Adaptive finite element analysis of elliptic problems with interface conditions for electromagnetic applications

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Abstract

In the context of variational formulations of elliptic problems, we study a three-dimensional generalized Poisson model with discontinuous diffusion and non-homogeneous mixed Dirichlet-Neumann conditions at the boundary and the interfaces between subdomains. An adaptive Galerkin finite element method is proposed, based on a posteriori error estimates proved in energy norm. As examples of elliptic problems involving interfaces, some electrostatic and magnetostatic cases are presented.

Key words - Elliptic problems, finite element methods, a posteriori error, electro- and magnetostatic models.

1 Introduction

Many problems in science and engineering can be represented in the mathematical form of systems of functional equations with given boundary and/or initial conditions. In most cases, such functional equations are partial or ordinary differential equations, so that evolution models are formulated by *initial boundary value problems* and stationary models by *boundary value problems*, both linear and non-linear [15, 38]. Besides differential models, certain physical phenomena are mathematically described by difference, integral or integro-differential equations. We here focus on differential modelling.

Thinking of the variety of phenomena modelled by ordinary and partial differential equations, typical linear stationary problems are based on Poisson's equation, Helmhotz's equation, equilibrium equations in elasticity, while non-linear examples are given by the Eikonal equation, the p-Laplacian equation, the minimal surface equation, and so on. As typical evolution problems, linear examples come from most diffusion-convection problems, the linear wave equation, the Fokker-Planck equation, Schrödinger's equation (with imaginary coefficient), Maxwell's equations, etc. Non-linear evolution problems are defined, for instance, by general reaction-diffusion equations, conservation laws, the Hamilton-Jacobi equation, Euler's equations for inviscid flow, Navier-Stokes equations for viscous flow, and many more. The above mentioned Helmhotz's

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equation and Schrödinger's equation are, moreover, typical examples of *eigenvalue* equations [18, 27, 34, 35, 42].

A crucial aspect in the formulation of a problem into a differential model, is to guarantee "good" regularity hypotheses on the problem domain and the data, so that the model (i) has a solution, (ii) the solution is unique, and, possibly, (iii) the solution depends on data in a continuous way. The first two conditions define what is called a *well-posed* problem. Besides, if also the third condition is satisfied, the problem is said to be *stable* [35].

Actually, the question of *existence* and *uniqueness* of the solution of a certain differential model is a non-trivial problem and only for a very limited number of "simple" equations on "simple" geometries the solution has been proved to exist and, possibly, to be unique. A basic question arises: which level of regularity is required, to call a certain function a *solution* for the problem under investigation? By definition, given a differential problem of order k, we know that a "classical" solution is required to be at least k times differentiable. Nevertheless, this demand of regularity gives raise to two fundamental problems. First, in most cases, theorems ensuring existence and uniqueness of such regular solutions are not known for the model under consideration. Second, such a strong regularity may even not be necessary in practise, owing to the physical significance of the solution in the real problem associated to the model. To recover real properties of the physics underlying a differential model, e.g. certain properties of discontinuity of electromagnetic fields or potentials in the transition across different materials, or shock waves in fluid dynamics processes regulated by conservation laws, we have to investigate a wider class of solutions, called *weak* or *generalized solutions* [18]. Moreover, by demanding weaker regularity requirements, it can be possible to prove more easily that a generalized solution exists, in fact, and is unique.

This is the typical approach followed when a *variational formulation* of the differential problem is introduced. In this case, the equation is transformed into an integral relation in suitable form, in which the unknown solution appears with a lower order of derivative, owing to some typical integral theorems. A weak solution for the current model will be then defined as a function that satisfies such integral relation for an appropriate set of "test" functions. Of course, the definition is given in such a way that any possible classical solution of the original problem is also (weak) solution of the corresponding variational formulation. In many cases, it is then possible to prove existence and uniqueness of the weak solution, resorting to arguments of *Hilbert space theory*, and introducing proper *Sobolev spaces* of functions with generalized derivatives "sufficiently" regular up to a certain order [35].

After the theoretical question of existence and uniqueness of the solution, at least in a weak sense, the interest is then to solve the problem. Since analytical solutions are very rarely obtainable, numerical methods take over analytical ones from giving approximate, *computable* solutions. Thus, these computed solutions will represent an approximation of the investigated physical properties in the real problem under consideration.

We here point out the term *approximation*: no equations describe physical reality exactly, namely the computed solution is subject to *error*. What we call error is, actually, the sum of several independent contributions due to the several phases of modelling, discretization and computation. Regarding this, an interesting development in Computational Mathematics is to measure the several contributions of the error, in order to estimate the accuracy of the numerical solution and suggest criteria to improve it. Later on we will turn back to this important aspect.

Typical numerical methods for the solution of differential models are *finite element*, *finite difference*, *spectral* and *boundary element methods* [33]. Both theoretical and practical reasons arise in the choice of the appropriate numerical technique, depending on the nature of the problem (e.g. the domain geometry, possible a priori known properties of the solution, and so on), but also on the available resources in terms of time and means. Sometimes, still now, constraints of low computational time and cost can force to use simpler discretization schemes at low order (e.g. finite difference methods). Problems on complex-shaped devices, or requiring high precision, however, address towards more robust and accurate numerical methods. Among them, finite element methods [28, 45] have recently become a standard computation tool.

The first finite element techniques were introduced by R. Courant in 1943, developed in the successive years for structural analysis applications in aircraft design (elasticity equations, plate equations, etc.). A solid mathematical background was developed in the 1960s for elliptic problems [13], then extended to parabolic and hyperbolic problems later. From then on, with the coming of automatic computation, this method has had a wide diffusion in many engineering applications: structural and solid state mechanics, fluid dynamics, nuclear engineering, heat conduction, convection-diffusion processes, petroleum engineering, reaction-diffusion processes, electromagnetism, wave propagation, integration circuits, and so on. Concerning models in electromagnetism [12, 25], finite element methods are currently utilized for a vast amount of static, quasi-static and time dependent problems, ranging from high frequency microwaves used in mobile communication to low frequency occurring in power energy [26, 32, 41].

Broadly speaking, finite element methods originate from approximation of a variational form in a discrete finite-dimensional subspace, in which the weak solution is approximated by a linear combination of proper orthogonal basis functions. Sets of basis functions are considered also in spectral methods but, in that case, basis functions have a global definition over all the domain (typically, they are orthogonal polynomials, trigonometric functions, etc.). Differently from the latter, basis functions in finite element methods are assumed to be *piecewise defined* polynomials, satisfying a *local compact support property*. In this case, the basis functions are naturally associated to a discrete set of nodes defined by a partition of the domain in small subregions (the so-called *finite elements*). In one-dimensional problems, elements are intervals. Typical element shapes in two dimensions are triangles and rectangles and, in three-dimensions, tetrahedra and bricks. Mixed shapes could also be considered in different regions of the domain. Besides, space-time elements could also be introduced, to solve initial boundary value problems when variational forms are defined by simultaneous

integration in space and time.

An area of increasing interest is given by *adaptive* finite element methods [3, 4, 5, 14, 30, 31, 44]. Differently from classical techniques, in adaptive methods the improvement of the solution accuracy is reached by means of a local refinement of the mesh, or a local increase of the polynomial order of the finite element solution, performed on a selection of mesh elements having large error indicators. Such indicators are computed from suitable *a posteriori error* estimates of the finite element solution obtained according to many different criteria, and measured by known quantities, namely the numerical solution, the mesh size and model data. An adaptive approach allows to reduce in a significant way the computational cost and time of the finite element calculation on the model.

As a typical differential model encountered in some engineering applications, we here consider a boundary value problem based on a generalized Poisson model with a discontinuous diffusion coefficient and non-homogeneous mixed Dirichlet and Neumann boundary conditions. In addition, further conditions are assigned at the interfaces between pairs of domain subregions. Notice that the problem of interfaces between material domains has not been particularly investigated in the literature. On the contrary, some engineering problems require to model the behaviour of certain physical quantities in the transition across adjacent materials having different properties. A typical application, leading to generalized Poisson's equations with interface conditions, is encountered in electrostatic and magnetostatic problems derived from Maxwell's equations. Our purpose, here, is to present an adaptive finite element analysis for this class of elliptic problems, in the abstract setting of Sobolev spaces, with a posteriori error estimation of the finite element solution.

The report is organized as follows. Section 2 defines the class of generalized Poisson models with discontinuous diffusion and mixed boundary and interface conditions of Dirichlet and Neumann type. As particular cases of the abstract model, some applications are presented in Section 3, describing electrostatic and magnetostatic problems derived from static Maxwell's equations when scalar potential formulations are used. The variational formulation of the model is then discussed in Section 4, and existence and uniqueness of the weak solution are proved by using the Lax-Milgram theorem. In Section 5, a Galerkin finite element solution is proposed, considering a piecewise linear discretization on tetrahedral mesh. Then, a formal error analysis follows. In Section 6, modelling and computational errors are first discussed. Focusing on the computational error due to the finite element solution, a priori and a posteriori approaches are discussed. In Section 7, finally, a posteriori error estimates are proved in energy norm, as quantitative indicators of local accuracy of the finite element solution to be used for adaptivity refinement purposes.

A possible algorithm of adaptive mesh refinement, based on the a posteriori error here introduced, will be discussed in [24], with a presentation of numerical results on some test cases.

2 The model problem

We first introduce in its abstract framework the class of boundary value problems studied in the present work.

Notation. In the sequel, the symbols $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural (0 included) numbers, respectively. C denotes a real positive constant, different at each occurrence. For any domain Ω in the Euclidean space $\mathbb{R}^n$, the symbols $\partial\Omega$, $\overline{\Omega}$ and $\dot{\Omega}$ denote the boundary, the closure and the interior part of Ω, respectively.

Let $\Omega_1 \subseteq R^3$ and $\Omega_2 \subseteq R^3$ be two open, disjoint, bounded and connected domains, Lebesgue measurable with Lipschitz continuous boundaries $\partial\Omega_1$ and $\partial\Omega_2$. We assume they are *adjacent*, i.e. the set $\Gamma_I = \partial\Omega_1 \cap \partial\Omega_2$ is a non-empty surface (called *interface*). Let us denote by $\Omega = \overline{\Omega_1 \dot{\cup} \Omega_2}$ the open global domain. We here consider the following class of second order boundary-interface value problems:

$$\begin{cases} -\nabla \cdot (\alpha \nabla u) = f & \text{in } \Omega, \\ u_1 = u_2 + d & \text{on } \Gamma_I, \\ \alpha_1 \nabla u_1 \cdot \mathbf{n}_1 + \alpha_2 \nabla u_2 \cdot \mathbf{n}_2 = p & \text{on } \Gamma_I, \\ u = u_D & \text{on } \Gamma_D, \\ \alpha \nabla u \cdot \mathbf{n} = q & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $\Gamma_D \subseteq \Gamma$ and $\Gamma_I \subseteq \Gamma$ are parts of the boundary $\Gamma = \partial\Omega$, such that $\Gamma_D \cup \Gamma_N = \Gamma$.

Problem (1) is a generalized Poisson model with non-homogeneous mixed Dirichlet-Neumann boundary and interface conditions, in the unknown $u = u(x, y, z)$. $\mathbf{n}_i$ and $\mathbf{n}$ denote the unit vectors normal to Γ_I and Γ pointing towards the exterior of the domains Ω_i, $i = 1, 2$, and Ω respectively, while u_i and α_i denote the values (or limit values) on Γ_I of the functions u and α from the sides of each Ω_i, $i = 1, 2$. The (real-valued) functions $\alpha = \alpha(x, y, z)$ and $f = f(x, y, z)$ defined on Ω, $d = d(x, y, z)$ and $p = p(x, y, z)$ on Γ_I, $q = q(x, y, z)$ on Γ_N and $u = u_D(x, y, z)$ on Γ_D are given data satisfying certain regularity properties. For the following, we assume that the diffusion coefficient $\alpha = \alpha(x, y, z)$ is a positive bounded function on Ω, Lipschitz continuous over each Ω_i, so that the linear Poisson operator $\mathcal{L}u = -\nabla \cdot (\alpha \nabla u)$ in (1) is uniformly elliptic on Ω. Further, we assume that (1) has a unique solution.

Concerning this, our analysis will be continued in a variational sense: in Section 4 it will be proved that a *weak* solution exists, in fact, and is unique. In the next section we will present some applications of the general model (1) to three-dimensional electro- and magnetostatic problems.

3 Applications in electromagnetism

We here focus on some engineering applications of the general problem introduced in Section 2. Typical electromagnetic models, in fact, can be described by boundary value problems (1), in which diffusion is discontinuous, piecewise defined on parts of the domain, and where also interface conditions have to be taken into account. Such electromagnetic models describe in static conditions the behaviour of fields, and associated potential quantities, on materials adjacent to each others and having different diffusion properties. In this case, interface conditions have to be included to model known behaviours of these fields and potentials in the transition between one medium and the other. Formally, the difference between electrostatic and magnetostatic models lies in the analytical expression of the data f, d, p, q and u_D appearing in (1), and their different physical significance related to electric or magnetic properties.

In the following, $\mathbf{E}$ denotes the electric field intensity, $\mathbf{D}$ the electric flux density, $\mathbf{H}$ is the magnetic field intensity and $\mathbf{B}$ is the magnetic flux density. These are the four electromagnetic fields related to each others by Maxwell's equations and constitutive relations. First, we observe that in stationary conditions electric and magnetic properties in Maxwell's equations decouple from each others so that the field equations take the form of two independent partial differential systems: the first one is expressed in terms of fields $\mathbf{E}$, $\mathbf{D}$, the second one in terms of $\mathbf{H}$, $\mathbf{B}$. Electrostatic and magnetostatic problems can then be studied separately, as described in the following. Throughout the document, when omitted, it is always intended that all mentioned material regions are open, bounded and connected sets, with boundaries and interfaces satisfying the"good" properties of regularity specified in Section 2.

A survey of electromagnetic fields can be read in [12, 25]. Concerning mathematical modelling of electromagnetic problems derived from Maxwell's equations, see [26, 32, 41], particularly focusing on finite element techniques.

3.1 Electrostatic models

Let $\Omega \subseteq R^3$ be the region of interest with boundary surface $\Gamma = \partial\Omega$, composed of dielectric materials, all linear, homogeneous and isotropic. The electric permittivity ϵ on Ω is then defined as a piecewise constant positive function, constant on each material. Without any loss of generality, we can assume that the domain Ω is partitioned into two dielectric materials Ω_1 and Ω_2, i.e. $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with interface $\Gamma_I = \partial\Omega_1 \cap \partial\Omega_2$. In the more general case, if several materials are contained in the regions Ω_i, the analysis that follows has to be carried out for each pair of adjacent materials. Besides, for the sake of simplicity, we consider materials not permanently polarized, and assume that no surface charges are imposed at the boundary and the interface.

As known, electrostatic Maxwell's equations are expressed in terms of $\mathbf{E}$ and $\mathbf{D}$, and take the form

$$\nabla \times \mathbf{E} = 0, \tag{2}$$
$$\nabla \cdot \mathbf{D} = \rho, \tag{3}$$

in the whole space, where ρ is the volume charge density. As no surface charges are imposed, at the interface Γ_I between materials Ω_1 and Ω_2 transition conditions for the fields $\mathbf{E}$ and $\mathbf{D}$ are expressed by

$$\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0, \tag{4}$$
$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0, \tag{5}$$

where $\mathbf{n}$ is the unit vector normal to Γ_I whose direction can be arbitrarily chosen, e.g. pointing from Ω_2 to Ω_1, and $\mathbf{E}_i$ ($\mathbf{D}_i$) denote the values (or limit values) of $\mathbf{E}$ ($\mathbf{D}$) on Γ_I from the side of Ω_i, for $i = 1, 2$. Conditions at the boundary Γ have to be given by considering *known* the electromagnetic state of any external medium. This means that fields are given by assignment of their normal or tangential components on the several parts of Γ. For instance, when Ω is surrounded by a perfectly conductor material, we have

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma_D, \tag{6}$$
$$\mathbf{n} \cdot \mathbf{D} = 0 \quad \text{on } \Gamma_N, \tag{7}$$

where Γ_D and Γ_N are parts of Γ such that $\Gamma = \Gamma_D \cup \Gamma_N$. Besides, to describe the macroscopic properties of each dielectric material, we consider the constitutive relation

$$\mathbf{D} = \epsilon \mathbf{E}. \tag{8}$$

Under these premises, since the electric field intensity $\mathbf{E}$ is irrotational in the whole simply connected space, from (2), a differentiable function V can be introduced, called *scalar electric potential*, such that

$$-\nabla V = \mathbf{E}. \tag{9}$$

Using the constitutive relation (8) in (3) and substituting (9), we get the *scalar Poisson's equation*

$$-\nabla \cdot (\epsilon \nabla V) = \rho, \tag{10}$$

describing the behaviour in static conditions of the scalar electric potential inside any dielectric medium, like Ω_1 and Ω_2 in our case. To obtain interface conditions in terms of V, we substitute (9) in (4) and get a Neumann-like condition

$$\epsilon_1 \nabla V_1 \cdot \mathbf{n}_1 + \epsilon_2 \nabla V_2 \cdot \mathbf{n}_2 = 0 \qquad \text{on } \Gamma_I, \tag{11}$$

where $\mathbf{n}_i$ denote the unit vectors normal to Γ_I pointing towards the exterior of the corresponding domains Ω_i, for $i = 1, 2$, and where V_i and ϵ_i denote the values (or limit values) of V and ϵ on Γ_I from the side of Ω_i, for $i = 1, 2$. The second interface condition is obtained by substituting relation (8) and definition (9) in condition (5). We have

$$\nabla (V_1 - V_2) \times \mathbf{n} = 0 \tag{12}$$

on Γ_I, hence

$$\nabla(V_1 - V_2) \cdot \mathbf{t} = 0, \tag{13}$$

where $\mathbf{t}$ is the unit vector tangent to Γ_I. Then, once fixed an arbitrary point $\mathbf{r}_0 \in \Gamma_I$, we integrate over any piecewise regular path $\gamma[\mathbf{r}_0, \mathbf{r}] \subseteq \Gamma_I$ for any point $\mathbf{r} \in \Gamma_I$, and get the following Dirichlet-like condition

$$V_1(\mathbf{r}) = V_2(\mathbf{r}). \tag{14}$$

In a similar way, boundary conditions are obtained from (6) and (7), leading to classical Dirichlet and Neumann conditions on the parts Γ_D and Γ_N, respectively.

Thus, considering all equations together, we obtain the following boundary value problem

$$\begin{cases} -\nabla \cdot (\epsilon_1 \nabla V_1) = \rho_1 & \text{in } \Omega_1, \\ -\nabla \cdot (\epsilon_2 \nabla V_2) = \rho_2 & \text{in } \Omega_1, \\ V_1 = V_2 & \text{on } \Gamma_I, \\ \epsilon_1 \nabla V_1 \cdot \mathbf{n}_1 + \epsilon_2 \nabla V_2 \cdot \mathbf{n}_2 = 0 & \text{on } \Gamma_I, \\ V_i = V_{D,i} & \text{on } \Gamma_{D,i}, \quad i = 1, 2, \\ \epsilon_i \nabla V_i \cdot \mathbf{n} = 0 & \text{on } \Gamma_{N,i}, \quad i = 1, 2, \end{cases} \tag{15}$$

where $\Gamma_{D,i} = \Gamma_D \cap \partial\Omega_i$ and $\Gamma_{N,i} = \Gamma_N \cap \partial\Omega_i$, for $i = 1, 2$. The unknown function in (15) represents the potential V on the global domain, originated from each component potential V_i on the material Ω_i with permittivity ϵ_i, for $i = 1, 2$. Source data of the problem are the volume charge densities ρ_i, as well as values $V_{D,i}$, for $i = 1, 2$, describing assigned potentials on parts $\Gamma_{D,i}$ of the boundary.

Model (15) is the simplest electrostatic problem leading to a generalized Poisson's equation, to be viewed inside the abstract class (1). In this case, the electric permittivity describes the diffusion property of dielectric materials. Here, no surface charges have been imposed at the interface between pairs of materials, so that the transition between potentials at the interface occurs continuously, i.e. $d = 0$ on Γ_I, using the notations of (1). Concerning the interface condition of Neumann type, here we have $p = 0$, since no polarized materials have been considered, as well as $q = 0$, since no parts of Γ_N exist bounding such type of materials.

However, a generalization to materials having permanent polarization $\mathbf{P}$ is straightforward. In this case, a contribution $\nabla \cdot \mathbf{P}$ has to be included in the right hand side of Poisson's equations associated to any permanently polarized material. Besides, terms $\mathbf{P} \cdot \mathbf{n}$ need to be added as right hand sides in the conditions of Neumann type for any interface and boundary part in contact with polarized materials, with $\mathbf{n}$ unit normal vector pointing away from them.

Once the electric potential V has been computed from the solution of model (15), and consequently the fields $\mathbf{E}$ and $\mathbf{D}$ become known, global quantities of physical interest can be then estimated, for example the electric energy. Given

a material domain $\mathcal{C} \subseteq R^3$, the *total electric energy* stored in $\mathcal{C}$ due to $\mathbf{E}$ is defined as

$$W = \frac{1}{2}\int_{\mathcal{C}} \mathbf{D}\cdot\mathbf{E}\, d\mathbf{x} = \frac{1}{2}\int_{\mathcal{C}} \epsilon E^2\, d\mathbf{x}. \tag{16}$$

3.2 Magnetostatic models

The intention is now to investigate the behaviour of magnetic fields in static conditions in a region of space containing prescribed currents. Let $\Omega \subseteq R^3$ be the region of interest with boundary Γ, composed of materials that are soft media, all linear, homogeneous and isotropic, so that the magnetic permeability μ is a piecewise constant positive function, i.e. constant on each material. Let us suppose that there exists a conductor domain $\Omega_s \subseteq \Omega$, also called *source region*, where a current flows with density $\mathbf{J}_s$. Let us partition the domain in two material regions Ω_1 and Ω_2 such that Ω_2 contains Ω_s and Ω_1 is simply connected, and denote by $\Gamma_I = \partial\Omega_1 \cap \partial\Omega_2$ the interface between them. Each of the domains Ω_i can be composed of several materials.

In static conditions, Maxwell's equations for the magnetic fields $\mathbf{H}$ and $\mathbf{B}$ are given by

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{17}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{18}$$

where $\mathbf{J}$ is the free current density, coincident with $\mathbf{J}_s$ in static conditions. In terms of the magnetic fields, the following conditions have then to be considered at the interface Γ_I:

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0, \tag{19}$$

$$\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \tag{20}$$

where $\mathbf{n}$ is the unit vector normal to Γ_I e.g. pointing from Ω_2 to Ω_1. Analogously to the electrostatic case, boundary conditions with a perfectly conductor exterior medium, are given by

$$\mathbf{n} \times \mathbf{H} = 0 \quad \text{on } \Gamma_D, \tag{21}$$

$$\mathbf{n} \cdot \mathbf{B} = 0 \quad \text{on } \Gamma_N, \tag{22}$$

where $\Gamma_D \subseteq \Gamma$, and $\Gamma_N = \Gamma - \Gamma_D$.

Two suitable scalar magnetic potentials can now be introduced in separate parts of the domain. From Helmhotz's theorem, in fact, we know that the magnetic field intensity $\mathbf{H}$ can always be written in the form

$$\mathbf{H} = \mathbf{H}_s + \mathbf{H}_m, \tag{23}$$

where $\mathbf{H}_s$ is solenoidal and represents the magnetic field intensity due to the imposed current $\mathbf{J}_s$, while $\mathbf{H}_m$ is irrotational and describes the magnetic field intensity due to induced magnetization in materials. It is possible then to define a scalar potential φ, called *reduced magnetic potential*, such that

$$\mathbf{H} = -\nabla\varphi + \mathbf{H}_s. \tag{24}$$

The field contribution $\mathbf{H}_s = \mathbf{H}_s(\mathbf{r})$ can be computed separately from $\mathbf{J}_s$, by using the *Biot-Savart law*

$$\mathbf{H}_s(\mathbf{r}) = \frac{1}{4\pi}\int_{\Omega_s} \mathbf{J}_s \times \frac{\mathbf{r}-\mathbf{r}'}{\mid \mathbf{r}-\mathbf{r}' \mid^3} \quad d\mathbf{r}', \tag{25}$$

where $\mathbf{r}$ is an arbitrary point in space (called *field point*) and $\mathbf{r}'$ is any point in the source region Ω_s (called *source point*). As regards this, see [23, 40] for a description of some numerical and analytical techniques for the integration of the Biot-Savart law. Then, considering equation (18) and the constitutive relation

$$\mathbf{B} = \mu\mathbf{H} \tag{26}$$

for soft materials, we get the following *magnetic Poisson's equation*

$$\nabla \cdot (\mu\nabla\varphi) = \nabla \cdot \mu\mathbf{H}_s. \tag{27}$$

On simply connected domains not containing source regions Ω_s, like Ω_1 in our case, it is possible to introduce another definition of scalar potential. Since $\nabla \times \mathbf{H} = 0$ in such domains, a function ϕ can be introduced such that

$$\mathbf{H} = -\nabla\phi, \tag{28}$$

called *total magnetic potential.* Therefore, the governing equations in regions like Ω_1 are given by

$$\nabla \cdot (\mu\nabla\phi) = 0. \tag{29}$$

In a *double potential formulation*, the approach is to use both ϕ and φ. In our assumptions, for instance, the total potential ϕ is defined on the region Ω_1 with no source currents, while φ is defined on the remaining region Ω_2, which is simply connected. For brevity, in the following Ω_1 will be also called the *total domain* and Ω_2 the *reduced domain.* According to this formulation, the interface Γ_I is the surface separating regions with different potential definitions. A possible advantage of this approach, differently from a complete formulation in φ, lies in the reduction of some cancellation errors that would occur in the reduced domain while computing the total field $\mathbf{H}$ by means of (24), especially for large permeabilities μ.

To express boundary/interface conditions in terms of the two potentials ϕ and φ, we use the constitutive relation (26) in (20), and substitute the definitions (28) and (24) of ϕ and φ in (19) and (20). The first condition becomes

$$(\mu_2\nabla\varphi - \mu_1\nabla\phi) \cdot \mathbf{n} = \mu_2\mathbf{H}_s \cdot \mathbf{n} \qquad \text{on } \Gamma_I, \tag{30}$$

where $\mathbf{n}$ is the unit vector normal to Γ_I, whose direction can be arbitrarily chosen (e.g. pointing away from the reduced domain). Further, the second interface condition becomes

$$[\nabla(\varphi - \phi) - \mathbf{H}_s] \times \mathbf{n} = 0. \tag{31}$$

Similarly to the electrostatic case, condition (31) can be now written in terms of tangential components. We first assume that Γ_I is connected. Then, fixed arbitrarily a point $\mathbf{r}_0 \in \Gamma_I$, we integrate over any piecewise regular path $\gamma[\mathbf{r}_0, \mathbf{r}] \subseteq \Gamma_I$ for any point $\mathbf{r} \in \Gamma_I$, and get the second total/reduced interface condition

$$\phi(\mathbf{r}) = \varphi(\mathbf{r}) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{H}_s \cdot d\mathbf{r} \qquad \text{on } \Gamma_I, \tag{32}$$

where it is assumed $\phi(\mathbf{r}_0) = \varphi(\mathbf{r}_0)$. Condition (32) describes a jump of discontinuity at the interface between total and reduced potential values, due to the source field. If Γ_I is disconnected, a condition like (32) has to be repeated on each single connected component.

If the total domain is composed of more soft materials, the corresponding interface conditions reduce to simpler homogeneous expressions. If $\Omega_{1,1} \subseteq \Omega_1$ is a material adjacent to a material $\Omega_{1,2} \subseteq \Omega_1$, we get

$$\begin{aligned} &\phi_1 = \phi_2, \\ &(\mu_{1,1} - \mu_{1,2}) \nabla\phi \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_{12}, \end{aligned} \tag{33}$$

where $\Gamma_{12} = \partial\Omega_{1,1} \cap \partial\Omega_{1,2}$, and ϕ_1 and ϕ_2 are the values (or limit values) of ϕ on Γ_{12} from the sides of $\Omega_{1,1}$ and $\Omega_{1,2}$ respectively. Analogous expressions follow in case of interface conditions between two soft materials in the reduced domain, by changing ϕ with φ.

In a similar way, boundary conditions can be obtained. If the reduced domain $\overline{\Omega}_2$ intersects the boundary, there will a part $\Gamma_{N,2} = \partial\Omega_2 \cap \Gamma_N$ such that a non-homogeneous Neumann condition holds, in the form

$$\nabla\varphi \cdot \mathbf{n} = \mathbf{H}_s \cdot \mathbf{n} \quad \text{on } \Gamma_{N,2}, \tag{34}$$

when $\Gamma_{N,2}$ is non-empty. Besides, if $\partial\Gamma_{D,2} = \partial\Omega_2 \cap \Gamma_D$ is non-empty and connected, we get the non-homogeneous Dirichlet condition

$$\varphi(\mathbf{r}) = \int_{\mathbf{r}_A}^{\mathbf{r}} \mathbf{H}_s \cdot d\mathbf{r}, \quad \varphi(\mathbf{r}_A) = 0, \quad \text{on } \Gamma_{D,2}, \tag{35}$$

where $\mathbf{r}_A \in \Gamma_{D,2}$ is a point fixed arbitrarily. If $\Gamma_{D,2}$ is disconnected, a condition like (35) has to be given on each single connected component. Similarly, we consider now the total domain $\overline{\Omega}_1$ when it intersects the boundary. Then, the Neumann condition takes the form

$$\nabla\phi \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_{N,1}, \tag{36}$$

with $\Gamma_{N,1} = \partial\Omega_1 \cap \Gamma_N$, when it is non-empty. If $\Gamma_{D,1} = \partial\Omega_1 \cap \Gamma_D$ is non-empty and connected, we get a non-homogeneous constant Dirichlet condition. In particular, when $\Gamma_{D,1} \cup \Gamma_I \cup \Gamma_{D,2}$ is connected and condition (35) has been imposed on $\Gamma_{D,2}$, for consistency we have to fix the condition

$$\phi(\mathbf{r}) = \int_{\mathbf{r}_A}^{\mathbf{r}_0} \mathbf{H}_s \cdot d\mathbf{r} \quad \text{on } \Gamma_{D,1}. \tag{37}$$

In the other cases, one may specify

$$\phi(\mathbf{r}) = \phi(\mathbf{r}_B) \quad \text{on } \Gamma_{D,1}, \tag{38}$$

where $\mathbf{r}_B$ is a point fixed arbitrarily in $\Gamma_{D,1}$, assuming for instance $\phi(\mathbf{r}_B) = 0$.

Finally, considering all equations and conditions together, we obtain the following differential boundary value problem on Ω:

$$\begin{cases} -\nabla \cdot (\mu_1 \nabla \phi) = 0 & \text{on } \Omega_1, \\ -\nabla \cdot (\mu_2 \nabla \varphi) = -\nabla \cdot (\mu_2 \mathbf{H}_s) & \text{on } \Omega_2, \\ \phi(\mathbf{r}) = \varphi(\mathbf{r}) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{H}_s \cdot d\mathbf{r}, \quad \phi(\mathbf{r}_0) = \varphi(\mathbf{r}_0) & \text{on } \Gamma_I, \\ \mu_1 \nabla \phi \cdot \mathbf{n}_1 + \mu_2 \nabla \varphi \cdot \mathbf{n}_2 = \mu_2 \mathbf{H}_s \cdot \mathbf{n}_2 & \text{on } \Gamma_I, \\ \phi(\mathbf{r}) = \int_{\mathbf{r}_A}^{\mathbf{r}_0} \mathbf{H}_s \cdot d\mathbf{r} & \text{on } \Gamma_{D,1}, \\ \varphi(\mathbf{r}) = \int_{\mathbf{r}_A}^{\mathbf{r}} \mathbf{H}_s \cdot d\mathbf{r}, \quad \varphi(\mathbf{r}_A) = 0 & \text{on } \Gamma_{D,2}, \\ \nabla \phi \cdot \mathbf{n} = 0 & \text{on } \Gamma_{N,1}, \\ \nabla \varphi \cdot \mathbf{n} = \mathbf{H}_s \cdot \mathbf{n} & \text{on } \Gamma_{N,2}, \end{cases} \tag{39}$$

in the unknowns ϕ and φ on their respective domains Ω_1 and Ω_2, where $\mathbf{r}_0 \in \Gamma_I$ and $\mathbf{r}_A \in \Gamma_{D,2}$ are fixed arbitrarily. In (39) $\mathbf{H}_s$ represents the source magnetic field intensity due to $\mathbf{J}_s$, computed by using the Biot-Savart law (25), as described in [23].

Differently from the simpler electrostatic case, in which the potential V is always continuous in the transition across the interface, model (39) presents a jump $d \neq 0$ of discontinuity between ϕ and φ, together with non-constant values on Γ_D, owing to line integrals of $\mathbf{H}_s$. Figure 1 shows a schematic 2D version of the model domain. The topology is valid also to describe a possible 2D representation of the domains in the electrostatic problem, when taking $\Omega_s = \emptyset$.

For the sake of simplicity, only soft materials have been considered here. If permanent magnets exist, the contribution due to the coercive field $\mathbf{H}_c$ has to be included. Even when considering permanent magnets, however, the magnetostatic model has form (1). In such a case, a contribution $\nabla \cdot \mu \mathbf{H}_c$ has to be added to the right hand side of Poisson's equations corresponding to permanent magnets. Boundary and interface conditions of Dirichlet type remain unchanged, while boundary and interface conditions of Neumann type have to include a contribution $\mu \mathbf{H}_c \cdot \mathbf{n}$ at the right hand side, where $\mathbf{n}$ is the unit normal vector whose direction is chosen using the same convention than in (30). This corresponds to a further non-zero contribution to the jump p across the interface, or to a non-zero value q on the part of boundary intersecting permanent

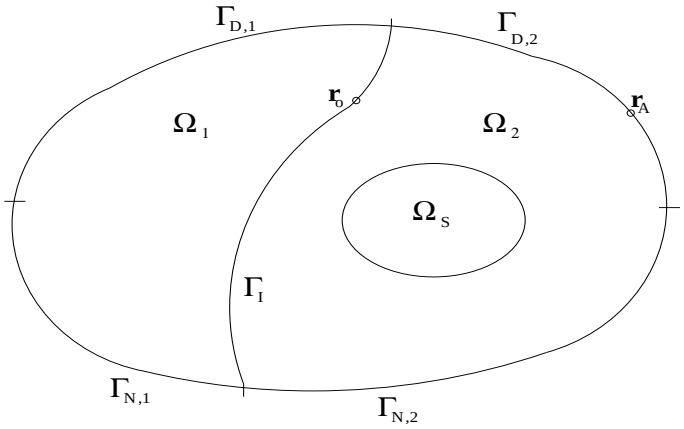

Figure 1: A two-dimensional projection of the model domain

magnet regions. See [24] for the complete formalization of the magnetostatic problem with permanent magnets.

Once the potentials ϕ and φ have been computed from the solution of model (39), and consequently the fields $\mathbf{H}$ and $\mathbf{B}$ are known, we can consider global quantities of physical interest, like the magnetic energy. The definition is given similarly to the electric energy. Given a material domain $\mathcal{D} \subseteq R^3$, assuming that $\mathbf{B}$ is linearly related to $\mathbf{H}$, the *total magnetic energy* stored in $\mathcal{D}$ due to $\mathbf{H}$ is

$$W = \frac{1}{2}\int_{\mathcal{D}} \mathbf{B}\cdot\mathbf{H}\, d\mathbf{x} = \frac{1}{2}\int_{\mathcal{D}} \mu H^2\, d\mathbf{x}. \tag{40}$$

In the next section, we intend to present the variational formulation of the general model (1), to be used for a finite element solution of the electrostatic and magnetostatic cases here introduced.

4 Variational formulation

In the sequel, the following notations will be used. Given an open Lebesgue measurable domain $\mathcal{D} \subset \mathbb{R}^3$ with Lipschitz continuous boundary $\partial\mathcal{D}$, for any integer $p = 1, 2, \ldots$, we denote by $L_p(\mathcal{D})$ the *Lebesgue space of order p*, i.e. the Banach space of all equivalence classes of real or complex-valued functions u Lebesgue integrable on $\mathcal{D}$, in which the p-powers of $|u|$ are Lebesgue integrable on $\mathcal{D}$. For each integer $m = 0, 1, 2, \ldots$, we denote by $H^m(\mathcal{D})$ the *Sobolev spaces of order m*, i.e. the Hilbert space of functions $u \in L_2(\mathcal{D})$ with distributional derivatives $D^\alpha u \in L_2(\mathcal{D})$, for any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n) \in \mathbb{N}^n$ such that $|\alpha| = \sum_{i=1}^n \alpha_i \leq m$. $H^1(\mathcal{D})$ is the most commonly used setting for variational formulations of second order problems. As subspaces, $H_0^1(\mathcal{D})$ will denote the space of functions $u \in H^1(\mathcal{D})$ with zero trace on $\partial\mathcal{D}$ and similarly, for any surface $\Gamma \subseteq \partial\mathcal{D}$, the space $H^1_{0,\Gamma}(\mathcal{D})$ will denote the set of functions $u \in H^1(\mathcal{D})$

having zero trace on Γ. Finally, we denote by $H^{1/2}(\Gamma)$ the Hilbert space of real or complex-valued functions v defined almost everywhere on Γ such that there exist functions $u \in H^1(\mathcal{D})$ with $u\,|_\Gamma = v$ (in the sense of trace). For any normed or pre-Hilbert space V here considered, when not otherwise specified, we will denote canonical norms and inner products by $\|\cdot\|_V$ and $(\cdot,\cdot)_V$, respectively. See [36] for a survey of Lebesgue spaces and [18, 35] for a study of Sobolev spaces and mathematical topics related to partial differential equations.

Let us consider the boundary value problem (1). Under *weak* conditions of regularity for f, d, p, q and u_D, we derive a variational formulation of model (1). Let us assume $f \in L_2(\Omega)$, $p \in L_2(\Gamma_I)$, $q \in L_2(\Gamma_N)$, $d \in H^{1/2}(\Gamma_I)$ and $u_D \in H^{1/2}(\Gamma_D)$. We obtain then the weak form of (1) by using a formal procedure similar to the one applied to the classical Poisson model with no interfaces [33], considering both domains Ω_1 and Ω_2. Thus, we multiply the differential equation in (1) by an arbitrary $v \in H_{0,\Gamma_D}(\Omega)$ and integrate over each Ω_i, $i = 1, 2$. Then, we apply Green's theorem to the two integral equalities and split the corresponding surface integrals over $\partial\Omega_i$ in the several parts of Γ_D, Γ_N and Γ_I defined by the relation

$$\partial\Omega_i = (\partial\Omega_i \cap \Gamma_D) \cup (\partial\Omega_i \cap \Gamma_N) \cup (\partial\Omega_i \cap \Gamma_I), \quad i = 1, 2. \tag{41}$$

Finally, using Neumann conditions both at the interface and the boundary and summing up the two equalities, we get the following weak form for problem (1):

(variational formulation) *Find* $u = u(x, y, z) \in L_2(\Omega)$, *where* $u\,|_{\Omega_i} = u_i \in H^1(\Omega_i)$ *for* $i = 1, 2$, *such that*

$$\int_\Omega \alpha \nabla u \cdot \nabla v d\mathbf{x} = \int_\Omega fv\ d\mathbf{x} + \int_{\Gamma_N} qv\ d\mathcal{S} + \int_{\Gamma_I} pv\ d\mathcal{S} \qquad \forall v \in H^1_{0,\Gamma_D}(\Omega), \tag{42}$$

and such that $u = u_D$ *on* Γ_D *and* $u_1 = u_2 + d$ *on* Γ_I (*in the sense of trace*).

In this model with interfaces, the solution $u = u(x, y, z)$ belongs to $L_2(\Omega)$ but not to $H^1(\Omega)$, because of the presence of a non-zero jump function d on Γ_I in the first interface condition of (1). Thus, as a natural setting for our weak solution it makes sense to introduce the following *piecewise* Sobolev space

$$H^1(\Omega_1, \Omega_2) = \{w \in L_2(\Omega) : w\,|_{\Omega_i} \in H^1(\Omega_i), i = 1, 2\}. \tag{43}$$

Denoting by $w_i = w\,|_{\Omega_i}$, for $i = 1, 2$, the component functions of any $w \in H^1(\Omega_1, \Omega_2)$, the definition

$$\|w\|_{H^1(\Omega_1,\Omega_2)} = \|w_1\|_{H^1(\Omega_1)} + \|w_2\|_{H^1(\Omega_2)} \tag{44}$$

can be then chosen as a natural norm. Coherently, we introduce the subspace $H^1_{0,\Gamma_D}(\Omega_1, \Omega_2)$ of functions $w \in H^1(\Omega_1, \Omega_2)$ whose trace γw on Γ_D vanishes. Similarly to spaces $H^1_0(\Omega)$ and $H^1_{0,\Gamma_D}(\Omega)$, for each $w \in H^1_{0,\Gamma_D}(\Omega_1, \Omega_2)$ it is easy to prove that the definition

$$\| w \|_{H^1_{0,\Gamma_D}(\Omega_1,\Omega_2)} = \| w_1 \|_{H^1_{0,\Gamma_D}(\Omega_1)} + \| w_2 \|_{H^1_{0,\Gamma_D}(\Omega_2)} \tag{45}$$

is another norm for $H^1_{0,\Gamma_D}(\Omega_1,\Omega_2)$, together with the induced norm from (44), where w_i are the component functions of w on Ω_i, $i = 1, 2$.

In (42), differently from the solution $u \in H^1(\Omega_1,\Omega_2)$, the test functions v belong to $H^1_{0,\Gamma_D}(\Omega) \subseteq H^1(\Omega)$, so that their trace γv is well-defined in Γ_I and Γ. Similarly to models without any interface conditions, it can be shown immediately that (42) is equivalent to the weak formulation of a model with Dirichlet boundary conditions that are *homogeneous* on Γ_D and such that *no jumps exist at the interface* Γ_I. Regarding this, let us consider the substitution

$$u = \tilde{u} + \tilde{u}_D, \tag{46}$$

where $\tilde{u}_D$ is an extension of u_D on Ω such that $\tilde{u}_D \in H^1(\Omega_1,\Omega_2)$ and the component functions $\tilde{u}_{D,1}$ on Ω_1 and $\tilde{u}_{D,2}$ on Ω_2 satisfy the condition $\tilde{u}_{D,1} = \tilde{u}_{D,2} + d$ on Γ_I. Then, it results $\tilde{u} \in H^1_{0,\Gamma_D}(\Omega)$. Now we introduce the bilinear form

$$a(w,v) = \int_\Omega \alpha \nabla w \cdot \nabla v \, d\mathbf{x} \qquad \forall w, v \in H^1_{0,\Gamma_D}(\Omega) \tag{47}$$

and, for a fixed function ϱ Lebesgue measurable on a Lipschitz continuous surface $\mathcal{S} \subseteq \partial\Omega$, the following notation of linear functional

$$< \varrho, v >_{\mathcal{S}} = \int_{\mathcal{S}} \varrho v \, d\mathcal{S} \qquad \forall v \in H^1_{0,\Gamma_D}(\Omega). \tag{48}$$

Then, denoting by $(\cdot,\cdot)_\Omega$ the usual L_2-scalar product on Ω, from (42) we get the following

(equivalent variational formulation) *Find* $\tilde{u} \in H^1_{0,\Gamma_D}(\Omega)$ *such that*

$$a(\tilde{u}, v) = b(v) \qquad \forall v \in H^1_{0,\Gamma_D}(\Omega), \tag{49}$$

where a *is given by* (47) *and*

$$b(v) = (f, v)_\Omega - (\alpha \nabla \tilde{u}_D, \nabla v)_\Omega + < q, v >_{\Gamma_N} + < p, v >_{\Gamma_I}, \tag{50}$$

with $< q, \cdot >_{\Gamma_N}$ *and* $< p, \cdot >_{\Gamma_I}$ *given by* (48).

The advantage of this equivalent formulation is that now the sought weak solution vanishes on Γ_D like the test functions v, and its component functions $\tilde{u}_i$ satisfy the condition $\tilde{u}_1 = \tilde{u}_2$ on Γ_I. The non-zero value at the Dirichlet boundary and the discontinuity at the interface have been transferred to a known function $\tilde{u}_D$.

Good properties hold for a and b. First, a is symmetric and continuous, i.e.

$$| a(w,v) | \leq C_1 \, \| w \|_{H^1_{0,\Gamma_D}(\Omega)} \| v \|_{H^1_{0,\Gamma_D}(\Omega)} \qquad \forall w, v \in H^1_{0,\Gamma_D}(\Omega), \tag{51}$$

as it follows from the Cauchy-Schwartz inequality, by taking $C_1 = \| \alpha \|_{L_\infty}$ (here, equal to $\max_\Omega \alpha$). Then, due to the uniform ellipticity, we have

$$a(v,v) \geq C_2 \, \| v \|^2_{H^1_{0,\Gamma_D}(\Omega)} \quad \forall v \in H^1_{0,\Gamma_D}(\Omega), \tag{52}$$

with $C_2 = \min_\Omega \alpha$ (in such a case, it is said that a is *coercive* on $H^1_{0,\Gamma_D}(\Omega)$). Besides, the linear operator b is continuous, i.e.

$$| b(v) | \leq C_3 \, \| v \|_{H^1_{0,\Gamma_D}(\Omega)} \quad \forall v \in H^1_{0,\Gamma_D}(\Omega), \tag{53}$$

as it follows from the boundedness property of the functionals $(f,v)_\Omega$, $< q, v >_{\Gamma_N}$, $< p, v >_{\Gamma_I}$ and $(\alpha \nabla \tilde{u}_D, \nabla v)_\Omega$, using Cauchy-Schwartz and Poincaré's inequalities, with $C_3 > 0$ depending on α, f, p, q, $\tilde{u}_D$, Ω_1 and Ω_2. Thus, from the Lax-Milgram theorem [18, 33], there exists a unique weak solution for the variational problem (49), as well as (42).

To conclude, property (52) guarantees that $a(v,v) \geq 0$, for each $v \in H^1_{0,\Gamma_D}(\Omega)$ (the property holds also in a larger space, like $H^1(\Omega)$). We can therefore introduce the equivalent norm

$$\| v \|_{\mathcal{E},\Omega} = \sqrt{a(v,v)} = \Big(\int_\Omega \alpha \nabla v \cdot \nabla v \, d\mathbf{x} \Big)^{1/2}, \tag{54}$$

which will be used many times in the following, called *energy norm*.

5 Finite element discretization

For a finite element discretization of the variational form (42), let $T_h = \{\tau\}$ be a *conformal* 3D triangulation of Ω, i.e. a partition of Ω in open triangular subdomains τ pairwise disjoint or having either a vertex, an edge or a face in common, such that $\overline{\Omega} = \bigcup \overline{\tau}$. Let us suppose that T_h is also *constrained* at the interface Γ_I, i.e. each element is disjoint from Γ_I or has a vertex, an edge or a face lying on it. The index h, called *mesh size*, measures the pointwise size of the elements in the triangulation, and has to satisfy some regularity properties. Typically, h is defined as the piecewise constant positive function such that $h \,|_\tau = \mathrm{diam}(\tau)$ for any element τ, i.e. it equals the longest edge of τ. Weaker assumptions for h can however be suggested. For instance, defining $h_\tau = \mathrm{diam}(\tau)$ for each $\tau \in T_h$, C. Johnson and K. Eriksson in [17] require that:

1) $h \in C^1(\overline{\Omega})$, it is positive and there exists $\nu > 0$ such that

$$| \nabla h(\mathbf{x}) | \leq \nu, \quad \forall x \in \overline{\Omega}; \tag{55}$$

2) there exists a constant $c_1 > 0$ such that

$$c_1 h_\tau^3 \leq \int_\tau d\mathbf{x}, \quad \forall \tau \in T_h; \tag{56}$$

3) there exists a constant $c_2 > 0$ such that

$$c_2 h_\tau \leq h(\mathbf{x}) \leq h_\tau, \quad \forall x \in \tau, \ \forall \tau \in T_h. \tag{57}$$

By means of these conditions it is required to have a sufficiently "good" mesh, that is: 1) sufficiently uniform with a not too large variation in size among the elements, 2) with not too sharp angles in the elements, and 3) whose element shapes are not too far from the equilateral one. In [9, 11] it is explained how the quality of the element shapes in the triangulation can affect the convergence of finite element solutions.

We can now turn to the discretization of (49), by using a *continuous Galerkin* finite element method. Let $V_h^r \subseteq H^1_{0,\Gamma_D}(\Omega)$ be the finite-dimensional space of continuous functions piecewise defined on Ω, vanishing on Γ_D, that are continuous polynomials of order r over each element in T_h. Then, by definition, the Galerkin finite element solution satisfies the variational form (49) for test functions considered in this finite-dimensional space, i.e. the following problem:

(discrete variational formulation) *Find* $\tilde{u}_h \in V_h^r$ *such that*

$$a(\tilde{u}_h, v_h) = b(v_h) \qquad \forall v_h \in V_h^r, \tag{58}$$

where a *and* b *are defined by* (47) *and* (50).

The discrete model $A\mathbf{u} = \mathbf{b}$ is then derived in a similar manner to models without interface. It is sufficient to impose the discrete condition (58) on test functions v_h belonging to the same finite-dimensional basis in V_h^r used for the finite representation of the discrete solution. The basis functions are chosen so that a local compact support property holds (e.g. in the piecewise linear case, the well-known *hat-functions* can be used). Together with the interior sources and the boundary conditions, in our case a contribution from the interface conditions has to be included in the right hand side $\mathbf{b}$. If a piecewise linear Galerkin method is applied, for instance, the number of degrees of freedom for the linear system is the number n of the interior and Neumann boundary nodes defined by the triangulation T_h, with interface nodes included (called *free nodes*). In the definition of the discrete model, the interface condition $u_1 = u_2 + d$ is inserted into the structure of the linear system, in order to compute directly the finite element approximation u_h of the solution u, instead of $\tilde{u}_h$. In the unknown vector $\mathbf{u} = (\widehat{u}_1, \widehat{u}_2, \cdots, \widehat{u}_n)^T$, the generic component $\widehat{u}_i = u_h(x_i, y_i, z_i)$ represents the value of the finite element solution u_h at the i-th free node of T_h. The components of $\mathbf{u}$ on nodes belonging to Γ_I are associated either to the function u_1 or to u_2, by arbitrary convention, as the value of the other one is derived immediately from the knowledge of the jump d.

Owing to the uniform ellipticity of the form $a(u, v)$, the stiffness matrix is non-singular, symmetric, positive definite and, due to the local support property, sparse. Therefore, the linear system is generally solved by iterative methods, e.g. Krylov subspace methods. Suitable preconditioners $\mathbf{B}$ can be applied in order to reduce from $\kappa(\mathbf{A})$ to $\kappa(\mathbf{BA})$ the condition number of the stiffness matrix, which would otherwise grow as the mesh size becomes finer. In [10] X.

Cai and K. Samuelsson suggest some methods, like additive Schwartz methods and multigrid techniques, that can be used as possible preconditioners for the stiffness matrix. Details involving implementation aspects of the discrete model (58) will be described in [24].

6 Error analysis

Broadly speaking, differential models can be written in the general form $A(u) = g$, where A is a *differential operator*, g represents *data* and u is the *solution.* In the representation of a physical problem by a *computable* mathematical form we introduce, however, perturbations and approximations. Generally, the operator A is modelled by $\widehat{A}$ and the datum g is approximated by $\widehat{g}$. Therefore, if $\widehat{u}$ is the exact solution of the approximated model $\widehat{A}(\widehat{u}) = \widehat{g}$ the difference $e_{\mathcal{M}} = u - \widehat{u}$ represents what we can call the *modelling error.* Then, depending on the discretization method and the computation algorithm, we also have to take into consideration the *computational error* $e_{\mathcal{C}} = \widehat{u} - U$ where U is the approximated solution of $\widehat{A}(\widehat{u}) = \widehat{g}$. The total error is thus the sum of both the modelling and computational errors:

$$e = u - U = (u - \widehat{u}) + (\widehat{u} + U) = e_{\mathcal{M}} + e_{\mathcal{C}}. \tag{59}$$

In this work, we intend to investigate the computational error, dominant part of the error when the computational model $\widehat{A}(\widehat{u}) = \widehat{g}$ is a "sufficiently" good approximation of the exact model $A(u) = g$. Here, the approximated solution U is intended to be the finite element solution u_h of the variational form (42), obtained by solving the discrete form (58). A future task for a global error analysis will be then to investigate other error contributions. In our model, which involves several steps of modelling and computation, the sources of "perturbation" arise in fact in modelling of data at more levels (computation of $\mathbf{H}_s$, discretization of $\mathbf{H}_s \cdot \mathbf{n}$ for the Neumann conditions, quadrature of $\int \mathbf{H}_s \cdot d\mathbf{r}$ for the Dirichlet conditions) as well as numerical computation (Galerkin finite element discretization, linear system solution).

Consider now the *finite element error*

$$e = u - u_h, \tag{60}$$

where u is the exact solution of a certain model and u_h is the finite element solution. Error estimates of (60) provide a measure of the accuracy and stability of the finite element solution. They can be given in two forms, as *a priori* and *a posteriori estimates.*

A priori error estimates for (60) are expressed in terms of the exact solution, so that they can give information about regularity properties of the exact solution and the order of convergence of the finite element method. Differently from the former, a posteriori estimates are expressed in terms of data, mesh size and finite element solution, i.e. they are defined by quantities that are all *computable.* The latter turn out to be useful to suggest operative criteria to

improve the solution accuracy by successive steps of finite element calculation in a *feed-back* process resorting to known or previously computed information. In this case, no a priori information of the exact solution are required.

In an *adaptive* finite element method, a posteriori error estimates are used inside a computational procedure that constructs a finite element discretization for a given problem according to two basic objectives [17]:

(a) the error of the approximate solution lies within a given tolerance;
(b) the finite element solution is obtained by a (nearly) minimal number of degrees of freedom.

By condition (a) we require that the method is *reliable*, and by condition (b) that it is also *efficient*.

The basic idea behind an adaptive finite element algorithm is the following. If a global a posteriori error estimate is lower than a given tolerance, the finite element solution is considered accurate (and the algorithm stops). Otherwise, local error indicators are computed on mesh entities (elements, nodes, or edges/faces) for a "local" improvement of the solution accuracy by refinement/correction of selected entities on which the error indicator results large. Regarding this, we can distinguish essentially three refinement approaches:

- *h-refinement*, decrease of the element size;
- *p-refinement*, increase of the order in the polynomial representation of the solution;
- *r-refinement*, movement/correction of node positions.

Once the refinement procedure has been performed on the mesh, a successive and more accurate finite element solution is then computed using the new definition of the degrees of freedom.

Many authors (I. Babuska, W.C. Rheimboldt, R.E. Bank, A. Weiser, O.C. Zienkiewicz, C. Johnson, R. Verfurth, J.E. Flaherty, and some others) have studied adaptive finite element methods according to a posteriori error indicators estimated by different criteria. See references [3, 4, 5, 14, 16, 17, 29, 30, 31, 44] and, for a more complete list, the ones quoted in [14, 21, 44].

Among the ones above mentioned, the first a posteriori error estimator has been suggested by Babuska and Bank in the 80's. In that case the error was computed from the solution of "perturbed" local variational problems, where residual quantities appeared as data for the error model. This idea has been used recently for static electromagnetic models in [2, 19], and for eddy current problems in [21]. Differently from Babuska's, in Johnson's approach error estimates are provided as upper bounds of the error in proper norms ($\| e \|_{L_2}$, $\| e \|_{H_1}$, $\| e \|_{\mathcal{E}}$, etc.), related to residual quantities occurring at the interior and the boundary of the elements, up to proper constants. Basically these error estimates are obtained by using the orthogonality properties of the Galerkin method together with standard finite element interpolation estimates, and they require an estimation of stability factors, which can be obtained by solving *dual problems* associated to the initial one.

According to these guidelines, in the next section we consider the abstract framework presented in Section 4 and extend some arguments proposed in [14, 17, 39] for classical Poisson problems, to prove some a posteriori estimates of (60) for the finite element solution of model (39).

7 A posteriori error estimation

Under the hypotheses and notations introduced in Section 4, we have seen that the model problem (1) can always be associated to a variational form (49) with the bilinear form a and the linear functional b defined in (47) and (50), which is equivalent to (42). Let therefore u be the exact solution of (42), obtained from the solution $\tilde{u}$ of (49) and relation (46). Similarly, let u_h be the discrete solution, computed from the Galerkin finite element solution $\tilde{u}_h$ of (58) and the definition

$$u_h = \tilde{u}_h + \tilde{u}_{D,h}, \tag{61}$$

where $\tilde{u}_{D,h}$ is the projection of $\tilde{u}_D$ in the finite-dimensional space $V^r_{h,\Gamma_I} \subseteq H^1(\Omega_1, \Omega_2)$ of functions that are continuous on Ω_1 and Ω_2, piecewise defined polynomials of order r over each element in T_h. As regards u and u_h in the original models, we use the same bilinear form a as defined in (47), and introduce a functional g obtained from b in (50) such that

$$g(v) = (f, v)_\Omega + <q, v>_{\Gamma_N} + <p, v>_{\Gamma_I} \qquad \forall v \in H^1_{0,\Gamma_D}(\Omega). \tag{62}$$

By construction, we know that u satisfies (42), i.e.

$$a(u, v) = g(v) \qquad \forall v \in H^1_{0,\Gamma_D}(\Omega), \tag{63}$$

together with the non-homogeneous Dirichlet conditions on Γ_D and Γ_I, and u_h satisfies the discrete form

$$a(u_h, v_h) = g(v_h) \qquad \forall v_h \in V^r_h. \tag{64}$$

Under these premises, from (63) and (64) it follows:

Proposition. *The finite element error* (60) *satisfies the "orthogonality" property*

$$a(e, v_h) = 0 \qquad \forall v_h \in V^r_h. \tag{65}$$

In Ω we now introduce the following *residual*

$$r = -\nabla \cdot (\alpha \nabla u) - f, \tag{66}$$

associated to the functional

$$<r, v> = a(u_h, v) - (f, v)_\Omega - <q, v>_{\Gamma_N} - <p, v>_{\Gamma_I}, \tag{67}$$

for each $v \in H^1_{0,\Gamma_D}(\Omega)$. Then

Lemma. *For all $v \in H^1_{0,\Gamma_D}(\Omega)$, $v_h \in V_h^r$, it holds:*

$$\begin{aligned} &< r, v >= -a(e, v), \\ &< r, v_h >= 0, \\ &< r, v >=< r, v - v_h > . \end{aligned} \tag{68}$$

Proof. The first one follows immediately from the linearity of a, the second one from (64) and the third one as a corollary, due to the linearity of $< r, \cdot >$.

Given the triangulation $T_h = \{\tau\}$, let $\mathcal{F}$ be the set of faces of all elements $\tau \in T_h$. In the following, for any face $f \in \mathcal{F}$ between a tetrahedron τ_f^+ and its adjacent τ_f^-, and any $\psi \in L^2(\overline{\tau_f^+} \cup \overline{\tau_f^-})$, we denote by ψ_f^+ and ψ_f^- the limit values of ψ on f from the two sides of τ_f^+ and τ_f^-, respectively, and by $[\psi]_f$ the jump of ψ across f in the direction of the unit vector $\mathbf{n}$ normal to f, defined by

$$\begin{aligned} &\psi_f^+ = \lim_{t \to 0^+} \psi(\mathbf{x} + t\mathbf{n}), \quad \psi_f^- = \lim_{t \to 0^-} \psi(\mathbf{x} - t\mathbf{n}), \quad \forall \mathbf{x} \in f, \\ &[\psi]_f = \psi_f^+ - \psi_f^- . \end{aligned} \tag{69}$$

Now it is possible to demonstrate the following a posteriori error estimate for (60):

Theorem 5.1. *The energy norm of the Galerkin finite element error* (60) *in the variational problem* (42) *satisfies the following property: there exists a constant $C > 0$ for which*

$$\| e \|_{\mathcal{E},\Omega} \leq C \| \tilde{h} R \|_{L_2(\Omega)}, \tag{70}$$

where $\tilde{h} = \tilde{h}(x)$ and $R = R(x)$ are the functions piecewise defined on each element τ such that

$$\begin{aligned} &\tilde{h}|_\tau = h_\tau, \\ &R|_\tau = |-\nabla \cdot (\alpha \nabla u_h) - f| + h_\tau^{-1/2} \mathrm{Meas}(\tau)^{-1/2} \, S_\tau, \end{aligned} \tag{71}$$

with $h_\tau = \mathrm{diam}(\tau)$, and

$$\begin{aligned} S_\tau = &\| \eta_\tau ([\alpha \nabla u_h \cdot \mathbf{n}] - p) \|_{L_2(\partial\tau \cap \Gamma_I)} \\ &+ \| \eta_\tau [\alpha \nabla u_h \cdot \mathbf{n}] \|_{L_2(\partial\tau \setminus (\Gamma \cup \Gamma_I))} + \| \alpha \nabla u_h \cdot \mathbf{n} - q \|_{L_2(\partial\tau \cap \Gamma_N)}, \end{aligned} \tag{72}$$

where $\{\eta_\tau\}_{\tau \in T_h}$ is a family of positive real numbers chosen in such a way that $0 < \eta_\tau < 1$ for all $\tau \in T_h$, with $\eta_{\tau_i} + \eta_{\tau_j} = 1$ if τ_i and τ_j are adjacent.

Proof. The demonstration is based on the elementwise decomposition of the integrals defining a and g by the triangulation T_h. First, we write

$$< p, v >_{\Gamma_I} = \sum_{f \in \mathcal{F}} \int_{f \cap \Gamma_I} pv \, d\mathcal{S}, \tag{73}$$

define

$$a_\tau(u, v) = \int_\tau \alpha \nabla u \cdot \nabla v \, d\mathbf{x} \qquad \forall v \in H^1_{0,\Gamma_D}(\Omega), \tag{74}$$

and consider the set decomposition

$$\partial\tau = (\partial\tau \cap \Gamma_D) \cup (\partial\tau \cap \Gamma_N) \cup (\partial\tau \cap \Gamma_I) \cup (\partial\tau \setminus (\Gamma \cup \Gamma_I)). \tag{75}$$

Then, we have

$$\begin{aligned}
< r, v > &= \sum_{\tau \in T_h} \big(a_\tau(u_h, v) - (f, v)_\tau - < q, v >_{\partial\tau \cap \Gamma_N} \big) - \sum_{f \in \mathcal{F}} < p, v >_{f \cap \Gamma_I} \\
&= \sum_{\tau \in T_h} \big((-\nabla \cdot (\alpha \nabla u_h) - f, v)_\tau + < \alpha \partial u_h / \partial n - q, v >_{\partial\tau \cap \Gamma_N} \\
&\quad + < \alpha \partial u_h / \partial n, v >_{\partial\tau \setminus \Gamma} \big) - \sum_{f \in \mathcal{F}} < p, v >_{f \cap \Gamma_I} \\
&= \sum_{\tau \in T_h} \big((-\nabla \cdot (\alpha \nabla u_h) - f, v)_\tau + < \alpha \nabla u_h \cdot \mathbf{n} - q, v >_{\partial\tau \cap \Gamma_N} \big) \\
&\quad + \sum_{f \in \mathcal{F}} \big(< [\alpha \nabla u_h \cdot \mathbf{n}]_f - p, v >_{f \cap \Gamma_I} + < [\alpha \nabla u_h \cdot \mathbf{n}]_f, v >_{f \setminus (\Gamma \cup \Gamma_I)} \big) \\
&= \sum_{\tau \in T_h} \big((-\nabla \cdot (\alpha \nabla u_h) - f, v)_\tau + < \alpha \nabla u_h \cdot \mathbf{n} - q, v >_{\partial\tau \cap \Gamma_N} \\
&\quad + < \eta_\tau([\alpha \nabla u_h \cdot \mathbf{n}] - p), v >_{\partial\tau \cap \Gamma_I} + < \eta_\tau [\alpha \nabla u_h \cdot \mathbf{n}], v >_{\partial\tau \setminus (\Gamma \cup \Gamma_I)} \big),
\end{aligned} \tag{76}$$

where $\{\eta_\tau\}_{\tau \in T_h}$ is a set of weights chosen so that $0 < \eta_\tau < 1$ for all $\tau \in T_h$, and $\eta_{\tau_i} + \eta_{\tau_j} = 1$ if τ_i and τ_j are adjacent. Considering now the third relation in (68), the Cauchy-Schwartz inequality and the monotony property for integrals, we obtain

$$\begin{aligned}
|< r, v >| \leq \sum_{\tau \in T_h} \Big(& \| -\nabla \cdot (\alpha \nabla u_h) - f \|_{L_2(\tau)} \| v - v_h \|_{L_2(\tau)} \\
& + \| \alpha \nabla u_h \cdot \mathbf{n} - q \|_{L_2(\partial\tau \cap \Gamma_N)} \| v - v_h \|_{L_2(\partial\tau)} \\
& + \| \eta_\tau([\alpha \nabla u_h \cdot \mathbf{n}] - p) \|_{L_2(\partial\tau \cap \Gamma_I)} \| v - v_h \|_{L_2(\partial\tau)} \\
& + \| \eta_\tau [\alpha \nabla u_h \cdot \mathbf{n}] \|_{L_2(\partial\tau \setminus (\Gamma \cup \Gamma_I))} \| v - v_h \|_{L_2(\partial\tau)} \Big).
\end{aligned} \tag{77}$$

Two fundamental inequalities have then to be used on elements τ and their boundary. The first one is a local version of the *trace inequality* [18, 39]: given a function $\omega \in H^1(\tau)$ there exists a constant $C_{\tau,0} > 0$ such that

$$\| \omega \|_{L_2(\partial\tau)} \leq C_{\tau,0} \big(h_\tau^{-1/2} \| \omega \|_{L_2(\tau)} + h_\tau^{1/2} \| \nabla \omega \|_{L_2(\tau)} \big). \tag{78}$$

The second one is a classical result from interpolation theory [14]: denoted by Ω_τ the union of the closure of elements adjacent to any element τ, for any function $\omega \in H^1(\Omega_\tau)$ there exists a constant $C_{\tau,1} > 0$ for which

$$\| D^{(m)}(\omega - \pi_h\omega) \|_{L_2(\tau)} \leq C_{\tau,1}\, h_\tau^{l-m} \| D^{(l)}\omega \|_{L_2(\Omega_\tau)}, \tag{79}$$

for $m = 0, 1,\ l = 1, 2$, where $\pi_h\omega$ is the nodal interpolant of ω at nodes of Ω_τ, with $D^{(0)}\omega = \omega$ and $D^{(1)}\omega = \nabla\omega$. In particular, it follows

$$\begin{aligned} \| v - \pi_h v \|_{L_2(\tau)} &\leq C_{\tau,1} h_\tau \| \nabla v \|_{L_2(\Omega_\tau)}, \\ \| \nabla(v - \pi_h v) \|_{L_2(\tau)} &\leq C_{\tau,1} \| \nabla v \|_{L_2(\Omega_\tau)}, \end{aligned} \tag{80}$$

hence

$$\| v - \pi_h v \|_{L_2(\partial\tau)} \leq C_{\tau,2}\, h_\tau^{1/2} \| \nabla v \|_{L_2(\Omega_\tau)}, \tag{81}$$

with $C_{\tau,2} = 2\, C_{\tau,0} C_{\tau,1}$. We substitute the first inequality of (80) and (81) in (77), and resume all the constants to obtain

$$\begin{aligned} |<r, v>| \leq C^{(0)} \sum_{\tau \in T_h} \Big(& h_\tau \| -\nabla\cdot(\alpha\nabla u_h) - f \|_{L_2(\tau)} + h_\tau^{1/2} \big(\| \eta_\tau [\alpha \nabla u_h \cdot \mathbf{n}] \|_{L_2(\partial\tau \setminus (\Gamma \cup \Gamma_I))} \\ & + \| \eta_\tau ([\alpha \nabla u_h \cdot \mathbf{n}] - p) \|_{L_2(\partial\tau \cap \Gamma_I)} + \| \alpha \nabla u_h \cdot \mathbf{n} - q \|_{L_2(\partial\tau \cap \Gamma_N)} \big) \Big) \| \nabla v \|_{L_2(\Omega_\tau)}, \end{aligned} \tag{82}$$

for a constant $C^{(0)} > 0$. Then, we consider the monotony property for the integration domains $\Omega_\tau \subseteq \Omega$, the upper bound

$$\sum_{i=1}^{n} a_i \leq \kappa_n \Big(\sum_{i=1}^{n} a_i^2 \Big)^{1/2}, \quad \forall a_i \geq 0, \quad i = 1, 2, \ldots, n, \tag{83}$$

which holds for some constants $\kappa_n > 0$, as well as the bounds $a^2 + b^2 \leq (a+b)^2 \leq 2a^2 + 2b^2$ for all $a,\ b \geq 0$. After some rearrangement, by introduction of the global notations (71) and (72), we get

$$|<r, v>| \leq C^{(1)} \| \tilde{h} R \|_{L_2(\Omega)} \| \nabla v \|_{L_2(\Omega)}, \tag{84}$$

for the constant $C^{(1)} = \sqrt{2}\, \kappa_n C^{(0)}$. Then, we choose $v = e$ and use the equality

$$|<r, e>| = \| e \|_{\mathcal{E},\Omega}^2, \tag{85}$$

obtained from the first equation in (68). To conclude, we consider the equivalence between norms $\| e \|_{\mathcal{E},\Omega}$ and $\| \nabla e \|_{L_2(\Omega)}$. In particular, from

$$\alpha_{min} \| \nabla e \|_{L_2(\Omega)}^2 \leq \| e \|_{\mathcal{E},\Omega}^2 \tag{86}$$

we get the upper bound in L_2-norm

$$\| \nabla e \|_{L_2(\Omega)} \leq \tilde{C}\, \| \tilde{h} R \|_{L_2(\Omega)}, \tag{87}$$

with $\tilde{C} = C^{(1)} / \alpha_{min}$. On the other hand, from

$$\alpha_{max} \| \nabla e \|^2_{L_2(\Omega)} \geq \| e \|^2_{\mathcal{E},\Omega} \tag{88}$$

relation (70) finally holds, by choosing $C = C^{(1)} \sqrt{\alpha_{max}} / \alpha_{min}$. *Q.E.D.*

The above estimation on Ω shows that the energy norm of the finite element error depends on the residual at the interior of elements, in a proportional way to the mesh size h_τ, as well as the residual at the inter-element jump of the normal derivative of the solution, in a proportional way to $h_\tau^{1/2}$.

For the numerical computation of (70) two important aspects have to be considered. The first one concerns the weights η_τ appearing in the pairwise contributions across element faces. Although an infinite number of values can be assigned to them, it is convenient to define them in such a way that the corresponding terms in the upper bound (70) become as small as possible. The simplest choice is to take $\eta_\tau = 1/2$, i.e. to consider an *equidistribution* of the error contribution due to jumps of the normal derivative. A better choice, however, is to define a different weight from element to element, taking into account the geometry of the problem domain, the possible discontinuity of the diffusion coefficient α across elements, as well as local behaviours of the finite element solution (e.g. gradients, etc.). Regarding this, a possible non-uniform definition of the weights η_τ is suggested in the following Theorem 5.2.

The second (crucial) aspect is to provide an appropriate estimation of the constants C and $\tilde{C}$. The proof of Theorem 5.1 suggests how to compute them, by their relation with the constants appearing in the trace and interpolation inequalities (78) and (79). Nevertheless, such estimates would be too large. A sharper estimation than (70) is rather obtained by separating the contributions of the interior residuals and the jumps of the normal derivative across the element faces with introduction of two different constants, instead of one, resorting to few modifications in the previous demonstration.

Taking into account these two practical difficulties, a finer error estimation is then suggested by modification of the global general upper bound (70), as follows.

Theorem 5.2. *The energy norm of the Galerkin finite element error* (60) *in the variational problem* (42) *satisfies the following property: there exist two constants* $C_1 > 0$ *and* $C_2 > 0$ *for which*

$$
\begin{aligned}
\| e \|^2_{\mathcal{E},\Omega} &= \| \alpha^{1/2} \nabla(u-u_h) \|^2_{L_2(\Omega)} \qquad (89)\\
&\le \sum_{\tau\in T_h} \Big(C_1 \| \alpha^{-1/2} h_\tau (-\nabla\cdot(\alpha\nabla u_h) - f) \|^2_{L_2(\tau)} \\
&\quad + C_2 \Big(\sum_{f\in(\partial\tau\setminus(\Gamma\cup\Gamma_I))} \| (\alpha^+ + \alpha^-)^{-1/2} h_f^{1/2} [\alpha\nabla u_h\cdot\mathbf{n}]_f \|^2_{L_2(f)} \\
&\quad + \sum_{f\in(\partial\tau\cap\Gamma_I)} \| (\alpha^+ + \alpha^-)^{-1/2} h_f^{1/2} ([\alpha\nabla u_h\cdot\mathbf{n}]_f - p) \|^2_{L_2(f)} \\
&\quad + \sum_{f\in(\partial\tau\cap\Gamma_N)} \| (\alpha^{-1/2} h_f^{1/2} (\alpha\nabla u_h\cdot\mathbf{n} - q) \|^2_{L_2(f)} \Big) \Big),
\end{aligned}
$$

where h_τ and h_f denote the element and face sizes, defined by $h_\tau = \mathrm{diam}(\tau)$ and $h_f = \mathrm{diam}(f)$ for each face f in τ, for $\tau \in T_h$, and where α^+ and α^- denote the limit values of α at each face f of τ from its interior and exterior respectively, according to notation (69).

Proof. The demonstration is similar to the one of Theorem 5.1. The fundamental difference consists in the specification of the weights η_τ and in a different definition of the constant factors. Again, we compute $< r, v >$. In the last right hand side of equation (76) we can divide and multiply by $\alpha^{1/2}$ the addends referred to τ and $\partial\tau\cap\Gamma_N$, and multiply and divide by $h_\tau^{1/2}$ the addends referred to $\partial\tau\cap\Gamma_N$, $\partial\tau\cap\Gamma_I$ and $\partial\tau\setminus(\Gamma\cup\Gamma_I)$. Then, we define the weights η_τ introduced in Theorem 5.1 so that

$$
\eta_\tau|_f = \frac{\alpha^+}{\alpha^+ + \alpha^-}, \qquad (90)
$$

for each face f of element τ. If we split in two factors the contributions of η_τ by the substitution

$$
\eta_\tau = \eta^*_\tau \sqrt{\alpha^+}, \qquad (91)
$$

with $\eta^*_\tau|_f = \sqrt{\alpha^+}/(\alpha^+ + \alpha^-)$, equation (77) takes the modified form

$$
\begin{aligned}
|<r,v>| \le \sum_{\tau\in T_h} \Big(& \| \alpha^{1/2}(v-v_h) \|_{L_2(\tau)} \, \| \alpha^{-1/2}(-\nabla\cdot(\alpha\nabla u_h) - f) \|_{L_2(\tau)} \\
& + \| \alpha^{1/2} h_\tau^{-1/2} (v-v_h) \|_{L_2(\partial\tau)} \, (\, \| \alpha^{-1/2} h_\tau^{1/2} (\alpha\nabla u_h\cdot\mathbf{n} - q) \|_{L_2(\partial\tau\cap\Gamma_N)} \qquad (92)\\
& + \| \eta^*_\tau h_\tau^{1/2} ([\alpha\nabla u_h\cdot\mathbf{n}] - p) \|_{L_2(\partial\tau\cap\Gamma_I)} + \| \eta^*_\tau h_\tau^{1/2} [\alpha\nabla u_h\cdot\mathbf{n}] \|_{L_2(\partial\tau\setminus(\Gamma\cup\Gamma_I))}) \Big).
\end{aligned}
$$

From (80) and (81) we obtain the inequalities

$$
\begin{aligned}
& \| \alpha^{1/2}(v - \pi_h v) \|_{L_2(\tau)} \le \tilde{C}_{\tau,1} h_\tau \| \nabla v \|_{L_2(\Omega_\tau)}, \qquad (93)\\
& \| \alpha^{1/2} h_\tau^{-1/2} (v - \pi_h v) \|_{L_2(\partial\tau)} \le \tilde{C}_{\tau,2} \| \nabla v \|_{L_2(\Omega_\tau)},
\end{aligned}
$$

where $\tilde{C}_{\tau,1} = C_{\tau,1}\alpha^{1/2}_{\max,\tau}$ and $\tilde{C}_{\tau,2} = C_{\tau,2}\alpha^{1/2}_{\max,\tau}$, with $\alpha_{\max,\tau}$ maximum diffusion value on $\overline{\tau}$. We now consider the upper bound

$$\frac{\sqrt{\alpha^+}}{\alpha^+ + \alpha^-} < \frac{1}{\sqrt{\alpha^+ + \alpha^-}}, \tag{94}$$

and split the boundary contributions on each element along the single faces. If we denote by $h_f = \mathrm{diam}(f)$ the size of each face f, for any tetrahedron $\tau \in T_h$ there exists a constant $\kappa_\tau > 0$ depending on the geometry of the tetrahedron such that $h_\tau \leq \kappa_\tau h_f$ for any face f of τ. Thus, summarizing all elemental constants $\tilde{C}_{\tau,1}$ and κ_τ in a global $\tilde{C}_1$, as well as $\tilde{C}_{\tau,2}$ and κ_τ in a $\tilde{C}_2$, we obtain

$$\begin{aligned} |<r, v>| \leq \sum_{\tau \in T_h} \Big(& \tilde{C}_1 \, \| \alpha^{-1/2} h_\tau (-\nabla \cdot (\alpha \nabla u_h) - f) \|_{L_2(\tau)} \\ & + \tilde{C}_2 \Big(\sum_{f \in (\partial\tau \setminus (\Gamma \cup \Gamma_I))} \| (\alpha^+ + \alpha^-)^{-1/2} h_f^{1/2} [\alpha \nabla u_h \cdot \mathbf{n}]_f \|_{L_2(f)} \\ & + \sum_{f \in (\partial\tau \cap \Gamma_I)} \| (\alpha^+ + \alpha^-)^{-1/2} h_f^{1/2} ([\alpha \nabla u_h \cdot \mathbf{n}]_f - p) \|_{L_2(f)} \\ & + \sum_{f \in (\partial\tau \cap \Gamma_N)} \| (\alpha^{-1/2} h_f^{1/2} (\alpha \nabla u_h \cdot \mathbf{n} - q) \|_{L_2(f)} \Big) \Big) \| \nabla v \|_{L_2(\Omega_\tau)}, \end{aligned} \tag{95}$$

as a modification of (82). Again, we choose $v = e$, apply equality (85) and consider bounds (86) and (88). Taking here $\| e \|^2_{\mathcal{E},\Omega}$ instead of $\| e \|_{\mathcal{E},\Omega}$ and using inequality (83), we finally obtain estimation (89) for suitable constants $C_1 > 0$ and $C_2 > 0$ related to $\tilde{C}_1^2$ and $\tilde{C}_2^2$ respectively. *Q.E.D.*

Giving a more detailed information than the general bound (70), estimation (89) shows that the elemental contribution of the error due to the interior residual is inversely proportional to $\alpha^{1/2}$ and proportional to the element size, while the local contributions on the element boundary take into account the discontinuity of diffusion across the faces, inversely proportional to $(\alpha^+ + \alpha^-)^{1/2}$ and proportional to the square root of the face sizes.

In the particular case without any interfaces, (89) coincides with the error estimate introduced by X. Cai and K. Samuelsson in [10]. This estimate could be considered as error indicator in a mesh refinement procedure for an adaptive Galerkin finite element solution, since a numerical estimation of the constants C_1 and C_2 has already been derived in [10] both in the 2D and 3D case. These estimates result in fact sharper than the large constants obtained from the trace and interpolation inequalities. The suggested values are $C_1 = 0.1$, $C_2 = 0.15$ for triangles and $C_1 = 0.07$, $C_2 = 0.55$ for tetrahedra. An algorithm of adaptive h-refinement will described in detail in [24], based on the a posteriori error in energy norm (89).

To conclude, alternative approaches of a posteriori error estimation could be proposed, depending on which quantity is considered for measuring the error on it. Here, we have focused mainly on the pure error of the potential, in order to evaluate directly the accuracy of the Galerkin finite element solution of the discrete form (58) associated to model (1). An alternative criterion could be to estimate the error on the "energy" defined by

$$\mathcal{E} = \int_{\mathcal{D}} |\nabla u|^2 \, d\mathbf{x} - \int |\nabla u_h|^2 \, d\mathbf{x} \tag{96}$$

or, better in our case with diffusion coefficients $\alpha = \epsilon$ or $\alpha = \mu$, by

$$\mathcal{E}_\alpha = \int_{\mathcal{D}} \alpha \, |\nabla u|^2 \, d\mathbf{x} - \int \alpha \, |\nabla u_h|^2 \, d\mathbf{x}, \tag{97}$$

for certain domains $\mathcal{D} \subseteq \mathcal{R}^3$ contained in Ω or Ω_i, $i = 1, 2$, related to physical materials of the model. In the electrostratic problem (15), as well as in the magnetostatic problem (39), in fact, the latter definition is related to a physical quantity, i.e. the electric or magnetic energy, respectively (16) or (40), by means of definition (9), or (24) and (28). This approach seems efficient for models derived from field formulations, as it has been shown in [6, 8] for 3D magnetostatic cases. As concerns scalar potential formulations, the new definition of error seems appropriate for domains where only *total* potentials are introduced, like in all electrostatic models, or in magnetostatic models having permanent magnets but *not* source currents. In the last case, the error $\mathcal{E}_\mu$ does not coincide completely with the effective error of the magnetic energy because of a *double* potential formulation that has recourse necessarily to a *reduced* potential φ in a part of the domain containing currents. In this magnetostatic case with imposed currents, in fact, also the contribution of the source field $\mathbf{H}_s$ has to be taken into account in the estimation of the magnetic energy.

8 Conclusions

In the present work, a three-dimensional generalized Poisson model with discontinuous diffusion has been studied, considering mixed non-homogeneous boundary and interface conditions of Dirichlet and Neumann type. Interface conditions have been included to simulate the behaviour of physical properties in the transition across adjacent materials having different diffusion properties. A typical application of this class of models is encountered in electrostatic and magnetostatic problems derived from Maxwell's equations by introducing scalar potential definitions.

The variational formulation of the model has been discussed, and existence and uniqueness of the weak solution have been proved by the Lax-Milgram theorem. A piecewise linear Galerkin finite element solution on tetrahedral mesh has been then proposed. A posteriori error estimates in energy norm have been derived, to provide criteria for an adaptive strategy of mesh refinement in order to increase the solution accuracy. Such estimates take into account error contributions at the interior of each element, due to the residual of the finite element solution, together with contributions at the element boundary, due to inter-element jumps in the normal derivative of the finite element solution.

Work [24] will present numerical results after implementation of the adaptive finite element solution and a posteriori error estimation here suggested, with application to some stationary electromagnetic problems.

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