On Γ-convergence in anisotropic Orlicz-Sobolev spaces

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0 Introduction

In the study of variational problems in applied mathematics the concept of variational convergence called Γ-convergence has come to be a very important tool. One reason is its compactness properties for general classes of functionals and topologies. In addition almost all other variational convergences follow as consequences of the Γ-convergence. For an introduction to the theory we refer to Dal Maso [5].

In this paper we study Γ-convergence for a class of lower semicontinuous functionals defined on the Orlicz-Sobolev class $W^1L_G(\Omega)$ defined below. There are many advantages of such a development. The analysis in Orlicz-Sobolev spaces uses properties like convexity and growth (Δ_2-property) in such a way that one can obtain variational solutions to larger classes of nonlinear problems than in usual Sobolev spaces, see e.g. [8].

The paper is organized as follows: In Section 1 we give some preliminary results on Γ-convergence and on Orlicz-Sobolev spaces. The main results are presented in Section 2. In particular we prove a Γ-compactness result (Theorem 2.2) for functionals defined on $W^1L_G(\Omega)$. The framework uses the localization method as presented in [5]. We also compare Γ-convergence and convergence of minima (Theorem 2.3 and Theorem 2.4). Section 3 is devoted to the proof of Theorem 2.2 and contains in particular an Orlicz-space version of the fundamental estimate. In Section 4, finally, we give some concluding remarks.

1 Preliminary results

Let X be a topological space and let $\mathcal{N}(x)$ denote the set of all open neighborhoods of $x \in X$. Further, let $\{F_h\}$ be a sequence of functions from X into $\overline{R}$.

Definition 1 The Γ-lower and Γ-upper limits of the sequence $\{F_h\}$ are the functions from X into $\overline{R}$ defined by

$$F'(x) = \Gamma - \liminf_{h\to\infty} F_h(x) = \sup_{\omega\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{z\in\omega} F_h(z)$$

and

$$F''(x) = \Gamma - \limsup_{h\to\infty} F_h(x) = \sup_{\omega\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{z\in\omega} F_h(z),$$

respectively. If these two limits coincide, i.e. if there exists a unique function $F : X \to \overline{R}$ such that

$$F = \Gamma - \liminf_{h\to\infty} F_h(x) = \Gamma - \limsup_{h\to\infty} F_h(x),$$

we say that the sequence $\{F_h\}$ Γ-converges to F.

Remark 1 By the definition its obvious that $\{F_h\}$ Γ-converges to F if and only if

$$\Gamma - \limsup_{h\to\infty} F_h \le F \le \Gamma - \liminf_{h\to\infty} F_h.$$

This means that Γ-convergence and lower semicontinuity are closely related concepts. We have the following sequential characterization of Γ-convergence, see [5], Proposition 8.1:

Theorem 1.1 *Let X be a separable metric space and let $\{F_h\}$ be a sequence of functionals from X into $\overline{R}$. Then*

(i) for every $x \in X$ and for every sequence $\{x_h\}$ converging to x,

$$F'(x) \le \liminf_{h\to\infty} F_h(x_h);$$

(ii) for every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that

$$F'(x) = \liminf_{h\to\infty} F_h(x_h);$$

(iii) for every $x \in X$ and for every sequence $\{x_h\}$ converging to x,

$$F''(x) \le \limsup_{h\to\infty} F_h(x_h);$$

(iv) for every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that

$$F''(x) = \limsup_{h\to\infty} F_h(x_h).$$

Consequently $\{F_h\}$ Γ-converges to a function $F \in X$ if and only if

(v) for every $x \in X$ and for every sequence $\{x_h\}$ converging to x,

$$F(x) \le \liminf_{h\to\infty} F_h(x_h)$$

and

(vi) for every $x \in X$ there exists a sequence $\{x_h\}$ converging to x such that

$$F(x) = \lim_{h\to\infty} F_h(x_h)$$

Moreover, Γ-convergence enjoys the following compactness property, see [5], Theorem 8.5:

Theorem 1.2 *Let X be a separable metric space. Then every sequence $\{F_h\}$ of functionals from X into $\overline{R}$ has a Γ-convergent subsequence.*

We recall that a Young function $A : [0, \infty) \to [0, \infty]$ is a function of the form

$$A(t) = \int_0^t a(x)dx$$

where the function $a : [0, \infty) \to [0, \infty]$ is increasing, left continuous and not identically zero and not identically infinity on the interval $(0, \infty)$.

The Orlicz space $L_A(\Omega)$ is the set of measurable functions f on Ω such that $\|f\|_{A,\Omega} < \infty$, where

$$\|f\|_{A,\Omega} = \inf \left\{ \theta > 0 : \int_\Omega A\left(\frac{|f(x)|}{\theta}\right) dx \leq 1 \right\}$$

(the Luxemburg norm on $L_A(\Omega)$)

A G-function $G : \mathbf{R^m} \to [\mathbf{0}, \infty]$ is a function with the following properties:

(i) $G(0) = 0$;

(ii) $\lim_{|x|\to\infty} G(x) = \infty, \qquad \left[x \in R^m : |x| = \left(\sum_{i=1}^m x_i^2\right)^{1/2}\right]$;

(iii) G is convex

(iv) G is symmetric i.e. $G(-x) = G(x)$, $x \in \mathbf{R^m}$;

(v) the set $G^{-1}(\infty) = \{x \in R^m; G(x) = \infty\}$ is separated from 0;

(vi) G is lower semi-continuous.

Additionally we will assume that G is monotonically increasing in each variable separately, that G and G^* (the convex polar) satisfies Δ_2 condition (this will guarantee that the separability and reflexivity of function spaces defined below, see [9]). The vector valued Orlicz-space $L_G(\Omega)$ is defined as follows:

Let G be a G-function and let Ω be a domain in $\mathbf{R^n}$, let $u_1, u_2, ..., u_m$ be real valued measurable functions defined on Ω and let $u = (u_1, u_2, ..., u_m)$ be a vector valued function. Then, u is said to belong to $L_G(\Omega)$ if there exists a $\lambda > 0$ such that

$$\int_\Omega G(\lambda u(x)) < \infty.$$

The space $L_G(\Omega)$ is equipped with a norm corresponding to the Luxemburg norm given by

$$\|u\|_{G,\Omega} = \inf\left\{\theta > 0 : \int_\Omega G\left(\frac{u}{\theta}\right) dx \leq 1\right\}.$$

There should not be any ambiguity for the same notations $L_A(\Omega)$ and $L_G(\Omega)$ used for Young function and G-function,respectively.

For a G-function G, the complementary function G_+^* is defined by

$$G_+^*(u) = \sup_{v_i \geq 0} (u \cdot v - G(v)),$$

where $u \cdot v = \sum_{i=1}^m u_i v_i$.

Let G be a G-function of $(n+1)$ variables. The anisotropic Orlicz-Sobolev space, denoted by $W^1 L_G(\Omega)$, is defined to be the space of weakly differentiable functions u for which $(u, Du) = (u, D_1 u, D_2 u, ..., D_n u)$ belongs to $L_G(\Omega)$. A norm for the space $W^1 L_G(\Omega)$ is given by

$$\|u\| = \|(u, Du)\|_{G,\Omega}.$$

For further details regarding Orlicz-Sobolev spaces we refer to the monographs [1] and [9].

Given two functions A and B, the notation $A \prec\prec B$ means that for every $\lambda > 0$

$$\lim_{t\to\infty} \frac{A(t)}{B(\lambda t)} = 0.$$

Let us recall the following imbedding result (see [7]).

Theorem 1.3 *Let Ω be a bounded domain in $\mathbf{R^n}$ with the cone property, let f be a continuous non-negative function on $[0, \infty)$ and let G be a G-function of $(n+1)$ variables on $[0, \infty)$ such that $G_+^*(0, f(s), f(s), ...f(s)) \leq s$. Furthermore, let A be a Young function given by*

$$A^{-1}(|t|) = \frac{1}{\eta} \int_0^{|t|} \frac{f^{-1}(s)}{s^{1/n}} ds$$

for some constant $\eta > 0$. If B is a Young function such that $B \prec\prec A$, then $W^1 L_G(\Omega)$ is compactly imbedded in $L_B(\Omega)$.

2 The main results

Let the function G be defined as above and let us define G_0 and B as

$$G_0(\xi_1,\ \xi_2,\dots,\ \xi_n) = G(0,\ \xi_1,\ \xi_2,\dots,\ \xi_n)$$

and

$$B(u) = G(u,\ u,\dots,\ u),$$

respectively, where we assume that B satisfies all the hypotheses of Theorem 1.3 above. We have the following compactness result:

Theorem 2.1 *Suppose that G satifies the Δ_2-condition. Then every sequence of functionals $F_h : L_B(\Omega) \to \overline{R}$ has a $\Gamma(L_B)$-convergent subsequence.*

Proof Since G satifies the Δ_2-condition, $L_B(\Omega)$ is separable, see e.g. Kufner et. al. [9], and thus the result follows from the compactness Theorem 1.2 above.

Let us now define the space $\mathcal{M} = \mathcal{M}(c,\beta)$ of Caratheodory functions $f : \Omega \times R^n \to [0,\ +\infty)$ satisfying the conditions:

(1) $f(x,\xi)$ is convex in ξ.

(2) $G_0(\xi_1,\dots,\xi_n) \leq f(x,\xi) \leq c(1 + G_0(\xi_1,\dots,\xi_n))$.

(3) G satisfies the Δ_2-condition with constant β.

Let us also define the class $\mathcal{F}(\mathcal{M})$ of functionals $F : L_B(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty)$ given by

$$F(u,A) = \int_A f(x, Du(x))dx,$$

for $f \in \mathcal{M}$ and $A \in \mathcal{A}(\Omega)$, where $\mathcal{A}(\Omega)$ denotes the family of all open subsets of Ω. We extend in the usual way the functionals to $+\infty$ on $L_B(\Omega) \setminus W^1L_G(\Omega)$.

The main objective is now to establish a result which says that the Γ-limit of a sequence

$$F_h(u, A) = \int_A f_h(x, Du(x))dx,$$

in $\mathcal{F}(\mathcal{M})$ has an integral representation

$$F_0(u, A) = \int_A \varphi(x, Du(x))dx, \tag{2.1}$$

where also $\varphi \in \mathcal{M}$.

The main result of this paper is the following compactness result:

Theorem 2.2 *For every sequence $\{F_h\}$ in $\mathcal{F}(\mathcal{M})$ there exists a subsequence $\{F_{h_k}\}$ and a functional $F_0 \in \mathcal{F}(\mathcal{M})$ such that $F_{h_k}(\cdot, A)$ $\Gamma(L_B)$-converges to F_0 for every $A \in \mathcal{A}(\Omega)$.*

Remark 2 $F_0(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$ and moreover, the local property of the Γ-limit shows that in the integral representation (2.1) the function $\varphi \in \mathcal{M}$ is independent of A.

Remark 3 By the definition of Γ-convergence it easily follows that

(i) F_0 is lower semicontinuous.

(ii) If H is continuous, then

$$F_0 + H = \Gamma(L_B) - \lim_h F_h + H.$$

Theorem 2.2 will be proven in the next section. We end this section by giving examples of the relationship between Γ-convergence and convergence of minima. Let F_h and F

belong to $\mathcal{F}(\mathcal{M})$ and let $H : L_B(\Omega) \to R$ be a continuous functional with the property that there exist some constants $c > 0$ and $b \in R$ such that

$$H(u) \geq c \int_\Omega B(u(x))dx - b \tag{2.2}$$

for all $u \in L_B(\Omega)$. Let us put

$$m_h = \inf_{u \in W^1 L_G(\Omega)} \{F_h(u) + H(u)\} \tag{2.3}$$

and

$$m = \inf_{u \in W^1 L_G(\Omega)} \{F(u) + H(u)\} \tag{2.4}$$

Theorem 2.3 *If $\{F_h\}$ Γ-converges to F in $L_B(\Omega)$ then m_h converges to m.*

Proof We recall that for any topological vector space X it holds that

$$\min_{x \in X} F(x) = \lim(\inf_{x \in X} F_h(x)) \tag{2.5}$$

whenever $\{F_h\}$ is a X-equi-coercive sequence of functionals which $\Gamma(X)$-converges to F (see e.g. [5] Theorem 7.8). The minima in (3) and (4) can be taken over $L_B(\Omega)$ instead of $W^1 L_G(\Omega)$. Moreover, by Remark 3, $\{F_h + H\}$ Γ-converges to $\{F + H\}$ in $L_B(\Omega)$. It holds that

$$F_h + H \geq k_1 \Psi - k_2$$

for some positive constants k_1 and k_2, where

$$\Psi(u) = \begin{cases} \int_\Omega G(u(x), Du(x))dx & \text{if } u \in W^1 L_G(\Omega) \\ +\infty & \text{otherwise} \end{cases} .$$

This follows from the fact that

$$\begin{aligned} G(u(x), Du(x)) &= G(\frac{1}{2}(2u(x)) + \frac{1}{2}0, \frac{1}{2}\overrightarrow{0} + \frac{1}{2}(2Du(x))) \leq \\ &\overset{convexity}{\leq} \frac{1}{2}G(2(u(x), 0, ..., 0)) + \frac{1}{2}G(2(0, Du(x))) \\ &\leq \beta(G((u(x), 0, ..., 0)) + G((0, Du(x)))) \\ &\overset{G\ increasing}{\leq} c(G((u(x), ..., u(x))) + G_0(Du(x))) \end{aligned}$$

Moreover, we observe that $\Psi(u) \leq 1$ if $\|u\| \leq 1$ (by the definition of the Luxenburg norm) and that $\|u\| \leq \Psi(u)$ if $1 < \|u\|$ (use that by the definition of the Luxenburg norm and by convexity $1 < \Psi(\frac{u}{\theta}) \leq \theta^{-1}\Psi(u)$ for all $1 < \theta < \|u\|$). Thus, the set $\{u : \Psi(u) \leq t\}$ is bounded in $W^1L_G(\Omega)$ for all $t > 0$. Moreover, by the imbedding result Theorem 1.3, it holds that $\overline{\{u : \Psi(u) \leq t\}}$ is compact in $L_B(\Omega)$ which implies that the sequence $\{F_h + H\}$ is equi-coercive in $L_B(\Omega)$. Consequently we obtain that $m_h \to m$ by replacing X by $L_B(\Omega)$, F_h by $F_h + H$ and F by $F + H$ in (5).

Theorem 2.4 *Assume that all hypoteses are satisfied as in Theorem 2.3 except that 2 is replaced by the assumption that there exists a bounded set K in $W^1L_G(\Omega)$ such that*

$$\inf_{u\in W^1L_G(\Omega)} \{F_h(u) + H(u)\} = \inf_{u\in U} \{F_h(u) + H(u)\}$$

for all h. Then, if $\{F_h\}$ Γ-converges to F in $L_B(\Omega)$ it holds that m_h converges to m.

Proof We recall that for any topological vector space X it holds that

$$\min_{x\in X} F(x) = \lim(\inf_{x\in X} F_h(x)) \tag{2.6}$$

whenever $\{F_h\}$ $\Gamma(X)$-converges to F and there exists a compact set K such that

$$\inf_{x\in X} \{F_h(u)\} = \inf_{x\in K} \{F_h(u)\}$$

for all h (see [5] Theorem 7.4.). Minimizing over $X = L_B(\Omega)$, and $K = \overline{U}$ and replacing F_h by $F_h + H$ and F by $= F + H$ in 6 we therefore obtain the desired result.

3 Some results related to Theorem 2.2 and its proof

The proof of Theorem 2.2 will be divided into a number of lemmas. Inspired by the pedagogical presentation in Dal Maso [5] we will establish the result by using localization and by proving that functionals $F \in \mathcal{F}(\mathcal{M})$ satisfies the fundamental estimate in Orlicz-Sobolev spaces. A neccesary condition for the integral representation (2.1) is that $F_0(u,\cdot)$

is a measure. For this purpose we introduce increasing set functions:

Definition A set function $\sigma : \mathcal{A}(\Omega) \to [0,+\infty]$ is called

(i) an increasing set function if $\sigma(\emptyset) = 0$ and $\sigma(A_1) \le \sigma(A_2)$ for $A_1 \subset A_2$.

(ii) subadditive if

$$\sigma(A_1 \cup A_2) \le \sigma(A_1) + \sigma(A_2),$$

for all $A_1,\ A_2 \in \mathcal{A}(\Omega)$.

(iii) superadditive if

$$\sigma(A_1 \cup A_2) \ge \sigma(A_1) + \sigma(A_2),$$

for all $A_1,\ A_2 \in \mathcal{A}(\Omega)$ with $A_1 \cap A_2 = \emptyset$.

(iv) inner regular if

$$\sigma(A) = \sup\{\sigma(B) :\ B \in \mathcal{A}(\Omega),\ B \subset\subset A\},$$

for all $A \in \mathcal{A}(\Omega)$.

Lemma 3.1 *Let $\sigma : \mathcal{A}(\Omega) \to [0,+\infty]$ be an increasing set function. The following statements are equivalent:*

(1) σ is is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on Ω;

(2) σ is subadditive, superadditive and inner regular;

(3) the set function

$$\nu(E) = \inf\{\sigma(A) :\ A \in \mathcal{A}(\Omega),\ E \subseteq A\}$$

is a Borel measure on Ω.

Proof See e.g. [5], Theorem 14.23.

We will use the properties of increasing set functions to obtain the integral representation of the Γ-limit F_0. We begin with

Lemma 3.2 *Let $\{F_h\}$ be a sequence of functionals in $\mathcal{F}(\mathcal{M})$. Suppose that for every $u \in W^1L_G(\Omega)$*

$$\sigma'(A) = \Gamma(L_B) - \liminf_h F_h(u, A)$$

and

$$\sigma''(A) = \Gamma(L_B) - \limsup_h F_h(u, A)$$

define inner regular increasing set functions. Then there exists a subsequence $\{F_{h_k}(u, A)\}$ which $\Gamma(L_B)$-converges for all $u \in W^1L_G(\Omega)$ and $A \in \mathcal{A}(\Omega)$.

Proof Consider the countable family $\mathcal{R}$ of all finite unions of open rectangles of Ω with rational vertices. For every fixed sequence $\{F_h\}$ we can use a diagonal procedure and Theorem 2.1 to extract a subsequence $\{F_{h_k}(u, R)\}$ which $\Gamma(L_B)$-converges for all $R \in \mathcal{R}$ and $u \in W^1L_G(\Omega)$. Now, let $A \in \mathcal{A}(\Omega)$ and $u \in W^1L_G(\Omega)$. By hypothesis $\sigma'(A)$ and $\sigma''(A)$ define inner regular increasing set functions. This gives

$$\begin{aligned}
&\Gamma(L_B) - \liminf_h F_{h_k}(u, A) = \sigma'(A) = \\
&= \sup\{\sigma'(B) : \; B \in \mathcal{A}(\Omega), \; B \subset\subset A\} \\
&= \sup\{\sigma'(R) : \; R \in \mathcal{R}(\Omega), \; R \subset\subset A\} \\
&= \sup\{\sigma''(R) : \; R \in \mathcal{R}(\Omega), \; R \subset\subset A\} \\
&= \sup\{\sigma''(B) : \; B \in \mathcal{A}(\Omega), \; B \subset\subset A\} \\
&= \sigma''(A) = \Gamma(L_B) - \limsup_h F_{h_k}(u, A).
\end{aligned}$$

We proceed by proving a fundamental estimate in L_B which will guarantee that the Γ-limits define inner regular increasing set functions.

Definition We say that F satisfies the L_B-fundamental estimate if for every A, A' and B in $\mathcal{A}(\Omega)$ with $A' \subset\subset A$ and $\alpha > 0$ there exists $M_\alpha > 0$ such that for all u, $v \in W^1L_G(\Omega)$ there exists a cut-off function ψ between A' and A such that

$$F(\psi u + (1-\psi)v, A' \cup B) \leq (1+\alpha)(F(u,A) + F(v,B)) + M_\alpha \int_{(A\cap B)\setminus A'} B(u-v)\,dx + \alpha.$$

Remark 4 Let A, $A' \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$. We say that ψ is a cut-off function between A' and A if ψ is smooth with compact support in A, $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on A'.

Lemma 3.3 *The class $\mathcal{F}(\mathcal{M})$ satisfies the L_B-fundamental estimate uniformly.*

Proof Let $F \in \mathcal{F}(\mathcal{M})$ and let A, A' and B in $\mathcal{A}(\Omega)$ with $A' \subset\subset A$. Define

$$\delta = \operatorname{dist}(A', \partial A)$$

and take $0 < \eta < \delta$ and $0 < r < \delta - \eta$. Let ψ be a cut-off function between

$$\{x \in A : \ \operatorname{dist}(x, A') < r\} \ \text{ and } \ \{x \in A : \ \operatorname{dist}(x, A') < r + \eta\},$$

with $|D\psi| \leq 2/\eta$. Define the sets

$$B_r^\eta = \{x \in B : \ r < \operatorname{dist}(x, A') < r + \eta\},$$

$$I_1 = \{x \in B : \operatorname{dist}(x, A') \geq r + \eta\}$$

and

$$I_2 = \{x \in A' \cup B : \operatorname{dist}(x, A') \leq r\}.$$

For u, $v \in W^1L_G(\Omega)$ a repeated use of the convexity and the Δ_2-property of G yield

$$F(\psi u + (1-\psi)v, A' \cup B)$$

$$
\begin{aligned}
&= \int_{A'\cup B} f(x, \psi Du + (1-\psi)Dv + (u-v)D\psi)dx \\
&= \int_{I_1} f(x, Dv)dx + \int_{I_2} f(x, Du)dx + \int_{B_r^\eta} f(x, \psi Du + (1-\psi)Dv + (u-v)D\psi)dx \\
&\le F(u, A) + F(v, B) + c\int_{B_r^\eta} (1 + G_0(\psi Du + (1-\psi)Dv + (u-v)D\psi))dx \\
&\le F(u, A) + F(v, B) + c\int_{B_r^\eta} (1 + G_0(2(\frac{1}{2}(\psi Du + (1-\psi)Dv) + \frac{1}{2}(u-v)D\psi)))dx \\
&\le F(u, A) + F(v, B) + c\int_{B_r^\eta} (1 + \beta G_0(\frac{1}{2}(\psi Du + (1-\psi)Dv)) + \frac{1}{2}(u-v)D\psi))dx \\
&\le F(u, A) + F(v, B) + c\int_{B_r^\eta} (1 + \frac{\beta\psi}{2} G_0(Du) + \frac{\beta(1-\psi)}{2} G_0(Dv)) + \frac{\beta^\kappa}{2} G_0((u-v)D\psi/|D\psi|))dx \\
&\le F(u, A) + F(v, B) + \frac{c\beta}{2}\int_{B_r^\eta} (1 + G_0(Du) + G_0(Dv))dx + \frac{c\beta^\kappa}{2}\int_{(A\cap B)\setminus A'} G_0((u-v)D\psi/|D\psi|)dx \\
&\le F(u, A) + F(v, B) + \frac{c\beta}{2}\int_{B_r^\eta} (1 + G_0(Du) + G_0(Dv))dx + \frac{c\beta^\kappa}{2}\int_{(A\cap B)\setminus A'} B(u-v)dx
\end{aligned}
$$

where $\kappa = 1 - \frac{\log\eta}{\log 2}$. Now define

$$\mu(U) = \frac{c\beta}{2}\int_U (1 + G_0(Du) + G_0(Dv))dx.$$

By the structure conditions

$$\mu(A\cap B) \le \frac{c\beta}{2}(m(A\cap B) + F(u, A) + F(v, B)).$$

Moreover, for every $N = 1,\ 2, \ldots$,

$$\mu(A\cap B) \ge \sum_{k=1}^{N} \mu(\{x \in B : \delta\frac{k-1}{N} < \mathrm{dist}(x, A') < \delta\frac{k}{N}\}).$$

Consequently, for every $N = 1,\ 2, \ldots$ there exists $k \in \{1, \ldots,\ N\}$ such that

$$\mu(\{x \in B : \delta\frac{k-1}{N} < \mathrm{dist}(x, A') < \delta\frac{k}{N}\}) \le \frac{c\beta}{2N}(m(A\cap B) + F(u, A) + F(v, B)).$$

Hence, for fixed $\alpha > 0$, by chosing

$$N \geq \frac{1}{\alpha} \max\{\frac{c\beta}{2} m(A \cap B), \frac{c\beta}{2}\}, \;\; \eta = \frac{\delta}{N} \;\text{ and }\; r = \frac{k-1}{N}\delta,$$

we obtain

$$M_\alpha = \frac{c\beta^\kappa}{2},$$

which depends only on A, A', B, c and β and can thus be chosen uniformly in the class $\mathcal{F}(\mathcal{M})$.

In the next two lemmas we apply the fundamental estimate to show that the Γ-limits satisfies the measure properties subadditivity and inner regularity.

Lemma 3.4 *Let $\{F_h\}$ be a sequence in $\mathcal{F}(\mathcal{M})$ which satisfies the L_B-fundamental estimate as $h \to \infty$. Then*

$$F'(u, A' \cup B) \leq F'(u, A) + F''(u, B)$$

and

$$F''(u, A' \cup B) \leq F''(u, A) + F''(u, B),$$

for all $u \in W^1L_G(\Omega)$ and A, A' and B in $\mathcal{A}(\Omega)$ with $A' \subset\subset A$.

Proof By Theorem 1.1 there exists two sequences $\{u_h\}$ and $\{v_h\}$ converging to u strongly in $L_B(\Omega)$ such that

$$F'(u, A) = \liminf_h F_h(u_h, A)$$

and

$$F''(u, B) = \limsup_h F_h(v_h, B).$$

If we now apply the L_B-fundamental estimate as $h \to \infty$ to the functions u_h and v_h with fixed $\alpha > 0$, there exist M_α and h_α such that for all $h > h_\alpha$ there exists a sequence of functions

$$w_h = \psi_h u_h + (1 - \psi_h) v_h,$$

where ψ_h are cut-off functions between A' and A such that

$$F_h(w_h, A' \cup B) \leq (1+\alpha)(F(u_h, A) + F(v_h, B)) + M_\alpha \int_{(A\cap B)\setminus A'} B(u_h - v_h)\, dx + \alpha,$$

Now $w_h \to u$ in $L_B(\Omega)$. Moreover, since convergence in $L_B(\Omega)$ implies B-mean convergence, see e.g. Kufner et. al. [9], p. 157, it follows that

$$\int_{(A\cap B)\setminus A'} B(u_h - v_h)\, dx \to 0.$$

Consequently,

$$\begin{aligned} F'(u, A' \cup B) &\leq \liminf_h F_h(w_h, A' \cup B) \\ &\leq (1+\alpha)(\liminf_h F_h(u_h, A) + \liminf_h F_h(v_h, B)) + \alpha \\ &= (1+\alpha)(F'(u, A) + F''(v, B)) + \alpha. \end{aligned}$$

Since α can be chosen arbitrarily the first inequality follows. The second inequality is proved the same way.

The last lemma concerns inner regularity of the Γ-limits.

Lemma 3.5 *Let $\{F_h\}$, F' and F'' be defined as in Lemma 3.4. Let $u \in W^1L_G(\Omega)$. If $F'(u,\cdot)$ and $F''(u,\cdot)$ are increasing set functions and if*

$$F''(u, A) \leq \tilde{C} \int_A (1 + G_0(Du))dx,$$

for all $A \in \mathcal{A}(\Omega)$, then $F'(u,\cdot)$ and $F''(u,\cdot)$ are inner regular and moreover $F''(u,\cdot)$ is subadditive.

Proof Since $\{F_h\}$ satisfies the L_B-fundamental estimate the proof follows along the line of Proposition 11.6 in [2], by taking Lemma 3.4 into account.

Proof of Theorem 2.2 We extend as above the functionals to $+\infty$ on $L_B(\Omega) \setminus W^1L_G(\Omega)$. By Lemma 3.3 $\{F_h\}$ satisfies the L_B-fundamental estimate. Therefore, by Lemma 3.5, the Γ-lower and Γ-upper limits define inner regular increasing set functions. Compactness thus follows from Lemma 3.2 and the measure properties again follows from Lemma 3.5 if we take Lemma 3.1 into account.

4 Some final comments and concluding remarks

Theorem 2.2 opens the possibility to find representations of the Γ-limit for large classes of interesting problems. In particular in the periodic case, i.e. when f_h is of the form

$$f_h(x,\xi) = f(hx,\xi),$$

it is possible, with the obvious modifications, to apply classical homogenization methods analogous to those presented in for instance Dal Maso [5]. Moreover, for the case when f_h is of the form

$$f_h(x,\xi) = f(x,hx,..,h^m x,\xi),$$

one can mimic the reiterated homogenization techniques presented in [3] and obtain homogenization results. Similar compactness and homogenization results are clearly also obtainable for corresponding nonlinear parabolic operators by combining the compactness result in this paper with the G-convergence and multi-scale convergence methods described in e.g. [4, 6, 10, 11, 12, 13]. These interesting questions will be discussed in a forthcoming paper.

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