CAPELLI IDENTITY AND RELATIVE DISCRETE SERIES OF THE LINE BUNDLE OVER TUBE DOMAINS

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ABSTRACT. We use the Capelli identity to give the explicit realization of some relative discrete series of L^2-space of sections of line bundle over a tube domain. This amounts to a geometric construction of some Opdam shift operators.

1. INTRODUCTION

In this paper we shall study the connection between the Capelli identity for tube type Hermitian symmetric spaces, and the problem of constructing explicitly the discrete spectrum of L^2- spaces of sections of line bundles over such domains. The main result is Theorem 4.2 which gives the explicit intertwining operator between the two models of holomorphic discrete series representations of the group. It is interesting to remark (as is done after Corollary 4.4) that this differential operator is actually of a very canonical type, namely that of a generalized gradient operator.

Let $D = G/K$ be an irreducible Hermitian symmetric space of non-compact type realized as a bounded symmetric domain in a complex vector space. The group K thus has one-dimensional center and the one-dimensional representations on K then induce homogeneous line bundles over D. In his paper [15] Shimeno gives the Plancherel decomposition for the L^2-space of sections of a homogeneous line bundle over D. There appear finitely many discrete parts in the decomposition; they are also called relative discrete series. It is proved in [15] that all the relative discrete parts are G-equivalent to holomorphic discrete series by identifying the infinitesimal character. For the unit ball in $\mathbb{C}^n$ this was proved also in [20] by explicit calculations. On the other hand the holomorphic discrete series have their standard module as weighted Bergman spaces of holomorphic functions on D. Thus it is of interest to find the explicit intertwining operators from the relative discrete series into the holomorphic discrete series. For the unit disk this is done in [19] via the holomorphic differential operator $(\frac{\partial}{\partial z})^l$ and the Bol's lemma, which asserts that

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the operator intertwines the actions of $G = SU(1,1)$ on some two line bundles over the unit disk. Later we realized that the those intertwining operators can also be constructed via the invariant Cauchy-Riemann operator, and consequently we find the intertwining operators for the unit ball in [13] and a general bounded symmetric domain in [21]; see also [12] for the case of the Riemann sphere.

The L^2-space of sections of the line bundle can be realized as a functions on the domain D. The corresponding L^2 space is then a weighted L^2-space on D. The corresponding weighted Bergman space of holomorphic functions on D is one of the relative discrete series, whereas the other relative discrete series consist of non-holomorphic scalar-valued functions on D. It is proved in [15] that they are G-equivalent to a holomorphic discrete series with the highest weight being irreducible representations of K in the symmetric tensors of the tangent space of D. Now some of those representations of K are one-dimensional, namely those corresponding to the Jordan determinant representation. In this present note we find the intertwining operator for the corresponding relative discrete series via the Cayley type operator, which are generalization of the differential operator $\frac{\partial}{\partial z}$.

2. Weighted L^2-space on Bounded symmetric domains

We briefly recall the bounded realization of a Hermitian symmetric space, see [5], and [9].

Let $D = G/K$ be a irreducible bounded symmetric domain of tube type in a complex vector space V of dimension d. The space V has a structure of a Jordan algebra. Let $\Delta(z)$ be the Jordan determinant function. We normalize a Hermitian inner product on V so that a minimal tripotent has norm 1, and denote $dm(z)$ the corresponding Lebesgue measure on V. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$, which is then also a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{+} + \mathfrak{p}^{-}$ be the decomposition of the complexification of $\mathfrak{g}^{\mathbb{C}}$ under adjoint action of the center of $\mathfrak{k}$, with $\mathfrak{p}^{+}$ being identified with the vector space V. We fix an element Z in the center of $\mathfrak{k}$ so that it has eigenvalue $\frac{2}{r}$ on the space $\mathfrak{p}^{+}$. Let $\gamma_1, \dots, \gamma_r$ be the Harish-Chandra strongly orthogonal roots. We fix a system of unit root vectors $\{e_1, e_2, \dots, e_r\}$ in V. We define a K-invariant polynomial $h(z)$ on V by

$$h(c_1 e_1 + \cdots + c_r e_r) = \prod_{j=1}^{r} (1 - |c_j|^2)$$

and let $h(z,w)$ be its polarization, which is holomorphic in z and anti-holomorphic in w. The Bergman reproducing kernel of D is then $ch(z,w)^{-p}$ for some positive constant c, where p is an integer called the genus of D.

Let $\alpha > -1$. We consider the weighted probability measure

$$dm_\nu(z) = C_\alpha h(z)^\alpha dm(z)$$

on D. Here C_α is a normalizing constant whose precise value will not concern us.

There is a unitary representation of G on $L^2(D, d\mu_\alpha)$ given by the formula

$$U_g^{(\nu)} : f(z) \mapsto f(g(z))(J_g(z))^{\frac{\nu}{p}} \qquad (g \in G) \tag{2.1}$$

where J_g stands for the Jacobian of the transformation g and

$$\nu = \alpha + p \tag{2.2}$$

3. Capelli identity

In this section we will use the results of Faraut - Koranyi to prove the following Capelli identity.

Let $\mathcal{P}$ be the space of all holomorphic polynomials on V. By a well known result of Hua and Schmid, the space $\mathcal{P}$ under the action of K is decomposed into irreducible subspaces $\mathcal{P}^{\underline{\mathbf{m}}}$ of signatures $\underline{\mathbf{m}} = m_1\gamma_1 + \cdots + m_r\gamma_r$, with $m_1 \geq \cdots \geq m_r \geq 0$.

Theorem 3.1. *The operator $\Delta(z)^l\Delta(\partial)^l$ acts on each K-space $\mathcal{P}^{\underline{\mathbf{m}}}$ as a scalar*

$$\Delta(z)^l\Delta(\partial)^l f = q_l(\underline{\mathbf{m}})f, \ \ f \in \mathcal{P}^{\underline{\mathbf{m}}},$$

with

$$q_l(\underline{\mathbf{m}}) = \prod_{k=1}^{r}(\frac{a}{2}(r-k) + 1 + m_k - l)_l = (-1)^{rl}\prod_{k=1}^{r}(-m_k - \frac{a}{2}(r-k))_l$$

Proof. For any $f \in \mathcal{P}^{\underline{\mathbf{m}}} \subset \mathcal{P}$,

$$f(z) = \int_{\mathfrak{p}^+} e^{(z,w)} f(w) e^{-(w,w)} dw.$$

We act on this equality by $\Delta(z)^l\Delta(\partial)^l$,

$$\begin{aligned}\Delta(z)\Delta(z)^l\Delta(\partial) f &= \int_{\mathfrak{p}^+} \Delta(z)^l\overline{\Delta(w)^l} e^{(z,w)} f(w) e^{-(w,w)} dw \\ &= \int_{\mathfrak{p}^+} \Delta(z)^l\overline{\Delta(w)^l}\sum_{\underline{\mathbf{m}}'} K_{\underline{\mathbf{m}}'}(z,w) f(w) e^{-(w,w)} dw\end{aligned}$$

Now the map $h(z) \mapsto \Delta(z)^l h(z)$ is, up to a constant, an intertwining maps from $\mathcal{P}^{\underline{\mathbf{m}}'}$ onto $\mathcal{P}^{\underline{\mathbf{m}}'+l}$, thus

$$\Delta(z)^l\overline{\Delta(w)^l} K_{\underline{\mathbf{m}}'}(z,w) = C(\underline{\mathbf{m}}', l) K_{\underline{\mathbf{m}}'+l}(z,w) \tag{3.1}$$

for some positive constant $C(\underline{\mathbf{m}}', l)$. Taking $z = w = e$ and using Lemma 3.1 and Theorem 3.4 in [5] we find

$$\begin{aligned} C(\underline{\mathbf{m}}', l) &= \frac{K_{\underline{\mathbf{m}}'}(e,e)}{K_{\underline{\mathbf{m}}'+l}(e,e)} \\ &= \frac{d_{\underline{\mathbf{m}}'}}{(n/r)_{\underline{\mathbf{m}}'}} \frac{(n/r)_{\underline{\mathbf{m}}'+l}}{d_{\underline{\mathbf{m}}'+l}} \\ &= \prod_{k=1}^{r} (\frac{a}{2}(r-k) + 1 + m'_k)_l \end{aligned}$$

Thus

$$\begin{aligned} \Delta(z)^l \Delta(\partial)^l f &= \int_{\mathfrak{p}^+} \sum_{\underline{\mathbf{m}}'} C(\underline{\mathbf{m}}', l) K_{\underline{\mathbf{m}}'+l}(z,w) f(w) e^{-(w,w)} dw \\ &= \int_{\mathfrak{p}^+} C(\underline{\mathbf{m}} - l, l) K_{\underline{\mathbf{m}}}(z,w) f(w) e^{-(w,w)} dw \\ &= C(\underline{\mathbf{m}} - l, l) f(z), \end{aligned}$$

where in the second last equality we use the fact that $f \in \mathcal{P}^{\underline{\mathbf{m}}}$ and Schur lemma. Thus

$$q_l(\underline{\mathbf{m}}) = C(\underline{\mathbf{m}} - l, l) = \prod_{k=1}^{r} (\frac{a}{2}(r-k) + 1 + m_k - l)_l.$$

This completes the proof. □ □

Remark 3.2. Theorem A has previously proved by Dib [3] and [1]. See also [18]. Our proof above is essentially the same as that in [1].

Remark 3.3. As is shown in [14], Proposition 1.2, the above formula is a reformulation of the main result of [7], which essentially calculates the Laplace transform of $\Delta^{\underline{\mathbf{m}}}$ by using the Gindikin Gamma function. Note that here we are still using a different ordering. (Our γ_j is Sahi's $2\varepsilon_{n-j}$ and $r = n$.)

4. Line bundle over the tube domain

In this section we will use the Capelli identity to realize explicitly some relative discrete series of L^2-space of sections of line bundle over a tube domains.

The irreducible decomposition of the space $L^2(D, \mu_\alpha)$ has been given by Shimeno [15]. It is proved there that all the relative discrete series appearing in the decomposition are holomorphic discrete series. We summarize the result there in the following.

Fix $\alpha > -1$ and let ν be as in (2.2). We define

$$k = \begin{cases} \frac{\alpha+1}{2} - 1 = \frac{\nu-p+1}{2} & \text{if } \alpha \text{ is an odd integer} \\ \left[\frac{\alpha+1}{2}\right] = \left[\frac{\nu-p+1}{2}\right] & \text{otherwise.} \end{cases} \tag{4.1}$$

Here $[t]$ stands for the integer part of $t \in \mathbb{R}$. Denote

$$D_\nu = \{\underline{\mathbf{m}} = \sum_{j=1}^{r} m_j\gamma_j, 0 \le m_1 \le \cdots \le m_r \le l\}$$

For any such Shimeno proved in [15], Theorem 5.10, that a relative discrete series (if any) in $L^2(D, \mu_\alpha)$ are equivalent to a holomorphic discrete series. We reformulate this result in the following.

Theorem 4.1. *(Shimeno [15]) For each $\underline{\mathbf{m}}$ in D_ν there exists a relative discrete series $A^2_{\underline{\mathbf{m}}}(D, \nu)$ appearing in $L^2(D, \mu_\alpha)$, and they are equivalent to holomorphic discrete series of the form with highest weights (under certain ordering of the root spaces of $\mathfrak{g}^{\mathbb{C}}$)*

$$\Lambda\Big|_{(\eta-)^{\mathbb{C}}} = \underline{\mathbf{m}} - \frac{\nu}{2}\sum_{j=1}^{r}\gamma_j, \quad \underline{\mathbf{m}} \in D_\nu$$

and $\Lambda(\eta \cap \mathfrak{k}_s) = 0$, $\Lambda(iZ) = -\nu$ in case $D = G/K$ is non-tube domain.

When $\underline{\mathbf{m}} = (l, \dots, l)$ we write the relative discrete series $A^2_{\underline{\mathbf{m}}}(D, \nu)$ by $A^2_l(D, \nu)$. Our main result is the following, denoting by $L^2_a(D, \nu - 2l)$ the subspace of holomorphic functions, i.e. the standard module for these discrete series.

Theorem 4.2. *With the notation (4.1) we have the operator*

$$S_l : f(z) \mapsto h(z, z)^{-(\nu - l - \frac{d}{r}}\Delta(\partial)^l(h(z, z)^{\nu - l - \frac{d}{r}}f(z))$$

is an intertwining operator $L^2_a(D, \nu - 2l)$ onto $A^2_l(D, \nu)$ for $l = 0, 1, \dots, k$.

To prove the theorem we need the following intertwining property of the Cayley-Capelli operator $\Delta(\partial)$, which was proved by Arazy [2], Theorem 6.4; see also [16], Lemma 7.1 and [8].

Theorem 4.3. *The Cayley-Capelli operator $\Delta(\partial)$ intertwines the action $U^{\frac{n}{r}-1}$ with $U^{\frac{n}{r}+1}$, namely*

$$\Delta(\partial)(J_g(z))^{\frac{\frac{n}{r}-1}{p}}f(gz)) = (J_g(z))^{\frac{\frac{n}{r}+1}{p}}(\Delta(\partial)f)(gz)$$

for holomorphic functions f on D and $g \in G$.

Note that since the operator $\Delta(\partial)$ is a holomorphic differential operator, thus the above result holds for all C^∞-functions f.

With this theorem we can establish the formal intertwining property of of S_l.

Corollary 4.4. *The operator S_l intertwines the action $U^{\nu-2l}$ with the action U^ν of G on C^∞-functions on D.*

Proof. In order to exhibit the intertwining property of the multiplication by $h(z,z)^c$, we introduce the notation (see [11])

$$U^{\nu,\kappa}(g) : f(z) \mapsto f(g(z))(J_g(z))^{\frac{\nu}{p}}(\overline{J_g(z)})^{\frac{\kappa}{p}} \qquad (g \in G)$$

Let T be the operator

$$T_\kappa : f(z) \mapsto f(z)h(z,z)^\kappa.$$

Then by the transformation property of $h(z,z)$ we know that T intertwines the action $U^{\nu,\kappa}$ with $U^{\nu-\kappa,0}$; see Lemma 5 in [11]. Now our operator $S_l = T_{-(\nu-l-\frac{d}{r})}\Delta(\partial)T_{\nu-l-\frac{d}{r}}$. The results follows by the above intertwining properties of the operator T_κ and that of $\Delta(\partial)$ in Theorem 4.3. □

The operator S_l can also be constructed geometrically via the covariant holomorphic differential operator; see [17] and [22] Indeed, consider the holomorphic line bundle on D defined via the action $U^{\nu-2l}$. Let ∇ be the Hermitian connection compatible with the complex structure and $\mathcal{D}$ the holomorphic part, so that $\nabla = \mathcal{D} + \bar{\partial}$. The operator $\mathcal{D}$ maps the line bundle to its tensor product with the holomorphic cotangent bundle; in another word, it maps functions to $V' = \mathfrak{p}^-$-valued functions, after trivializing the bundles. The power $\mathcal{D}^{rl}$ the maps to the symmetric tensor $\otimes^{lr}V'$ of V'. However there is a distinguished K-component in the symmetric tensor, namely the one-dimensional representation with highest weight $l(\gamma_1 + \cdots + \gamma_r)$ (disregarding the center action of K). Let P_l the the orthogonal projection onto the component. Then we have

$$P_l\mathcal{D}^{rl} = cS_l$$

for some non-zero constant c; see Lemma 4.3 in [17] and Lemma 3.2 in [22]. As an intertwining operator from one line bundle to other, the operator S_l maps in particular the spherical functions for the line bundle $U^{\nu-2l}$ to those for U^ν, and thus are the hypergeometric shift operators [6]. So our result gives a geometric construction of some of the Opdam shift operators. It has not been known before that there is a geometric interpretation of the shift operators [10].

We now prove our Theorem 4.2.

Proof. We have established the formal intertwining property of the operator S_l. We prove now that it maps into $A^2_l(D,\nu)$. Note that with the condition on l implies that the weighted space $L^2_a(D,\nu-2l)$ is non-trivial. Take $f \in L^2_a(D,\nu-2l)$ to be the constant function 1. We calculate its image, namely,

$$h(z,z)^{-(\nu-l-\frac{d}{r}}\Delta(\partial)^l(h(z,z)^{\nu-l-\frac{d}{r}})$$

We consider first $\Delta(z)^l\Delta(\partial)^l$. We use the Faraut-Koranyi expansion [5] of the reproducing kernel

$$h(z,w)^{-\kappa}=\sum_{\underline{\mathbf{m}}}(\kappa)_{\underline{\mathbf{m}}}K_{\underline{\mathbf{m}}}(z,w),$$

so that

$$\Delta(z)^l\Delta(\partial)^l h(z,z)^{\nu-l-\frac{d}{r}}=\sum_{\underline{\mathbf{m}}}(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}}\Delta(z)^l\Delta(\partial)^l K_{\underline{\mathbf{m}}}(z,z). \tag{4.2}$$

With the notation in the proof of Theorem 3.1 we have

$$\Delta(z)^l\Delta(\partial)^l K_{\underline{\mathbf{m}}}(z,z)=C(\underline{\mathbf{m}}-l,l)K_{\underline{\mathbf{m}}}(z,z)$$

which vanishes whenever $m_r<l$. We write therefore $\underline{\mathbf{m}}=\underline{\mathbf{m}}'+l$ and each term in the above summation is

(4.3)

$$\begin{aligned}(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}}C(\underline{\mathbf{m}}-l,l)K_{\underline{\mathbf{m}}}(z,z)&=(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}'+l}C(\underline{\mathbf{m}}',l)K_{\underline{\mathbf{m}}'+l}(z,w)\\&=(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}'+l}\Delta(z)^l\overline{\Delta(w)^l}K_{\underline{\mathbf{m}}'}(z,z),\end{aligned}$$

by (3.1). However clearly,

$$(-(\nu-l-\frac{d}{r}))_{\underline{\mathbf{m}}'+l}=(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}(-(\nu-l-\frac{d}{r})+l)_{\underline{\mathbf{m}}'},$$

the summation (4.2) is then

$$\begin{aligned}&(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}\Delta(z)^l\overline{\Delta(z)^l}\sum_{\underline{\mathbf{m}}'}(-(\nu-l-\frac{d}{r})+l)_{\underline{\mathbf{m}}'}K_{\underline{\mathbf{m}}'}(z,z)\\&=(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}\Delta(z)^l\overline{\Delta(z)^l}h(z,z)^{\nu-2l-\frac{d}{r}},\end{aligned} \tag{4.4}$$

where we have used again the Faraut-Koranyi expansion. From this it follows that

$$h(z,z)^{-(\nu-l-\frac{d}{r}}\Delta(\partial)^l(h(z,z)^{\nu-l-\frac{d}{r}})=(-(\nu-l-\frac{d}{r}))_{(l,\dots,l)}\frac{\overline{\Delta(z)^l}}{h(z,z)^l}.$$

which is nonzero, and its $L^2(D,\alpha)$-norm is dominated by

$$\int_D h(z)^{\alpha-2l}dm(z)$$

which is finite since $\alpha-2l>-1$, by our assumption on l. That is the function $S_l f$ is in $A_l^2(D,\nu)$. Now both $L_a^2(D,\nu-2l)$ onto $A_l^2(D,\nu)$ are irreducible representations of G (and of $\mathfrak{g}^{\mathbb{C}}$). Thus the operator S_l is an unitary intertwining onto $A_l^2(D,\nu)$, up to a non-zero constant. □

The above proof actually also implies

Corollary 4.5. *The highest weight vector of $A_l^2(D,\nu)$ is*

$$h(z,z)^{-(\nu-l-\frac{d}{r}}\Delta(\partial)^l h(z,z)^{\nu-l-\frac{d}{r}}=\prod_{j=1}^{r}(-(\nu-\frac{d}{r})-\frac{a}{2}(j-1)+l)_l\frac{\overline{\Delta(z)}^l}{h(z,z)^l}$$

This result has also been proved previously in [4] by using tensor product arguments.

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