv 1-\mathrm{e}^{-x}$.

For the process X in $\mathbb{R}^2$ we have

$$X(t)=(X_1(t),X_2(t))=\Big(X_1(t),\,\sqrt{3}\big(X_1(t)-2\int_{-\infty}^{t}\mathrm{e}^{\frac{3}{2}(s-t)}X_1(s)\,ds\big)\Big), \tag{2.2}$$

where X_1 is a stationary Gauss Markov (Ornstein Uhlenbeck) process with covariance function $\mathbf{Cov}\{X_1(s),X_1(s+t)\}=\mathrm{e}^{-|t|/2}$. By routine calculations, we get

$$\mathbf{Var}\left\{\begin{pmatrix}X_1(s)\\X_2(s)\\X_1(s+t)\\X_2(s+t)\end{pmatrix}\right\}=\begin{pmatrix}1&0&\mathrm{e}^{-\frac{1}{2}t}&\sqrt{3}\,(\mathrm{e}^{-\frac{3}{2}t}-\mathrm{e}^{-\frac{1}{2}t})\\0&1&0&\mathrm{e}^{-\frac{3}{2}t}\\\mathrm{e}^{-\frac{1}{2}t}&0&1&0\\\sqrt{3}\,(\mathrm{e}^{-\frac{3}{2}t}-\mathrm{e}^{-\frac{1}{2}t})&\mathrm{e}^{-\frac{3}{2}t}&0&1\end{pmatrix} \tag{2.3}$$

for $t>0$. Hence X_2 is also a stationary Gauss Markov process and X is stationary.

The four eigenvalues of the variance matrix in (2.3) are given by

$$1 \pm \mathrm{e}^{-\frac{3}{2}t} \sqrt{2\,\mathrm{e}^{2t} - 3\,\mathrm{e}^{t} + 2 \pm 2\,(\mathrm{e}^{t} - 1)\sqrt{\mathrm{e}^{2t} - \mathrm{e}^{t} + 1}}\,.$$

(This can be conveniently verified with mathematical programme packages like for example `Mathematica`.) The two largest of these eigenvalues are at most

$$\lambda_1(t) = \lambda_2(t) \equiv 1 + \mathrm{e}^{-\frac{3}{2}t} \sqrt{2\,\mathrm{e}^{2t} - 3\,\mathrm{e}^{t} + 2 + 2\,(\mathrm{e}^{t} - 1)\sqrt{\mathrm{e}^{2t} - \mathrm{e}^{t} + 1}} \leq 2 - C_1 t$$

for $t \in [0,h]$, for some constant $C_1 = C_1(h) \in [0,(2h)^{-1}]$. The two smallest eigenvalues are at most $\lambda_3 = \lambda_4 \equiv 1$. Using Lemma 1 we readily obtain

$$\begin{aligned}
\mathbf{P}\big\{|X(0)|^2 > u,\ |X(qt)|^2 > u\big\} &\leq \mathbf{P}\big\{|X(qt)|^2 + |X(0)|^2 > 2u\big\} \\
&\leq K\,\exp\left\{\frac{6\cdot(1+2)}{2\,(1 - C_1 qt)}\right\} \mathrm{e}^{-u/\lambda_1(qt)} \\
&\leq K\,\mathrm{e}^{18}\,\mathrm{e}^{-\frac{1}{4}C_1 t}\,\mathrm{e}^{-\frac{1}{2}u} \\
&= K\,\mathrm{e}^{18}\,\mathrm{e}^{-\frac{1}{4}C_1 t}\,\mathbf{P}\{|X(0|^2 > u\} \quad \text{for } u \geq 4 \text{ and } qt \leq h.
\end{aligned} \tag{2.4}$$

It follows that Albin (1990, Condition B) holds for the process $|X|^2 = \{|X(t)|^2\}_{t\in\mathbb{R}}$.

By (2.3), $(X_1(t)\ \ X_2(t))^T$ is independent of

$$\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} \equiv \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} - \begin{pmatrix} \mathrm{e}^{\frac{1}{2}t} & \sqrt{3}\,(\mathrm{e}^{\frac{3}{2}t} - \mathrm{e}^{\frac{1}{2}t}) \\ 0 & \mathrm{e}^{\frac{3}{2}t} \end{pmatrix} \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}$$

for $t > 0$, where

$$\mathbf{Var}\{Z_1(t)\} = 3\mathrm{e}^{3t} - 6\mathrm{e}^{2t} + 4\mathrm{e}^{t} - 1 \leq 3\,(\mathrm{e}^{3t} - 1) \leq 9\,t\,\mathrm{e}^{3t} \quad \text{and} \quad \mathbf{Var}\{Z_2(t)\} = \mathrm{e}^{3t} - 1.$$

By elementary matrix algebra we have

$$\left|\begin{pmatrix} \mathrm{e}^{\frac{1}{2}t} & \sqrt{3}\,(\mathrm{e}^{\frac{3}{2}t} - \mathrm{e}^{\frac{1}{2}t}) \\ 0 & \mathrm{e}^{\frac{3}{2}t} \end{pmatrix} \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} - \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix}\right|^2 \leq (4 + \sqrt{3})\,(\mathrm{e}^{\frac{3}{2}t} - 1)^2 u^2$$

when $|X(0)| \leq u$. Moreover, we have

$$\sqrt{1 + 2\eta/u} - 1 \geq \tfrac{1}{2}\eta/u \geq \tfrac{1}{4}\eta/u + \sqrt{4 + \sqrt{3}}\,(\mathrm{e}^{\frac{3}{32}\eta/u} - 1) \geq \tfrac{1}{4}\eta/u + \sqrt{4 + \sqrt{3}}\,(\mathrm{e}^{\frac{3}{2}qt} - 1)$$

for $0 < \sqrt{t} < \eta < \frac{1}{16} < u$. From this we conclude that

$$\begin{aligned}
&\mathbf{P}\big\{|X(qt)|^2 > u + 2\eta,\ |X(0)|^2 \leq u\big\} \\
&\leq \mathbf{P}\Big\{|X(qt) - X(0)| > \tfrac{1}{2}\eta/\sqrt{u},\ |X(qt)|^2 > u,\ |X(0)|^2 \leq u\Big\}
\end{aligned}$$

$$\le \mathbf{P}\Big\{|Z(qt)|\ge \tfrac{1}{2}\eta/\sqrt{u}-\sqrt{4+\sqrt{3}}\,(\mathrm{e}^{\frac{3}{2}qt}-1)\sqrt{u},\ |X(qt)|^2>u\Big\}$$
$$\le \Big(\mathbf{P}\big\{Z_1(qt)^2\ge \tfrac{1}{32}\eta^2/u\big\}+\mathbf{P}\big\{Z_2(qt)^2\ge \tfrac{1}{32}\eta^2/u\big\}\Big)\,\mathbf{P}\{|X(0)|^2>u\}$$
$$\le 2\,\mathbf{P}\big\{\mathrm{N}(0,1)^2\ge \tfrac{1}{9\cdot 32}\mathrm{e}^{-3qt}\eta^2/t\big\}\,\mathbf{P}\{|X(0)|^2>u\} \qquad \text{for}\quad 0<\sqrt{t}<\eta<\tfrac{1}{16}<u.$$

Hence, by Albin (1992, Proposition 2), $|X|^2$ satisfies Albin (1990, Condition C).

By routine calculations, we have, for any event A,

$$\mathbf{P}\big\{A\,\big|\,\tfrac{1}{2}(|X(0)|^2-u)>0\big\} = \int_{x=0}^{x=\infty}\int_{\theta=0}^{\theta=2\pi} \mathbf{P}\big\{A\,\big|\,X(0)=\sqrt{u+2x}\,(\cos(\theta),\sin(\theta))\big\}\,\frac{d\theta dx}{2\pi\,\mathrm{e}^{x}}.$$

Since $\hat{Z}_1(s)\equiv X_1(s)-\mathrm{e}^{-\frac{1}{2}s}X_1(0)$ is independent of $X_1(0)$ for $s\ge 0$, and $X_2(0)$ is independent of $X(s)$ for $s\ge 0$, (2.3) further shows that, with obvious notation (writing $=_D$ for equality of finite dimensional distributions),

$$\big(|X(t)|^2\,\big|\,X(0)=(y,z)\big)$$
$$=_D \Big(X_1(t)^2+3\Big(X_1(t)-\mathrm{e}^{-\frac{3}{2}t}(y-\tfrac{1}{\sqrt{3}}z)-2\textstyle\int_0^t \mathrm{e}^{\frac{3}{2}(s-t)}X_1(s)\,ds\Big)^2\,\Big|\,X_1(0)=y\Big)$$
$$=_D \big(\hat{Z}_1(t)+\mathrm{e}^{-\frac{1}{2}t}y\big)^2+3\Big(\hat{Z}_1(t)-(\mathrm{e}^{-\frac{1}{2}t}-\mathrm{e}^{-\frac{3}{2}t})y+\tfrac{1}{\sqrt{3}}\mathrm{e}^{-\frac{3}{2}t}z-2\textstyle\int_0^t \mathrm{e}^{\frac{3}{2}(s-t)}\hat{Z}_1(s)\,ds\Big)^2.$$

Since the finite dimensional distributions of $\sqrt{u}\,\hat{Z}_1(qt)$ converge to those of $W(t)$ as $u\to\infty$, it is now a straightforward matter to deduce that

$$\Big(\tfrac{1}{2}(|X(t)|^2-u)\,\Big|\,X(0)=\sqrt{u+2x}\,(\cos(\theta),\sin(\theta))\Big)$$
$$\to_D (\cos(\theta)+\sqrt{3}\sin(\theta))\,W(t)-\tfrac{1}{2}(\cos(\theta)+\sqrt{3}\sin(\theta))^2t+x \qquad \text{as}\quad u\to\infty$$
$$=_D \sqrt{2}\,W\big(\tfrac{1}{2}(\cos(\theta)+\sqrt{3}\sin(\theta))^2t\big)-\tfrac{1}{2}(\cos(\theta)+\sqrt{3}\sin(\theta))^2t+x$$

(where $\to_D$ is convergence of finite dimensional distributions). Picking random variables $\varTheta$ and U with uniform distribution over $[0,2\pi]$ and unit mean exponential distribution, respectively, such that $\varTheta$, U and W are independent, we thus get

$$\big(\tfrac{1}{2}(|X(t)|^2-u)\,\big|\,\tfrac{1}{2}(|X(0)|^2-u)>0\big)\to_D \sqrt{2}\,W(\tfrac{1}{2}\varPsi t)-\tfrac{1}{2}\varPsi t+U,$$

where $\varPsi=(\cos(\varTheta)+\sqrt{3}\sin(\varTheta))^2$. Hence $|X|^2$ satisfies Albin (1990, Condition A(0)).

By application of Albin (1990, Theorem 1) to the process $|X|^2$, we obtain

$$\frac{q(u)\,\mathbf{P}\big\{\sup_{t\in[0,h]}|X(t)|^2>u\big\}}{h\,\mathbf{P}\{|X(0)|^2>u\}}$$

$$\to \lim_{a\downarrow 0} \frac{1}{a}\mathbf{P}\Big\{\sup_{k\geq 1} \sqrt{2}\, W(\tfrac{1}{2}\Psi ak) - \tfrac{1}{2}\Psi ak + U \leq 0\Big\}$$
$$= \int_0^{2\pi} \lim_{\hat a\downarrow 0} \frac{1}{\hat a}\mathbf{P}\Big\{\sup_{k\geq 1} \sqrt{2}\, W(\hat a k) - \hat a k + U \leq 0\Big\}\ \frac{(\cos(\theta)+\sqrt{3}\sin(\theta))^2}{2}\,\frac{d\theta}{2\pi}$$
$$= \lim_{\hat a\downarrow 0} \frac{1}{\hat a}\mathbf{P}\Big\{\sup_{k\geq 1} \sqrt{2}\, W(\hat a k) - \hat a k + U \leq 0\Big\} \qquad \text{as} \quad u\to\infty.$$

Here well-known results for extremes of Gaussian processes [e.g., Albin (2000c)] show that the limit on the right-hand side equals $H_1 = 1$, where $\{H_\alpha\}_{\alpha\in(0,2]}$ are the the famous constants introduced by Pickands (1969). $\square$

3. The case when $\alpha=\alpha(u)\downarrow 0$**.** Here we establish the double exponential law

$$\lim_{u\to\infty} \mathbf{P}\big\{\sup_{t\in[\exp\{-u^{-1}e^{u/2}\},1]} \tfrac{1}{2}(|Y(t)|^2-u)\leq x\big\} = \exp\{-e^{-x}\} \qquad \text{for} \quad x\in\mathbb{R}.$$

This result follows immediately from our second theorem:

Theorem 2. *For the process* $X(t)=Y(e^t)$, $t\in\mathbb{R}$, *with* Y *given by* (1.1), *we have*

$$\lim_{u\to\infty} \mathbf{P}\big\{\sup_{t\in[0,\,u^{-1}e^{u/2}]} \tfrac{1}{2}(|X(t)|^2-u)\leq x\big\} = \exp\{-e^{-x}\} \qquad \textit{for} \quad x\in\mathbb{R}.$$

The proof relies on Albin (1990, Theorem 5), which in turn builds directly on Leadbetter and Rootzén (1982, Theorem 4.3).

Proof of Theorem 2. By Theorem 1 together with Albin (1990, Theorems 2.c and 5), it is enough to verify that Conditions D$'$ and D(0) of Albin (1990) hold: By (2.3), $(X_1(0)\ \ X_2(0))^T$ is independent of

$$\begin{pmatrix}\hat Z_1(t)\\ \hat Z_2(t)\end{pmatrix} \equiv \begin{pmatrix}X_1(t)\\ X_2(t)\end{pmatrix} - \begin{pmatrix} e^{-\frac{1}{2}t} & 0\\ \sqrt{3}\,(e^{-\frac{3}{2}t}-e^{-\frac{1}{2}t}) & e^{-\frac{3}{2}t}\end{pmatrix}\begin{pmatrix}X_1(0)\\ X_2(0)\end{pmatrix}$$

for $t>0$. Here (2.3) shows that

$$\mathbf{Var}\left\{\begin{pmatrix}\hat Z_1(t)\\ \hat Z_2(t)\end{pmatrix}\right\} = \begin{pmatrix} 1-e^{-t} & \sqrt{3}\,(e^{-t}-e^{-2t})\\ \sqrt{3}\,(e^{-t}-e^{-2t}) & 1-3e^{-t}+6e^{-2t}-4e^{-3t}\end{pmatrix}$$

for $t>0$, the eigenvalues of which are given by

$$(1-e^{-t})\left(1-e^{-t}+2e^{-2t}\pm 2e^{-t}\sqrt{1-e^{-t}+e^{-2t}}\right) \leq 1+3e^{-t}.$$

Also notice that

$$|\hat Z(t)| \geq |X(t)| - C_2 e^{-\frac{1}{2}t}|X(0)| \geq (1-C_2 e^{-\frac{1}{2}t})\,|X(0)| \qquad \text{when} \qquad |X(t)|\geq|X(0)|,$$

where $C_2=(4+\sqrt{3})^{1/2}$. It follows that

$$\begin{aligned}\mathbf{P}\big\{|X(0)|^2>u,\,|X(t)|^2>u\big\} &= 2\,\mathbf{P}\big\{|X(t)|^2\geq |X(0)|^2>u\big\}\\ &\leq 2\,\mathbf{P}\big\{|\hat{Z}(t)|>(1-C_2\mathrm{e}^{-\frac{1}{2}t})\sqrt{u}\big\}\,\mathbf{P}\{|X(0)|^2>u\}\\ &\leq 2\,\mathbf{P}\big\{(1+3\mathrm{e}^{-t})|X(0)|^2>(1-C_2\mathrm{e}^{-\frac{1}{2}t})^2u\big\}\,\mathbf{P}\{|X(0)|^2>u\}\\ &\leq 2\,\exp\bigg\{-\frac{(1-C_2\mathrm{e}^{-\frac{1}{2}t})^2u}{2\,(1+3\mathrm{e}^{-t})}\bigg\}\,\mathbf{P}\{|X(0)|^2>u\}.\end{aligned}$$

Picking a constant $h>0$ such that $1-C_2\mathrm{e}^{-\frac{1}{2}h}>0$, this gives

$$\sum_{k=[h/(aq)]+1}^{[2\ln(u)/(aq)]}\mathbf{P}\big\{|X(aqk)|^2>u\,\big|\,|X(0)|^2>u\big\}\leq 4\,\frac{\ln(u)}{aq}\,\exp\bigg\{-\frac{(1-C_2\mathrm{e}^{-\frac{1}{2}h})^2u}{2\,(1+3\mathrm{e}^{-h})}\bigg\}\to 0 \tag{3.1}$$

as $u\to\infty$ for each choice of $a>0$. Moreover, we have

$$\sum_{k=[2\ln(u)/(aq)]+1}^{[\lambda\,\mathrm{e}^{u/2}]}\mathbf{P}\big\{|X(aqk)|^2>u\,\big|\,|X(0)|^2>u\big\}\leq 2\lambda\mathrm{e}^{u/2}\exp\bigg\{-\frac{(1-C_2/u)^2u}{2\,(1+3/u^2)}\bigg\}\to 0$$

as $u\to\infty$ and $\lambda\downarrow 0$ (in that order), for each $a>0$. This together with (3.1), show that Albin (1990, Conditions D′) holds for $|X|^2$.

Inserting $X_1(t)=_D\int_{-\infty}^t \mathrm{e}^{\frac{1}{2}(s-t)}dW(s)$ in (2.2), routine calculations give

$$X(t)=_D\Big(\textstyle\int_{-\infty}^t\mathrm{e}^{\frac{1}{2}(s-t)}dW(s),\ \sqrt{12}\int_{-\infty}^t(\mathrm{e}^{\frac{3}{2}(s-t)}-\frac{1}{2}\mathrm{e}^{\frac{1}{2}(s-t)})\,dW(s)\Big).$$

Given constants $a>0$ and $0<\lambda<\tau<\infty$, pick $s_1<\ldots<s_p<t_1<\ldots<t_{p'}$ in $\big\{aqk: k\in\mathbb{Z},\,0\leq akq\leq\tau u^{-1}\mathrm{e}^{\frac{1}{2}u}\big\}$ with $t_1-s_p\geq\lambda u^{-1}\mathrm{e}^{\frac{1}{2}u}$. We shall verify that

$$\lim_{u\to\infty}\bigg(\mathbf{P}\Big\{\bigcap_{i=1}^{p}\{|X(s_i)|^2\leq u\},\bigcap_{j=1}^{p'}\{|X(t_j)|^2\leq u\}\Big\}\\ -\mathbf{P}\Big\{\bigcap_{i=1}^{p}\{|X(s_i)|^2\leq u\}\Big\}\mathbf{P}\Big\{\bigcap_{j=1}^{p'}\{|X(t_j)|^2\leq u\}\Big\}\bigg)=0. \tag{3.2}$$

To that end we introduce the truncated process

$$X(r;t)=_D\Big(\textstyle\int_r^t\mathrm{e}^{\frac{1}{2}(s-t)}dW(s),\ \sqrt{12}\int_r^t(\mathrm{e}^{\frac{3}{2}(s-t)}-\frac{1}{2}\mathrm{e}^{\frac{1}{2}(s-t)})\,dW(s)\Big)\qquad\text{for}\quad r<t.$$

Notice that the components of $\hat{X}(r;t)=X(t)-X(r;t)$ have variances

$$\mathbf{Var}\{\hat{X}_1(r;t)\}=\mathrm{e}^{r-t}\quad\text{and}\quad\mathbf{Var}\{\hat{X}_2(t)\}=4\mathrm{e}^{3(r-t)}-6\,\mathrm{e}^{2(r-t)}+3\mathrm{e}^{r-t}\leq 3\mathrm{e}^{r-t}. \tag{3.3}$$

Since $X(s_p;t_1),\ldots,X(s_p;t_{p'})$ are independent of $X(s_1),\ldots,X(s_p)$, we have

$$\mathbf{P}\Big\{\bigcap_{i=1}^{p}\{|X(s_i)|^2\le u\},\bigcap_{j=1}^{p'}\{|X(t_j)|^2\le u\}\Big\}$$

$$\le \mathbf{P}\Big\{\bigcap_{i=1}^{p}\{|X(s_i)|^2\le u\}\Big\}\mathbf{P}\Big\{\bigcap_{j=1}^{p'}\{|X(s_p;t_j)|\le\sqrt{u}+\tfrac{\varepsilon}{\sqrt{u}}\}\Big\}+\mathbf{P}\Big\{\bigcup_{j=1}^{p'}\{|\hat{X}(s_p;t_j)|>\tfrac{\varepsilon}{\sqrt{u}}\}\Big\}$$

$$\le \mathbf{P}\Big\{\bigcap_{i=1}^{p}\{|X(s_i)|^2\le u\}\Big\}\mathbf{P}\Big\{\bigcap_{j=1}^{p'}\{|X(t_j)|\le\sqrt{u}+\tfrac{2\varepsilon}{\sqrt{u}}\}\Big\}+2\sum_{j=1}^{p'}\mathbf{P}\{|\hat{X}(s_p;t_j)|>\tfrac{\varepsilon}{\sqrt{u}}\}$$

$$\le \mathbf{P}\Big\{\bigcap_{i=1}^{p}\{|X(s_i)|^2\le u\}\Big\}\mathbf{P}\Big\{\bigcap_{j=1}^{p'}\{|X(t_j)|^2\le u\}\Big\}+p'\,\mathbf{P}\big\{\sqrt{u}\le|X(0)|\le\sqrt{u}+\tfrac{2\varepsilon}{\sqrt{u}}\big\}$$

$$+\,2p'\,\mathbf{P}\{|\hat{X}(s_p;t_1)|>\tfrac{\varepsilon}{\sqrt{u}}\}$$

for $\varepsilon>0$. Since $p'\le a^{-1}\tau\,\mathbf{P}\{|X(0)|^2>u\}$, (2.1) and (3.3), together with the fact that $t_1-s_p\ge\lambda u^{-1}\mathrm{e}^{\frac{1}{2}u}$, readily show that the limessuperior of the difference in (3.2) is at most $a^{-1}\tau(1-\mathrm{e}^{-2\varepsilon})$. Sending $\varepsilon\downarrow 0$ it follows that the limessuperior in (3.2) is at most 0. In an entirely analogous way, using the same truncation technique as before, it is seen that the limesinferior of the difference in (3.2) is at least 0. Hence (3.2) holds, meaning that Albin [1990, Condition D(0)] holds. □

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