MULTIPLIERS OF SPHERICAL HARMONICS AND ENERGY OF MEASURES ON THE SPHERE

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ABSTRACT. We consider the operator, $f(\Delta)$ for Δ the Laplacian, on spaces of measures on the sphere in $\mathbb{R}^d$, show how to determine a family of approximating kernels for this operator assuming certain technical conditions are satisfied, and give estimates for the L^2-norm of $f(\Delta)\mu$ in terms of the energy of the measure μ. We derive a formula, analogous to the classical formula relating the energy of a measure on $\mathbb{R}^d$ with its Fourier transform, comparing the energy of a measure on the sphere with the size of its spherical harmonics. An application is given to pluriharmonic measures.

1. INTRODUCTION

When Δ is the Laplacian on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ the action of an operator, $f(\Delta)$, can be defined on various function spaces using the spectral theorem. We consider the action of this operator on the space of measures. The operator acts as a convolution operator, and in the case when $f(t) \to 0$ when $t \to \infty$ it is natural to expect that the operator is well-behaved. As a result, there are standard techniques which can often be used for studying the operator $f(\Delta)$ on smooth function spaces. However, the operator is not, in general, well enough behaved to apply these techniques to spaces of measures. Consequently, the study of the action of $f(\Delta)$ on spaces of measures consists of the study of particular functions f, with explicitly computed kernels of the convolution, and typically involves the consideration of a family of kernels which approximate the kernel of convolution.

In section 2 of this article we begin with a function f which satisfies certain conditions, and show how to calculate an explicit form of a kernel (or family of approximating kernels) for the operator $f(\Delta)$ acting on measures on the unit sphere in $\mathbb{R}^d = \mathbb{R}^{2n}$. Conditions are given which ensure that $f(\Delta)\mu$ belongs to L^2.

In section 3 we consider integral operators which arise from a Riesz potential and prove that for all $0 < t < d-1$,

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{d\mu(x)d\mu(y)}{|x-y|^t} \sim \|\mu_0\|_2^2 + \sum_{k=1}^{\infty} k^{t-d+1} \|\mu_k\|_2^2 \tag{1.1}$$

where μ_k is the projection of the measure μ on the spherical harmonics of degree k. As the spherical harmonics are the analogue of the Fourier transform for measures on the sphere this result is in the same spirit as the following classical relationship between the Fourier transform of a measure on $\mathbb{R}^d$ and its energy (see [8], [5]):

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$$I_t(\mu) \equiv \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{d\mu(x)d\mu(y)}{|x-y|^t} = c\int |x|^{t-d} |\widehat{\mu}(x)|^2\, dx. \tag{1.2}$$

A similar formula is also known for measures on the torus ([6]). Because the size of the t-energy of a measure, $I_t(\mu)$, is closely related to geometric properties of the measure the classical relationship has proven to be very useful. For example, it has been applied to study the Hausdorff dimension of projections and intersections, distance sets, and the average rate of decay of the Fourier transform (c.f. [2], [7], [11] and the references cited therein).

Formula (1.1) can easily be seen to be true if μ is a sufficiently smooth function (c.f. [9]). However, the arguments are more delicate if this is not the case. In particular, if the measure has a singular component, then either side of (1.1) could be infinite, and in this case the formula should be understood to mean that both parts are simultaneously infinite.

The formula is applied to estimate the size of the coefficients of pluriharmonic measures in section 4.

2. Multipliers on spherical harmonics

In this section we assume d is an even integer and we set $m = (d-2)/2$.

Consider the Laplacian Δ on $\mathbb{S}^{d-1}$. This is a self-adjoint, negative operator whose eigenvalues are given by $\lambda_k = -k(k+d-2)$, for $k \in \mathbb{N}$, with the corresponding eigenfunctions being the spherical harmonics of order k. If f is a complex-valued function defined on the spectrum of Δ, then we can define the operator $f(\Delta)$ by using the spectral theorem. We are interested in understanding when $f(\Delta)$ acts on a space of measures, where by a measure we mean a finite, positive, Borel measure supported on the unit sphere in $\mathbb{R}^d$, where $d > 2$.

A multiplier on the spherical harmonics has the property that it acts on the spherical harmonics $\{Y_k\}$ by $T(Y_k) = m_k Y_k$ for some sequence $\{m_k\}$. An example of a multiplier is the integral operator,

$$T(f) = \int_{S^{d-1}} k(x\cdot y) f(y)dy,$$

where k is a continuous function. The associated sequence of scalars $\{m_k\}$ can be found by the Funk-Hecke formula (c.f. [9], p.11). The operator $f(\Delta)$ is also a multiplier on the spherical harmonics, with the scalars being given by $\{f(\lambda_k)\}$.

Given a sequence $\{m_k\}$ we define a function f on the spectrum of Δ by $f(\lambda_k) = m_k$. Assuming certain technical conditions are satisfied, we will describe a process for determining a kernel, $K(x,y)$, so that

$$f(\Delta)(\mu) = \int_{S^{d-1}} K(x,y)d\mu(y).$$

Bounds for the L^2-norm of $f(\Delta)(\mu)$ will be given in terms of the *t-energy* of the measure μ, $I_t(\mu)$, which is defined as

$$I_t(\mu) \equiv \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^t}.$$

We formally define the kernel K by

$$K(x,y) \equiv \sum_{k=0}^{\infty} f(\lambda_k) Z_y^{(k)}(x) \text{ for } x,y \in \mathbb{S}^{d-1}$$

where $Z_y^{(k)}(x)$ are the zonal harmonic functions. The zonal harmonics have a simple expression in terms of the ultraspherical (or Gegenbauer) polynomials on $[-1,1]$, $P_k^m(t)$, namely

$$Z_y^{(k)}(x) = c_d(k) P_k^m(x \cdot y)$$

for suitable constants $c_d(k)$ (see [12], IV.2). If we set $a(k+1) = f(\lambda_k)c_d(k)$, then we can write $K(x,y) = P(x \cdot y)$ where

$$P(t) \equiv \sum_{k=0}^{\infty} a(k+1) P_k^m(t),\ t \in [-1,1].$$

It is well known that $P_k^m(t)$ is the coefficient of z^k in the series expansion of the function $g_t(z) = (1-2tz+z^2)^{-m}$. This function is analytic on $\Omega_r = \{z \in \mathbb{C} : |z| < r\}$ for any $r < 1$, consequently

$$P_k^m(t) = \frac{1}{2\pi i} \int_{\partial\Omega_r} g_t(z) z^{-k-1} dz$$

where $\partial\Omega_r$ is the boundary of Ω_r. Our first result uses this fact to compute $P(t)$.

Proposition 2.1. *Suppose that $Q(z) = \sum_{k=0}^{\infty} a(k) z^k$ is analytic in a neighbourhood of the unit disk and $P(t) = \sum_{k=0}^{\infty} a(k+1) P_k^m(t)$ for $t \in [-1,1]$. Then*

$$\begin{aligned}(2.1)\ P(t) \quad = \quad & \frac{-1}{(m-1)!}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-(t+i\sqrt{1-t^2}))^{-m}Q(\frac{1}{z})]|_{z=t-i\sqrt{1-t^2}} + \right.\\ & \left.\frac{d^{m-1}}{dz^{m-1}}[(z-(t-i\sqrt{1-t^2}))^{-m}Q(\frac{1}{z})]|_{z=t+i\sqrt{1-t^2}}\right).\end{aligned}$$

Proof. If we let Q_N and P_N denote the N'th partial sums of Q and P respectively, then clearly

$$P_N(t) = \frac{1}{2\pi i} \int_{\partial\Omega_r} g_t(z) Q_N(\frac{1}{z}) dz.$$

Let U be a neighbourhood of $t_0 \in [-1,1]$ and assume $\omega_U \subseteq \Omega_{r,r'} = \{z \in \mathbb{C} : r < |z| < r'\}$ is a neighbourhood of the arc $\{t \pm i\sqrt{1-t^2} : t \in \overline{U}\}$ on which $Q(1/z)$ is analytic. Notice that for each N,

$$\frac{1}{2\pi i} \int_{\partial\Omega_{r,r'}} g_t(z) Q_N(\frac{1}{z}) dz \to -P_N(t)$$

as $r' \to \infty$ since the integral over the larger circle converges to zero.

Since the function $g_t(z)Q_N(1/z)$ is meromorphic in $\Omega_{r,r'}$ with only two poles, $t \pm i\sqrt{1-t^2}$, it is holomorphic in $\Omega_{r,r'} \setminus \omega_U$ provided $t \in U$, and therefore its integral over the boundary of $\Omega_{r,r'} \setminus \omega_U$ vanishes. Thus

$$P_N(t) = \frac{-1}{2\pi i} \int_{\partial\omega_U} g_t(z) Q_N(\frac{1}{z}) dz\ .$$

As $Q(z)$ is analytic in a neighbourhood of the unit circle, its partial sums converge uniformly for all z such that $1/z \in \partial\omega_U$. Because $g_t(z)$ is uniformly bounded for $z \in \partial\omega_U$ and $t \in U$, we can pass to the limit in the integral, uniformly in t. Hence

$$P_N(t) \to P(t) = \frac{-1}{2\pi i}\int_{\partial\omega_U} g_t(z)Q(\frac{1}{z})dz \text{ as } N \to \infty$$

uniformly in $t \in [-1,1]$.

For any fixed t the integrand defining P has only two singularities inside the domain ω_U, poles of order m at $t \pm i\sqrt{1-t^2}$, hence the Residue theorem can be applied to yield

$$P(t) = -\left(Res_{t+i\sqrt{1-t^2}}(g_t(z)Q(1/z)) + Res_{t-i\sqrt{1-t^2}}(g_t(z)Q(1/z))\right).$$

Evaluating this gives the desired result. ∎

Recall that for a measure μ, the projection of μ onto the space of spherical harmonics of degree k is given by

$$\mu_k(x) = \int Z_y^{(k)}(x)d\mu(y).$$

The L^2 norm of μ_k can be computed as $\int\int Z_y^{(k)}(x)d\mu(x)d\mu(y)$ and the L^2 norm of μ is given by $\sum_{k=0}^{\infty}\|\mu_k\|_2^2$.

Corollary 2.2. *Suppose $Q(z) = \sum_{k=0}^{\infty} a(k)z^k$ is analytic in a neighbourhood of the unit disk. If $f(\lambda_k) = a(k+1)/c_d(k)$, then the operator $f(\Delta)$ is given by the continuous kernel $K(x,y) = \sum f(\lambda_k)Z_y^{(k)}(x)$, i.e., for any measure μ on $\mathbb{S}^{d-1}$ we have $\int K(x,y)d\mu(y) = f(\Delta)\mu$. Moreover,*

$$\int\int K(x,y)d\mu(y)d\mu(x) = \sum_{k=0}^{\infty} f(\lambda_k)\|\mu_k\|_2^2.$$

Proof. The arguments above imply that $K_N(x,y) \to K(x,y)$ uniformly in $x, y \in \mathbb{S}^{d-1}$. Since

$$\int K_N(x,y)d\mu(y) = \sum_{k=0}^{N} f(\lambda_k)\mu_k$$

and

$$\int\int K_N(x,y)d\mu(y)d\mu(x) = \sum_{k=0}^{N} f(\lambda_k)\|\mu_k\|_2^2,$$

the result follows. ∎

Theorem 2.3. *Let $d = 2m+2 > 2$ be an even integer and let $3-d < s < 2$. Suppose $Q(z) = \sum_{k=1}^{\infty} a(k)z^k$ is analytic in a neighbourhood of the disk. If there is a constant C_1 such that for each $j = 0, 1, ..., m-1$*

$$\left|\frac{d^j}{dz^j}Q(z)|_{z=t\pm i\sqrt{1-t^2}}\right| \le C_1(1-t^2)^{-(s+j)/2} \text{ for } t \in [-1,1],$$

then there is a constant C_2 depending only on d, s and C_1 so that for any finite, positive measure μ on $\mathbb{S}^{d-1}$ we have

$$\left(\|\mu_0\|_2^2 + \left|\sum_{k=1}^{\infty}\frac{a(k+1)}{c_d(k)}\|\mu_k\|_2^2\right|\right) \le C_2 I_{s+d-3}(\mu).$$

Proof. Throughout the proof C will denote a constant depending only on d, s and C_1, which may vary. We will continue to use the notation outlined above. In particular, $P(t) = \sum a(k+1)P_k^m(t)$ and $K(x,y) = P(x \cdot y)$. The key idea is to examine the order of the singularities of $P(t)$ at ± 1. For the duration of the proof we will denote the poles of g_t as $z_0 = t + i\sqrt{1-t^2}$ and $z_1 = t - i\sqrt{1-t^2}$. Note that $|z_0| = |z_1| = 1$.

We begin by calculating (2.1); the proof of this lemma is entirely elementary. The notation $m_{(j)}$ denotes the product $m(m+1)\cdots(m+j-1)$.

Lemma 2.4. *For $N \geq 0$ and $k \geq 1$,*

$$\frac{d^N}{dz^N}\left((z-b)^{-m}Q(\frac{1}{z})\right) = \sum_{j=0}^{N}\binom{N}{j}(-1)^j m_{(j)}(z-b)^{-(m+j)}\frac{d^{N-j}}{dz^{N-j}}\left(Q(\frac{1}{z})\right),$$

and

$$\frac{d^k}{dz^k}\left(Q(\frac{1}{z})\right) = \sum_{n=1}^{k} a_n^{(k)} z^{-k-n} Q^{(n)}(\frac{1}{z})$$

where $a_n^{(k)}$ are suitable coefficients and $Q^{(n)}$ denotes the n'th derivative of Q.

Applying the residue formula (2.1) and the lemma (with $N = m-1$) one can see that $P(t)$ is equal to

$$\frac{-1}{(m-1)!}\sum_{(k,l)=(0,1),(1,0)}\left(\begin{array}{c} \sum_{j=0}^{m-2} b_j \sum_{n=1}^{m-1-j} a_n^{(m-1-j)} z_k^{-m+j-n+1} Q^{(n)}(\frac{1}{z_k})(z_k - z_l)^{-(m+j)} \\ +b_{m-1}(z_k - z_l)^{-(2m-1)}Q(\frac{1}{z_k}) \end{array}\right)$$

where $b_j = \binom{m-1}{j}(-1)^j m_{(j)}$. Since $|z_k - z_l| = 2\sqrt{1-t^2}$, the assumption on the derivatives of Q implies that for $n \leq m-1-j$,

$$\begin{aligned} |Q^{(n)}(1/z_k)(z_k - z_l)^{-(m+j)}| &\leq C\left(\sqrt{1-t^2}\right)^{-(m+j)-(s+n)} \\ &\leq C\left(\sqrt{1-t^2}\right)^{-(2m-1+s)}. \end{aligned}$$

It follows that

$$|P(t)| \leq C\left(1-t^2\right)^{-(m+(s-1)/2)}.$$

Thus

$$\int\int |K(x,y)|\, d\mu(x)d\mu(y) \leq C\int\int \frac{d\mu(x)d\mu(y)}{((1+x\cdot y)(1-x\cdot y))^{m+(s-1)/2}}.$$

Since x, y are on the sphere, $2 \pm 2x\cdot y = |x \pm y|^2$. Hence $\int\int |K(x,y)|$ is dominated by

$$C\int\int\left(|x+y|^{-(s+2m-1)} + |x-y|^{-(s+2m-1)}\right) d\mu(x)d\mu(y).$$

Define $\widetilde{\mu}(E) = \mu(E) + \mu(-E)$. As $\mu \geq 0$ it is clear that $I_t(\mu) \leq I_t(\widetilde{\mu})$ for any t. Using the relationship between energy and the Fourier transform (1.2) it is also easy to see that $4I_t(\mu) \geq I_t(\widetilde{\mu})$. Hence the previous corollary implies

$$\left|\sum_{k=0}^{\infty}\frac{a(k+1)}{c_d(k)}\|\mu_k\|_2^2\right| \leq \int\int |K(x,y)|\, d\mu(x)d\mu(y) \leq CI_{s+2m-1}(\mu).$$

∎

Remark 2.1. *Using facts found in* [12], *IV.2. one can compute* $c_d(k)$. *It is well known that* $Z_x^{(k)}(x) = a_k w_{d-1}^{-1}$, *where* $w_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ *is the surface area of* $\mathbb{S}^{d-1}$ *and* a_k *is the dimension of the space of spherical harmonics of degree* k:

$$a_k = \frac{(d+k-3)!(d+2k-2)}{(d-2)!k!} \text{ for } k \geq 2,\ a_0 = 1, a_1 = d.$$

As $P_k^m(x \cdot x) = P_k^m(1)$ *is the coefficient of* z^k *in the Taylor series expansion for* $(1-z)^{2-d}$ *we have* $P_k^m(1) = \binom{d+k-3}{k}$. *Thus*

$$c_d(k) = \frac{Z_x^{(k)}(x)}{P_k^m(x \cdot x)} = \frac{\Gamma(d/2)(d+2k-2)}{(d-2)2\pi^{d/2}} \sim k.$$

Remark 2.2. *If we take* $Q(z) = (1-z^2)^n$ *and* $d > n+3$, *then the theorem implies*

$$\left| \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{c_d(2k-1)} \|\mu_{2k}\|_2^2 \right| \leq C_1 I_{d-3-n}(\mu).$$

Next, we demonstrate how this approach may be modified to handle the situation when the power series, $Q(z) = \sum a(k) z^k$, is analytic only on the interior of the disk. In this case some additional hypothesis is required; positivity of the scalars $a(k)$ is sufficient.

Corollary 2.5. *Suppose* $Q(z) = \sum_{k=1}^{\infty} a(k) z^k$ *is analytic in the open unit disk and* $a(k) \geq 0$ *for all* k. *For* $R > 1$ *let* $Q_R(z) = Q(z/R)$. *If there is a constant* C_1 *such that for each* $j = 0, 1, ..., m-1$ *and* $R > 1$

$$\left| \frac{d^j}{dz^j} Q_R(z)|_{z = t \pm i\sqrt{1-t^2}} \right| \leq C_1 (1-t^2)^{-(s+j)/2} \text{ for all } t \in [-1,1],$$

then there is a constant C_2, *depending on* d, s *and* C_1, *so that for any finite, positive measure* μ *on* $\mathbb{S}^{d-1}$ *we have*

$$\left(\|\mu_0\|_2^2 + \sum_{k=1}^{\infty} \frac{a(k+1)}{k} \|\mu_k\|_2^2 \right) \leq C_2 I_{s+d-3}(\mu).$$

Proof. As Q_R is analytic in a neighbourhood of the disk, the coefficients $a(k+1)$ are positive and $c_d(k) \sim k$, the theorem yields

$$\left(\|\mu_0\|_2^2 + \sum_{k=1}^{\infty} \frac{R^{-(k+1)} a(k+1)}{k} \|\mu_k\|_2^2 \right) \leq C_2 I_{s+d-3}(\mu)$$

for some constant C_2. Letting $R \to 1$ gives the desired result. ∎

This can be used to bound the L^2 norm of $f(\Delta)\mu$ in terms of the energy of the measure μ.

Corollary 2.6. *Suppose* f *is real-valued and set* $a(k+1) = (f(\lambda_k))^2 c_d(k)$. *If the hypotheses of the previous corollary are satisfied with* $Q(z) = \sum_{k=1}^{\infty} a(k) z^k$, *then for any measure* μ *on* $\mathbb{S}^{d-1}$ *we have*

$$\|f(\Delta)\mu\|_2^2 = \sum_{k=0}^{\infty} (f(\lambda_k))^2 \|\mu_k\|_2^2 \leq C_2 I_{s+d-3}(\mu).$$

Corollary 2.7. *For any $s \in [0,2)$ and even integer $d = 2m+2 > 2$ there is a constant $C = C(s,d)$ such that for all finite, positive measures μ on $\mathbb{S}^{d-1}$ we have*

$$\left(\|\mu_0\|_2^2 + \sum_{k=1}^{\infty} k^{-s}\,\|\mu_k\|_2^2\right) \le C I_{d-1-s}(\mu).$$

Proof. Consider $Q(z) = (1-z)^{s-2}$. The Taylor series coefficients of Q are non-negative and asymptotically equivalent to k^{1-s}. Since $|R - z_k|^2 \ge 1-t$ the result follows from Corollary 2.5. ∎

Example 2.1. *Suppose Q corresponds to the projection onto the harmonics of order divisible by n and the dimension $d = 4$. In this case formula (2.1) reduces to*

$$P(t) = \frac{2i}{\sqrt{1-t^2}}(Q(t - i\sqrt{1-t^2}) - Q(t + i\sqrt{1-t^2})).$$

Because f is real-valued, this further simplifies to $P(t) = \frac{4}{\sqrt{1-t^2}}\operatorname{Im}(Q(t+i\sqrt{1-t^2}))$. Hence the associated family of approximating kernels is given by $K_R(x,y) = P_R(x\cdot y)$ where

$$P_R(t) = \frac{8}{\pi^2\sqrt{1-t^2}} R^{n-1}\operatorname{Im}\left(z\frac{R^n + (n-1)z^n}{(R^n - z^n)}\right),$$

for $z = t + i\sqrt{1-t^2}$.

3. Energy and spherical harmonics

Corollary 2.7 is a partial generalization of the classical formula (1.2) relating the energy of a measure on $\mathbb{R}^d$ with its Fourier transform. In this section we use properties of the heat kernel to obtain a more complete generalization. We no longer require d to be an even integer.

Theorem 3.1. *For each $0 < s < d-1$ there are constants $a, b > 0$ such that*

$$a I_s(\mu) \le \|\mu_0\|_2^2 + \sum_{k=1}^{\infty} k^{s-d+1}\|\mu_k\|_2^2 \le b I_s(\mu) \tag{3.1}$$

for all finite, positive, Borel measures μ supported on the unit sphere $\mathbb{S}^{d-1}$ with $d > 2$.

Before beginning the proof we note that as the sphere is a compact manifold of positive Ricci curvature, the heat kernel H is unique and has the form $H(t,x,y) = \sum_{k=0}^{\infty} e^{\lambda_k t} Z_y^{(k)}(x)$ where, as before, $Z_y^{(k)}(x)$ are the zonal harmonic functions of degree k with pole at y and $\lambda_k = -k(k+d-2)$ are the eigenvalues of the Laplacian.

We will use the following well-known facts about heat kernels. Throughout this section $d(x,y)$ refers to the interior metric on the sphere, a metric equivalent to the usual Euclidean metric.

Theorem 3.2. *([3], 5.5.1) The heat kernel $H(t,x,y)$ is a strictly positive C^∞ function on $(0,\infty)\times\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}$.*

Theorem 3.3. *([3], 5.5.6, 5.6.1) For all $0 < \delta < 1$ there exist positive constants C, c, depending only on d and δ, such that*

$$\begin{aligned} c(\min\{t,1\})^{-(d-1)/2} e^{-d(x,y)^2/4(1-\delta)t} &\le H(t,x,y) \\ &\le C(\min\{t,1\})^{-(d-1)/2} e^{-d(x,y)^2/4(1+\delta)t} \end{aligned}$$

for all $t > 0$ and $x, y \in \mathbb{S}^{d-1}$.

Proof. We begin by considering $E_t(\mu) \equiv \int H(t,x,y)d\mu(x)d\mu(y)$ for $t > 0$. It is known ([12], p.144) that the function $|Z_y^{(k)}(x)|$ is bounded by a polynomial in k, uniformly in x and y. As $e^{t\lambda_k}$ decreases exponentially in k for every fixed $t > 0$, the series in spherical harmonics for $H(t,x,y)$ converges uniformly in x, y. Thus

$$\int H(t,x,y)d\mu(y) = \sum_{k=0}^{\infty} e^{t\lambda_k}\mu_k(x).$$

As $|\mu_k(x)|$ is dominated by $\|Z_y^{(k)}(x)\|_\infty \|\mu\|$, similar arguments establish that

$$E_t(\mu) = \sum_{k=0}^{\infty} e^{t\lambda_k} \int (\int Z_y^{(k)}(x)d\mu(y))d\mu(x) = \sum_k e^{t\lambda_k}\|\mu_k\|_2^2.$$

Taking $\delta = \frac{1}{2}$, Theorem 3.3 gives the estimates

$$c(\min\{1,t\})^{-(d-1)/2}e^{-d(x,y)^2/2t} \le H(t,x,y) \le C(\min\{1,t\})^{-(d-1)/2}e^{-d(x,y)^2/6t}.$$

Thus

$$\begin{aligned} & c\int\int(\min\{1,t\})^{-(d-1)/2}e^{-d(x,y)^2/2t}d\mu(x)d\mu(y) \\ \le \quad & \sum_k e^{t\lambda_k}\|\mu_k\|_2^2 \le C\int\int(\min\{1,t\})^{-(d-1)/2}e^{-d(x,y)^2/6t}d\mu(x)d\mu(y), \end{aligned}$$

for $t > 0$, where the constants c, C depend only on d.

Next, multiply both sides of the inequalities by the positive function

$$\psi(t) = \begin{cases} t^{-1+s/2} & \text{if } t < 1 \\ t^{-1+(s-d+1)/2} & \text{if } t \ge 1 \end{cases},$$

and integrate over t in $(0,\infty)$. As all integrands are positive, we can change the order of integration to obtain

$$\begin{aligned} c\int\int K_2(x,y)d\mu(x)d\mu(y) \quad &\le \quad \sum_k \int_0^\infty \psi(t)e^{t\lambda_k}\|\mu_k\|_2^2 dt \\ &\le \quad C\int\int K_6(x,y)d\mu(x)d\mu(y), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} K_A(x,y) \quad &= \quad \int_0^\infty t^{-1+(s-d+1)/2}e^{-d(x,y)^2/At}dt \\ &= \quad (\frac{d(x,y)^2}{A})^{(s-d+1)/2}\int_0^\infty \tau^{-1+(s-d+1)/2}e^{-1/\tau}d\tau \\ &= \quad A^{(d-s-1)/2}(d(x,y))^{s-d+1}\Gamma(\frac{d-1-s}{2}), \end{aligned} \tag{3.3}$$

for $A = 2, 6$.

Suppose $k \neq 0$. We have

$$\begin{aligned}\int_0^\infty \psi(t)e^{t\lambda_k}dt &= \int_0^1 t^{-1+s/2}e^{t\lambda_k}dt + \int_1^\infty t^{-1+(s-d+1)/2}e^{t\lambda_k}dt \\ &= |\lambda_k|^{-s/2}\int_0^{|\lambda_k|} \tau^{-1+s/2}e^{-\tau}d\tau + \int_1^\infty t^{-1+(s-d+1)/2}e^{t\lambda_k}dt.\end{aligned}$$

The second term in the final sum is clearly bounded by $|\lambda_k|^{-1}e^{\lambda_k}$. As the function $f(x) = x^{-1+s/2}e^{-x}$ decreases when $x \geq s/2-1$ and $|\lambda_k| \geq d-1 > s/2-1$ it follows that

$$\int_1^\infty t^{-1+(s-d+1)/2}e^{t\lambda_k}dt \leq e^{1-d}(d-1)^{-1+s/2}|\lambda_k|^{-s/2}.$$

Since the first integral dominates $\int_0^1 \tau^{-1+s/2}e^{-\tau}d\tau \geq 2e^{-1}s^{-1}$, the first term in the sum satisfies

$$|\lambda_k|^{-s/2}2e^{-1}s^{-1} \leq |\lambda_k|^{-s/2}\int_0^{|\lambda_k|} \tau^{-1+s/2}e^{-\tau}d\tau \leq \Gamma(\frac{s}{2})|\lambda_k|^{-s/2}.$$

When $k = 0$, then

$$\int_0^\infty \psi(t)dt = \frac{2}{s} + \frac{2}{d-1-s}.$$

Combined with (3.2) and (3.3), these results imply that for $0 < s < d-1$ we have

$$\begin{aligned}&c2^{(d-s-1)/2}\Gamma\left(\frac{d-s-1}{2}\right) I_{d-1-s}(\mu) \\ \leq\ &\left(\frac{1}{s} + \frac{1}{d-1-s}\right)\|\mu_0\|_2^2 + \frac{1}{s}\sum_{k=1}^\infty |\lambda_k|^{-s/2}\|\mu_k\|_2^2 \\ \leq\ &C6^{(d-s-1)/2}\Gamma\left(\frac{d-s-1}{2}\right) I_{d-1-s}(\mu),\end{aligned}$$

where the constants $c, C > 0$ depend only on the dimension d. Since $|\lambda_k| \sim k^2$ we obtain the desired formula. ∎

Remark 3.1. *1. We thank Prof. L. Colzani for suggesting this approach which improved upon a previous version of this paper. Colzani has also communicated to us that another proof of Theorem 3.1 can be given by using the Poisson kernel, $(1-r^2)\,|rx-y|^{-d}$, rather than the heat kernel, and that this would give explicit constants. We prefer to give the heat kernel approach as it readily generalizes to other manifolds.*

2. Another alternative is to consider the kernels $|x-\beta y|^{-t}$ and let $\beta \to 1$. S. Eilertsen has communicated to us that for $0 < \beta < 1$

$$\int\int \frac{1}{|x-\beta y|^t}d\mu(x)d\mu(y) = \sum_{k=0}^\infty c_k(\beta)\|\mu_k\|_2^2$$

where as $\beta \to 1$ the coefficients $c_k(\beta)$ tend to

$$c_k \equiv \omega_{d-1} \frac{\Gamma(t/2+k)}{\Gamma(d+k-1-t/2)} \frac{\Gamma(d/2)\Gamma(d-t-1)}{\Gamma(t/2)\Gamma((d-t)/2))} .$$

(Here ω_{d-1} is the surface area of $\mathbb{S}^{d-1}$.) If $t > d-2$, then $c_k(\beta)$ increases to c_k, hence one can obtain the precise result $I_t(\mu) = \sum_{k=0}^{\infty} c_k \|\mu_k\|_2^2$. If $t \leq d-2$ one can only conclude that $I_t(\mu) \geq \sum_{k=0}^{\infty} c_k \|\mu_k\|_2^2$.

4. The size of pluriharmonic measures

We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and consider the set of positive functions which are pluriharmonic in the unit ball. The Taylor series around the origin of such a function can be presented as $c_0 + \sum_{|\alpha| \geq 1} c_\alpha z^\alpha + \overline{c_\alpha z^\alpha}$. Its boundary value on the sphere is a finite measure called a pluriharmonic measure.

4.1. Size of coefficients.

We have the following estimate on the size of the coefficients of pluriharmonic functions.

Proposition 4.1. *For any integer $n \geq 2$ there is a constant C such that if $f(z) = c_0 + \sum_{|\alpha| \geq 1} c_\alpha z^\alpha + \overline{c_\alpha z^\alpha}$ is a positive, pluriharmonic function defined on the unit ball in $\mathbb{C}^n$ and $0 < \varepsilon < 1$, then*

$$\sum_{k=1}^{\infty} k^{-n-\varepsilon} \left(\sum_{|\alpha|=k} \frac{\alpha!}{|\alpha|!} |c_\alpha|^2 \right) \leq C\varepsilon^{-1} c_0^2 < \infty.$$

Proof. As remarked in [1], if μ is any pluriharmonic measure, then $\mu(B(a,r)) \leq C_n r^{2n-2} \mu(\mathbb{S}^{2n-1})$ for all $a \in \mathbb{S}^{2n-1}$, $r > 0$. It is shown in [5], p. 65 that this implies

$$I_{2n-2-\varepsilon}(\mu) \leq \left(C_n(2n-2)\varepsilon^{-1} + 1\right) \left(\mu(\mathbb{S}^{2n-1})\right)^2$$

for every $\varepsilon > 0$.

When μ is the boundary value of f, then $\mu_k = \sum_{|\alpha|=k} c_\alpha z^\alpha + \overline{c_\alpha z^\alpha}$ when $k \geq 1$, and therefore (see [10], p.16)

$$\|\mu_k\|_2^2 = \sum_{|\alpha|=k} \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} 2 |c_\alpha|^2 \sim k^{1-n} \sum_{|\alpha|=k} \frac{\alpha!}{|\alpha|!} |c_\alpha|^2 .$$

Also, $\mu(\mathbb{S}^{2n-1}) = f(0) = c_0$ because f is harmonic in the ball. Now apply Theorem 3.1 (or Corollary 2.7) with $s = 2n - \varepsilon - 2$. ∎

4.2. Hausdorff dimension.

The Hausdorff dimension of a measure μ is defined as

$$\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\}.$$

It is known that if $I_t(\mu) < \infty$, then $\dim_H \mu \geq t$ (c.f. [5]).

In [4] an example is constructed of a singular, pluriharmonic, probability measure μ on the sphere in $\mathbb{C}^n$ with spherical harmonics satisfying $\|\mu_k\|_2 \leq ck^{-1/2}$ for $k \geq 1$. It is an easy consequence of formula (3.1) that such a measure has Hausdorff dimension $2n-1$, the maximum possible.

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