THE FOKKER-PLANCK OPERATOR AS AN ASYMPTOTIC LIMIT IN ANISOTROPIC MEDIA

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ABSTRACT. We derive the Fokker-Planck operator describing a highly forward peaked scattering process in the linear transport equation, in anisotropic media, as a formal asymptotic limit of the exact integral operator. The resulting operator, being both convective and diffusive in angle and energy variables, reduces the degenerate nature of the Fokker-Planck system compared to the isotropic media case where the corresponding limit is only convective in energy and diffusive in angle.

1. INTRODUCTION

The particle flow through a background medium is described by a linear transport equation in phase space viz:

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \Omega \cdot \nabla_{\mathbf{x}}\psi + \sigma\psi = \int_0^\infty dE' \int_{S^2} d\Omega' \sigma_s \psi + Q, \tag{1.1}$$

associated with appropriate initial and boundary conditions which we will not concern us in this note. Here

- $\psi \equiv \psi(\mathbf{x}, E, \Omega, t) = vf(\mathbf{x}, E, \Omega, t)$ is the current function
- f is the distribution of particles in phase space $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}$,
- t in the time variable
- $\Omega = \mathbf{v}/|\mathbf{v}|$,
- $v = |\mathbf{v}|$,
- E is the energy variable ($E = \frac{1}{2}mv^2$),
- $Q \equiv Q(\mathbf{x}, E, \Omega, t)$ is the external source term
- $\sigma = \sigma(\mathbf{x}, E, \Omega, t)$ is the total cross-section
- $\sigma_s = \sigma_s(\mathbf{x}, E' \to E, \Omega' \to \Omega, t)$,
  is the scattering kernel describing the probability of scattering from a pre-collision particle energy and direction (E', Ω') to a post collision coordinates (E, Ω). Below, in σ_s, we shall replace "$\to$" by a "," sign unless we want to indicate the explicit pre and post collision variables.

In the case of isotropic background media σ_s depends only on $\Omega' \cdot \Omega$, rather than Ω' and Ω separately, then the total cross-section σ may be assumed to be independent of the direction Ω. This case is investigated by Pomraning in [22]. Asymptotic analysis, based on Enskog type expansions is given in a pioneer work by Larsen and Keller [19]. Similar studies can be found in

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[24], [1], [7], [8] and [9]. See also [2] and [12] for some related mathematical studies. In all these studies, the energy dependence is explicitly included.

In anisotropic media the micro-structure causes the mean free path of the particles to become dependent on their direction of motion with respect to some fixed axis. The equation which results is similar to the one-speed Boltzmann transport equation but has cross-sections which are functions of direction. In some applications, the value of this consideration is more evident when the anisotropy of the medium is caused by its atomic structure (e.g., in radiation therapy, see Jette [17]), or method of fabrication (e.g., in semiconductor devices) rather than any physical heterogeneity. Usually for velocity dependent cross-sections the energy and directional components are not separated, see Cercignani [10]. In this study, however, we are in particular interested in the order of the, limiting, convective and diffusive terms with respect to each velocity component. Here are some classical examples on the influence of the direction dependent cross-sections, see, e.g. Duderstadt and Martin [13] and Williams [26]: (i) The particle distribution arising from a *plane source emitting particles* at a well defined angle into an infinite medium. (ii) The *albedo and Milne problems* for a half-space solved by Weiner-Hopf technique and connected by an asymptotic method to give a full description of transmission through a thick slab. (iii) Solution of integral form of the Boltzmann equation reduced to a diffusion like situation resembling the *rod model of Wing*, see [20]. (iv) The energy spectrum of particles slowing down from a high energy source by *elastic collisions* in an infinite homogeneous system.

In charged particle transport the scattering kernel σ_s is highly forward peaked about both a zero energy transfer and a zero direction change, and the number of scattering collisions is very large. Therefore, the *scattering mean free path* is very small. This problem is the subject of asymptotic studies in [19]. To study this problem numerically using deterministic approaches is very much involved since a reliable algorithm in this case, requires a mesh size of order of the mean free path (i.e., very small). This implies an unrealistically fine degree of numerical resolution, see studies in [3],[4], [5], and [6]. Likewise, a stochastic (Monte-Carlo) simulation would be time consuming since a very large number of scattering interactions must be followed for each particle before it demise out of the system by either absorption or leakage, see [18]. To circumvent these difficulties Chardraseckhar [11] suggested to replace the integral scattering operator in the transport equation with a differential Fokker-Planck operator. As a consequence of this replacement, the dominant (large) in and out scattering terms cancel, thus the mean free path is substantially increased and the system is semi-rarefied. In this setting a concise and heuristic derivation of the Fokker-Planck operator, in isotropic media case, is given by Pomraning [22], who has also formalized the derivation procedure as an asymptotic limit of the integral transport operator.

In this note we extend Pomraning's approach to the case of anisotropic background media. Note that in both isotropic and anisotropic background media the scattering may be referred as being anisotropic, see [21] and [27]. However, in the former case the scattering kernel is a function of only $\Omega \cdot \Omega'$, while in the latter case σ_s would depend on Ω and Ω', separately. The emphasis of this difference will be more clear in Sections 4 and 5 below. In the anisotropic media the resulting asymptotic limit would contain *compatible* convective and diffusive terms in both angle and energy variables, (with respect to the smallness parameter ε and (δ, γ) describing peaking of the scattering kernel in energy (ε) and direction (δ, γ), respectively). We shall denote the scattering mean free path by the piecewise constant function Δ where, assuming a homogenization, we transfer the (E, Ω) dependence to the cross-sections. We let $\mathcal{O}(\Delta) = \mathcal{O}(\delta) = \mathcal{O}(\gamma) = \mathcal{O}(\varepsilon)$.

An outline of this note is as follows: in Section 2 we derive Fourier cosine moments of the anisotropic scattering kernel. In section 3 we reformulate the transport equation expanded in surface harmonics. Section 4 is devoted to a formal derivation of the Fokker-Planck operator through the higher moments expansions for isotropic media with anisotropic scattering. In Section 5 we give a general asymptotic approach for the anisotropic background media. The operators obtained in Sections 4 and 5 are "convective-diffusive" in both angle and energy variables. Finally, in our concluding Section 6, we comment on the effects of the strength of the forward peakedness of the scattering kernel.

2. Isotropic media anisotropic scattering

In this section we consider the scattering kernel $\sigma_s(\mathbf{x}, E' \to E, \Omega' \to \Omega, t)$ as a function of $(\Omega \cdot \Omega')$ and derive its Fourier cosine moments. The surface harmonic developments for both transport and Fokker-Planck equations, in isotropic media, are based on these moments. See [21] and [22] for the scattering functions depending linearly on $(\Omega \cdot \Omega')$, (referred as *linearly anisotropic scattering*), and [1] and [24] in some general case of higher order moments. In an scattering event the particle must possess some final energy and angle, and we have the relationship.

$$\begin{aligned} \sigma_s(\mathbf{x}, E', \Omega', t) &= \int_0^\infty dE \int_{S^2} \sigma_s(\mathbf{x}, E' \to E, \Omega' \to \Omega, t) d\Omega \\ &= \int_0^\infty dE \int_{-1}^1 (1-\mu^2)^{1/2} d\mu \int_0^{2\pi} d\chi \sigma_s(\mathbf{x}, E' \to E, \mu' \to \mu, \chi' \to \chi, t), \end{aligned} \tag{2.1}$$

where we used the standard spherical coordinates viz; $\Omega = (\theta, \chi)$ and $\mu = \cos\theta$.

Below we represent the right-hand side of (1.1), i.e. the so-called in-scattering term, as an expansion over its surface harmonic components by using the *addition formula* for Legendre polynomials, see [16]. In what follows, for the notational simplicity we shall omit writing the $\mathbf{x}$ and t dependence keeping in mind that this is for our convenience and we have no, e.g., space homogeneity assumption.

$$\begin{aligned} &\sigma_s(E' \to E, \Omega' \to \Omega) = \sigma_s(E' \to E, \mu' \to \mu, \chi' \to \chi) \\ &= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sigma_{s\ell}(E', E) \Big[P_\ell(\mu') P_\ell(\mu) + 2 \sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\mu') P_\ell^m(\mu) \cos m(\chi' - \chi) \Big] \end{aligned} \tag{2.2}$$

where

$$P_\ell^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m P_\ell(\mu)}{d\mu^m} = \frac{(1-\mu^2)^{m/2}}{2^\ell \ell!} \frac{d^{\ell+m}}{d\mu^{\ell+m}} (\mu^2 - 1)^\ell, \tag{2.3}$$

$$\|P_\ell^m\|^2 = \Big(\frac{2}{2\ell+1}\Big) \frac{(\ell+m)!}{(\ell-m)!}, \tag{2.4}$$

By formula (2.2) σ_s is expanded on $P(\Omega \cdot \Omega')$ and we have a symmetry in angle according to

$$\sigma_s(E' \to E, \Omega' \to \Omega) = \sigma_s(E' \to E, \Omega \to \Omega'). \tag{2.5}$$

The relation (2.5) combined with the general assumption of the *reciprocity relation* or *detailed balance*, (see Cercignani [10] and Williams [25] for exact definitions),

$$f(\mathbf{v}') \sigma_s(E' \to E, \Omega' \to \Omega) = f(\mathbf{v}) \sigma_s(E \to E', -\Omega \to -\Omega'), \tag{2.6}$$

and an additional symmetry assumption on f leads to some cancellations. However, in the present setting these cancellations do not have substantial simplifying effects. Tracing these cancellations would require further algebraic labour with no or very little impact on the final result.

We derive $\sigma_{s\ell}(E', E)$ as an integral representation of $\sigma_s(E' \to E, \Omega' \to \Omega)$. To this approach we start writing the q-th Fourier cosine moments of σ_s, where we use

$$\int_0^{2\pi} \cos m(\chi' - \chi) \cos q\chi' d\chi' = \begin{cases} 2\pi, & q = m = 0, \\ \pi \cos q\chi, & q = m \geq 1, \\ 0, & q \neq m, \end{cases} \tag{2.7}$$

to obtain that

$$
\begin{aligned}
&\int_0^{2\pi} \sigma_s(E' \to E, \Omega' \to \Omega) \cos q\chi' d\chi' \\
&= \sum_{\ell=q}^{\infty} \Big(\frac{2\ell+1}{4\pi}\Big)\sigma_{s\ell}(E',E)\alpha_q \frac{(\ell-q)!}{(\ell+q)!} P_\ell^q(\mu')P_\ell^q(\mu)\pi\alpha_q^{-1}\cos q\chi \\
&= \sum_{\ell=q}^{\infty} \Big(\frac{2\ell+1}{4}\Big)\frac{(\ell-q)!}{(\ell+q)!}\sigma_{s\ell}(E',E)P_\ell^q(\mu')P_\ell^q(\mu)\cos q\chi,
\end{aligned} \tag{2.8}
$$

where we have defined α_q as

$$
\alpha_q = \begin{cases} 1, & q = 0, \\ 2, & q \geq 1. \end{cases} \tag{2.9}
$$

Recall that for each positive integer q, $\{P_\ell^q\}_{\ell=q}^{\infty}$ is an orthogonal basis for $L^2(-1,1)$, see Folland [14] Theorem 6.7. Now we multiply (2.8) by $P_k^q(\mu')$, $k \geq q$ and integrate over $\mu' \in (-1,1)$, to get

$$
\begin{aligned}
&\int_{-1}^{1} P_k^q(\mu')d\mu' \int_0^{2\pi} \sigma_s(E' \to E, \Omega' \to \Omega)\cos q\chi' d\chi' \\
&\qquad = \sum_{\ell=q}^{\infty} \Big(\frac{2\ell+1}{2}\Big)\frac{(\ell-q)!}{(\ell+q)!}\sigma_{s\ell}(E',E)P_\ell^q(\mu)\cos q\chi \\
&\qquad \times \int_{-1}^{1} P_\ell^q(\mu')P_k^q(\mu')d\mu' = \sigma_{sk}(E',E)P_k^q(\mu)\cos q\chi.
\end{aligned} \tag{2.10}
$$

where we used (2.4). Further, multiplying (2.10) by $P_k^q(\mu)\cos q\chi$ and integrating over $\mu \in (-1,1)$ and $\chi \in (0, 2\pi)$, and using (2.4) once again, we obtain

$$
\begin{aligned}
&\int_{-1}^{1} P_k^q(\mu)d\mu \int_{-1}^{1} P_k^q(\mu')d\mu' \int_0^{2\pi} \cos q\chi' d\chi' \int_0^{2\pi} \cos q\chi d\chi \sigma_s(E' \to E, \Omega' \to \Omega) \\
&= \sigma_{sk}(E',E)\int_{-1}^{1} P_k^q(\mu)P_k^q(\mu)d\mu \int_0^{2\pi}\cos^2 q\chi d\chi \\
&= \sigma_{sk}(E',E)\Big(\frac{2\pi}{2k+1}\Big)\frac{(k+q)!}{(k-q)!}\alpha_q^{-1}.
\end{aligned} \tag{2.11}
$$

Thus, we have for $k \geq q$ that

$$
\begin{aligned}
\sigma_{sk}(E',E) = \Big(\frac{2k+1}{2\pi}\Big)\frac{(k-q)!}{(k+q)!}\alpha_q \int_{-1}^{1} P_k^q(\mu)d\mu \int_{-1}^{1} P_k^q(\mu')d\mu' \int_0^{2\pi}\cos q\chi d\chi \\
\times \int_0^{2\pi} \cos q\chi' d\chi' \sigma_s(E' \to E, \mu' \to \mu, \chi' \to \chi).
\end{aligned} \tag{2.12}
$$

The corresponding Legendre coefficients for the linearly anisotropic scattering reads

$$
\sigma_{sk}(E',E) = 2\pi \int_{-1}^{1} d\omega P_k(\omega)\sigma_s(E' \to E, \omega), \tag{2.13}
$$

where $\omega = \Omega \cdot \Omega'$ and the scattering cross-section corresponding to (2.1) is now independent of the directional variable and simply reads as follows

$$
\sigma_s(\mathbf{x}, E', t) = 2\pi \int_0^{\infty} dE \int_{-1}^{1} d\omega \sigma_s(E' \to E, \omega). \tag{2.14}
$$

3. Surface harmonic expansion of the transport equation

In this section we continue with the case of *isotropic media anisotropic scattering kernel* and expand the solution $\psi(E,\Omega)$ of (1.1) in surface harmonics $Y_{nk}(\Omega)$ with adjoint $Y^*_{nk}(\Omega)$, viz

$$\psi(E,\Omega)=\sum_{n=0}^{\infty}\sum_{k=-n}^{n}\Big(\frac{2n+1}{4\pi}\Big)a_{nk}\psi_{nk}(E)Y_{nk}(\Omega), \tag{3.1}$$

where

$$a_{nk}=\frac{(n-|k|)!}{(n+|k|)!},\qquad \psi_{nk}(E)=\int_{S^2}d\Omega Y^*_{nk}(\Omega)\psi(E,\Omega), \tag{3.2}$$

$$Y_{nk}(\Omega)=Y_{nk}(\mu,\chi)=P_n^k(\mu)e^{ik\chi}. \tag{3.3}$$

Thus, using (2.2) and (3.1) we may expand the energy integrand in (1.1) as follows

$$\begin{aligned} I:=&\int_{S^2}d\Omega'\sigma_s(E'\to E,\Omega'\to\Omega)\psi(E',\Omega')\\ =&\sum_{\ell=0}^{\infty}\Big(\frac{2\ell+1}{4\pi}\Big)\sum_{n=0}^{\infty}\sum_{k=-n}^{n}\Big(\frac{2n+1}{4\pi}\Big)a_{nk}\sigma_{s\ell}(E',E)\psi_{nk}(E')\times\\ &\int_{S^2}\Big(\sum_{m=0}^{\ell}\alpha_m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(\mu')P_\ell^m(\mu)\cos m(\chi'-\chi)\Big)Y_{nk}(\Omega')\,d\Omega'. \end{aligned} \tag{3.4}$$

The orthogonality condition for surface harmonics reads as

$$\int_{S^2}d\Omega\,Y_{nk}(\Omega)Y^*_{\ell m}(\Omega)=\frac{4\pi}{(2n+1)}\cdot\frac{1}{a_{nk}}\delta_{n\ell}\delta_{mk}, \tag{3.5}$$

where $Y^*_{\ell m}(\Omega)$ is the complex conjugate of $Y_{\ell m}(\Omega)$. Using (3.5) the integral in (3.4) is computed as

$$\begin{aligned} J:=&\int_{S^2}d\Omega'\Big(\sum_{j=-\ell}^{\ell}a_{\ell j}Y_{\ell j}(\Omega)Y^*_{\ell j}(\Omega')\Big)Y_{nk}(\Omega')=\sum_{j=-\ell}^{\ell}a_{\ell j}Y_{\ell j}(\Omega)\frac{4\pi}{2n+1}\frac{1}{a_{nk}}\delta_{n\ell}\delta_{jk}\\ =&\,a_{\ell k}Y_{\ell k}(\Omega)\frac{4\pi}{2n+1}\frac{1}{a_{nk}}\delta_{n\ell}=\frac{4\pi}{2n+1}Y_{nk}(\Omega)\delta_{n\ell}. \end{aligned} \tag{3.6}$$

Inserting (3.6) in (3.4), we get

$$I=\sum_{n=0}^{\infty}\sum_{k=-n}^{n}\Big(\frac{2n+1}{4\pi}\Big)a_{nk}Y_{nk}(\Omega)\sigma_{sn}(E',E)\psi_{nk}(E'). \tag{3.7}$$

Thus, in isotropic media, the transport equation (1.1) with an anisotropic in-scattering term expanded over surface harmonic bases functions can be written in the following form:

$$\begin{aligned} &\frac{1}{v}\frac{\partial\psi(E,\Omega)}{\partial t}+\Omega\cdot\nabla_{\mathbf{x}}\psi(E,\Omega)+\sigma(E,\Omega)\psi(E,\Omega)\\ &=\sum_{n=0}^{\infty}\sum_{k=-n}^{n}\Big(\frac{2n+1}{4\pi}\Big)a_{nk}Y_{nk}(\Omega)\int_0^{\infty}\sigma_{sn}(E',E)\psi_{nk}(E')dE'+Q(E,\Omega), \end{aligned} \tag{3.8}$$

with $\sigma_{sn}(E',E)$ given by (2.12).

4. Asymptotic expansion of the anisotropic scattering kernel

In this section we consider higher moments expansion of the scattering kernel compared with the *zeroth moment* studied in [22]. Equation (3.8) forms the starting point of the Fokker-Planck development for isotropic media with anisotropic scattering. At the end of this development, using the surface harmonics expansions, we shall be able to eliminate the dominant in and out scattering terms and, with a positivity assumption on the resulting "zeroth order terms", rewrite the remaining part on the right hand side of (3.8) as a Fokker-Planck operator acting on $\psi(E,\Omega)$. To this approach we assume that the particles transport in a bounded domain of the characteristic size $\mathcal{O}(1)$. The scattering mean free path (Δ) is assumed to be small, $\mathcal{O}(\varepsilon)$, and consequently the scattering cross-section being the reciprocal of the scattering mean free path is large, i.e., $\sigma_s(E,\Omega)\gg 1$. For isotropic media Δ may assumed to be constant, whereas in anisotropic media (Section 5) we use a *local average* on each homogeneous part of the domain (local homogenization), again denoted by Δ, and the (E,Ω) dependence is transfered to the scattering cross-section and kernel. Without such an assumption we have to deal with somewhat more involved algebraic labour, see, e.g. [26]. Thus using the scaling

$$\sigma_s(E,\Omega)=\frac{\hat{\sigma}_s(E,\Omega)}{\Delta}, \tag{4.1}$$

we have $\hat{\sigma}_s(E,\Omega)=\mathcal{O}(1)$ and $\Delta\ll 1$. We apply the same scaling to the scattering kernel

$$\sigma_s(E'\to E;\Omega'\to\Omega)=\sigma_s(E'\to E;\mu'\to\mu;\chi'\to\chi),$$

and use the fast variables: $(\xi,\eta,\zeta)\in\mathbb{R}\times\widetilde{\mathcal{S}}^2$; defined by

$$\xi=\frac{E'-E}{\varepsilon};\quad \varepsilon\ll 1, \tag{4.2}$$

$$\eta=\frac{\mu'-\mu}{\delta};\quad \delta\ll 1, \tag{4.3}$$

$$\zeta=\frac{\chi'-\chi}{\gamma};\quad \gamma\ll 1. \tag{4.4}$$

Thus we scale the scattering kernel as

$$\begin{aligned}\sigma(E',E;\mu',\mu;\chi',\chi)&:=\frac{1}{\Delta}\hat{\sigma}_s\Big(E',\frac{E'-E}{\varepsilon};\mu',\frac{\mu'-\mu}{\delta};\chi',\frac{\chi'-\chi}{\gamma}\Big)\\&=\frac{1}{\Delta}\hat{\sigma}_s(E',\xi;\mu',\eta;\chi',\zeta),\end{aligned} \tag{4.5}$$

where $\hat{\sigma}_s(E',\xi;\mu',\eta;\chi',\zeta)$ is $\mathcal{O}(1)$ and the partial derivatives $\partial\hat{\sigma}_s/\partial\xi$, $\partial\hat{\sigma}_s/\partial\eta$ and $\partial\hat{\sigma}_s/\partial\zeta$ are assumed to be $\mathcal{O}(1)$ as $\varepsilon,\delta,\gamma\to 0$. Physically the smallness parameters ε and (δ,γ) are the measures for the peaking of the scattering kernel in energy and direction, respectively. In this way we have that the scattering cross-section is large, and the scattering kernel is highly peaked about $E=E'$ and $\Omega=\Omega'$. We also introduce the absorption cross-section $\sigma_a(E,\Omega)$ defined by

$$\sigma_a(E,\Omega):=\sigma(E,\Omega)-\sigma_s(E,\Omega).$$

Now we insert the Legendre coefficients $\sigma_{sn}(E',E)$, given by (2.12), and the scalings (4.1) and (4.5) in the transport equation (3.8) and get, for any positive integer q with $n\geq q$, a scaled transport equation according to

$$\begin{aligned}
&\frac{1}{v}\frac{\partial\psi(E,\Omega)}{\partial t}+\Omega\cdot\nabla_x\psi(E,\Omega)+\Big[\sigma_a(E,\Omega)+\frac{\hat{\sigma}_s(E,\Omega)}{\Delta}\Big]\psi(E,\Omega)\\
&\quad=\frac{1}{\Delta}\sum_{n=q}^{\infty}\sum_{k=-n}^{n}\Big(\frac{2n+1}{4\pi}\Big)\Big(\frac{2n+1}{2\pi}\Big)\frac{(n-q)!}{(n+q)!}\alpha_q a_{nk}Y_{nk}(\Omega)\times\\
&\int_0^\infty dE'\psi_{nk}(E')\int_{-1}^{1}P_n^q(\mu')d\mu'\int_{-1}^{1}P_n^q(\mu)\int_0^{2\pi}\cos q\chi' d\chi'\int_0^{2\pi}\cos q\chi d\chi\\
&\quad\times\hat{\sigma}_s\Big(E',\frac{E'-E}{\varepsilon},\mu',\frac{\mu'-\mu}{\delta},\chi,\frac{\chi'-\chi}{\gamma}\Big)+Q(E,\Omega).
\end{aligned}\tag{4.6}$$

We seek the asymptotic limit of the scaled transport equation (4.6) as the smallness parameters Δ,ε and (δ,γ) tend to zero. To this approach, given a positive integer q, for $n\geq q$ we consider the integral

$$\begin{aligned}
K&=\frac{1}{\Delta}\int_0^\infty dE'\int_{-1}^{1}d\mu' P_n^q(\mu')\int_{-1}^{1}d\mu P_n^q(\mu)\int_0^{2\pi}d\chi'\cos q\chi'\int_0^{2\pi}d\chi\cos q\chi\\
&\quad\times\hat{\sigma}_s\Big(E',\frac{E'-E}{\varepsilon};\mu',\frac{\mu'-\mu}{\delta};\chi',\frac{\chi'-\chi}{\gamma}\Big)\psi_{nk}(E')\\
&=\frac{\varepsilon\delta\gamma}{\Delta}\int_{-E/\varepsilon}^{\infty}d\xi\int_{(-1-\mu)/\delta}^{(1-\mu)/\delta}P_n^q(\mu+\delta\eta)\int_{-1}^{1}d\mu P_n^q(\mu)\int_{-\chi/\gamma}^{(2\pi-\chi)/\gamma}\cos q(\chi+\gamma\zeta)d\zeta\\
&\quad\times\int_0^{2\pi}d\chi\cos q\chi\hat{\sigma}_s(E+\varepsilon\xi,\xi;\mu+\delta\eta,\eta;\chi+\gamma\zeta,\zeta)\psi_{nk}(E+\varepsilon\xi),
\end{aligned}\tag{4.7}$$

where, in the second equality, we changed the integration variables from E' and $\Omega'=(\mu',\chi')$ to ξ,η and ζ according to (4.2)-(4.4). Thus

$$K=\Big(\frac{\varepsilon\delta\gamma}{\Delta}\int_{-E/\varepsilon}^{\infty}d\xi\int_{(-1-\mu)/\delta}^{(1-\mu)/\delta}d\eta\int_{-1}^{1}P_n^q(\mu)d\mu\int_{-\chi/\gamma}^{(2\pi-\chi)/\gamma}d\zeta\int_0^{2\pi}\cos q\chi d\chi\Big)\mathcal{F}_{nk}(\mathbf{r}),\tag{4.8}$$

where

$$\mathcal{F}_{nk}(\mathbf{r}):=P_n^q(\mu+\delta\eta)\cos q(\chi+\gamma\zeta)\hat{\sigma}_s(E+\varepsilon\xi,\xi;\mu+\delta\eta,\eta;\chi+\gamma\zeta,\zeta)\psi_{nk}(E+\varepsilon\xi).\tag{4.9}$$

We now Taylor expand $\mathcal{F}_{nk}$ about $\varepsilon=\delta=\gamma=0$; where carrying up to quadratic terms in smallness parameters yields

$$\mathcal{F}_{nk}(\mathbf{r})=\{\mathbf{T}\}_{E\mu\chi}^{\xi\eta\zeta}\mathcal{F}_{nk}(\mathbf{r}_0)+\mathcal{O}\Big(\sum_{i+j+\nu=3}\varepsilon^i\delta^j\gamma^\nu\Big)\mathcal{F}_{nk}\Big(\alpha\mathbf{r}_0+(1-\alpha)\mathbf{r}\Big),\tag{4.10}$$

where

$$\begin{aligned}
\{\mathbf{T}\}_{E\mu\chi}^{\xi\eta\zeta}:=&\Big[1+\varepsilon\xi\frac{\partial}{\partial E}+\delta\eta\frac{\partial}{\partial\mu}+\gamma\zeta\frac{\partial}{\partial\chi}+\frac{1}{2}(\varepsilon\xi)^2\frac{\partial^2}{\partial E^2}+\frac{1}{2}(\delta\eta)^2\frac{\partial^2}{\partial\mu^2}\\
&+\frac{1}{2}(\gamma\zeta)^2\frac{\partial^2}{\partial\chi^2}+\varepsilon\delta\xi\eta\frac{\partial^2}{\partial E\partial\mu}+\varepsilon\gamma\xi\zeta\frac{\partial^2}{\partial E\partial\chi}+\delta\gamma\eta\zeta\frac{\partial^2}{\partial\mu\partial\chi}\Big],
\end{aligned}\tag{4.11}$$

further $0\leq\alpha\leq1$ and

$$\mathcal{F}_{nk}(\mathbf{r}_0):=P_n^q(\mu)\cos q\chi\psi_{nk}(E)\hat{\sigma}_s(E,\xi;\mu,\eta;\chi,\zeta).$$

To proceed, we assume that we can replace the lower limit of integration over ξ by $-\infty$, and that the error we make in doing this is $\mathcal{O}(\varepsilon^3)$ or smaller. This is certainly legitimate if the scattering kernel falls off exponentially from its peak at $\xi=0$. However, if the kernel falls off algebraically at too weak a rate, this replacement may increase the error in the Fokker-Planck treatment over the $\mathcal{O}(\varepsilon^3)$ error indicated in (4.10). The error introduced by the above replacement makes our treatment in here as an asymptotic, rather than a convergence approach. As a consequence of this replacement is that we may use, in our asymptotic derivations, the commutativity assumption

$$(\varepsilon\xi)^p \frac{\partial^p}{\partial E^P} = \frac{\partial^p}{\partial E^P}(\varepsilon\xi)^p, \qquad p = 1, 2. \tag{4.12}$$

Relation (4.12) is useful, e.g., in zero moment expansion ($q \equiv 0$) it yields analogue expressions for the coefficients of differential operators in angle and energy variables (which is not the case in higher moments) leading to a unified operator representation. For simplicity we shall use the "formal" notation

$$\int_{S^2 \otimes \tilde{S}^2} d\eta d\mu d\zeta d\chi := \int_{(-1-\mu)/\delta}^{(1-\mu)/\delta} d\eta \int_{-1}^{1} d\mu \int_{-\chi/\gamma}^{(2\pi-\chi)/\gamma} d\zeta \int_0^{2\pi} d\chi, \tag{4.13}$$

Thus inserting (4.10) in (4.8) and using the above assumption, leading to (4.12), on ξ we may write

$$\begin{aligned} K = \frac{\varepsilon\delta\gamma}{\Delta} \int_{-\infty}^{\infty} d\xi \int_{S^2 \otimes \tilde{S}^2} P_n^q(\mu)\cos(q\chi) \cdot \Big[\Big(1 + \frac{\partial}{\partial E}\varepsilon\xi + \delta\eta \frac{\partial}{\partial \mu} + \gamma\zeta \frac{\partial}{\partial \chi} \\ + \frac{1}{2}\frac{\partial^2}{\partial E^2}(\varepsilon\xi)^2 + \frac{1}{2}(\delta\eta)^2 \frac{\partial^2}{\partial \mu^2} + \frac{1}{2}(\gamma\zeta)^2 \frac{\partial^2}{\partial \chi^2} \\ + \delta\eta \frac{\partial^2}{\partial E \partial \mu}\varepsilon\xi + \gamma\zeta \frac{\partial^2}{\partial E \partial \chi}\varepsilon\xi + \delta\gamma\eta\zeta \frac{\partial^2}{\partial \mu \partial \chi} \Big) \mathcal{F}_{nk}(\mathbf{r}_0) \\ + \mathcal{O}\Big(\sum_{i+j+\nu=3} \varepsilon^i \delta^j \gamma^\nu \Big) \mathcal{F}_{nk}(\alpha\mathbf{r}_0 + (1-\alpha)\mathbf{r}) \Big] d\eta d\mu d\zeta d\chi. \end{aligned} \tag{4.14}$$

We now change integration variables in (4.13) from ξ, η, ζ to E', μ', χ' according to

$$\xi = \frac{E - E'}{\varepsilon}, \tag{4.15}$$

$$\eta = \frac{\mu' - \mu}{\delta}, \tag{4.16}$$

$$\zeta = \frac{\chi' - \chi}{\gamma}, \tag{4.17}$$

Note that equations (4.16) and (4.17) are identical to equations (4.3) and (4.4) respectively, but equation (4.15) is not identical to equation (4.2) (the E and E' are interchanged). For isotropic background media this is motivated by the angular reversibility relation (2.5), whereas for anisotropic media it is included in the asymptotic procedure. We use the simplifying notation

$$\int_{S^2 \otimes S^2} d\mu' d\mu d\chi' d\chi := \int_{-1}^{1} d\mu' \int_{-1}^{1} d\mu \int_0^{2\pi} d\chi' \int_0^{2\pi} d\chi, \tag{4.18}$$

and define

$$\mathcal{G}_\Delta(\mathbf{r}_0) := P_n^q(\mu)\cos(q\chi) \times \frac{1}{\Delta}\hat{\sigma}_s\Big(E, \frac{E - E'}{\varepsilon}; \mu', \frac{\mu' - \mu}{\delta}; \chi', \frac{\chi' - \chi}{\gamma}\Big).$$

Now recalling (4.9)-(4.13); and using (4.15)-(4.17), the equation (4.14) can be written as follows:

$$
\begin{aligned}
K = \int_{-\infty}^{\infty} dE' \int_{S^2\otimes S^2} & P_n^q(\mu)\cos(q\chi)\Big[\Big(1+\frac{\partial}{\partial E}(E-E')+(\mu'-\mu)\frac{\partial}{\partial \mu}+(\chi'-\chi)\frac{\partial}{\partial \chi} \\
& +\frac{1}{2}\frac{\partial^2}{\partial E^2}(E-E')^2+\frac{1}{2}(\mu'-\mu)^2\frac{\partial^2}{\partial \mu^2}+\frac{1}{2}(\chi'-\chi)^2\frac{\partial^2}{\partial \chi^2} \\
& +(\mu'-\mu)\frac{\partial^2}{\partial E\partial \mu}(E-E')+(\chi'-\chi)\frac{\partial^2}{\partial E\partial \chi}(E-E') \\
& +(\mu'-\mu)(\chi'-\chi)\frac{\partial^2}{\partial \mu\partial \chi}\Big)\Big(\mathcal{G}_\Delta(\mathbf{r}_0)\psi_{nk}(E)\Big) \\
& +\mathcal{O}\Big(\sum_{i+j+\nu=3}\varepsilon^i\delta^j\gamma^\nu\Big)\mathcal{G}_\Delta(\alpha\mathbf{r}_0+(1-\alpha)\mathbf{r})\psi_{nk}(\alpha E+(1-\alpha)E')\Big]d\mu' d\mu d\chi' d\chi.
\end{aligned} \tag{4.19}
$$

Below we shall frequently use the symmetry relation (2.5) and the identity

$$
\begin{aligned}
\mathcal{G}_\Delta(\mathbf{r}_0) &= P_n^q(\mu)\cos(q\chi)\frac{1}{\Delta}\hat{\sigma}_s\Big(E,\frac{E-E'}{\varepsilon};\mu',\frac{\mu'-\mu}{\delta};\chi',\frac{\chi'-\chi}{\gamma}\Big) \\
&= P_n^q(\mu)\cos(q\chi)\sigma_s(E,E';\mu',\mu;\chi',\chi) \stackrel{!}{=} P_n^q(\mu)\cos(q\chi)\sigma_s(E,E';\mu,\mu';\chi,\chi').
\end{aligned} \tag{4.20}
$$

Note that the bottom limit of integration on the E' integrals can be replaced by zero since the scattering kernel vanishes for negative E' (the probability of scattering to a negative energy is zero). In the remaining of this section we do not retain the explicit cross-term derivatives. This is because the standard Fokker-Planck operator does not possess cross-term differentiations. In a similar case Hagedorn [15] has given a full development retaining all the involved terms up to a significantly high order of accuracy. Now we split the right hand side of (4.19) as

$$
\begin{aligned}
K(E) :=& \Big[\mathcal{K}_{nq}^0(E)+\mathcal{K}_{nq}^E(E)+\mathcal{K}_{nq}^\mu(E)+\mathcal{K}_{nq}^\chi(E)\Big]\Big(\psi_{nk}(E)\Big) \\
&+\mathcal{O}\Big(\sum_{i_0+j_0+\nu_0=2}\frac{\varepsilon^{i_0}\delta^{j_0}\gamma^{\nu_0}}{\Delta}\Big)+\mathcal{O}\Big(\sum_{i_1+j_1+\nu_1=3}\frac{\varepsilon^{i_1}\delta^{j_1}\gamma^{\nu_1}}{\Delta}\Big), \\
& i_0,j_0,\nu_0=0,1; \quad i_1,j_1,\nu_1=0,1,2,3,
\end{aligned} \tag{4.21}
$$

where

$$
\begin{aligned}
\mathcal{K}_{nq}^0(E) :=& \Big(\int_0^\infty dE'\int_{-1}^1 d\mu[P_n^q(\mu)]^2\int_0^{2\pi}d\chi\cos^2 q\chi\int_{-1}^1 d\mu'\int_0^{2\pi}d\chi' \\
& \sigma_s\Big(E,E',\mu,\mu',\chi,\chi'\Big)\Big) := \Lambda_{n,2}^q\sigma_s(E,E',\mu,\mu',\chi,\chi'),
\end{aligned} \tag{4.22}
$$

$$
\mathcal{K}_{nq}^E(E) :=\Lambda_{n,2}^q\Big\{\Big[\frac{\partial}{\partial E}(E-E')+\frac{1}{2}\frac{\partial^2}{\partial E^2}(E-E')^2\Big]\Big(\sigma_s(E,E',\mu,\mu',\chi,\chi')\Big)\Big\}, \tag{4.23}
$$

$$
\mathcal{K}_{nq}^\mu(E) :=\Lambda_{n,1}^q\Big\{\Big[(\mu'-\mu)\frac{\partial}{\partial \mu}+\frac{1}{2}(\mu'-\mu)^2\frac{\partial^2}{\partial \mu^2}\Big]\Big(P_n^q(\mu)\cos q\chi\sigma_s(E,E',\mu,\mu',\chi,\chi')\Big)\Big\}, \tag{4.24}
$$

with

$$
\Lambda_{n,r}^q := \int_0^\infty dE'\int_{-1}^1 d\mu[P_n^q(\mu)]^r\int_0^{2\pi}d\chi\cos^r q\chi\int_{-1}^1 d\mu'\int_0^{2\pi}d\chi', \qquad r=1,2, \tag{4.25}
$$

and $\mathcal{K}^{\chi}_{nq}(E)$ is defined analogously as $\mathcal{K}^{\mu}_{nq}(E)$. Inserting K in (4.6) we thus have

$$\begin{aligned}
&\frac{1}{v}\frac{\partial\psi(E,\Omega)}{\partial t}+\Omega\cdot\nabla_x\psi(E,\Omega)+\Big[\sigma_a(E,\Omega)+\sigma_s(E,\Omega)\Big]\psi(E,\Omega)\\
&=\sum_{n=q}^{\infty}\sum_{k=-n}^{n}\Big(\frac{2n+1}{4\pi}\Big)\Big(\frac{2n+1}{2\pi}\Big)\frac{(n-q)!}{(n+q)!}\alpha_q a_{nk}Y_{nk}(\Omega)\times\\
&\Big[\mathcal{K}^0_{nq}(E)+\mathcal{K}^E_{nq}(E)+\mathcal{K}^{\mu}_{nq}(E)+\mathcal{K}^{\chi}_{nq}(E)\Big]\Big(\psi_{nk}(E)\Big)+Q(E,\Omega)\\
&+\mathcal{O}\Big(\sum_{i_0+j_0+\nu_0=2}\frac{\varepsilon^{i_0}\delta^{j_0}\gamma^{\nu_0}}{\Delta}\Big)+\mathcal{O}\Big(\sum_{i_1+j_1+\nu_1=3}\frac{\varepsilon^{i_1}\delta^{j_1}\gamma^{\nu_1}}{\Delta}\Big),\\
&i_0,j_0,\nu_0=0,1;\quad i_1,j_1,\nu_1=0,1,2,3.
\end{aligned}\tag{4.26}$$

Now the idea is: (i) to move in $Y_{nk}(\Omega)$ inside $\mathcal{K}^E_{nq}(E)$ (in front of σ_s) and (ii) to move out $\psi_{nk}(E)$ in the remaining terms outside $\Lambda^q_{n,r}$ operators. In this way, for the "zeroth moment", (corresponding to the choice of $q\equiv 0$), the contribution from $\mathcal{K}^0_{nq}(E)$ will be cancelled by a principal part of $\sigma_s(E,\Omega)$ on the left hand side. This would correspond to the cancellation of the large in and out scattering terms. Likewise the $\mathcal{K}^E_{nq}(E)$ term would result in a convection-diffusion term in energy variable with the convection of order $\mathcal{O}(\varepsilon/\Delta)$ and the diffusion of order $\mathcal{O}(\varepsilon^2/\Delta)$. These terms coincide with the corresponding terms in Pomraning's [22] treatment of the isotropic media case. In the case of general moments, (i.e. for $q>0$, integer), the terms involving $\mathcal{K}^0_{nq}(E)$ and $\mathcal{K}^E_{nq}(E)$ would have similar behaviour as above. However, because of the presence of $Y_{nk}(\mu,\chi)$ in (4.6), the contributions from the terms corresponding to $\mathcal{K}^{\mu}_{nq}(E)$ and $\mathcal{K}^{\chi}_{nq}(E)$ are somewhat different. For these terms we may write

$$\begin{aligned}
Y_{nk}(\Omega)\mathcal{K}^{\mu}_{nq}(E):=&\Lambda^q_{n,1}\Big[(\mu'-\mu)\frac{\partial}{\partial\mu}+\frac{1}{2}(\mu'-\mu)^2\frac{\partial^2}{\partial\mu^2}\Big]\times\\
&\Big(Y_{nk}(\Omega)P^q_n(\mu)\cos q\chi\sigma_s(E,E',\Omega,\Omega')\Big)+\mathcal{L}^{\mu}_{nq}(E,\Omega),
\end{aligned}\tag{4.27}$$

and analogously introduce the rest term $\mathcal{L}^{\chi}_{nq}(E,\Omega)$ in representing $Y_{nk}(\Omega)\mathcal{K}^{\chi}_{nq}(E)$ in a similar form as (4.27). Inserting $Y_{nk}(\Omega)\mathcal{K}^{\mu}_{nq}(E)$ and $Y_{nk}(\Omega)\mathcal{K}^{\chi}_{nq}(E)$ in (4.26), yields (i) *a correctly ordered angular diffusion* of $\mathcal{O}\Big((\delta^2+\gamma^2)/\Delta\Big)$ compared to $\mathcal{O}\Big(\delta/\Delta\Big)$ in [22], and (ii) a convective term of $\mathcal{O}\Big((\delta+\gamma)/\Delta\Big)$, corresponding to the contributions of the first order terms involving $(\mu'-\mu)$ and $(\chi'-\chi)$. Such a convective term in angle is missing in the zero moment representation of [22]. Further, we point out that, the diffusion part of the result of [22] is based on the differential equation $\Big(n(n+1)+\nabla^2_{\Omega}\Big)Y_{nk}(\Omega)=0$, satisfied by the surface harmonics (see also Section 6, below). More specifically in the Taylor expansion, of the Legendre polynomial $P_n(\omega)$ about $\omega=1$, the second term $\delta P'_n(1)=\delta n(n+1)\sim-\delta\nabla^2_{\Omega}$ is of order $\mathcal{O}(\delta)$. This gives rise to a full diffusion of order $\mathcal{O}(\delta/\Delta)=\mathcal{O}(1)$, with $\delta\sim\Delta$. In general, (when higher moments are involved), the diffusion, involving second order derivatives, has the correct order of $\mathcal{O}\Big((\delta^2+\gamma^2)/\Delta\Big)\sim\mathcal{O}(\delta+\gamma)$. The serious draw-back in the higher moment representation is related to the decrease of the absorption term $\sigma_a(E,\Omega)$ corresponding to the contributions from $\mathcal{L}^{\mu}_{nq}(E,\Omega)$ and $\mathcal{L}^{\chi}_{nq}(E,\Omega)$:

$$\begin{aligned}
\bar\sigma_a(E,\Omega)\psi(E,\Omega):=&\Big(\sigma_a(E,\Omega)+\sigma_s(E,\Omega)\Big)\psi(E,\Omega)-\sum_{n=q}^{\infty}\sum_{k=-n}^{n}\frac{(2n+1)^2}{8\pi^2}\frac{(n-q)!}{(n+q)!}\times\\
&\alpha_q a_{nk}\Big[Y_{nk}(\Omega)\Lambda^q_{n,2}\sigma_s(E,E',\Omega,\Omega')+\mathcal{L}^{\mu}_{nq}(E,\Omega)+\mathcal{L}^{\chi}_{nq}(E,\Omega)\Big]\psi_{nk}(E).
\end{aligned}\tag{4.28}$$

Thus the higher moments expansion simply requires either an assumption on, or a guarantee of the positivity of $\bar{\sigma}_a$. In either case this is a strong and cumbersome condition to fulfil. We skip the details and, assuming positive $\bar{\sigma}_a$, summarise the results of this section in the following q-th moment asymptotic Fokker-Planck expansion of the linear transport equation in isotropic background media with anisotropic scattering:

$$
\begin{aligned}
&\frac{1}{v}\frac{\partial\psi(E,\Omega)}{\partial t} + \Omega\cdot\nabla_x\psi(E,\Omega) + \bar{\sigma}_a(E,\Omega)\psi(E,\Omega) \\
&\quad= \sum_{n=q}^{\infty}\sum_{k=-n}^{n}\int_0^{\infty} dE'\Big\{\Big[\frac{\partial}{\partial E}(E-E') + \frac{1}{2}\frac{\partial^2}{\partial E^2}(E-E')^2\Big]\cdot \\
&\qquad\quad \Big(\psi_{nk}^q(E,\Omega)\Gamma_{n,2}^q\sigma_s(E,E',\Omega,\Omega')\Big) \\
&\quad + \Gamma_{n,1}^q\Big[(\mu'-\mu)\frac{\partial}{\partial\mu} + \frac{1}{2}(\mu'-\mu)^2\frac{\partial^2}{\partial\mu^2} + (\chi'-\chi)\frac{\partial}{\partial\chi} + \frac{1}{2}(\chi'-\chi)^2\frac{\partial^2}{\partial\chi^2}\Big]\cdot \\
&\qquad\quad \Big(\psi_{nk}^q(E,\Omega)P_n^q(\mu)\cos q\chi\,\sigma_s(E,E',\Omega,\Omega')\Big)\Big\} \\
&\quad + Q(E,\Omega) + \mathcal{O}\Big(\sum_{i_0+j_0+\nu_0=2}\frac{\varepsilon^{i_0}\delta^{j_0}\gamma^{\nu_0}}{\Delta}\Big) + \mathcal{O}\Big(\sum_{i_1+j_1+\nu_1=3}\frac{\varepsilon^{i_1}\delta^{j_1}\gamma^{\nu_1}}{\Delta}\Big), \\
&\qquad\quad i_0, j_0, \nu_0 = 0,1; \quad i_1, j_1, \nu_1 = 0,1,2,3,
\end{aligned} \tag{4.29}
$$

where

$$
\psi_{nk}^q(E,\Omega) := \frac{(2n+1)^2}{8\pi^2}\frac{(n-q)!}{(n+q)!}\alpha_q a_{nk} Y_{nk}(\Omega)\psi_{nk}(E), \tag{4.30}
$$

and

$$
\Gamma_{n,r}^q := \int_{-1}^{1} d\mu[P_n^q(\mu)]^r \int_0^{2\pi} d\chi\cos^r q\chi \int_{-1}^{1} d\mu' \int_0^{2\pi} d\chi', \qquad r = 1,2, \tag{4.31}
$$

is the energy integrand in $\Lambda_{n,r}^q$. Now we assume that the integrations do not change orderings. Moreover since ξ, η and ζ are $\mathcal{O}(1)$ variables, then according to relations (4.15)-(4.17); $(E-E') = \mathcal{O}(\varepsilon)$, $(\mu'-\mu) = \mathcal{O}(\delta)$, $(\chi'-\chi) = \mathcal{O}(\gamma)$ and $\sigma_s(E,E',\Omega,\Omega') = \mathcal{O}(1/\Delta)$. Hence we have obtained both convective and diffusive terms in both energy and angular variables of magnitudes

$$
(E-E')\sigma_s(E,E',\Omega,\Omega') = \mathcal{O}\Big(\frac{\varepsilon}{\Delta}\Big), \quad \text{(order of convection in energy)} \tag{4.32}
$$

$$
(E-E')^2\sigma_s(E,E',\Omega,\Omega') = \mathcal{O}\Big(\frac{\varepsilon^2}{\Delta}\Big), \quad \text{(order of diffusion in energy)} \tag{4.33}
$$

$$
\Big[(\mu'-\mu) + (\chi'-\chi)\Big]\sigma_s(E,E',\Omega,\Omega') = \mathcal{O}\Big(\frac{\delta+\gamma}{\Delta}\Big), \quad \text{(order of convection in angle)} \tag{4.34}
$$

$$
\Big[(\mu'-\mu)^2 + (\chi'-\chi)^2\Big]\sigma_s(E,E',\Omega,\Omega') = \mathcal{O}\Big(\frac{\delta^2+\gamma^2}{\Delta}\Big), \quad \text{(order of diffusion in angle).} \tag{4.35}
$$

We emphasis that all these orderings are in the integrand, and the assumption we made in writing (4.32) through (4.35): that the integration do not change the ordering is a severe restriction. The same holds true for taking the $\mathcal{O}\Big(\sum_{i_0+j_0+\nu_0=2}(\varepsilon^{i_0}\delta^{j_0}\gamma^{\nu_0})/\Delta\Big)$, $i_0, j_0, \nu_0 = 0,1$ and $\mathcal{O}\Big(\sum_{i_1+j_1+\nu_1=3}(\varepsilon^{i_1}\delta^{j_1}\gamma^{\nu_1})/\Delta\Big)$, $i_1, j_1, \nu_1 = 0,1,2,3$, terms out of the integrals involved. It is not obvious that the integration do not change the ordering since σ_s itself contains the smallness parameters ε, δ and γ. The situation again depends upon the rate of fall-off of the scattering kernel from its peaks in energy and direction. For exponential fall-off the relations (4.32)-(4.35) are correctly ordered, but for algebraic fall-off the order of one or more of these terms may be larger than indicated. This observation again places a restriction on the scattering kernel for the Fokker-Planck differential operator to be an asymptotic limit of the exact integral operator.

As we pointed out earlier, in the equation (4.29), the dominant part of in and out scattering term; $\sigma_s(E,\Omega)\psi(E,\Omega)$ has cancelled out. Further, a positivity assumption for $\bar{\sigma}_a(E,\Omega)$, defined in (4.28), is included. In the resulting Fokker-Planck equation, the convective terms in energy and direction are of the same order: $\mathcal{O}(\varepsilon/\Delta)=\mathcal{O}(\delta/\Delta)=\mathcal{O}(\gamma/\Delta)=\mathcal{O}(1)$; likewise the diffusion terms in energy and direction variables; are of order $\mathcal{O}(\varepsilon^2/\Delta)=\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\delta^2/\Delta)=\mathcal{O}(\gamma^2/\Delta)=\mathcal{O}(\varepsilon)$, respectively. In this setting all the involved differential operators are correctly ordered which is not the case in the zeroth moment expansion.

5. The Fokker-Planck development in anisotropic media

The studies in previous sections are indicating that the surface harmonics expansion, because of their symmetry structure, are not suitable for problems considered on anisotropic media. Any effort in this direction must start by expanding the scattering kernel σ_s, and such an expansion corresponding to (2.2), for the isotropic case, i.e,

$$\sigma_s(E'\to E,\Omega'\to\Omega)=\sum_{mn} C_{mn}Y_{mn}(\Omega'\cdot\Omega),$$

would read as

$$\sigma_s(E'\to E,\Omega'\to\Omega)=\sum C_{mn}^{m'n'}Y_{mn}(\Omega)Y_{m'n'}(\Omega'), \tag{5.1}$$

in anisotropic case. Dealing with these new coefficients $C_{mn}^{m'n'}$ would lead to both lengthy and cumbersome formalism than those of Sections 3 and 4. It appears, however, that in this case for asymptotic treatment of the scattering kernel $\hat{\sigma}_s$ a direct approach is more appropriate. Below we shall use the same scalings as (4.1) and (4.5) with the fast variables introduced in (4.2)-(4.4) and write the integral on the right hand side of (1.1) as

$$\begin{aligned}\int_0^\infty dE'\int_{S^2}d\Omega'\sigma_s\psi=&\frac{1}{\Delta}\int_0^\infty dE'\int_{-1}^1 d\mu'(1-(\mu')^2)^{1/2}\int_0^{2\pi}d\chi'\\ &\hat{\sigma}_s\Big(E',\frac{E'-E}{\varepsilon};\mu',\frac{\mu'-\mu}{\delta};\chi',\frac{\chi'-\chi}{\gamma}\Big)\psi(E',\mu',\chi')\\ =&\frac{\varepsilon\delta\gamma}{\Delta}\int_{-E/\varepsilon}^\infty d\xi\int_{(-1-\mu)/\delta}^{(1-\mu)/\delta}\Big(1-(\mu+\delta\eta)^2\Big)^{1/2}d\eta\int_{-\chi/\gamma}^{(2\pi-\chi)/\gamma}d\zeta\\ &\hat{\sigma}_s(E+\varepsilon\xi,\xi;\mu+\delta\eta,\eta;\chi+\gamma\zeta,\zeta)\psi(E+\varepsilon\xi,\mu+\delta\eta,\chi+\gamma\zeta),\end{aligned}\tag{5.2}$$

where, as in (4.7), in the second equality, we changed the integration variables from E' and Ω' to the fast variables ξ,η and ζ. Thus we define

$$\widetilde{K}:=\Big(\frac{\varepsilon\delta\gamma}{\Delta}\int_{-E/\varepsilon}^\infty d\xi\int_{(-1-\mu)/\delta}^{(1-\mu)/\delta}d\eta\int_{-\chi/\gamma}^{(2\pi-\chi)/\gamma}d\zeta\Big)\widetilde{\mathcal{F}}(\mathbf{r}), \tag{5.3}$$

where

$$\widetilde{\mathcal{F}}(\mathbf{r}):=\Big(1-(\mu+\delta\eta)^2\Big)^{1/2}\hat{\sigma}_s(E+\varepsilon\xi,\xi;\mu+\delta\eta,\eta;\chi+\gamma\zeta,\zeta)\psi(E+\varepsilon\xi,\mu+\delta\eta,\chi+\gamma\zeta). \tag{5.4}$$

Taylor expanding $\widetilde{\mathcal{F}}$ about $\varepsilon=\delta=\gamma=0$; and as in Section 4, carrying up to quadratic terms in smallness parameters yields

$$\widetilde{\mathcal{F}}(\mathbf{r})=\{\mathbf{T}\}_{E\mu\chi}^{\xi\eta\zeta}\widetilde{\mathcal{F}}(\mathbf{r}_0)+\mathcal{O}\Big(\sum_{i+j+\nu=3}\varepsilon^i\delta^j\gamma^\nu\Big)\widetilde{\mathcal{F}}\Big(\alpha\mathbf{r}_0+(1-\alpha)\mathbf{r}\Big), \tag{5.5}$$

where $\{\mathbf{T}\}_{E\mu\chi}^{\xi\eta\zeta}$ is defined by (4.11), $0 \le \alpha \le 1$ and

$$\widetilde{\mathcal{F}}(\mathbf{r}_0) := (1-\mu^2)^{1/2}\hat{\sigma}_s(E,\xi;\mu,\eta;\chi,\zeta)\psi(E,\mu,\chi). \tag{5.6}$$

Now by the same argument as in Section 4, (on the change of lower integration limit for ξ : $-E/\varepsilon \to -\infty \to 0$), and also using the angular and cross-term differentiation counterparts of (4.12), we finally have that

$$\begin{aligned}\widetilde{K} =&\widetilde{\sigma}_s(E,\Omega)\psi(E,\Omega) + \frac{\partial}{\partial E}\Big[\mathcal{R}_1(E,\Omega)\psi(E,\Omega)\Big] + \frac{\partial^2}{\partial E^2}\Big[\mathcal{R}_2(E,\Omega)\psi(E,\Omega)\Big]\\ &+ \frac{\partial}{\partial \mu}\Big[\mathcal{S}_1(E,\Omega)\psi(E,\Omega)\Big] + \frac{\partial^2}{\partial \mu^2}\Big[\mathcal{S}_2(E,\Omega)\psi(E,\Omega)\Big] + \frac{\partial}{\partial \chi}\Big[\mathcal{T}_1(E,\Omega)\psi(E,\Omega)\Big]\\ &+ \frac{\partial^2}{\partial \chi^2}\Big[\mathcal{T}_2(E,\Omega)\psi(E,\Omega)\Big] + \frac{\partial^2}{\partial E\partial \mu}\Big[\mathcal{U}(E,\Omega)\psi(E,\Omega)\Big] + \frac{\partial^2}{\partial E\partial \chi}\Big[\mathcal{V}(E,\Omega)\psi(E,\Omega)\Big]\\ &+ \frac{\partial^2}{\partial \mu\partial \chi}\Big[\mathcal{W}(E,\Omega)\psi(E,\Omega)\Big] + \mathcal{O}\Big(\sum_{i+j+\nu=3}\frac{\varepsilon^i\delta^j\gamma^\nu}{\Delta}\Big), \quad i,j,\nu=0,1,2,3.\end{aligned} \tag{5.7}$$

Note that in (5.7) for the highly forward peaked scattering with $(\mu-\mu')=\mathcal{O}(\delta)$, we have that

$$\widetilde{\sigma}_s(E,\Omega) = \int_0^\infty dE'\int_{-1}^1 d\mu'\int_0^{2\pi} d\chi'(1-\mu^2)^{1/2}\sigma_s(E,E';\mu',\mu;\chi',\chi), \tag{5.8}$$

is of the same order as $\sigma(E,\Omega)$, defined in (2.1), with the corresponding variable weight function $(1-\mu'^2)^{1/2}$ replaced by the constant $(1-\mu^2)^{1/2}$. Further we have that for $k=1,2$,

$$\mathcal{R}_k(E,\Omega) = \frac{1}{k}\int_0^\infty dE'(E-E')^k\int_{-1}^1 d\mu'\int_0^{2\pi} d\chi'(1-\mu^2)^{1/2}\sigma_s(E,E';\mu,\mu';\chi,\chi') = \mathcal{O}\Big(\frac{\varepsilon^k}{\Delta}\Big),$$

$$\mathcal{S}_k(E,\Omega) = \frac{1}{k}\int_0^\infty dE'\int_{-1}^1 d\mu'(\mu'-\mu)^k\int_0^{2\pi} d\chi'(1-\mu^2)^{1/2}\sigma_s(E,E';\mu,\mu';\chi,\chi') = \mathcal{O}\Big(\frac{\delta^k}{\Delta}\Big),$$

$$\mathcal{T}_k(E,\Omega) = \frac{1}{k}\int_0^\infty dE'\int_{-1}^1 d\mu'\int_0^{2\pi} d\chi'(\chi'-\chi)^k(1-\mu^2)^{1/2}\sigma_s(E,E';\mu,\mu';\chi,\chi'), = \mathcal{O}\Big(\frac{\gamma^k}{\Delta}\Big).$$

Similarly the cross-differentiating terms, which we did not retain in Section 4, are now

$$\mathcal{U}(E,\Omega) = \int_0^\infty dE'(E-E')\int_{-1}^1 d\mu'(\mu'-\mu)\int_0^{2\pi} d\chi'(1-\mu^2)^{1/2}\sigma_s(E,E';\mu,\mu';\chi,\chi') = \mathcal{O}\Big(\frac{\varepsilon\delta}{\Delta}\Big),$$

$$\mathcal{V}(E,\Omega) = \int_0^\infty dE'(E-E')\int_{-1}^1 d\mu'\int_0^{2\pi} d\chi'(\chi'-\chi)(1-\mu^2)^{1/2}\sigma_s(E,E';\mu,\mu';\chi,\chi') = \mathcal{O}\Big(\frac{\varepsilon\gamma}{\Delta}\Big),$$

$$\mathcal{W}(E,\Omega) = \int_0^\infty dE'\int_{-1}^1 d\mu'(\mu'-\mu)\int_0^{2\pi} d\chi'(\chi'-\chi)(1-\mu^2)^{1/2}\sigma_s(E,E';\mu,\mu';\chi,\chi') = \mathcal{O}\Big(\frac{\delta\gamma}{\Delta}\Big).$$

The main features of this simple derivation are as follows:

(i) Here we have not omitted the cross-differentiation terms.

(ii) In scaling back from the fast variables to the original ones, we have used (4.16)-(4.17) rather than (4.3)-(4.4), (the latter one is more natural in our case). We may motivate our choice of fast variables assuming either a "detailed balance" of the type (2.6) or a periodicity assumption of the form

$$\int_{-1+2\mu}^{1+2\mu} d\mu'\int_{2\chi}^{2\pi-2\chi}\widetilde{\mathcal{F}}(\mathbf{r}_0)d\chi' \sim \int_{-1}^1 d\mu'\int_0^{2\pi}\widetilde{\mathcal{F}}(\mathbf{r}_0)d\chi'.$$

Otherwise, to derive a result similar to (5.7) a formalism, (as that of Section 4 leading to (4.27)), involving the integration limits in angle would be necessary. Now the emerge of convection-diffusion operators and their orderings, in all velocity components, is evident. Hence inserting $\widetilde{K}$ of (5.7) into (5.2) for the right hand side of the transport equation we obtain the corresponding Fokker-Planck operator containing also cross-differentiation terms. The details are similar to the corresponding parts in Section 4 and therefore are omitted.

6. Conclusion

In our derivation of the Fokker-Planck operator we have obtained diffusion both in angle and energy. Diffusion terms are now correctly ordered (of $\mathcal{O}(\varepsilon^2/\Delta) = \mathcal{O}(\varepsilon)$ and $\mathcal{O}((\delta^2+\gamma^2)/\Delta) = \mathcal{O}(\varepsilon)$ in energy and angle, respectively) compared to the zeroth moment studies where the angular diffusion term is obtained from a Sturm-Liouville eigenvalue problem satisfied by surface harmonics:

$$\Big[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} + \Big(\frac{1}{1-\mu^2}\Big)\frac{\partial^2}{\partial\chi^2} + n(n+1)\Big]Y_{nm}(\mu,\chi) = 0, \tag{6.1}$$

where the eigenvalues $n(n+1) = P_n'(1)$ being first order derivatives in Taylor expansions have the first order coefficients resulting to diffusion of $\mathcal{O}(\delta/\Delta) = \mathcal{O}(\varepsilon)$.

As a result of assuming "highly forward peaked" scattering we obtained our Fokker-Planck limits in Sections 4 and 5. The same treatment is not applicable, e.g., to the Henyey-Greenstein kernel

$$\sigma_s(E', E, \Omega'\cdot\Omega) := \frac{\sigma_s(E')}{2\pi} f(\Omega'\cdot\Omega)\delta(E'-E). \tag{6.2}$$

Here δ is the Dirac delta-function, and $f = (g + 2r(\partial_r g))/2$, with g being the generating function for the Legendre polynomials:

$$g(r,\omega) := \frac{1}{(1-2r\omega+r^2)^{1/2}} = \sum_{n=0}^{\infty} r^n P_n(\omega), \quad \omega = \Omega'\cdot\Omega, \tag{6.3}$$

$$f(r,\omega) := \frac{1-r^2}{2(1-2r\omega+r^2)^{3/2}} = \sum_{n=0}^{\infty}\Big(\frac{2n+1}{2}\Big) r^n P_n(\omega). \tag{6.4}$$

From (6.4) we can see, in terms of a scaled variable, that the rate of algebraic fall off of this kernel from its peak, (as $r \to 1$), at $\omega = \Omega'\cdot\Omega \sim 1$ is $\mathcal{O}(\lambda^{-3/2})$, with $\lambda = (1-\omega)/(1-r)^2$. This fall off rate is too weak and therefore the Henyey-Greenstein scattering kernel does not possess a Fokker-Plank asymptotic limit, see [22] for further discussion.

Another point of concern is the vulnerability of assuming simple homogenization made by transferring the velocity dependence of the mean free path Δ to the cross-sections. Similar assumption can lead to significant error in the asymptotic study of the screened Rutherford pencil beam problems, (see [23]), with hetrogeneities and the forward peaked scattering kernel defined by

$$\sigma_s(\omega) := \frac{\kappa(1+\kappa)}{\pi(1+2\kappa-\omega)^2}, \quad \omega = \Omega'\cdot\Omega, \tag{6.5}$$

where κ is the screening parameter.

In summary:

(i) A necessary but not sufficient condition for equations (4.19) and (5.7) to lead to Fokker-Planck asymptotic limit of equations (3.8) and (1.1), respectively, is that the scattering kernel be peaked in both angle and energy.

(ii) The additional sufficient condition is that this peaking be either exponential or strongly algebraic.

This conclusion is in common for both anisotropic media and higher moments expansion of isotropic media cases of this paper, as well as the "zeroth moment" expansion of the isotropic media case studied by Pomraning [22].

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