A note on Urankar's semi-analytical computation of the Biot-Savart law by elliptic integrals

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Abstract

The problem of three-dimensional integration of the Biot-Savart law, with its singularity $O(|\mathbf{r} - \mathbf{r}'|^{-3})$ in the integrand function, is considered. When the integration domain is a torus with rectangular cross section and arbitrary azimuthal length, a modified semi-analytical Urankar's method is proposed, in which the Biot-Savart law is expressed in terms of elementary functions, Jacobian elliptic functions and complete/incomplete elliptic integrals of the first, second and third kind. Analytical formulas are presented in an adapted form with respect to the original expressions from Urankar, and an optimized computation scheme is suggested, based on a combination of modified Urankar's formulas with one-dimensional numerical integration, efficient for massive computation of the Biot-Savart law on a large number of field points in arbitrary space positions.

Key words - 3D integration, elliptic integrals, Jacobian elliptic functions, finite volume approximation, Biot-Savart law.

1 Introduction

As an important engineering application resorting to mathematical methods, we here consider the problem of large scale magnetic field computation, which leads to the mathematical problem of three-dimensional analytical and numerical integration on complex conductor geometries. The fundamental integral here considered is the Biot-Savart law, frequently used in electromagnetic applications, defining the magnetic field generated in space by imposed currents.

Computation of electromagnetic fields requires a complex analysis and accurate evaluation. In the design of electromagnetic devices and in the prediction of electromagnetic phenomena, the analysis needs to be carried out with accurate computation of source and induced electromagnetic fields, keeping into account several aspects: the physical properties of the materials, the geometric shapes and constraints, the boundary/interface conditions, the magnitude of involved forces, the knowledge of source currents with determination of its critical limits,

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and electromagnetic induction phenomena.

Many electromagnetic problems require to compute the magnetic field due to imposed currents as autonomous physical datum or as a "partial" information embodied inside a differential model representing a specific electromagnetic problem. In fact, mathematical models based on Maxwell's equations, the fundamental differential equations governing all macroscopic electromagnetic phenomena, can in fact lead to (possibly initial) boundary value problems where the source magnetic field appears as a datum in the right hand side contribution of the differential equations or as a quantity in implicit form in the interface and boundary conditions (for instance, see [10]). In this case the source magnetic field needs to be estimated during the initial modelling phase preceding the effective model solution.

The purpose of this work is to focus on Biot-Savart's integral from a mathematical and computational point of view, by suggesting some efficient analytical and numerical methods to compute the magnetic field due to source currents. Another integral expression that can be used is Ampere's circuital law, in which the field definition is given in implicit form. Here, Biot-Savart's integral will be mainly investigated, because of the explicit definition of the source magnetic field, and due to the presence of a singularity of type $O(|\mathbf{r} - \mathbf{r}'|^{-3})$ in the integrand function, which makes the integral interesting from a mathematical point of view. The suggested integration methods keep into account the shape of the conductor geometry. While for complex shaped conductor regions only numerical or semi-numerical integrations are possible, for coils geometries an analytical approach can be used, expressed in terms of elementary functions, Jacobian elliptic functions and complete/incomplete elliptic integrals of the first, second and third kind. The only numerical contribution in this procedure is given by estimation of elliptic integrals, for which efficient algorithms already exist in literature. This "semi-analytical" approach is the kernel of our work.

The report is organized as follows. Section 2 presents an overview of the main integration methods that have been investigated in literature for the computation of Ampere's law and, especially, some numerical integration techniques that can be applied to the Biot-Savart law. Section 3 discusses the case in which the conductor is a coil with rectangular cross section, for which a semi-analytical integration of the Biot-Savart law can be derived, what will be called the *modified semi-analytical Urankar's method*. On this regard, notice that Urankar's method has been already investigated in literature: the purpose of this section is rather to correct some wrong values reported in previous references, to specify values for limit field positions, not originally considered, and to improve the efficiency of the method from a computational point of view, especially for a massive computation on a large number of field points (as it is required, for instance, in finite element computations). Section 4 introduces briefly some ideas and suggestions for a computation algorithm based on the semi-analytical procedure here presented, with notes on possible algorithms for the numerical estimation of elliptic integrals.

To test practically the efficiency of the modified semi-analytical Urankar's scheme, in a future work some examples and numerical results will be presented on significant test cases, for a comparison among several integration techniques of the Biot-Savart law in terms of accuracy and computation time.

2 Numerical integration methods for Ampere's and Biot-Savart laws: an overview

The total magnetic field intensity can be considered as the sum of two different contributions, one generated by imposed source currents and the other due to induced magnetization of the background materials. The rigorous argument is Helmhotz's theorem, which asserts that the magnetic field intensity $\mathbf{H}$ can be always partitioned as

$$\mathbf{H} = \mathbf{H}_s + \mathbf{H}_m, \tag{1}$$

where $\mathbf{H}_s$ is solenoidal (the magnetic field intensity due to prescribed currents), while $\mathbf{H}_m$ is irrotational (the magnetic field intensity due to induced magnetization). The source field $\mathbf{H}_s$ can be computed by the *Biot-Savart law.* Consider an open domain $\Omega \subseteq \mathbb{R}^3$. Given a source current density $\mathbf{J}_s$ defined on a region $\Omega_s \subseteq \Omega$, such law tells that for any point $\mathbf{r} \in \Omega$ the resulting field $\mathbf{H}_s = \mathbf{H}_s(\mathbf{r})$ is given by

$$\mathbf{H}_s = \frac{1}{4\pi} \int_{\Omega_s} \mathbf{J}_s \times \nabla \big(\frac{1}{\mid \mathbf{r} - \mathbf{r}' \mid} \big) \; d\mathbf{r}' \tag{2}$$

where $\mathbf{r}'$ denotes each source point.

Another approach could be to define implicitly the source field by Ampere's circuital law. This one can be easily obtained in its integral form by applying Stoke's theorem to the static source relation $\nabla \times \mathbf{H}_s = \mathbf{J}_s$, giving raise to

$$\oint_\gamma \mathbf{H}_s \cdot d\ell = \int_\mathcal{S} \mathbf{J}_s \cdot d\mathcal{S} \tag{3}$$

for any closed piecewise regular path $\gamma \subseteq \Omega$, being $\mathcal{S}$ a piecewise regular surface such that $\partial \mathcal{S} = \gamma$ [8].

The purpose in the present note is to investigate several numerical and analytical methods that can be used to compute the magnetic field intensity $\mathbf{H}_s$ due to imposed currents, resorting to integration of Ampere's circuital law (3) or the Biot-Savart law (2).

On this regard, notice that a possible drawback in Ampere's law is the implicit form of $\mathbf{H}_s$ definition. However, many methods have been investigated in literature for a numerical estimation of (3). In some cases, the source magnetic field has to be estimated as a modelling datum inside a more extended electromagnetic problem, e.g. a boundary value problem by potential formulation in the frame of a finite element computation. In this case, a typical choice of curves γ and surfaces $\mathcal{S}$ is to use sequences of edges (trees) and faces (loops of faces) of the elements used in the associated finite element mesh. This 'discrete' choice of curves through edge-trees derived from the finite element mesh can then represent a successive advantage because information can be reused for the computation of specific line integrals of $\mathbf{H}_s$ obtained as data for the interface and boundary conditions of Dirichlet type. A description of some techniques for the numerical integration of Ampere's circuital law, as well as the Biot-Savart law, is contained in [14] and references quoted therein.

The present work rather focuses on Biot-Savart's integral, giving an explicit $\mathbf{H}_s$ definition. Expression (2) can, equivalently, be written as

$$\mathbf{H}_s(\mathbf{r}) = \frac{1}{4\pi} \int_{\Omega_s} \mathbf{J}_s \times \frac{\mathbf{r} - \mathbf{r}'}{\mid \mathbf{r} - \mathbf{r}' \mid^3} \quad d\mathbf{r}', \tag{4}$$

where Ω_s is the current carrying region, $\mathbf{r}$ and $\mathbf{r}'$ are any field point and source point, respectively.

Different methods for the computation of the Biot-Savart law have been developed by many authors. Some of them require assumptions and restrictions on the shape of the conductors (e.g. coils, bars, etc. in which currents flow along the direction of a specific curve). Among the numerical approaches, in the so-called *filament approximation* the conductor is a coil whose cross-sectional area is negligible, while in the *sheet approximation*, it is the thickness of the coil to be negligible. These methods can then be applied to conductors having more general shape by splitting them in finite filaments or finite sheets, and mixed approaches can be used according to the local geometry. A more efficient and faster semi-analytical approach to be used with coils, as suggested by Ciric in [9], is the so-called *surface-source approximation*, based on formulas containing only elementary functions, in which the coil is approximated by the union of straight segments having rectangular or polygonal cross sections whose faces along the current directions are trapezoidal. When coils have a big inner radius, the method is faster and more accurate than the above mentioned techniques. On the other hand, when coils have high curvature the segment decomposition has to be chosen finer, with consequent increase of the computational cost and time.

For more general geometries than coils the *finite volume approximation* is suggested, based on subdivision of the domain into small elements (the so-called finite volumes). On each volume domain a numerical integration scheme with a proper order (typically, a Gauss-Legendre quadrature) is applied, and contributions from all the elements are then summed up. If the Biot-Savart law needs to be computed in the context of magnetostatic boundary value problems to be solved by finite element methods (see for instance [10]), the efficiency of this finite volume approach can be tested by choosing as small volumes the elements of the domain triangulation used for the finite element model. In such a case, it is required that the current carrying regions are modelled by the mesh generator, i.e. the triangulation is constrained to the conductor boundary. Anyway, although this method allows to compute the Biot-Savart law for any conductor shape, a possible drawback is that it is quite time and memory consuming when a large number of volumes is used to increase the accuracy of the computed $\mathbf{H}_s$ values. Besides, in the finite volume approximation as well as in all techniques performed by numerical integration, a particular care is required when points $\mathbf{r}$ in field regions approach points $\mathbf{r}'$ in conductor regions, because of the singularity $1/|\mathbf{r}-\mathbf{r}'|^3$ in the integrand function of (4).

In order to overcome these drawbacks of numerical integration techniques, an alternative method is suggested by Urankar [15, 16, 17, 18, 19, 20, 21] based on a sequence of analytical formulas, which are valid for various types of conductor geometries. This method has been fairly investigated in literature: equivalent expressions of Urankar's formulas can be found, for instance, in [3, 11, 2]. Their analytical expression is presented (and corrected) in the next section, for the significant case of a coil conductor having a rectangular cross section.

3 Analytical integration of the Biot-Savart law for coils with rectangular cross section

Current carrying regions in real electromagnetic devices have mainly the shape of coils whose geometry may be made, in general, of finite circular arcs and/or straight segments. The magnetic source field due to the current is then obtained by summing up the partial fields generated by each part. For all the basic coil geometries (circular filaments, cylinders, coils with rectangular or n-sided polygonal cross sections) Urankar has given an analytical representation of the source field $\mathbf{H}_s$ in terms of elementary functions, Jacobian elliptic functions and complete/incomplete elliptic integrals of the first, second and third kind. Urankar's results are here reported for a current carrying region having the shape of torus with rectangular cross section, or an azimuthal restriction of it [17], as they have been used in the magnetostatic models studied in [10]. For implementation purposes, we also suggest a slightly modified semi-numerical version of Urankar's procedure in order to improve the computations when they have to be done on a large number of field points lying in arbitrary position with respect to the coil. Following an analogous procedure, similar formulas can be derived also to compute the magnetic vector potential $\mathbf{A}$ and, moreover, a generalization for coils having n-sided polygonal cross section can be obtained. Details of this generalization are contained in [19].

Consider in (4) an azimuthal source current density $\mathbf{J}_s$, constant in magnitude, whose domain Ω_s is the circular torus with rectangular cross section given by

$$\Omega_s = \{(r', \varphi', z') : \ R_1 \leq r' \leq R_2, \ \varphi_1 \leq \varphi' \leq \varphi_2, \ Z_1 \leq z' \leq Z_2\} \tag{5}$$

being known the radii R_1 and R_2, the angles φ_1 and φ_2, and the heights Z_1 and Z_2 (see Figure 1). For the following let $\mathbf{r}' = (r', \varphi', z')$ denote the generic source point, and $\mathbf{r} = (r, \varphi, z)$ the generic field point. For physical reasons it is always $-\pi \leq \varphi \leq \pi$ and, without any loss of generality, it is possible to assume

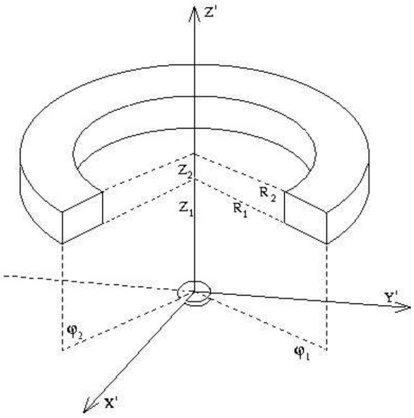

Figure 1: Circular coil with rectangular cross section

$-2\pi \leq \varphi_1 \leq \varphi_2 \leq 2\pi$. Rewriting in cylindrical coordinates, after integration over r' and z', the source magnetic field intensity $\mathbf{H}_s$ can be written as

$$\mathbf{H}_s(r,\varphi,z) = \frac{J_s}{4\pi}\big(\widetilde{\mathbf{H}}(R_2,\varphi,Z_2) - \widetilde{\mathbf{H}}(R_1,\varphi,Z_2) + \widetilde{\mathbf{H}}(R_2,\varphi,Z_1) - \widetilde{\mathbf{H}}(R_1,\varphi,Z_1)\big), \tag{6}$$

where the cylindrical components of $\widetilde{\mathbf{H}}$ are given by

$$\begin{aligned}
\widetilde{H}_r(r,\varphi,z) &= \int_{\vartheta_1}^{\vartheta_2} \cos\vartheta \left(D(\vartheta) + r\cos\vartheta \;\sinh^{-1}\beta_1(\vartheta)\right) d\vartheta \\
\widetilde{H}_\vartheta(r,\varphi,z) &= \int_{\vartheta_1}^{\vartheta_2} \sin\vartheta \left(D(\vartheta) + r\cos\vartheta \;\sinh^{-1}\beta_1(\vartheta)\right) d\vartheta, \\
\widetilde{H}_z(r,\varphi,z) &= \int_{\vartheta_1}^{\vartheta_2} (\gamma \sinh^{-1}\beta_1(\vartheta) - r\cos\vartheta \sinh^{-1}\beta_2(\vartheta) - r\sin\vartheta \arctan\beta_3(\vartheta))\, d\vartheta
\end{aligned} \tag{7}$$

with the definitions

$$\begin{aligned}
&\gamma = z' - z, \quad \vartheta = \varphi' - \varphi, \quad \vartheta_i = \varphi - \varphi_i \quad \text{for } i = 1,2, \\
&B^2(\vartheta) = {r'}^2 + r^2 - 2rr' \cos\vartheta, \qquad D^2(\vartheta) = \gamma^2 + B^2(\vartheta), \\
&G^2(\vartheta) = \gamma^2 + r^2 \sin^2\vartheta, \qquad \beta_1(\vartheta) = (r' - r\cos\vartheta)/G(\vartheta), \\
&\beta_2(\vartheta) = \gamma/B(\vartheta), \qquad \beta_3(\vartheta) = \gamma(r' - r\cos\vartheta)/(r\sin\vartheta D(\vartheta)).
\end{aligned} \tag{8}$$

Notice that in (7) terms do not become singular or indeterminate when $\varphi' = \varphi$. Let us substitute then $\alpha = (\pi - \vartheta)/2$, and $\alpha_i = (\pi - \vartheta_i)/2$. Considering in (7) the presence of odd and even integrands, the source field can be rewritten in the compact form

$$\widetilde{H}_l(r,\varphi,z) = \sum_{i=1}^{2}(-1)^{i+1}(\delta_{l\varphi} + \delta_{lm}\mathrm{sgn}\,\alpha_i)\,\widehat{H}_l(r,|\,\alpha_i\,|,z), \quad l = r,\varphi,z, \tag{9}$$

with $m = l$ if $l = r, z$ and $m \neq l$ if $l = \varphi$, where δ_{ij} denotes the Kronecker symbol. The expression of functions $\widehat{H}_l$ is given in the following.

First, let us introduce the notation for the *incomplete* elliptic integrals of the first, second and third kind, with argument ϑ, modulus k and characteristic n (see [1, 6, 4, 5]). For any real k such that $-1 < k < 1$, they are defined as the functions $E(\vartheta, k)$, $F(\vartheta, k)$ and $\Pi(\vartheta, n, k)$ respectively, such that

$$\begin{aligned} E(\vartheta,k) &= \int_0^{\vartheta} \left(1 - k^2 \sin^2\phi\right)^{1/2} d\phi \\ F(\vartheta,k) &= \int_0^{\vartheta} \frac{1}{\left(1 - k^2\sin^2\phi\right)^{1/2}}\, d\phi \\ \Pi(\vartheta,n,k) &= \int_0^{\vartheta} \frac{1}{(1 - n\sin^2\phi)\left(1 - k^2\sin^2\phi\right)^{1/2}}\, d\phi. \end{aligned} \tag{10}$$

where $E(k) = E(\pi/2, k)$, $F(k) = F(\pi/2, k)$ and $\Pi(n, k) = \Pi(\pi/2, n, k)$ are the so-called *complete* elliptic integral of the first, second and third kind, having argument $\pi/2$ and modulus k. Notice that in the definition of the integrals E, F and Π a notation coherent with Abramowitz and Stegun [1] has been here used, i.e. the sign for the parameter n in the integral $\Pi(\vartheta, n, k)$ is opposite to the one used by Bulirsch [4, 5]. Let us then define $\mathrm{sn}\,u$, $\mathrm{cn}\,u$ and $\mathrm{dn}\,u$ as the three basic Jacobian elliptic functions [1, 12] with amplitude $\mathrm{am}\,u = |\,\alpha\,|$ and $\mathrm{am}\,u_i = |\,\alpha_i\,|$, modulus k being implicit, i.e. such that

$$\begin{aligned} \mathrm{sn}\,u &= \sin\alpha, & \mathrm{sn}\,u_i &= \sin\alpha_i, \\ \mathrm{cn}\,u &= \cos\alpha, & \mathrm{cn}\,u_i &= \cos\alpha_i, \\ \mathrm{dn}\,u &= (1 - k^2\sin^2\alpha)^{1/2}, & \mathrm{dn}\,u_i &= (1 - k^2\sin^2\alpha_i)^{1/2}. \end{aligned} \tag{11}$$

Under these premises, two cases have then to be distinguished. By the initial assumption $-2\pi \leq \varphi' \leq 2\pi$, it follows $-\pi \leq \alpha, \alpha_i \leq 2\pi$. First, let us consider the case $|\,\alpha_i\,| \leq \pi/2$, describing a field point whose azimuthal coordinate lies internally to the azimuthal width of the coil arc. Evidently, if the coil describes a complete angle 2π this condition is fulfilled for all field points. Then, a double integration by parts of (7) is done when the argument is $|\,\alpha_i\,|$. After some trigonometric transformations and rearrangement algebra, it is possible to obtain a form that can be easily rewritten in terms of elliptic integrals and Jacobian elliptic functions, the first ones resulting in fact to be defined in the chosen range for $|\,\alpha_i\,|$. With regard to this, we introduce $c^2 = \gamma^2 + r^2$, $b = r + r'$, $a^2 = \gamma^2 + b^2$, $k^2 = 4rr'/a^2$, and the parameters $n_1 = 2r/(r - c)$, $n_2 = 2r/(r + c)$, and $n_3 = 4rr'/b^2$, together with the function

$$v(k) = \frac{1 + k^2(\gamma^2 - br)}{2rr'}, \tag{12}$$

and define $\theta_i = \mid \alpha_i \mid$. Then, terms $\widehat{H}_l$ in (9) can be expressed as

$$\begin{aligned}
\widehat{H}_r(r, \theta_i, z) &= D_r(\theta_i) + r\Im(\theta_i) - \frac{a}{2r} r'[E(\theta_i, k) - v(k)F(\theta_i, k)] \\
&\quad - \frac{1}{4ar} \sum_{p=1}^{3} (-1)^p Q_r(n_p)\Pi(\theta_i, n_p, k) + 2r \operatorname{sn} u_i \operatorname{cn} u_i \operatorname{dn} u_i \\
\widehat{H}_\varphi(r, \theta_i, z) &= D_\varphi(\theta_i) + 2\gamma\Im(\theta_i) - \frac{a}{2r} \operatorname{dn} u_i (b - 2r \operatorname{sn}^2 u_i) \\
&\quad - \frac{1}{4ar} \sum_{p=1}^{3} (-1)^p Q_\varphi(n_p) I(n_p) \\
\widehat{H}_z(r, \theta_i, z) &= D_z(\theta_i) + 2\gamma\Im(\theta_i) - \frac{3a}{4r}\gamma k^2 F(\theta_i, k) \\
&\quad - \frac{1}{4ar} \sum_{p=1}^{3} (-1)^p Q_z(n_p)\Pi(\theta_i, n_p, k)
\end{aligned} \tag{13}$$

where

$$I(n_p) = n_p \int_0^{u_i} \frac{\operatorname{sn} u \operatorname{cn} u \, du}{1 - n_p \operatorname{sn}^2 u}, \tag{14}$$

whose analytical representation in terms of Jacobian elliptic functions is given by

$$I(\xi) = \begin{cases}
-\frac{|\xi|^{1/2}}{2(k^2-\xi)^{1/2}} \ln \frac{[(k^2-\xi)^{1/2} - |\xi|^{1/2} \operatorname{dn} u]^2}{1-\xi \operatorname{sn}^2 u}, & \xi < 0, \\
\frac{1}{\operatorname{dn} u}, & \xi > 0,\ \xi = k^2, \\
\frac{\xi^{1/2}}{2(\xi-k^2)^{1/2}} \ln \frac{[(\xi-k^2)^{1/2} + \xi^{1/2} \operatorname{dn} u]^2}{1-\xi \operatorname{sn}^2 u}, & \xi > 0,\ \xi > k^2, \\
-\frac{\xi^{1/2}}{2(k^2-\xi)^{1/2}} \arcsin \frac{2\xi^{1/2} \operatorname{dn} u (k^2-\xi)^{1/2}}{k^2\ |1-\xi \operatorname{sn}^2 u|}, & \xi > 0,\ \xi < k^2,
\end{cases} \tag{15}$$

The other terms appearing in (13) are defined as follows. Defining $\tilde{\beta}_i(\alpha) = \beta_i(\pi - 2\alpha)$ for $i = 1, 2, 3$, we have

$$\Im(\alpha) = \int_0^\alpha \sinh^{-1} \tilde{\beta}_1(\alpha) d\alpha, \tag{16}$$

$$D_l(\alpha) = \begin{cases}
\frac{1}{4} r \sin 4\alpha \sinh^{-1} \tilde{\beta}_1(\alpha) & \\
\frac{1}{4} r \cos 4\alpha \sinh^{-1} \tilde{\beta}_1(\alpha), & l = r, \varphi, z, \\
r \sin 2\alpha \sinh^{-1} \tilde{\beta}_2(\alpha) + r \cos 2\alpha \tan^{-1} \tilde{\beta}_3(\alpha) &
\end{cases} \tag{17}$$

$$Q_l(n_p) = \delta_{pq}[r' - (-1)^p c]R_l(n_p) + \delta_{p3}\delta_{lz}\gamma b(r' - r)n_p, \quad l = r, \varphi, z, \tag{18}$$

with $q = p$, if $p = 1, 2$, and

$$R_l(n_p) = \begin{cases} n_p\gamma^2 c/r \\ (-1)^p(c^2 + \gamma^2), \\ -2\gamma c\, n_p, \end{cases} \quad l = r, \varphi, z. \tag{19}$$

In the computations, since $n_1 \to -\infty$ for field points having $\gamma = 0$, i.e. $z = z'$, consider in (13) the limit values $\lim_{z\to z'} Q_r(n_1(z)) = 0$ and $\lim_{z\to z'} Q_z(n_1(z)) = 0$.

Let us consider now the second case, when $|\alpha_i| > \pi/2$, for which the above elliptic integrals are not defined. The following integration domains are distinguished: $\pi/2 < |\alpha_i| \leq \pi$, $\pi < \alpha_i \leq 3\pi/2$ and $3\pi/2 < \alpha_i \leq 2\pi$. In the two first ranges, we can define $\theta_i = \pi - |\alpha_i|$, while in the last one $\theta_i = 2\pi - \alpha_i$, so that again $0 < |\theta_i| \leq \pi/2$. By splitting the domains and using symmetry properties of the corresponding integrands, one easily gets

$$\begin{aligned} &\widehat{H}_\varphi(|\alpha_i|) = \widehat{H}_\varphi(|\theta_i|), \\ &\widehat{H}_l(|\alpha_i|) = \begin{cases} 2\widehat{H}_l(\pi/2) - \operatorname{sgn}\theta_i\, \widehat{H}_l(|\theta_i|), & \pi/2 < |\alpha_i| \leq 3\pi/2, \\ 4\widehat{H}_l(\pi/2) - \widehat{H}_l(\theta_i), & 3\pi/2 < \alpha_i \leq 2\pi, \end{cases} \end{aligned} \tag{20}$$

for $l = r, z$.

Formulas (13) can be simplified in the axisymmetric case. When the coil has the total azimuthal length 2π radians, the φ-component of $\widehat{\mathbf{H}}$ vanishes, while both its r- and z-components do not depend on the angle φ. Therefore, choosing arbitrarily the value of φ, e.g. $\varphi = 0$, and thus $\alpha_i = \pm\pi/2$ for $i = 1, 2$, and considering that $D_r(\pi/2) = 0$, equations (13) reduce to

$$\begin{aligned} &\widehat{H}_r(r, \pi/2, z) = r\Im(\pi/2) - \frac{a}{2r}r'[E(k) - v(k)F(k)] - \frac{1}{4ar}\sum_{p=1}^{3}(-1)^p Q_r(n_p)\Pi(n_p, k) \\ &\widehat{H}_\varphi(r, \pi/2, z) = 0 \\ &\widehat{H}_z(r, \pi/2, z) = D_z(\pi/2) + 2\gamma\Im(\pi/2) - \frac{3a}{4r}\gamma k^2 F(k) - \frac{1}{4ar}\sum_{p=1}^{3}(-1)^p Q_z(n_p)\Pi(n_p, k) \end{aligned} \tag{21}$$

being $E(k)$, $F(k)$ and $\Pi(n_p, k)$ the complete elliptic integral of the first, second and third kind with argument $\pi/2$ and modulus k, and

$$D_z(\pi/2) = -\pi r/2 \operatorname{sgn}\gamma\,[1 + \operatorname{sgn}(r' - r)]. \tag{22}$$

In case the field point is lying on the z axis, i.e. $r = 0$, the expression of the source field is reduced to

$$\widehat{H}_r(0, \pi/2, z) = \widehat{H}_\varphi(0, \pi/2, z) = 0, \quad \widehat{H}_z(0, \pi/2, z) = \pi \,|\gamma|\, \sinh^{-1}(r'/\gamma). \quad (23)$$

Regarding these last expressions, it has to be remarked that in (22) the term sgn($\pm\pi/2$) (miss-print) has been removed from (22c) in [17], while in (23) $|\gamma|$ has been corrected at place of γ appearing in (23) of [17]. In the computations, moreover, for field points having $\gamma = 0$, i.e. $z = z'$, consider the limit value $\lim_{z \to z'} \widehat{H}_z(0, \pi/2, z) = 0$.

4 Computational aspects

Except for the integral $\Im(\alpha)$ in (16), the formulas presented in Section 3 have all been expressed in terms of elementary functions, Jacobian elliptic functions and elliptic integrals. Since the Jacobian elliptic functions can be reduced to trigonometric functions by (11), also the expression for the integral $I(n_p)$ is known analytically in (15).

Many efficient algorithms exist for the computation of the elliptic integrals of the first, second kind and third kind. Classic approaches, well known from literature, are the Landen transformation for incomplete integrals of first kind, the Bartky transformation for integrals of the second kind and Bulirsch's extension to integrals of the third kind [6, 4, 5]. Another more recent method is the unified Carlson's algorithm valid for all the three integrals, as it is suggested in [13]. An advantage of Carlson's procedure is that the cancellation errors occurring in the previous methods are reduced in a significant way.

Moreover, these formulas have the advantage to be valid on all field points with much greater accuracy than any of the integration methods suggested in Section 2, especially on the critic field regions close to the source regions. For implementation purposes, a particular care has rather to be used in the choice of the algorithm for the computation of the elliptic integrals. A straightforward application of Carlson's algorithm, for instance, even if it is more precise than other methods, turns out to be not defined for special points, corresponding to the limit positions (called *critic curves*) $z = Z_i$ and $r = R_i$, or $z = Z_i$ and $r = r_i^\star$, where $r_i^\star = R_i/(4R_i - 1)$, $i = 1, 2$, for which some of the $\mathcal{R}_c$, $\mathcal{R}_d$, $\mathcal{R}_f$, $\mathcal{R}_j$ functions introduced in Carlson's procedure are not defined [7].

In order to overcome these drawbacks, a semi-numerical modification of Urankar's integrals can be performed for an optimized algorithm when field points (r, φ, z) are in these limit positions, consisting of numerical quadrature of the one-dimensional integrals (7) in relative azimuthal coordinate. Elsewhere, the analytical approach can be used, e.g. resorting to Carlson's procedure for the computation of elliptic integrals. In a future work we intend to discuss some numerical results after implementation of the integration schemes here introduced.

5 Conclusions and future work

The present work deals with the problem of three-dimensional integration of the Biot-Savart law, for the computation of the magnetostatic field generated in space by imposed currents. The subject can be inserted in the frame of typical problems of solution/computation of volume integrals with a complex shaped integration domain or presenting singularities in the integrand function.

Numerical finite volume schemes can be used when the integration domain, i.e. the current carrying region, is complex shaped, e.g. a composite integration based on Gauss-Legendre quadrature on tetrahedral conductor subdomains. When the domain is a torus with rectangular cross section and arbitrary azimuthal length, like in the typical case of coils, the Biot-Savart law leads to analytical expressions that can be written in terms of elementary functions, Jacobian elliptic functions and complete/incomplete elliptic integrals of the first, second and third kind. For computation purposes, a mixed computation scheme is suggested by combining such analytical expressions with one-dimensional numerical integration, depending on field point positions inside and outside current carrying regions. This approach seems to be efficient for problems requiring massive computation on a large number of field points, as it occurs in finite element computations of associated magnetostatic boundary value problems.

A future work intends to present numerical results on significant test cases, for a comparison among several integration methods of the Biot-Savart law here considered, i.e (a) the modified semi-analytical Urankar's method, (b) numerical integration of one-dimensional integrals in relative azimuthal coordinate (e.g. by composite trapezoidal rule), and (c) finite volume approximation with composite three-dimensional Gauss-Legendre quadrature. For a more complete analysis, another interesting task could be to test the efficiency of the modified semi-analytical approach in comparison with a famous method used for coils, Ciric's surface-source approximation. Concerning finite volume integration techniques, a modified "weighted" Gaussian quadrature could be used for the local integration on each subdomain, instead of the classic Gauss-Legendre quadrature. Weights should be chosen in a proper way to handle the $O(\mid \mathbf{r} - \mathbf{r}' \mid^{-3})$ singularity in the integrand function of the Biot-Savart law. Tetrahedra or other subdomain shapes could be tested, choosing them properly adapted to the geometry of the overall conductor domain.

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