A note on two-scale limits of differential operators

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Abstract

In this note we characterize the two-scale limits of the differential operators curl, div and the time derivative. Analogously as for the gradients we obtain splittings into global and local curls, divergences and time derivatives, respectively.

1 Introduction

The two-scale convergence method introduced by Nguetseng [10] and further developed by Allaire [1] and many other thereafter has proved to be a very powerful and efficient tool in the study of homogenization of partial differential equations. A crucial result is the characterization of the two-scale limit of the gradient. In this note we review this result and prove analogous results for the curl, div and time derivative operators.

Let us consider a bounded sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 1$. By the weak sequential compactness in $L^2(\Omega)$ there exists a subsequence, still denoted $\{u_\varepsilon\}$, such that, as $\varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\varphi(x)\,dx \to \int_\Omega u(x)\varphi(x)\,dx, \tag{1}$$

for all test functions $\varphi \in C_0^\infty(\Omega)$ and we call u the weak limit of the sequence $\{u_\varepsilon\}$. In general the convergence in (1) is not strong, i.e. we do not have

$$\|u_\varepsilon - u\|_{L^2(\Omega)} \to 0, \tag{2}$$

with the only a priori information that $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$. Two typical reasons for this are the presence of oscillations or concentrations in the sequence $\{u_\varepsilon\}$. None of these features are captured in the limit process (1).

Inspired by the ad hok assumption that u_ε admits an asymptotic expansion

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \ldots, \tag{3}$$

where all the u_i's in the power series are assumed to be periodic in the second argument Nguetseng [10] extended the class of test functions to functions with two scales. In order to capture oscillations he considered test functions $\varphi = \varphi(x,y)$ of the class $C_0^\infty(\Omega; C_{per}^\infty(T^n))$, where T^n is the unit torus in $\mathbb{R}^n$. The subindex *per* stands for periodicity (with period T^n). The extension of ususal weak convergence to weak two-scale convergence reads

Definition 1 *A sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ is said to two-scale converge to a function $u_0 = u_0(x,y)$ in $L^2(\Omega \times T^n)$ if*

$$\int_\Omega u_\varepsilon(x)\varphi(x, \frac{x}{\varepsilon})\,dx \to \int_\Omega \int_{T^n} u_0(x,y)\varphi(x,y)\,dydx, \tag{4}$$

for all test functions $\varphi \in C_0^\infty(\Omega; C_{per}^\infty(T^n))$.

In the sequel we will frequently use the notation

$$u_\varepsilon \stackrel{2s}{\rightharpoonup} u_0.$$

The basic compactness result, due to Nguetseng [10], reads

Theorem 1 *For every bounded sequence* $\{u_\varepsilon\}$ *in* $L^2(\Omega)$ *there exist a subsequence and a function* u_0 *such that*

$$u_\varepsilon \stackrel{2s}{\rightharpoonup} u_0.$$

There are various generalizations of Theorem 1. The general versions in Holmbom [7], see also Holmbom, Svanstedt and Wellander [8], do not assume any periodicity and imply Theorem 1 as a special case. A fundamental result for applications to homogenization problems is the characterization of two-scale limits of functions in $H^1(\Omega)$:

Theorem 2 *Assume that* $\{u_\varepsilon\}$ *is a bounded sequence in* $H^1(\Omega)$. *Then*

$$u_\varepsilon \stackrel{2s}{\rightharpoonup} u_0.$$

and

$$\nabla u_\varepsilon \stackrel{2s}{\rightharpoonup} \nabla_x u_0 + \nabla_y u_1.$$

Moreover, $u_0 = u_0(x) \equiv u(x)$, *where* u *is the weak limit in (1) and* $u_1 = u_1(x,y) \in L^2(\Omega; H^1_{per}(T^n))$.

Remark 1 *From Theorem 2 it also follows, see [7] that*

$$\frac{u_\varepsilon - u_0}{\varepsilon} \stackrel{2s}{\rightharpoonup} u_1.$$

Remark 2 *By taking the gradient of (3) and by employing the chain rule one obtains*

$$\nabla u_\varepsilon = \varepsilon^{-1}\nabla_y u_0(x, \frac{x}{\varepsilon}) + \varepsilon^0 \left(\nabla_x u_0(x, \frac{x}{\varepsilon}) + \nabla_y u_1(x, \frac{x}{\varepsilon})\right) + \ldots.$$

These leading order terms are in agreement with the result of Theorem 2. Thus Theorem 2 justifies rigorously the existence of the first two terms in the expansion (3) and also the fact that the leading order term in (3) contains no oscillations for H^1*-functions.*

By introducing more scales in the test functions one can capture more hidden scales in the sequence $\{u_\varepsilon\}$. We recall that a sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ is said to multiscale converge to $u = u(x, y_1, y_2, \ldots, y_n)$ in $L^2(\Omega \times Y_1 \times Y_2 \times \cdots \times Y_n)$ if

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \ldots, \frac{x}{\varepsilon^n})\, dx =$$

$$\int_\Omega \int_{Y_1} \int_{Y_2} \cdots \int_{Y_n} u(x, , y_1, y_2, \ldots, y_n)\varphi(x, , y_1, y_2, \ldots, y_n)\, dx dy_1 dy_2 \cdots dy_n,$$

for all admissible test functions $\varphi \in L^2(\Omega; C^\infty_{per}(Y_1 \times Y_2 \times \cdots \times Y_n))$.

Remark 3 *In [2] Allaire and Briane prove compactness:*
Let $\{u_\varepsilon\}$ be a uniformly bounded sequence in $L^2(\Omega)$. Then there exists a subsequence and a function $u = u(x, y_1, y_2, \ldots, y_n)$ in $L^2(\Omega \times Y_1 \times Y_2 \times \cdots \times Y_n)$ such that u_ε multiscale converges to u.
They also prove the multiscale analogue of Theorem 2:
Let $\{u_\varepsilon\}$ be a uniformly bounded sequence in $H^1(\Omega)$. Then there exist subsequences such that

$$u_\varepsilon \to u_0 = u_0(x)$$

and

$$\nabla u_\varepsilon \to \nabla_x u_0(x) + \nabla_{y_1} u_1(x, y_1) + \nabla_{y_2} u_2(x, y_1, y_2) + \ldots + \nabla_{y_n} u_n(x, y_1, y_2, \ldots, y_n),$$

in the multiscale sense.

Remark 4 *In [8] it is proved that*

$$\frac{u_\varepsilon - u_0 - \varepsilon u_1 - \cdots - \varepsilon^{n-1} u_{n-1}}{\varepsilon^n} \to u_n,$$

in the multiscale sense, c.f. Remark 1.

Remark 5 *More recent work on multiscale convergence and reiterated homogenization for quasilinear elliptic problems can be found in Lions et. al [9] and for parabolic problems in Holmbom, Svanstedt and Wellander [8].*

2 Some vector notations

Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. We consider the curl and div of v defined as

$$\operatorname{curl} v = \nabla \times v = (\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2})$$

and

$$\operatorname{div} v = \nabla \cdot v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.$$

We also define the following function spaces:

$$G := \{v \in L^2(\Omega : \mathbb{R}^3) : v = \nabla \xi,\ \xi \in H^1(\Omega)\},$$

$$H_{curl} := \{v \in L^2(\Omega : \mathbb{R}^3) : \operatorname{curl} v \in L^2(\Omega : \mathbb{R}^3)\},$$

$$H := \{v \in H_{curl} : v = \nabla \times \psi,\ \psi \in H^1(\Omega)\}$$

and

$$H_{div} := \{v \in L^2(\Omega : \mathbb{R}^3) : \operatorname{div} v \in L^2(\Omega)\}.$$

The spaces are H_{curl} and H_{div} are equipped with their graph norms, i.e.

$$\|v\|_{H_{curl}} = \|v\|_{L^2(\Omega:\mathbb{R}^3)} + \|\nabla \times v\|_{L^2(\Omega:\mathbb{R}^3)}$$

and

$$\|v\|_{H_{div}} = \|v\|_{L^2(\Omega:\mathbb{R}^3)} + \|\operatorname{div} v\|_{L^2(\Omega)}.$$

We recall some results on decomposition of the space L^2. It goes back to Helmholtz [6] from 1870 that any smooth vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$ can be decomposed as

$$v = \nabla \times \Psi + \nabla \Phi, \tag{5}$$

In fact by considering the vector Laplacian

$$\Delta \psi = \nabla(\operatorname{div} \psi) - \nabla \times (\nabla \times \psi)$$

the decomposition in (5) is valid with the choice

$$\Psi = \nabla \times \psi \text{ and } \Phi = \operatorname{div} \psi.$$

for some vector potentials Ψ, $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ and a scalar potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$.

The following Hodge decompositions of L^2 holds, see e.g. Cessenat [4]:

Lemma 1

$$L^2(\Omega : \mathbb{R}^3) = \{v \in H_{div} : \operatorname{div} v = 0\} \oplus G. \tag{6}$$

$$L^2(\Omega : \mathbb{R}^3) = H \oplus \{v \in H_{curl} : \nabla \times v = 0\}. \tag{7}$$

For more results on decompositions we refer the reader to e.g. Galdi [5] and the references therein.

Remark 6 *In the proof of Theorem 2 one uses the orthogonal decomposition (6) and below we will use the orthogonal decomposition (7) in the splitting of the curl in Theorem 3.*

3 The Main Results

Theorem 3 *Assume that $\{u_\varepsilon\}$ is a bounded sequence in H_{curl}. Then,*

$$u_\varepsilon \overset{2s}{\rightharpoonup} u_0(x, y),$$

$$\nabla \times u_\varepsilon \overset{2s}{\rightharpoonup} \nabla_x \times u_0(x, y) + \nabla_y \times u_1(x, y) \tag{8}$$

and

$$\nabla \times u_\varepsilon \rightharpoonup \nabla \times u(x) \tag{9}$$

weakly in $L^2(\Omega : \mathbb{R}^3)$ where

$$u(x) = \int_{T^n} u_0(x, y)\, dy, \tag{10}$$

i.e. the weak limit in $L^2(\Omega : \mathbb{R}^3)$ of the sequence $\{u_\varepsilon\}$. Moreover, $u_0(x, y)$ can be decomposed as

$$u_0(x, y) = u(x) + \nabla_y \Phi(x, y),$$

for some scalar potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$.

Theorem 4 *Assume that* $\{u_\varepsilon\}$ *is a bounded sequence in* H_{div}. *Then,*

$$u_\varepsilon \overset{2s}{\rightharpoonup} u_0(x,y),$$

$$\operatorname{div} u_\varepsilon \overset{2s}{\rightharpoonup} \operatorname{div} u(x) + \operatorname{div}_y u_1(x,y) \tag{11}$$

and

$$\operatorname{div} u_\varepsilon \rightharpoonup \operatorname{div} u(x) \tag{12}$$

weakly in $L^2(\Omega)$ *where*

$$u(x) = \int_{T^n} u_0(x,y)\,dy, \tag{13}$$

i.e. the weak limit in $L^2(\Omega)$ *of the sequence* $\{u_\varepsilon\}$ *and where the two-scale limit* $u_0(x,y)$ *of the sequence* $\{u_\varepsilon\}$ *is of the form*

$$u_0(x,y) = u(x) + \nabla_y \times \Psi(x,y) + \nabla_y \xi(x,y),$$

for some vector potential $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ *and smooth scalar function* ξ *which is harmonic in the variable* y.

Theorem 5 *Assume that* $\{u_\varepsilon\}$ *is a bounded sequence in* $H^1(0,T)$. *Then*

$$\mathrm{d_t}\, u_\varepsilon \overset{2s}{\rightharpoonup} \mathrm{d_t}\, u_0 + \mathrm{d}_\tau\, u_1, \tag{14}$$

where $u_0 = u_0(t)$ *in* $L^2(0,T)$ *and* $u_1 = u_1(t,\tau)$ *in* $L^2(0,T;H^1_{per}(0,T))$.

Remark 7 *The results of Theorems 3, 4 and 5 can be generalized to the case of* N *scales. We refer to [12].*

4 Proofs

Proof of Theorem 3 Since $u_\varepsilon \in H_{curl}$ we can apply Theorem 1 and conclude that there exists a vector field $\chi = \chi(x,y)$ such that

$$\nabla \times\, u_\varepsilon(x) \overset{2s}{\rightharpoonup} \chi(x,y).$$

If we choose test functions $\varphi \in C_0^\infty(\Omega : C_{per}^\infty(T^n : \mathbb{R}^3))$ such that $\nabla_y \times \varphi = 0$ we get

$$\int_\Omega (\nabla \times u_\varepsilon(x)) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_{T^n} (\nabla_x \times u_0(x,y)) \cdot \varphi(x,y)\, dy dx$$

by Stokes theorem and the compact support of φ. This means that

$$(\chi(x,y) - \nabla_x \times u_0(x,y)) \cdot \varphi(x,y) = 0$$

for all test functions $\varphi \in C_0^\infty(\Omega : C_{per}^\infty(T^n : \mathbb{R}^3))$ with $\nabla_y \times \varphi = 0$. According to Lemma 1 we conclude that there exists a function u_1 such that

$$\nabla_y \times u_1(x,y) = \chi(x,y) - \nabla_x \times u_0(x,y)$$

in $L^2(\Omega; L^2_{per}(T^n : \mathbb{R}^3))$. Finally, by taking the limit of

$$\varepsilon \int_\Omega (\nabla \times u_\varepsilon(x)) \cdot \varphi\left(x, \frac{x}{\varepsilon}\right) dx$$

we obtain, by using Stokes' theorem and the compact support of φ,

$$0 = \int_\Omega \int_{T^n} (\nabla_y \times u_0(x,y)) \cdot \varphi(x,y)\; dy dx$$

This implies that

$$u_0(x,y) = \nabla_y \tilde{\Phi}(x,y),$$

for some scalar potential $\tilde{\Phi}(x,y)$. The weak convergence (9), which is proved in Wellander [13], says that

$$\int_{T^n} \chi(x,y)\, dy = \nabla \times u(x),$$

where,

$$u(x) = \int_{T^n} u_0(x,y)\, dy.$$

This means that we can express $u_0(x,y)$ as

$$u_0(x,y) = u(x) + \nabla_y \Phi(x,y),$$

for some scalar potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$. $\square$.

Proof of Theorem 4 In order to prove (11) we first observe that since u_ε belongs to H_{div} we can apply Theorem 1 and conclude that there exists a function $\eta = \eta(x, y)$ such that

$$\operatorname{div} u_\varepsilon(x) \stackrel{2s}{\rightharpoonup} \eta(x, y)$$

Let us now define

$$f(x, y) = \eta(x, y) - \int_{T^n} \eta(x, y)\, dy$$

and consider the Poisson's equation

$$\begin{cases} -\operatorname{div}_y(\nabla_y \phi(x, y)) = f(x, y) & \text{in } \Omega \times T^n, \\ \phi(x, \cdot) \in H^1_{per}(T^n), \quad x \in \Omega. \end{cases} \tag{15}$$

According to the Fredholm alternative, see e.g. [11], (15) has a unique solution since

$$\int_{T^n} f(x, y)\, dy = 0.$$

We also have, see [13], that

$$\int_{T^n} \eta(x, y)\, dy = \operatorname{div} u(x)$$

where

$$u(x) = \int_{T^n} u_0(x, y)\, dy.$$

By defining

$$u_1(x, y) = \nabla_y \phi(x, y)$$

we therefore conclude that

$$\eta(x, y) = \operatorname{div} u(x) + \operatorname{div}_y u_1(x, y).$$

The decomposition of $u_0(x, y)$ follows by taking the limit of

$$\varepsilon \int_\Omega \operatorname{div} u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right)\, dx.$$

We obtain

$$0 = \int_\Omega \int_{T^n} \operatorname{div}_y u_0(x, y) \varphi(x, y)\, dy dx$$

by Gauss' theorem and the compact support of φ. This means that

$$u_0(x,y) = \nabla_y \times \tilde{\Psi}(x,y) + \nabla_y \tilde{\xi}(x,y),$$

for some vector potential $\tilde{\Psi}(x,y)$ and smooth scalar potential $\tilde{\xi}$. If we combine this with the weak limit $u(x)$ we conclude that we can express $u_0(x,y)$ as

$$u_0(x,y) = u(x) + \nabla_y \times \Psi(x,y) + \nabla_y \xi(x,y),$$

for some vector potential Ψ and smooth scalar function ξ which is harmonic in the variable y. □

The arguments in the proof of Theorem 5 are analogous but much simpler and we omit them here.

Remark 8 *The results of Theorem 4 and Theorem 5 are very useful in the homogenization of Maxwell systems, see [13]. Another important application is the homogenization of Navier-Stokes systems in the vector potential - vorticity formulation, see [3].*

References

[1] G. Allaire, *Homogenization and two-scale convergence,* SIAM Journal of Mathematical Analysis, Vol. 23, No.6, (1992), 1482-1518.

[2] G. Allaire and M. Briane, *Multiscale convergence and reiterated homogenization*, Proc. Royal Soc. Edinburgh, Vol. 126, (1996), 297-342.

[3] B. Birnir, P. Newton and N. Svanstedt, *Twoscale asymptotics, vorticity and turbulent transport*, submitted.

[4] M. Cessenat, *Mathematical methods in electromagnetism*, Series on Advances in mathematics for applied sciences, vol. 41, World Scientific, 1996.

[5] G. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations, Vol I*, Springer Tracts in Natural Philosophy, vol. 38, Springer Verlag, 1994.

[6] H. Helmholtz, *Ueber die Theorie der Elektrodynamik. Erste Abhandlung. Ueber die Bewegungsgleichungen der Elektricität fur Ruhende Leitenede Körper*, J. Reine Ang. Math., 72, (1870), 57-129.

[7] A. Holmbom, *Some modes of convergence and their applications to optimal composites design*, Thesis, Luleå University, 1996.

[8] A. Holmbom, N. Svanstedt and N. Wellander *Multiscale convergence and reiterated homogenization of parabolic problems*, manuscript 2001.

[9] J.-L. Lions, D. Lukkassen, L.-E. Persson and P. Wall. *Reiterated homogenization of nonlinear monotone operators.*, Chin. Ann. Math., Ser. B , 22, 1, (2001), 1-12.

[10] G. Nguetseng, *A general Convergence Result For a Functional Related to the Theory of Homogenization,* SIAM Journal of Mathematical Analysis, Vol. 20, No.3, (1989), 608-623.

[11] L. Persson, L-E Persson, N. Svanstedt and J. Wyller, *The homogenization method - An introduction*, Studentlitteratur Publ., 1993.

[12] N. Svanstedt and N. Wellander, *Further results on two-scale limits of differential operators*, manuscript 2001.

[13] N. Wellander, *Homogenization of the Maxwell equations, Case 1: Linear Theory*, Appl Math. 46, (2001), 29-51.