HOMOGENEOUS MULTIPLICATION OPERATORS ON BOUNDED SYMMETRIC DOMAINS

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Dedicated to Jaak Peetre on the occasion of his sixty fifth birthday

ABSTRACT. Let $D = G/K$ be an irreducible bounded symmetric domain in its Harish-Chandra form in $\mathbb{C}^d$ and let $\mathcal{H}_\nu(D)$ the analytic continuation of weighted Bergman spaces of holomorphic functions on D. We consider the d-tuple $M = (M_1, \cdots, M_d)$ of multiplication operators by coordinate functions and study its spectral properties. We prove that the operator M is bounded for all ν in the continuous Wallach set. We find a necessary condition on the parameter ν for which the domain D is a k-spectral set of the tuple M. In particular, in the case where D is the unit ball in $\mathbb{C}^d$, we find the values of ν for which each multiplication operator M_j is hyponormal or subnormal or the tuple M is subnormal.

1. INTRODUCTION

The spectral theory of single operators, in particular of operators on function spaces, has been a highly developed subject. Among other important results is the *von Neumann inequality*: if T is a contraction on a Hilbert space H, then $\|f(T)\| \le \|f\|_\infty$ for any polynomial f, where $\|f\|_\infty$ is the maximum of the module of the polynomial f on the unit circle. This result can also be reformulated as follows. Consider the Hardy space H^2 on the unit disk and the multiplication operator M by the coordinate function z. Then $\|f(T)\| \le \|f(M)\|$ for any polynomial f. In his paper [6] Arveson proved a version of the von Neumann inequality for a contractive tuple of operators on a Hilbert space; see also [16] and [2]. He found a distinguished tuple of operators on a function space on the unit ball of $\mathbb{C}^d$, called the *symmetric Fock space*, which dominates any other contractive tuple of commuting operators. More precisely, let $\mathcal{H}_1(B^d)$ be the Hilbert space of holomorphic functions on B^d with reproducing kernel $(1 - \langle z, w\rangle)^{-1}$ and let $M = (M_1, \ldots, M_d)$ be the corresponding multiplication operators on $\mathcal{H}_1(B^d)$ by coordinate functions z_j. A tuple $T = (T_1, \ldots, T_d)$ of commuting operators on a Hilbert space H is called a *contractive tuple* if $\|T_1x_1 + \cdots + T_dx_d\|^2 \le \|x_1\|^2 + \cdots + \|x_d\|^2$ for all $x_1, \ldots, x_d \in H$. Arveson proved that the operator norm $\|f(T)\|$ for a polynomial f is dominated by the operator norm of $f(M)$, $\|f(T)\| \le \|f(M)\|$. Several other remarkable properties of the Hilbert space $\mathcal{H}_1(B^d)$ are also established [6]. Thus the operator tuple M on $\mathcal{H}_1(B^d)$

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has some distinguished properties. In this paper we will study the spectral property of a family of multiplication tuples on the holomorphic function spaces with the reproducing kernel $(1 - \langle z, w\rangle)^{-\nu}$ on the unit ball, for $0 < \nu < \infty$. We shall study also similar problems on a general irreducible bounded symmetric domain. For a matrix domain of type I those tuples have been studied earlier by Gagchi and Misra [7]. In particular, we answer here some of their open problems.

2. Preliminaries

In this section we recall some basic facts about bounded symmetric domains and fix the notation. We will be very brief and refer for the necessary background to [12], [13] and [17] and references therein.

Let D be an irreducible bounded symmetric domain, realized as the open unit ball in a Jordan triple V of finite rank r. Let $\{z\bar{v}u\} = \frac{1}{2}Q(z,u)\bar{v}$ be the triple product and $Q(z,u)$ the polarization of the quadratic map $Q(z)$. (Our triple product differs by a factor of $\frac{1}{2}$ from that of [17].) Let G be the identity component of the group of biholomorphic automorphisms of D, and let K be the isotropic subgroup of $0 \in D$. Then $D \equiv G/K$ is a hermitian symmetric space. We normalize a K-invariant inner product on V and denote $|v|$ the corresponding norm so that a minimal tripotent has norm 1. It follows immediately by the Peirce decomposition for an element $z \in V$ that if $z \in D$ then $|z| \le \sqrt{r}$. We fix a frame $\{e_1, e_2, \dots, e_r\}$ of minimal pairwise orthogonal tripotents and let $e = e_1 + \cdots + e_r$.

Let

$$V = \sum_{0 \le i \le j \le r} \oplus V_{ij}$$

be the joint *Peirce decomposition* of V with respect to the frame $\{e_1, \dots, e_r\}$. The integers

$$a = \dim V_{ij}, \quad b = \dim V_{0i}, \quad 0 < i < j \le r \tag{2.1}$$

are called the characteristic multiplicities, and they are independent of the decomposition. The Peirce decomposition with respect to the maximal tripotent e is

$$V = V_1 + V_{\frac{1}{2}}, \quad \text{with } V_1 = \sum_{1 \le i \le j \le r} V_{ij}, \quad , V_{\frac{1}{2}} = \sum_{j=1}^{r} V_{0j},$$

and

$$d_1 = \dim V_1 = \frac{1}{2} r(r-1)a + r, \quad d_{\frac{1}{2}} = \dim V_{\frac{1}{2}} = rb, \quad d = d_1 + d_{\frac{1}{2}}.$$

The integer

$$p = \frac{2d_1 + d_{\frac{1}{2}}}{r} = 2 + a(r-1) + b$$

will be called the *genus* of D.

For each fixed j let Δ_j the *determinant polynomial* of the Jordan algebra $A_j := \sum_{1\le i\le j} V_{ij}$, and extend it to a polynomial on V via the orthogonal projection onto A_j. A tuple $\underline{\mathbf{m}} = (m_1, \dots, m_r)$ of integers with $m_1 \ge m_2 \ge \dots \ge m_r \ge 0$ is called a *signature*. For each signature $\mathbf{m}$ let

$$\Delta_{\underline{\mathbf{m}}}(z) = \Delta_1(z)^{m_1-m_2} \dots \Delta_{r-1}(z)^{m_{r-1}-m_r} \Delta_r^{m_r}(z)$$

be the associated *conical polynomial*, and let $\mathcal{P}_{\underline{\mathbf{m}}} = \operatorname{span}\{\Delta_{\mathbf{m}} \circ k; k \in K\}$. It is known that the $\mathcal{P}_{\underline{\mathbf{m}}}$ are irreducible and mutually inequivalent under K, and that the space $\mathcal{P}(V)$ of all holomorphic polynomials on V admits the direct sum decomposition

$$\mathcal{P}(V) = \sum_{\underline{\mathbf{m}}} \oplus \mathcal{P}^{\underline{\mathbf{m}}}(V), \tag{2.2}$$

where the summation ranges over all signatures $\underline{\mathbf{m}}$.

Let us denote $L = \{l \in K; l \cdot e = e\}$ the isotropic subgroup of K at e. Thus $S = K/L$ is the *Shilov boundary* of D. For each signature $\underline{\mathbf{m}}$ let

$$\phi_{\underline{\mathbf{m}}}(w) = \int_L \Delta_{\underline{\mathbf{m}}}(lw) dl \tag{2.3}$$

be the associated *spherical polynomial*. It is known that $\phi_{\underline{\mathbf{m}}}$ is the only L-invariant polynomial in $\mathcal{P}_{\underline{\mathbf{m}}}$ for which $\phi_{\underline{\mathbf{m}}}(e) = 1$. See [14], [21] and [12].

Consider the *Fock space* $\mathcal{F}$ of entire functions on V with the inner product

$$(f, g)_{\mathcal{F}} = \frac{1}{\pi^d} \int_V f(z)\overline{g(z)} e^{-|z|^2} dm(z).$$

The reproducing kernel of $\mathcal{F}$ is then $e^{(z,w)}$. Let $K_{\underline{\mathbf{m}}}(z, w)$ be the reproducing kernel of the space $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ with the Fock space norm. Then (2.2) implies that

$$e^{(z,w)} = \sum_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, w).$$

The Bergman reproducing kernel of D with respect to the normalized Lebesgue measure is $h(z, w)^{-p}$, where $h(z, w)$ is an irreducible sesqui-holomorphic polynomials on $D \times D$ so that $h(z, z) = \prod_{j=1}^r (1 - s_j(z)^2)$, where $\{s_j(z)\}_{j=1}^r$ are the *singular numbers* of z. For $\nu > p - 1$ consider the weighted Bergman space $\mathcal{H}_\nu(D)$ of holomorphic functions f on D so that

$$\|f\|_\nu^2 = c_\nu \int_D |f(z)|^2 h(z, z)^{\nu - p} dm(z) < \infty,$$

where c_ν is a normalization constant so that the function 1 has norm 1. Its reproducing kernel is $h(z, w)^{-\nu}$. In terms of the decomposition (2.2) we have

$$h(z, w)^{-\nu} = \sum_{\underline{\mathbf{m}}} (\nu)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, w), \tag{2.4}$$

where $(\nu)_{\underline{\mathbf{m}}}$ is the generalized *Pochhammer symbol*:

$$(\nu)_{\underline{\mathbf{m}}} = \prod_{j=1}^{r}(\nu - \frac{a}{2}(j-1))_{m_j} = \prod_{j=1}^{r}\prod_{l=1}^{m_j}(\nu - \frac{a}{2}(j-1)+l-1).$$

The reproducing kernel $h(z,w)^{-\nu}$ is positive definite and thus defines a Hilbert spaces of holomorphic functions on D, for a large set of ν. More precisely, $h(z,w)^{-\nu}$ is positive definite exactly when ν is in the so-called *Wallach set*

$$W(D) = \{0, \frac{a}{2}, \dots, \frac{a}{2}(r-1)\} \cup (\frac{a}{2}(r-1), \infty).$$

See [12] and references therein. For $\nu > \frac{a}{2}(r-1)$, we still have

$$\mathcal{H}_\nu(D) = \sum_{\underline{\mathbf{m}}} \oplus \mathcal{P}^{\underline{\mathbf{m}}}(V), \tag{2.5}$$

in the Hilbert space sense. The reproducing kernel of $\mathcal{H}_\nu(D)$ is $h(z,w)^{-\nu}$, and the group G acts isometrically on $\mathcal{H}_\nu(D)$ by means of the projective representation

$$\pi_\nu(g)(f)(z) := \left(J(g^{-1})(z)\right)^{\frac{\nu}{p}} f(g^{-1}(z)). \tag{2.6}$$

Here and bellow $J(g)(z) := \mathrm{Det}(g'(z))$ denotes the complex Jacobian of $g \in G$ at the point $z \in D$.

Fix an orthonormal basis $\{v_j\}_{j=1}^d$ of V. For any $z \in D$ let $z_j = \langle z, v_j\rangle$ be its coordinates in the fixed basis $\{v_j\}$. Thus $z = \sum_{j=1}^d z_j v_j$.

Lemma 2.1. *With the above notation we have $|z_j| < \sqrt{r}$ for all $z \in D$.*

Indeed $|z_j| = |\langle z, v_j\rangle| \le |z||v_j| \le \sqrt{r}$, since $|v_j| = 1$.

We let M_j be the multiplication operator on the Hilbert space $\mathcal{H}_\nu(D)$ by the coordinate function z_j. Note that the tuple $M = (M_1, \dots, M_d)$ is K-invariant, namely if $k \in K$ and $M' = (M'_1, \dots, M'_d)$ is the tuple of multiplication operators by the coordinate functions associated with the basis $\{w_j\}_{j=1}^d$, with $w_j = kv_j$, then M and M' are unitarily equivalent: $M'_j f = (M_j(f \circ k)) \circ k^{-1}$ for all j.

In order to study the operator tuple M we use the idea of Arveson and consider the Hilbert space

$$\mathcal{H}_\nu(D) \otimes V' = \mathcal{H}_\nu(D) \otimes \mathbb{C}^d = \overbrace{\mathcal{H}_\nu(D) \oplus \cdots \oplus \mathcal{H}_\nu(D)}^{d \text{ copies}};$$

the space $V' = \mathbb{C}^d$ is viewed here as the cotangent space of D, with the co-adjoint action of K, $(k \cdot v')(w) = v'(k^{-1}w)$ for $v' \in V'$ and $w \in V$. Thus K acts on the tensor space naturally by

$$k(f \otimes v')(z) = f(k^{-1}z) \otimes k'v.$$

Let $\widetilde{M} : \mathcal{H}_\nu(D) \otimes V' \mapsto \mathcal{H}_\nu(D)$ be defined by

$$\widetilde{M}(f \otimes v')(z) = f(z)v'(z). \tag{2.7}$$

Clearly, $\widetilde{M}$ intertwines the actions of K. We observe also that

$$\widetilde{M}\widetilde{M}^* = \sum_{j=1}^{d} M_j M_j^* \tag{2.8}$$

and that $\widetilde{M}\,\widetilde{M}^*$ is K-invariant:

$$\widetilde{M}\,\widetilde{M}^*(f \circ k) = (\widetilde{M}\,\widetilde{M}^* f) \circ k \tag{2.9}$$

for all $k \in K$ and $f \in \mathcal{H}_\nu(D)$.

3. Rank one case: the unit ball in $\mathbb{C}^d$

We shall study bellow the contractivity, hyponormality and subnormality and k-spectral property of the multiplication tuples. We consider now the rank one case, namely the case where D is the open unit ball B^d in $\mathbb{C}^d$. The ideas developed here will be used latter to treat the higher rank cases. We recall first some definition; see [6], [10].

Definition 3.1. A bounded operator M on a Hilbert space H is called *subnormal* if there exists a Hilbert space K and a normal operator N on K so that $H \subset K$, N has H as its invariant subspace and its restriction of N is M. M is called *hyponormal* if $[M^*, M] \geq 0$. A tuple $T = (T_1, \ldots, T_d)$ of commuting operators on a Hilbert space H is called a *contractive tuple* if

$$\|T_1 x_1 + \cdots + T_d x_d\|^2 \leq \|x_1\|^2 + \cdots + \|x_d\|^2$$

for all $x_1, \ldots, x_d \in H$. A tuple $T = (T_1, \ldots, T_d)$ of commuting operators on a Hilbert space H is called a *subnormal tuple* is there exists a tuple $N = (N_1, \ldots, N_d)$ of commuting normal operators on a Hilbert space $K \supset H$, so that H is an invariant subspace of N_j and such that $T_j = N_{j_{|H}}$ for all j.

Let

$$G = SU(1,d) := \{g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(d+1, \mathbb{C}); g^*\, J\, g = J \text{ and } \det(g) = 1\},$$

where $J := \begin{pmatrix} 1 & 0 \\ 0 & -I_d \end{pmatrix}$, and A, B, C, D are 1×1, $1 \times d$, $d \times 1$ and $d \times d$ complex matrices, respectively. G acts on B^d via fractional linear transformations

$$gz = (Az + B)(Cz + D)^{-1}, \quad \text{where } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ z \in B^d,$$

K consists of linear transformation $z \mapsto AzD^{-1}$, with $\det(A)\det(D) = 1$.

We present first an elementary observation about Möbius-invariant Hilbert spaces on the unit disk B^1. Let $SU(1,1)$ be the Möbius group. It acts isometrically on

$\mathcal{H}_\nu(B^1)$ via (2.6), namely

$$\pi_\nu(g)f(z) = f(\frac{az+b}{cz+d})(cz+d)^{-\nu}, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The following result is a corollary of a more general result of Arazy and Fisher (see e.g. [4]). We include an elementary proof for the sake of completeness.

Lemma 3.2. *Let H be a Hilbert space of holomorphic functions on the unit disk B^1. Suppose that the space $\mathcal{P} = \mathcal{P}(B^1)$ of polynomials is dense in H and that there exists some $\nu > 0$ so that H is isometrically invariant under the group action π_ν of $SU(1,1)$. Then $H = \mathcal{H}_\nu(B^1)$ in the sense that there exists a positive constant c so that $\|f\|_H = c\|f\|_\nu$.*

This result extends to uniformly bounded actions of $SU(1,1)$ by π_ν; the conclusion is then that $H = \mathcal{H}_\nu(B^1)$ with equivalent norms. See [4].

Proof. By our assumption it is clear that the polynomials $\{z^n\}$ form an orthogonal basis, since the subgroup of rotations in $SU(1,1)$ acts on the one-dimensional subspaces $\mathbb{C}z^n$, $n \in \mathbb{N}$, with different characters. Next we calculate the norm of z^n. For this purpose we calculate infinitesimal action of the Lie algebra $\mathfrak{g}$ of the group $SU(1,1)$ on the subspace $\mathcal{P}$. Take $\xi = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ we see that

$$\pi_\nu(\xi)z^n = nz^{n-1} - (\nu+n)z^{n+1};$$

so that $\mathcal{P}$ is invariant under the action $\pi_\nu(\xi)$ so that, by the unitarity of $\pi_\nu(\exp(t\xi))$, $\pi_\nu(\xi)$ is a skew-symmetric operator. Therefore

$$(\pi_\nu(\xi)z^n, z^{n+1}) = -(z^n, \pi_\nu(\xi)z^{n+1})$$

which reads

$$(\nu+n)\|z^{n+1}\|^2 = (n+1)\|z^n\|^2$$

or

$$\frac{\|z^{n+1}\|^2}{\|z^n\|^2} = \frac{n+1}{\nu+n}.$$

By induction we get

$$\|z^n\|^2 = \frac{n!}{(\nu)_n}\|1\|^2 = c\|z^n\|_\nu^2..$$

This proves our claim with $c = \|1\|^2$. ☐

The following result is elementary and can be proved by direct calculation.

Lemma 3.3. *The multiplication operator M_z by z on $\mathcal{H}_\nu(B^1)$ is bounded for all $\nu > 0$. The following condition are equivalent*

(a): $\nu \geq 1$;

(b): *M_z is a contraction;*

(c): *M_z is a hypernormal;*

(d): *M_z is a subnormal.*

We consider first the subnormality and hyponormality of the multiplication operators M_j on $\mathcal{H}_\nu(B^d)$. We fix $j = 1$ and consider the operator M_1. The element $e_1 = (1, 0, \ldots, 0)$ is a minimal (and maximal) tripotents of V. The corresponding Peirce decomposition relative to e_1 is then

$$V = V_1 \oplus V_{\frac{1}{2}}$$

where $V_1 = \mathbb{C}e_1$ and $V_{\frac{1}{2}} = \{(0, z_2, \ldots, z_d); z_j \in \mathbb{C}, j = 2, \ldots, d\} \equiv \mathbb{C}^{d-1}$. Thus

$$\mathcal{P}(V) = \mathcal{P}(V_1) \otimes \mathcal{P}(V_{\frac{1}{2}}) \tag{3.1}$$

We recall further the embedding ρ of $SU(1,1) \times SU(d-1)$ into $G = \mathrm{Aut}(B^d) \equiv SU(1,d)$. If $g_1 \in SU(1,1)$ is the Möbius transformation $g_1(z_1) = \frac{z_1 - u}{1 - z_1\bar{u}}$, then $g := \rho(g_1) \in G$ is given by

$$g(z_1, z') = (\frac{z_1 - u}{1 - z_1\bar{u}}, \frac{z'}{1 - z\bar{u}}). \tag{3.2}$$

If $g_1(z_1) = e^{2i\theta}z_1 \forall z_1 \in B^1$ then $g := \rho(g_1)$ is given by $g(z_1, z') = (e^{2i\theta}z_1, e^{i\theta}z_2, \ldots, e^{i\theta}z_d)$. The elements of $SU(d-1)$ are embedded into $SU(1,d)$ trivially: if $D \in SU(d-1)$, then $\rho(D) \in SU(1,d)$ is the mapping $z = (z_1, z') \mapsto (z_1, Dz)$. It is easily checked that ρ is a monomorphism, and hence we can view $SU(1,1) \times SU(d-1)$ as a subgroup of $SU(1,d)$. We shall need bellow the elementary fact that the Jacobian of $g = (g_1, D) \in SU(1,1) \times SU(d-1)$ is

$$J_g(z) = J_{g_1}(z_1)^{d+1} \det(D), \quad \forall z \in B^d.$$

The polynomial space $\mathcal{P}(V_{\frac{1}{2}})$ decomposes under the action of $SU(d-1)$ into irreducible invariant subspaces as

$$\mathcal{P}(V_{\frac{1}{2}}) = \sum_{s=0}^{\infty} \mathcal{P}^s(V_{\frac{1}{2}}),$$

where for each $s \in \mathbb{N}$ $\mathcal{P}^s(V_{\frac{1}{2}})$ is the space of homogeneous polynomials of degree s. Consequently

$$\mathcal{P}(V) = \sum_{s=0}^{\infty} \mathcal{P}(V_1) \otimes \mathcal{P}^s(V_{\frac{1}{2}}). \tag{3.3}$$

This decomposition will be the main tool of our investigation of the properties of M_1. We let $\mathcal{P}^s_\nu(V_1)$ be $\mathcal{P}^s(V_1)$ equipped with the norm induced by $\mathcal{H}_\nu(B^d)$, namely we view a polynomial on V_1 also as a polynomial on V. The completion of (3.3) in the space $\mathcal{H}_\nu(B^d)$ is identified in the following Proposition. Here $\mathcal{H}_{\nu+s}(B^1) \otimes \mathcal{P}^s_\nu(V_{\frac{1}{2}})$ is the Hilbert-space tensor product.

Proposition 3.4. *Let $\nu \geq 0$. The Hilbert space $\mathcal{H}_\nu(B^d)$ is decomposed under the isometric action π_ν (2.6) of $SU(1,1) \times SU(d-1)$ into the direct sum*

$$\mathcal{H}_\nu(B^d) = \sum_{s=0}^{\infty} \mathcal{H}_{\nu+s}(B^1) \otimes \mathcal{P}_\nu^s(V_1)$$

of irreducible subspaces. Moreover, each component $\mathcal{H}_{\nu+s}(B^1) \otimes \mathcal{P}_\nu^s(V_1)$ is a function module of $H^\infty(B^1)$, and the decomposition is equivariant. Namely, if P_s is the orthogonal projection onto $\mathcal{H}_{\nu+s}(B^1) \otimes \mathcal{P}_\nu^s(V_1)$, then

$$P_s(mf) = mP_s(f)$$

for any $m \in H^\infty(B^1)$.

Proof. We first prove that $\mathcal{H}_{\nu+s}(B^1) \otimes \mathcal{P}_\nu^s(V_{\frac{1}{2}})$ are indeed subspaces of $\mathcal{H}_\nu(B^d)$. Consider the closed linear subspace L_s generated by the holomorphic functions $F \in \mathcal{H}_\nu(B^d)$ of the form $F(z) = f(z_1)p(z')$, where $f(z_1)$ are polynomials in z_1 and $p \in \mathcal{P}^s(V_{\frac{1}{2}})$ are polynomial of z'. We prove that $L_s = \mathcal{H}_{\nu+s}(B^1) \otimes \mathcal{P}_\nu^s(V_{\frac{1}{2}})$. Clearly both spaces have the span of the above polynomials $F(z) = f(z_1)p(z')$ as their dense subspace, so we need only to identify the norm in L_s. Fix any two polynomials $f_1(z_1)$ and $f_2(z_1)$ of z_1 and $h_j(z'), h_2(z') \in P_s(V_{\frac{1}{2}})$ let $F_j(z) = f_j(z_1)h_j(z') \in L_s$ and consider the inner product $\langle F_1, F_2\rangle_\nu$; it defines a $SU(d-1)$-invariant quadratic form on $\mathcal{P}_\nu^s(V_{\frac{1}{2}})$, which is an irreducible representation of $SU(d-1)$. Thus by Schur's lemma, we have

$$\langle F_1, F_2\rangle_\nu = \langle f_1h_1, f_2h_2\rangle_\nu = C_{\nu,s}(f_1, f_2)(h_1, h_2)_\nu \tag{3.4}$$

where $C_{\nu,s}(f_1, f_2)$ is independent of h_1, h_2. We claim that

$$C_{\nu,s}(f_1, f_2) = (f_1, f_2)_{\nu+s}$$

where the right hand side is the inner product in $\mathcal{H}_{\nu+s}(B^1)$. Indeed, fix $h_1 = h_2 = h \neq 0$ and consider the subspace $L_s(h)$ of L_s of the form $F(z) = f(z_1)h(z')$. Take elements $g = (g_1, I) \in SU(1,1) \times SU(d-1) \in SU(1,d)$ as in (3.2). We have

$$\pi_\nu(g^{-1})F(z) = f(g_1z_1)p(z')(1 - z_1\bar{u})^{-\nu-s}.$$

Clearly $L_s(h)$ is $SU(1,1) \times SU(d-1)$-invariant and the quadratic form $C_{\nu,s}(f_1, f_2)$ defines an $SU(1,1)$-invariant norm on the subspace. Hence by Lemma 3.1 we see that $C_{\nu,s}(f_1, f_2)$ is a constant multiple of $(f_1, f_2)_{\nu+s}$; the constant is 1 by the formula (3.4)). Thus

$$\langle F_1, F_2\rangle_\nu = (f_1 \otimes h_1, f_2 \otimes h_2).$$

Now the remaining claims are immediate consequences. □

As a corollary we find

Corollary 3.5. *Let $\nu > 0$. Then each operator M_j, $j = 1, \dots, d$ is hyponormal or subnormal if and only if $\nu \geq 1$.*

Indeed, in the above decomposition of $\mathcal{H}_\nu(B^d)$, all subspaces $\mathcal{H}_{\nu+s}(B^1)\otimes\mathcal{P}_\nu^s(V_{\frac{1}{2}})$ are invariant under M_1, and thus they are reducible invariant subspaces of M_1. M_1 is hypernormal if and only if it is hypernormal on each subspace $\mathcal{H}_{\nu+s}(B^1)\otimes\mathcal{P}_\nu^s(V_{\frac{1}{2}})$. The result then follows from Lemma 3.2.

Remark 3.6. Proposition 3.3 can also be proved by using the integral formula of the invariant norm in $\mathcal{H}_\nu(B^d)$ given in [3]. If $F\in\mathcal{H}_\nu(B^d)$ then

$$(F,F)_\nu=\frac{1}{(\nu)_l}\int_{B^d}(R+\nu)_l F(z)\overline{F(z)}d\mu_{\nu+l}(z)$$

where l is a positive integer such that $\nu+l>d$, and $d\mu_{\nu+l}(z)$ is the probability measure $c_{\nu+l}(1-|z|^2)^{\nu+l-d-1}dm(z)$. If F is a sum of the functions of the form $f(z_1)p(z')$, then we can calculate the norm of F be separation of variables. However this proof is not so easy to generalize to higher rank domains; also the above proof is somewhat easier.

We study now the tuple M and its relation to the operator $\widetilde{M}$ defined by (2.7).

Lemma 3.7. *Let $\nu>0$. Then*

$$\widetilde{M}\widetilde{M}^*=\sum_{j=1}^d M_jM_j^*=\sum_{m=1}^\infty\frac{m}{\nu+m-1}P_m,$$

where P_m is the orthogonal projection onto the space $\mathcal{P}^m$.

Proof. By Schur's lemma the operator $\widetilde{M}\widetilde{M}^*$ is a constant multiple of the identity operator on each polynomial space $\mathcal{P}^m$. To find the constant we compute $\widetilde{M}\widetilde{M}^*$ on the function $z_1^m\in\mathcal{P}^m$. Since M_j are weighted shifts, we have $M_1^*1=0$ and

$$M_1^*z_1^m=\frac{\|z_1^m\|^2}{\|z_1^{m-1}\|^2}=\frac{m}{\nu+m-1},\quad m\geq 1$$

□

Corollary 3.8. *Let $\nu>0$. Then*

$$\|\widetilde{M}\widetilde{M}^*\|=\begin{cases}\frac{1}{\nu}, & \nu<1\\ 1, & \nu<1.\end{cases}.$$

Thus, the tuple M is contractive on $\mathcal{H}_\nu(B^d)$ if and only if $\nu\geq 1$.

In this paper [6] Arveson defined the notion of a null contractive tuple. Consider the normal completely bounded map P on $B(\mathcal{H}_\nu(B^d))$: $\mathcal{C}(A)=\sum_{j=1}^d M_jAM_j^*$. M is called *null contractive* if $\mathcal{C}^\infty(I):=\lim_{k\to\infty}\mathcal{C}^k(1)=0$ in the strong operator topology. Thus we have

Lemma 3.9. *If $\nu\geq 1$ then*

$$\mathcal{C}^k(I)=\sum_{m=k}^\infty\frac{(m)_k}{(\nu+m-1)_k}P_m.$$

In particular, $\mathcal{C}^\infty(I)=\lim_{k\to\infty}\mathcal{C}^k(I)=0$, and M is null contractive.

Recall that a function module $\mathcal{H}$ of $H^\infty(B^d)$ is called contractive if $\|fh\| \le \|f\|_\infty \|h\|$ for all polynomial f and $h \in \mathcal{H}$; see [11].

Theorem 3.10. *Let $\nu \ge 1$. Each operator M_j is subnormal on $\mathcal{H}_\nu(B^d)$. If $1 \le \nu < d$ then $M = (M_1, \dots, M_n)$ is not a subnormal tuple and $\mathcal{H}_\nu(B^d)$ is not a contractive module of $H^\infty(B^d)$.*

Proof. The first part follows immediately from Theorem 3.2. Note that by [7] B^d is the joint spectrum of M, so if M on $\mathcal{H}_\nu(B^d)$ is subnormal then $\mathcal{H}_\nu(B^d)$ is a contractive module of $H^\infty(B^d)$. We need only prove that $\mathcal{H}_\nu(B^d)$ is not a contractive module of $H^\infty(B^d)$. Let ψ_m be a Ryll-Wojtaszczyk polynomial in the space $\mathcal{P}^m$ of homogeneous polynomials, so that by their construction,

$$\|\psi_m\|_d = \|\psi_m\|_{L^2(\partial B^d)} = 1, \quad \|\psi_m\|_\infty \le C;$$

see [20] and [9]. Note here that $\|\cdot\|_d$ is the Hardy space norm. Consider the function

$$f(z) = \sum_{m=0}^{\infty} c_m \psi_m^{(1)}(z)$$

with $\{c_m\}$ a sequence of complex numbers such that

$$\sum_{m=0}^{\infty} |c_m| \le 1, \quad \sum_{m=0}^{\infty} |c_m|^2 \frac{(d)_m}{m!} = \infty.$$

The existence of such a sequence follows easily from the fact that $\sup \frac{(d)_m}{m!} = \infty$ and the closed graph theorem. Thus $f \in H^\infty(B^d)$ and $\|f\|_\infty \le 1$. Suppose $\nu < d$ and that $\mathcal{H}_\nu(B^d)$ is a contractive module of $H^\infty(B^d)$. So that we have

$$\|f\|_\nu^2 = \|f1\|_\nu^2 \le \|f\|_\infty \le 1 \tag{3.5}$$

However

$$\begin{aligned} \|f\|_\nu^2 &= \sum_{m=0}^{\infty} |c_m|^2 \|\psi_m^{(1)}\|_\nu^2 = \sum_{m=0}^{\infty} |c_m|^2 \frac{\|\psi_m^{(1)}\|_\nu^2}{\|\psi_m^{(1)}\|_d} \|\psi_m^{(1)}\|_d \\ &= \sum_{m=0}^{\infty} |c_m|^2 \frac{\|\psi_m^{(1)}\|_\nu^2}{\|\psi_m^{(1)}\|_d} \\ &= \sum_{m=0}^{\infty} |c_m|^2 \frac{(d)_m}{m!} = \infty, \end{aligned}$$

which contradicts (3.5). This completes the proof. ◻

When $\nu = 1$ this result is proved in [6], Corollary 1. Our proof here is somewhat more conceptional by using the Ryll-Wojtaszczyk polynomials.

The proof above actually implies also the following result.

Corollary 3.11. *Let $\nu \geq 1$. The unit ball B^d (considered as the Taylor spectrum of $M = (M_1, \dots, M_n)$) is an k-spectral set of M, namely $\|f(M)\| \leq k \sup\{|f(z)|; z \in B^d\}$ for some $k > 0$ and all polynomials f, if and only if $\nu \geq d$.*

Thus, the e operator tuples M on the spaces $\mathcal{H}_\nu(B^d)$ $(1 \leq \nu < d)$ provides a natural family of commuting subnormal operators which do not form a subnormal tuple. Earlier, Lubin [18] and Abrahames [1] provided other examples of weighted shifts with the same property. We may well expect that the tuple M will provide counterexamples to some other known problems.

4. Boundedness properties of the multiplication operators

We will assume henceforth that D is an irreducible bounded symmetric domain of rank $r \geq 2$ as introduced in Section 2. First we study the boundedness properties.

Theorem 4.1. *The operators $M_1, \dots, M_d$ are bounded on $\mathcal{H}_\nu(B^d)$ if $\nu > \frac{a}{2}(r-1)$.*

We shall actually estimate the norm of the operator $M_1 M_1^* + \dots M_d M_d^*$, which is diagonal under the Schmid decomposition (2.5), and find its eigenvalues. First we recall a recurrence formula from [24]; see also [15], Proposition 5 (where a recurrence formula for the Macdonald q-Jack symmetric polynomials is obtained).

Lemma 4.2. *The following recurrence formula holds*

$$\phi_1(z)\phi_{\underline{\mathbf{m}}}(z) = \frac{1}{r}\sum_{j=1}^{r} c_{\underline{\mathbf{m}}}(\underline{\mathbf{m}} + \gamma_j)\phi_{\underline{\mathbf{m}}+\gamma_j}(z)$$

where

$$c_{\underline{\mathbf{m}}}(j) = \prod_{k \neq j} \frac{m_j - m_k + \frac{a}{2}(1 + k - j)}{m_j - m_k + \frac{a}{2}(k - j)},$$

with the convention that $c_{\underline{\mathbf{m}}}(j) = 0$ if $\underline{\mathbf{m}} + \gamma_j$ is not a signature.

The first part of the next result is proved in Lemma 3.1 in [12], which together with the uniqueness of $\phi_{\underline{\mathbf{m}}}(z)$ implies the second part, and the third part follows from [22], Lemma 2.6. Denote $d_{\underline{\mathbf{m}}}$ the dimension of $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ and $d_{\underline{\mathbf{m}}}(V_1)$ for the dimension of $\mathcal{P}^{\underline{\mathbf{m}}}(V_1)$.

Lemma 4.3. *The following recurrence formula holds*

$$K_{\underline{\mathbf{m}}}(z, e) = \frac{1}{\|\phi_{\underline{\mathbf{m}}}\|^2_{\mathcal{F}(V)}}\phi_{\underline{\mathbf{m}}}(z)$$

and

$$\|\phi_{\underline{\mathbf{m}}}\|^2_{\mathcal{F}(V)} = \frac{(\frac{d}{r})_{\underline{\mathbf{m}}}}{d_{\underline{\mathbf{m}}}}. \tag{4.1}$$

In fact the polynomial $\phi_{\underline{\mathbf{m}}}(z)$, for $z = z_1 + z_{\frac{1}{2}} \in V_1 \otimes V_{\frac{1}{2}}$, is independent of the variables $z_1 \in V_{\frac{1}{2}}$ and thus we have

$$\|\phi_{\underline{\mathbf{m}}}\|^2_{\mathcal{F}(V)} = \|\phi_{\underline{\mathbf{m}}}\|^2_{\mathcal{F}(V_1)} = \frac{(\frac{d_1}{r})_{\underline{\mathbf{m}}}}{d_{\underline{\mathbf{m}}}(V_1)}. \tag{4.2}$$

The dimension $d_{\underline{\mathbf{m}}}(V_1)$ is given by

$$d_{\underline{\mathbf{m}}}(V_1) = \prod_{1 \le j < k \le r} \frac{m_j - m_k + \frac{a}{2}(k-j)}{\frac{a}{2}(k-j)} \frac{B(m_j - m_k, \frac{a}{2}(k-j-1)+1)}{B(m_j - m_k, \frac{a}{2}(k-j+1))}$$

where B is the ordinary Beta-function.

Recall that the notation $\widetilde{M}$ was defined in (2.7).

Proposition 4.4. *The operator $\widetilde{M}\widetilde{M}^*$ acts on $\mathcal{H}_\nu(B^d)$ as a diagonal operator under the decomposition (2.5). Namely, for every signature $\mathbf{m}$,*

$$\tilde{M}\tilde{M}^* f = \tau(\underline{\mathbf{m}}) f, \quad f \in \mathcal{P}^{\underline{\mathbf{m}}}(V), \tag{4.3}$$

where

$$\tau(\underline{\mathbf{m}}) = \sum_{j=1}^{r} \frac{\frac{a}{2}(r-j) + m_j}{\nu - \frac{a}{2}(j-1) + m_j - 1} \prod_{l \neq j} \frac{m_j - m_l + \frac{a}{2}(l-j-1) + 1}{m_j - m_l + \frac{a}{2}(l-j)} \tag{4.4}$$

if $\underline{\mathbf{m}} \neq 0 = (0, \dots, 0)$, and $\tau(0) = 0$.

Proof. That $\widetilde{M}\widetilde{M}^*$ is a diagonal operator follows immediately from Schur's lemma and (2.9). To find the eigenvalue corresponding to the signature $\mathbf{m}$ we note that

$$\tau(\underline{\mathbf{m}}) f(w) = (\widetilde{M}\widetilde{M}^* f, K_w(z)) = (f, \widetilde{M}\widetilde{M}^* K_w(z)) \tag{4.5}$$

and

$$\widetilde{M}\widetilde{M}^* K_w(z) = \sum_{j=1}^{d} M_j M_j^* K_w(z) = \sum_{j=1}^{d} z_j \bar{w}_j K(z,w) = \langle z, w \rangle K(z,w)$$

By the Faraut-Koranyi expansion (2.4) we have

$$\langle z, w \rangle K(z,w) = \sum_{\underline{\mathbf{m}}} (\nu)_{\underline{\mathbf{m}}} \langle z, w \rangle K_{\underline{\mathbf{m}}}(z,w). \tag{4.6}$$

Clearly

$$\langle z, w \rangle K_{\underline{\mathbf{m}}}(z,w) = \sum_{j=1}^{r} \beta_{\underline{\mathbf{m}}}(j) K_{\underline{\mathbf{m}}+\gamma_j}(z,w)$$

for some constants $\beta_{\underline{\mathbf{m}}}(\gamma_j)$. To find these constants we let $w = e$. Using Lemma 4.3, the above formula reads

$$r\phi_{\gamma_1} \frac{1}{\|\phi_{\underline{\mathbf{m}}}\|^2_{\mathcal{F}(V_2)}} \phi_{\underline{\mathbf{m}}}(z,w) = \sum_{j=1}^{r} \beta_{\underline{\mathbf{m}}}(j) \frac{1}{\|\phi_{\underline{\mathbf{m}}+\gamma_j}\|^2_{\mathcal{F}(V_2)}} \phi_{\underline{\mathbf{m}}+\gamma_j}(z,w).$$

Comparing this with Lemma 4.2 we find

$$\beta_{\underline{\mathbf{m}}}(j) = \frac{\|\phi_{\underline{\mathbf{m}}+\gamma_j}\|^2_{\mathcal{F}(V_2)}}{\|\phi_{\underline{\mathbf{m}}}\|^2_{\mathcal{F}(V_2)}} c_{\underline{\mathbf{m}}}(j). \tag{4.7}$$

Continuing with (4.6), we have

$$\begin{aligned}\langle z, w\rangle K(z,w) &= \sum_{\underline{\mathbf{m}}} (\nu)_{\underline{\mathbf{m}}} \sum_{j=1}^{r} \beta_{\underline{\mathbf{m}}}(j) K_{\underline{\mathbf{m}}+\gamma_j}(z,w) \\ &= \sum_{\underline{\mathbf{m}}} \left(\sum_{j=1}^{r} \beta_{\underline{\mathbf{m}}-\gamma_j}(j) (\nu)_{\underline{\mathbf{m}}-\gamma_j} \right) K_{\underline{\mathbf{m}}}(z,w).\end{aligned}$$

Substituting this into (4.5) we find that, since $f \in \mathcal{P}^{\underline{\mathbf{m}}}(V)$ and $(\nu)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z,w)$ is the reproducing kernel of the space $\mathcal{P}^{\underline{\mathbf{m}}}(V)$,

$$\begin{aligned}\tau(\underline{\mathbf{m}}) f(w) &= (f, (\cdot, w) K_w(\cdot)) \\ &= \left(\frac{1}{(\nu)_{\underline{\mathbf{m}}}} \sum_{j=1}^{r} (\nu)_{\underline{\mathbf{m}}-\gamma_j} \beta_{\underline{\mathbf{m}}-\gamma_j}(j) \right) (f, (\nu)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(\cdot, w) \\ &= \left(\frac{1}{(\nu)_{\underline{\mathbf{m}}}} \sum_{j=1}^{r} (\nu)_{\underline{\mathbf{m}}-\gamma_j} \beta_{\underline{\mathbf{m}}-\gamma_j}(j) \right) f(w).\end{aligned}$$

Namely

$$\tau(\underline{\mathbf{m}}) = \sum_{j=1}^{r} \frac{(\nu)_{\underline{\mathbf{m}}-\gamma_j}}{(\nu)_{\underline{\mathbf{m}}}} \beta_{\underline{\mathbf{m}}-\gamma_j}(j). \tag{4.8}$$

The coefficient $\beta_{\underline{\mathbf{m}}-\gamma_j}(\underline{\mathbf{m}})$ can be found via (4.7) and Lemma 4.3,

$$\begin{aligned}&\beta_{\underline{\mathbf{m}}-\gamma_j}(j) = \frac{(\frac{d_2}{r})_{\underline{\mathbf{m}}}}{(\frac{d_2}{r})_{\underline{\mathbf{m}}-\gamma_j}} \frac{d_{\underline{\mathbf{m}}-\gamma_j}(V_2)}{d_{\underline{\mathbf{m}}}(V_2)} c_{\underline{\mathbf{m}}-\gamma_j}(j) \\ &= (\frac{a}{2}(r-j) + m_j) c_{\underline{\mathbf{m}}-\gamma_j}(j) \frac{d_{\underline{\mathbf{m}}-\gamma_j}(V_2)}{d_{\underline{\mathbf{m}}}(V_2)} \\ &= (\frac{a}{2}(r-j) + m_j) c_{\underline{\mathbf{m}}-\gamma_j}(j) \prod_{k \neq j} \frac{m_j - 1 - m_k + \frac{a}{2}(k-j)}{m_j - m_k + \frac{a}{2}(k-j)} \frac{m_j - 1 - m_k + \frac{a}{2}(k-j-1)}{m_j - 1 - m_k + \frac{a}{2}(k-j+1)} \\ &= (\frac{a}{2}(r-j) + m_j) \prod_{l \neq j} \frac{m_j - m_l + \frac{a}{2}(l-j-1) + 1}{m_j - m_l + \frac{a}{2}(l-j)},\end{aligned}$$

which then implies our result in view of (4.8). □

Proof of Theorem 4.1. We proof that the operator $\widetilde{M}\widetilde{M}^*$ is bounded. This implies that $\widetilde{M}$ and M_j are bounded. Since $\|\widetilde{M}\widetilde{M}^*\| = \sup_{\mathbf{m}} \tau(\mathbf{m})$ it suffices to estimate the numbers $\tau(\underline{\mathbf{m}})$. We claim first that

$$\prod_{l\neq j}\frac{m_j-m_l+\frac{a}{2}(l-j-1)+1}{m_j-m_l+\frac{a}{2}(l-j)}\leq\begin{cases}2^{r-j} & a=1\\ 1 & a=2\\ 2^{j-1} & a\geq 3\end{cases}. \tag{4.9}$$

Indeed the left hand side can be written as

$$\prod_{1\leq l<j}\frac{m_l-m_j+\frac{a}{2}(j-l+1)-1}{m_l-m_j+\frac{a}{2}(j-l)}\prod_{j<l\leq r}\frac{m_j-m_l+\frac{a}{2}(l-j-1)+1}{m_j-m_l+\frac{a}{2}(j-l)}$$

where all the factors are nonnegative. If $a=1$ each term in the first product is dominated by 1, and each term in the second product by 2, thus the whole product by 2^{r-j}; if $a\geq 3$ the same is true with the majorant being 2 and respectively 1; when $a=2$ all the terms are 1. This proved our claim. Finally, observe that when $\nu>\frac{a}{2}(r-1)$

$$\frac{\frac{a}{2}(r-1)-\frac{a}{2}(j-1)+m_j}{\nu-\frac{a}{2}(j-1)+m_j-1}\leq 2. \tag{4.10}$$

Summing the two inequalities (4.9) and (4.10) we obtain

$$\tau(\underline{\mathbf{m}})\leq\begin{cases}2^{r+1} & a=1\\ 2r & a=2\\ 2(2^r-1) & a\geq 3\end{cases}.$$

□

5. Ryll-Wojtaszczyk polynomials and function module properties of the multiplication operators

In this section we construct polynomials in certain space $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ whose Hardy space norm are bounded from below and which are uniformly bounded; we refer the reader again to [20], [9] and references therein for the significance of those polynomials in the study of function theory on the unit ball of $\mathbb{C}^d$.

Proposition 5.1. *Let* $\underline{\mathbf{m}}=(2l,2l,\dots,2l)$. *Consider the identity map* J *from* $P^{\underline{\mathbf{m}}}\subset C(S)$ *into* $P^{\underline{\mathbf{m}}}(V)\subset L^2(S)$. *Then*

$$\|J\|\geq\frac{1}{2}\sqrt{\pi}e^{-rb}$$

We use now the idea of Rudin (see [20], Appendix II). Consider the projection $P_{\mathbf{m}}$ from $C(S)$ onto $\mathcal{P}^{\underline{\mathbf{m}}}(V)$

$$P_{\underline{\mathbf{m}}}f(z)=(\frac{d}{r})_{\underline{\mathbf{m}}}\int_S K_{\underline{\mathbf{m}}}(z,w)f(w)d\sigma(w)$$

Lemma 5.2. *Let* $\underline{\mathbf{m}} = (2l, \dots, 2l)$. *The projection* $P_{\underline{\mathbf{m}}}$ *from* $C(S)$ *onto* $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ *(with the norm in* $C(S)$*) satisfies*

$$\|P_{\underline{\mathbf{m}}}\| \le e^{rb},$$

where b *is the Peirce multiplicity (2.1).*

Proof. By the K-invariance of the kernel $K_{\underline{\mathbf{m}}}$, i.e. $K_{\underline{\mathbf{m}}}(kz, kw) = K_{\underline{\mathbf{m}}}(z, w)$, it is clear that the norm of $P_{\underline{\mathbf{m}}}$ is given by

$$\|P_{\underline{\mathbf{m}}}\| = (\frac{d}{r})_{\underline{\mathbf{m}}} \int_S |K_{\underline{\mathbf{m}}}(z, w)| d\sigma(w)$$

and that this integral is independent of $z \in S$. Take $z = e$, the maximal tripotent. By Lemma 3.1 and Theorem 3.4 in [12] we have

$$(\frac{d}{r})_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(w, e) = (\frac{d}{r})_{\underline{\mathbf{m}}} \frac{d_{\underline{\mathbf{m}}}}{(\frac{d}{r})_{\underline{\mathbf{m}}}} \phi_{\underline{\mathbf{m}}}(w) = d_{\underline{\mathbf{m}}} \phi_{\underline{\mathbf{m}}}(w).$$

Thus, recalling (2.3), we obtain

$$\|P_{\underline{\mathbf{m}}}\| = d_{\underline{\mathbf{m}}} \int_S |\phi_{\underline{\mathbf{m}}}(w)| d\sigma(w) \le d_{\underline{\mathbf{m}}} \int_S |\Delta_{\underline{\mathbf{m}}}(w)| d\sigma(w). \tag{5.1}$$

Let now $\underline{\mathbf{m}} = (2l, \dots, 2l) = 2\underline{\mathbf{l}}$ with $\underline{\mathbf{l}} = (l, \dots, l)$, then

$$\int_S |\Delta_{\underline{\mathbf{m}}}(w)| d\sigma(w) = \int_S |\Delta_l(w)^2| d\sigma(w) = \int_S |\Delta_l(w)|^2 d\sigma(w) = \|\Delta_l(w)\|_S^2$$

The computation of this integral (for any $\underline{\mathbf{l}}$) can be performed by using the results of Faraut-Koranyi [12] and Upmeier [22]. Indeed Corollary 3.5 in [12] gives

$$\frac{\|\Delta_{\underline{\mathbf{l}}}(w)\|_S^2}{\|\Delta_{\underline{\mathbf{l}}}(w)\|_{\mathcal{F}}} = \frac{1}{(\frac{d}{r})_{\underline{\mathbf{l}}}},$$

where $\|\cdot\|_S$ stands for the norm in $L^2(S)$ and $\|\cdot\|_{\mathcal{F}}$ the norm in the the Fock space; the norm $\|\Delta_{\underline{\mathbf{l}}}(w)\|_{\mathcal{F}}$ of $\Delta_{\underline{\mathbf{l}}}(w)$ is the same as that of the its restriction on the tube domain $D \cap V_1$ of D (see Lemma 2.5 and Lemma 2.6 in [22], or [13]), from which we get

$$\|\Delta_{\underline{\mathbf{l}}}(w)\|_{\mathcal{F}}^2 = (\frac{d_1}{r})_{\underline{\mathbf{l}}}.$$

Thus

$$\|\Delta_{\underline{\mathbf{l}}}(w)\|_S^2 = \frac{(\frac{d_1}{r})_{\underline{\mathbf{l}}}}{(\frac{d}{r})_{\underline{\mathbf{l}}}}.$$

Furthermore, the dimension $d_{\underline{\mathbf{m}}}$ for $\underline{\mathbf{m}} = 2\underline{\mathbf{l}}$ is, by Lemma 2.7 [22],

$$d_{\underline{\mathbf{m}}} = \frac{\prod_{j=1}^r (1 + b + \frac{a}{2}(j-1))_{2l}}{\prod_{j=1}^r (1 + \frac{a}{2}(j-1))_{2l}}$$

Therefore the inequality (5.1) becomes

$$\|P_{\underline{\mathbf{m}}}\| \le \frac{\prod_{j=1}^r (1 + b + \frac{a}{2}(j-1)_{2l}}{\prod_{j=1}^r (1 + \frac{a}{2}(j-1))_{2l}} \frac{(\frac{d_1}{r})_{\underline{\mathbf{l}}}}{(\frac{d}{r})_{\underline{\mathbf{l}}}} = \prod_{j=1}^r \frac{(\frac{a}{2}(j-1) + b + l)_l}{(\frac{a}{2}(j-1) + l)_l)}$$

which is further dominated by e^{rb}. ☐

Lemma 5.3. *Let $\{p_j, j = 1, \dots, d_{\underline{\mathbf{m}}}\}$ be an orthonormal basis of $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ in the space $L^2(S)$. Then we have*

$$\sum_{j=1}^{d_{\underline{\mathbf{m}}}} |p_j(z)|^2 = d_{\underline{\mathbf{m}}}, \quad z \in S$$

Indeed, $\{p_j/(\frac{d}{r})_{\mathbf{m}}^{\frac{1}{2}}, j = 1, \dots, d_{\underline{\mathbf{m}}}\}$ is an orthonormal basis of $\mathcal{P}^{\mathbf{m}}$ in the Fischer inner product. Hence, for every $z \in S$

$$\sum_{j=1}^{d_{\underline{\mathbf{m}}}} |p_j(z)|^2 = d_{\underline{\mathbf{m}}} = (\frac{d}{r})_{\mathbf{m}} K_{\mathbf{m}}(z, z) = (\frac{d}{r})_{\mathbf{m}} \frac{\phi_{\mathbf{m}}(e)}{\|\phi_{\mathbf{m}}\|_{\mathcal{F}}^2} = d_{\mathbf{m}}.$$

We remark that in the case where $D = B^d$ Lemma 5.3 is established in Lemma 1 of [20], Appendix II.

The next lemma is Lemma 2 of [20], Appendix II.

Lemma 5.4. *Let B^n be the (Hilbert) open unit ball in $\mathbb{C}^n$, and consider on its boundary ∂B^n the normalized $n-1$ dimensional volume measure $d\sigma(\xi)$. Let $N = \binom{n+k-1}{n-1}$ be the dimension of the space $\mathcal{P}^k(B^n)$ of homogeneous polynomials of degree k on B^n. Then the operator*

$$Tf(z) = N \int_{\partial B^n} f(\xi) \langle z, \xi \rangle^k \, d\sigma(\xi)$$

projects $C(\partial B^n)$ onto $\mathcal{P}^k(B^d)$ (with the norm induced from $C(\partial B^n)$), and

$$\|T\| = \frac{\Gamma(n+k)\Gamma(1+\frac{k}{2})}{\Gamma(1+k)\Gamma(n+\frac{k}{2})}$$

If T' is any other projection of $C(\partial B^n)$ onto $\mathcal{P}^k(B^n)$, then $\|T'\| \geq \|T\|$.

Note that when $k = 1$, the above norm is

$$\|T\| = \frac{\Gamma(d+1)\Gamma(1+\frac{1}{2})}{\Gamma(2)\Gamma(n+\frac{1}{2})} > \frac{1}{2}\sqrt{\pi n} \tag{5.2}$$

which is what we are going to use.

With the above lemmas the proof of Proposition 5.1 can now be carried over by almost the same method in [20]; we give a sketch here and refer the reader to the above reference for details.

Proof of Proposition 5.1. We apply Lemma 5.4 with $n = d_{\underline{\mathbf{m}}}$. Let $\{p_j(z), j = 1, \dots n\}$ be as in Lemma 5.3 and let $\Phi(z) = (p_1(z), \dots, p_n(z))$. Thus Φ maps S into $\sqrt{n}\, \partial B^n$. Define

$$Q : C(\partial B^n) \mapsto C(S), \quad F(z) = \sqrt{n}\, F(\frac{\Phi(z)}{\sqrt{n}}), \quad z \in S.$$

Thus $\|Q\| = \sqrt{n}$. The restriction of Q to $\mathcal{P}^1(B^n) \subset C(\partial B^n)$ is an isometric mapping onto $\mathcal{P}^{\underline{\mathbf{m}}}(D) \subset L^2(S)$, and thus Q^{-1} defines an isometric operator from $\mathcal{P}^{\underline{\mathbf{m}}}(D) \subset$

$L^2(S)$ onto $\mathcal{P}^1_\infty(B^d) \subset C(S)$, and considered as an operator between the spaces $\|Q^{-1}\| = 1$. Introduce now the operator

$$Y = Q^{-1}TP_{\underline{\mathbf{m}}}Q : C(\partial B^n) \stackrel{Q}{\to} C(S) \stackrel{P_{\underline{\mathbf{m}}}}{\to} \mathcal{P}^{\underline{\mathbf{m}}}(D) \stackrel{J}{\to} \mathcal{P}^{\underline{\mathbf{m}}}(D) \stackrel{Q^{-1}}{\to} \mathcal{P}^1(B^n)) \subset C(\partial B^n).$$

Clearly Y is a projection and Lemma 5.4 for $k = 1$ implies that

$$\frac{1}{2}\sqrt{\pi n} < \|Y\|$$

On the other hand

$$\|Y\| \leq \|Q^{-1}\| \cdot \|J\| \cdot \|P_{\underline{\mathbf{m}}}\| \cdot \|Q\| \leq e^{rb}\sqrt{n}\|J\|$$

Combining the two inequality proves our result. □

As an application we get

Corollary 5.5. *For every $\underline{\mathbf{m}} = 2\underline{\mathbf{l}} = 2(l, \ldots, l)$, $l \geq 0$, there exists polynomials $W_l \in \mathcal{P}^{\underline{\mathbf{m}}}(V)$ such that*

$$\|W_l\|_{L^\infty(S)} \leq 1, \quad \|W_l\|_{L^2(S)} = \|W_l\|_{\frac{d}{r}} \geq \frac{1}{2}\sqrt{\pi}e^{-rb}.$$

We will call the $\{W_l\}_{l=0}^\infty$ RW-polynomials; see again [20].

Proposition 5.6. *The closed domain $\bar{D}$ as the Taylor spectrum of $M = (M_1, \ldots, M_n)$ is a k-spectral set of M on $\mathcal{H}_\nu(D)$ only if $\nu > \frac{d}{r}$.*

Proof. Suppose, to get a contradiction, that $\frac{a}{2}(j-1) < \nu < \frac{d}{r}$ and that D is a k-spectral set of M. For each $\underline{\mathbf{m}} = 2\underline{\mathbf{l}} = 2(l, \ldots, l)$ let W_l be the corresponding RW-polynomial. Consider the polynomials

$$f_N(z) = \sum_{l=1}^{N} c_l W_l(z)$$

where $\{c_l\}$ is a sequence such that

(5.3)
$$\sum_{l=1}^\infty |c_l| = 1, \quad \sum_{l=1}^\infty |c_l|^2 l^{(\frac{d}{r}-\nu)r} = \infty.$$

Thus

$$|f_N(z)| \leq \sum_{l=1}^k |c_l| \leq 1, \quad z \in D$$

by the construction of the RW-polynomials W_l. By our assumption

$$\|f_N(S)1\|_\nu \leq k \sup_{z\in D} |f_N(z)| \|1\|_\nu = k \sup_{z\in D} |f_N(z)| \leq k$$

Consider the left hand side. The polynomials W_l are pairwise orthogonal, since they are in different polynomials spaces, thus

$$\|f_N(S)1\|_\nu^2 = \sum_{l=1}^\infty |c_l|^2 \frac{(\frac{d}{r})_{2\underline{\mathbf{l}}}}{(\nu)_{2\underline{\mathbf{l}}}}$$

By Stirling's formula we have

$$\frac{(\frac{d}{r})_{2\underline{l}}}{(\nu)_{2\underline{l}}} \approx l^{(\frac{d}{r}-\nu)r}.$$

Thus the left hand side is unbounded, and we obtain a contradiction. □

6. Subnormality properties of d-tuple $(M_1, \cdots, M_d)$

We study now the joint subnormality property of the tuple M. We set

$$\nu_j = p - 1 - \frac{a}{2}(j-1) = \frac{d}{r} + \frac{a}{2}(r-j).$$

Let $\partial_j D$ be the j-th boundary orbit of D. The topological boundary (in the Euclidean space V) ∂D is a union of G-orbits $\partial_j D$, i.e.

$$\partial D = \cup_{j=1}^{r} \partial_j D, \quad \text{where } \partial_j D = G \cdot u_j, \quad u_j = \sum_{i=1}^{j} e_i.$$

Theorem 6.1. *The tuple $M = (M_1, \ldots, M_n)$ is subnormal precisely when ν is in the set*

$$\{\nu_1, \ldots, \nu_r\} \cup (p-1, \infty)$$

The result can be proved by using the results in [19] and by almost the same techniques as developed in [7], so we will be very brief. The next result is proved in [19], Theorem 5.2.6 in the context of unbounded realization of $D = G/K$; see also [5].

Theorem 6.2. *For each $j = 1, \ldots, r$ there exists a probability measure $d\mu_j$ on $\partial_j D$ so that $\mathcal{H}_\nu(D)$ for $\nu = \nu_j$ can be identified as a subspace of $L^2(\partial_j D, d\mu_j)$; the measure $d\mu_j$ is quasi-invariant, that is*

$$d\mu_j(gz) = |J_g(z)|^{2\frac{\nu_j}{p}} d\mu_j(z), \quad z \in \partial_j D.$$

We prove now Theorem 6.1.

Proof. If $\nu > p-1$ then clearly the multiplication by coordinate functions on $L^(D, d\mu_\nu)$ is a normal extension of M, thus M is subnormal. If ν is one of the ν_j then the result follows by the Theorem 6.2. Suppose now that M is subnormal on $\mathcal{H}_\nu(D)$. Lemma 5.1 in [7] implies that there exists a probability measure $d\sigma$ on an orbit of G in $\bar{D}$ such that

$$d\sigma(gz) = |J_g(z)|^{2\frac{\nu}{p}} d\mu(z)$$

for an appropriate semi-invariant measure μ on this G-orbit. Precisely, If the orbit is D, then σ must be the measure $d\mu_\nu = c_\nu h(z,z)^{\nu-p} dm(z)$, which is a finite measure if and only if $\nu > p-1$. If the orbit if one of the $\partial_j D$, $j = 1, \ldots, r$, then σ is proportional to the measure μ_j on $\partial_j D$; see [8], Chapter 7, §2.6, Corollary 1, or [23], Theorem 5.9. Let $F(z)$ be as in [7] the proportionality function. If $\nu \neq \nu_j$, then (see [7]) for any $u \in \partial_j D$

$$|J_g(u)| = 1, \quad g \in G_u$$

where G_u is the stabilizer of u in G. Take $u = u_j = \sum_{i=1}^{j} e_i$. Let $a \in V_1(e) \cap A$, where A is the point set of the quadratic map $Q(e) : \bar{V} \mapsto V$; see [17]. Then $g = \exp(\xi_a) \in G_u$, see [17], Theorem 9.1.5. However

$$J_g(u) = h(\tanh a, \tanh a)^{\frac{p}{2}} h(\tanh a, u)^{-p},$$

see Proposition 9.8, loc. cit.. Clearly $J_g(u)$ is not unimodular for all $a \in V_1(e) \cap A$, and is actually unbounded. This finishes the proof. □

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